Sigui $M = \mathbb{R}^3$ $\qquad F: M \to M, \; F(x,y,z) = (z,x,y)$

$$X = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \qquad Y = -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x} \qquad Z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$g = dx\otimes dx + dy\otimes dy + dz\otimes dz$$

Calculeu $g\otimes X,\; g\otimes Y,\; g\otimes Z$

Calculeu $X^\flat = c^1_1(g\otimes X) \equiv i_X g,\quad Y^\flat = c^1_1(g\otimes Y),\quad Z^\flat = c^1_1(g\otimes Z)$

Calculeu $X^\flat \otimes X, \ldots$

Calculeu $g(X,X) = \langle X^\flat, X\rangle \qquad g(X,Y), \ldots$

Calculeu $F^*(g) = g$

Calculeu $F^*(X) = Z \quad F^*(Y) = X \quad F^*(Z) = Y \qquad F_*(X) \quad F_*(Y) \quad F_*(Z)$

Calculeu $F^*(X^\flat), \ldots$

Comproveu $F^*(g(X,Y)) = F^*(g)\,(F^*(X), F^*(Y))$

Comproveu $F^*(c^1_1(g\otimes X)) = c^1_1(F^*(g)\otimes F^*(X))$

Calculeu $dX^\flat,\; dY^\flat,\; dZ^\flat$

Comproveu $F^*(dX^\flat) = d\,F^*(X^\flat) \ldots$

Calculeu $[X,Y] = -Z, \ldots [Y,Z] = -X, \quad [Z,X] = -Y$

Comproveu $F_*[X,Y] = [F_*(X), F_*(Y)], \ldots$

Calculeu $X^\flat \wedge Y^\flat$ $\qquad$ Comproveu $d(X^\flat\wedge Y^\flat) = dX^\flat\wedge Y^\flat - X^\flat\wedge dY^\flat$

$d(X^\flat\wedge Y^\flat)$

Comproveu $F^*(d(X^\flat\wedge Y^\flat)) = d(F^*(X^\flat\wedge Y^\flat))$

$$g \otimes X = (dx \otimes dx + dy \otimes dy + dz \otimes dz) \otimes \left(-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)$$

$$= -z\, dx \otimes dx \otimes \frac{\partial}{\partial y} + y\, dx \otimes dx \otimes \frac{\partial}{\partial z}$$

$$-z\, \underline{dy \otimes dy \otimes \frac{\partial}{\partial y}} + y\, dy \otimes dy \otimes \frac{\partial}{\partial z}$$

$$-z\, dz \otimes dz \otimes \frac{\partial}{\partial y} + y\, \underline{dz \otimes dz \otimes \frac{\partial}{\partial z}}$$

$$X^\flat = c^1_1(g \otimes X) = -z\, dy + y\, dz \qquad Y^\flat = -x\, dz + z\, dx \quad Z^\flat = -y\, dx + x\, dy$$

$$X^\flat \otimes X = (-z\, dy + y\, dz) \otimes \left(-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)$$

$$= z^2\, dy \otimes \frac{\partial}{\partial y} + y^2\, dz \otimes \frac{\partial}{\partial z} - yz\left(dy \otimes \frac{\partial}{\partial z} + dz \otimes \frac{\partial}{\partial y}\right)$$

$$g(X,X) = c^{12}_{12}(g \otimes X \otimes X) = \langle X^\flat, X \rangle = c^1_1(X^\flat \otimes X)$$

$$\langle X^\flat, X \rangle = \left\langle -z\, dy + y\, dz, -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right\rangle = y^2 + z^2$$

$$g(X,Y) = (dx \otimes dx + dy \otimes dy + dz \otimes dz)\left(-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right) = -xy$$

Per als càlculs amb F, denotem per (x,y,z) les coordenades naturals del $\mathbb{R}^3$ de partida, i per (x',y',z') les mateixes coordenades a l'espai d'arribada (a fi de facilitar la comprensió i evitar confusions; no és imprescindible).

$$F(x,y,z) = (z,x,y)$$

de manera que $F^*(x') = z \quad F^*(y') = x \quad F^*(z') = y$

Així $F^*(dx') = dF^*(x') = dz \qquad F^*(dy') = dx \qquad F^*(dz') = dy$

$$F^*(dx' \otimes dx') = F^*(dx') \otimes F^*(dx') = dz \otimes dz \quad \ldots$$

$$F^*(g) = F^*(dx' \otimes dx' + dy' \otimes dy' + dz' \otimes dz') =$$
$$= dz \otimes dz + dx \otimes dx + dy \otimes dy = g$$

$$F^*(X^\flat) = F^*(-z'dy' + y'dz')$$
$$= -F^*(z')\,F^*(dy') + F^*(y')\,F^*(dz')$$
$$= -y\,dx + x\,dy = Z^\flat$$

Podem expressar el càlcul de $F_*(X)$ de dues maneres.

Usant les coordenades naturals de TM (i identificant $T\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$) tenim

$$TF(x,y,z;\dot{x},\dot{y},\dot{z}) = (z,x,y;\dot{z},\dot{x},\dot{y})$$
$$X(x,y,z) = (x,y,z;0,-z,y)$$

i així podem calcular $F_*(X) = TF \circ X \circ F^{-1}$:

$$F_*(X)(x',y',z') = TF \circ X \circ F^{-1}(x',y',z') = TF \circ X(y',z',x')$$
$$= TF(y',z',x';0,-x',z') = (x',y',z';z',0,-x') = Y(x',y',z')$$

o sigui $F_*(X) = Y$

Anàlogament $F_*(Y) = Z$, $F_*(Z) = X$.

El pull-back és l'invers, i directament veiem $F^*(X) = Z$, $F^*(Y) = X$, $F^*(Z) = Y$.

Alternativament, calculem la jacobiana de F, que és la matriu de TF en les bases de vectors tangents coordenats:

$$JF(x,y,z) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Si $p \in M$ i $q = F(p)$, tenim doncs:

$$T_pF \cdot \left.\frac{\partial}{\partial x}\right|_p = \left.\frac{\partial}{\partial y'}\right|_q \qquad T_pF \cdot \left.\frac{\partial}{\partial y}\right|_p = \left.\frac{\partial}{\partial z'}\right|_q \qquad T_pF \cdot \left.\frac{\partial}{\partial z}\right|_p = \left.\frac{\partial}{\partial x'}\right|_q$$

Calculem $F_*(X)$:

$$\begin{aligned} F_*(X)(x',y',z') &= T_pF \cdot X(F^{-1}(x',y',z')) \\ &= T_pF \cdot X(y',z',x') \\ &= T_pF \cdot \left(-x' \left.\frac{\partial}{\partial y}\right|_p + z' \left.\frac{\partial}{\partial z}\right|_p\right) \\ &= -x' \left.\frac{\partial}{\partial z'}\right|_q + z' \left.\frac{\partial}{\partial x'}\right|_q = Y(x',y',z') \end{aligned}$$

Comprovació de $F^*(g(X,Y)) = F^*(g)(F^*(X), F^*(Y))$

$$\overset{\shortparallel}{F^*(-x'y') = -F^*(x')F^*(y') = -zx} \qquad \overset{\shortparallel}{g(Z,X) = -zx}$$

Comprovació de $F^*(c^1_1(g \otimes X)) = c^1_1(F^*(g) \otimes F^*(X))$

$$\overset{\shortparallel}{F^*(X^\flat) = Z^\flat} \qquad \overset{\shortparallel}{c^1_1(g \otimes Z) = Z^\flat}$$

$$[X,Y] = \left[-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z},\ -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right] = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = -Z$$

$$F_*[X,Y] \overset{?}{=\!=} [F_*(X), F_*(Y)]$$

$$\overset{\shortparallel}{F_*(-Z) = -X} \qquad \overset{\shortparallel}{[Y,Z] = \ldots = -X}$$

$$dX^\flat = d(-z\,dy + y\,dz)$$
$$= -dz\wedge dy + dy\wedge dz = 2\,dy\wedge dz$$

$$dY^\flat = 2\,dz\wedge dx \qquad dZ^\flat = 2\,dx\wedge dy$$

$$F^*(dX^\flat) = 2\,F^*(dy)\wedge F^*(dz) = 2\,dx\wedge dy$$
$$dF^*(X^\flat) = d\,Z^\flat = 2\,dx\wedge dy$$

$$X^\flat\wedge Y^\flat = (-z\,dy + y\,dz)\wedge(-x\,dz + z\,dx)$$
$$= xz\,dy\wedge dz - z^2\,dy\wedge dx - xy\,\cancel{dz\wedge dz} + yz\,dz\wedge dx$$

$$d(X^\flat\wedge Y^\flat) = z\,dx\wedge dy\wedge dz - 2z\,dz\wedge dy\wedge dx + z\,dy\wedge dz\wedge dx$$
$$= 4z\,dx\wedge dy\wedge dz$$

$$F^*(dX^\flat\wedge Y^\flat) = 4\,F^*(z)\,F^*(dx\wedge dy\wedge dz)$$
$$= 4y\,dz\wedge dx\wedge dy = 4y\,dx\wedge dy\wedge dz$$
$$dF^*(X^\flat\wedge Y^\flat) = d(zy\,dx\wedge dy - y^2\,dx\wedge dz + xy\,dy\wedge dz)$$

Comprovació de $d(X^\flat\wedge Y^\flat) = dX^\flat\wedge Y^\flat - X^\flat\wedge dY^\flat$

$$dX^\flat\wedge Y^\flat - X^\flat\wedge dY^\flat = (2\,dy\wedge dz)\wedge(-x\,dz + z\,dx) - (-z\,dy + y\,dz)\wedge(2\,dz\wedge dx)$$
$$= 2z\,dy\wedge dz\wedge dx + 2z\,dy\wedge dz\wedge dx$$
$$= 4z\,dx\wedge dy\wedge dz = d(X^\flat\wedge Y^\flat)$$

Sigui $M = \mathbb{R}^3$

$$X = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \qquad Y = -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x} \qquad Z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$g = dx \otimes dx + dy \otimes dy + dz \otimes dz$$

$$\omega = dx \wedge dy \wedge dz$$

Calcular el flux F_X de X

$$F_X^{t*}(Y)$$

$$F_X^{t*}(g)$$

$$F_X^{t*}(\omega)$$

Calcular $\mathcal{L}_X Y$

$$\mathcal{L}_X g$$

$$\mathcal{L}_X \omega$$

- Flux de $X = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$

$$\left\{\begin{array}{l}\dot{x} = 0\\ \dot{y} = -z\\ \dot{z} = y\end{array}\right. \longrightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1 & & \\ & \cos t & -\sin t\\ & \sin t & \cos t\end{pmatrix}\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}$$

Per tant $F_X : \mathbb{R}\times M \longrightarrow M$ és

$$F_X(t; x,y,z) = (x,\ y\cos t - z\sin t,\ y\sin t + z\cos t)$$

i $F_X^t : M \longrightarrow M$ són els isomorfismes lineals que tenen per jacobiana $\begin{pmatrix}1 & & \\ & \cos t & -\sin t\\ & \sin t & \cos t\end{pmatrix}$

- Càlcul de $F_X^{t*}(Y)$

$$F_X^{t*}(Y)(x,y,z) = \left(T_{(x,y,z)}F_X^t\right)^{-1}\cdot Y\left(F_X^t(x,y,z)\right)$$

$$\begin{pmatrix}1 & & \\ & \cos t & \sin t\\ & -\sin t & \cos t\end{pmatrix}\begin{pmatrix}y\sin t + z\cos t\\ 0\\ -x\end{pmatrix} = \begin{pmatrix}y\sin t + z\cos t\\ -x\sin t\\ -x\cos t\end{pmatrix}$$

$$F_X^{t*}(Y) = (y\sin t + z\cos t)\frac{\partial}{\partial x} - x\sin t\frac{\partial}{\partial y} - x\cos t\frac{\partial}{\partial z} = \cos t\, Y - \sin t\cdot Z$$

- Càlcul de $\mathcal{L}_X Y = \left.\frac{d}{dt}\right|_{t=0} F_X^{t*}(Y)$

$$\parallel$$

$$\left.\left(-\sin t\, Y - \cos t\, Z\right)\right|_{t=0} = -Z$$

que coincideix amb $[X,Y]$.

- Alternativament podem calcular primer

$$F_X^{t*}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x} \qquad F_X^{t*}\left(\frac{\partial}{\partial y}\right) = \cos t\frac{\partial}{\partial y} - \sin t\frac{\partial}{\partial z} \qquad F_X^{t*}\left(\frac{\partial}{\partial z}\right) = \sin t\frac{\partial}{\partial y} + \cos t\frac{\partial}{\partial z}$$

i doncs

$$F_X^{t*}(Y) = F_X^{t*}\left(-x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right) = -F_X^{t*}(x)\,F_X^{t*}\left(\frac{\partial}{\partial z}\right) + F_X^{t*}(z)\,F_X^{t*}\left(\frac{\partial}{\partial x}\right) =$$

$$= -x\left(\sin t\frac{\partial}{\partial y} + \cos t\frac{\partial}{\partial z}\right) + (y\sin t + z\cos t)\frac{\partial}{\partial x} =$$

$$= \cos t\left(-x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right) - \sin t\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \cos t\, Y - \sin t\, Z$$

$F_X^{t*}(x) = x$

$F_X^{t*}(y) = y\cos t - z\sin t$

$F_X^{t*}(z) = y\sin t + z\cos t$

$F_X^{t*}(dx) = d\,F_X^{t*}(x) = dx$

$F_X^{t*}(dy) = d\,F_X^{t*}(y) = \cos t\, dy - \sin t\, dz$

$F_X^{t*}(dz) = \sin t\, dy + \cos t\, dz$

$F_X^{t*}(dx\otimes dx) = F_X^{t*}(dx)\otimes F_X^{t*}(dx) = dx\otimes dx$

$$F_X^{t*}(dy\otimes dy) = (\cos t\, dy - \sin t\, dz)\otimes(\cos t\, dy - \sin t\, dz)$$
$$= \cos^2 t\, dy\otimes dy + \sin^2 t\, dz\otimes dz - \cos t \sin t\,(dy\otimes dz + dz\otimes dy)$$

$$F_X^{t*}(dz\otimes dz) = \sin^2 t\, dy\otimes dy + \cos^2 t\, dz\otimes dz + \cos t\sin t\,(dy\otimes dz + dz\otimes dy)$$

$$F_X^{t*}(g) = F_X^{t*}(dx\otimes dx + dy\otimes dy + dz\otimes dz) = dx\otimes dx + dy\otimes dy + dz\otimes dz = g$$

$$F_X^{t*}(\omega) = F_X^{t*}(dx\wedge dy\wedge dz) = F_X^{t*}(dx)\wedge F_X^{t*}(dy)\wedge F_X^{t*}(dz)$$
$$= dx\wedge(\cos t\, dy - \sin t\, dz)\wedge(\sin t\, dy + \cos t\, dz)$$
$$= \cos^2 t\, dx\wedge dy\wedge dz - \sin^2 t\, dx\wedge dz\wedge dy =$$
$$= dx\wedge dy\wedge dz = \omega$$

La invariancia de g y ω por F_X^t equivale a $\mathcal{L}_X g = 0$, $\mathcal{L}_X \omega = 0$.

$\mathcal{L}_X dx = d\,\mathcal{L}_X x = 0 \qquad \mathcal{L}_X dy = d\,\mathcal{L}_X y = d(-z) = -dz \qquad \mathcal{L}_X dz = d\,\mathcal{L}_X z = dy$

$\mathcal{L}_X(dx\otimes dx) = 0 \qquad \mathcal{L}_X(dy\otimes dy) = (\mathcal{L}_X dy)\otimes dy + dy\otimes(\mathcal{L}_X dy) = -dz\otimes dy - dy\otimes dz$

$\mathcal{L}_X(dz\otimes dz) = dy\otimes dz + dz\otimes dy$

$\mathcal{L}_X g = \mathcal{L}_X(dx\otimes dx + dy\otimes dy + dz\otimes dz) = 0$

$\mathcal{L}_X \omega = \mathcal{L}_X(dx\wedge dy\wedge dz) = 0\wedge dy\wedge dz + dx\wedge(-dz)\wedge dz + dx\wedge dy\wedge dy = 0$

Sigui $M = \mathbb{R}^3$

Considerem-hi les coordenades cartesianes (x,y,z) i les coordenades esfèriques (r,θ,ϕ) ordinàries. Estan relacionades per

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & \end{pmatrix}$$

Considerem

$$X = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \qquad Y = -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x} \qquad Z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$g = dx\otimes dx + dy\otimes dy + dz\otimes dz$$

$$\omega = dx\wedge dy\wedge dz$$

Expressem-los en coordenades esfèriques

Usarem
$$dx = \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi$$
$$dy = \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi$$
$$dz = \cos\theta\, dr - r\sin\theta\, d\theta$$

Ens caldrà la matriu inversa de la jacobiana: $\det\left(\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right) = r^2\sin\theta$ i

$$\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \frac{1}{r\sin\theta}\begin{pmatrix} r\sin^2\theta\cos\phi & r\sin^2\theta\sin\phi & r\cos\theta\sin\theta \\ \sin\theta\cos\theta\cos\phi & \sin\theta\cos\theta\sin\phi & -\sin^2\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

Així podem expressar

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$X = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} =$$

$$= -r\cos\theta\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\theta}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$+ r\sin\theta\sin\phi\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) =$$

$$= -\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}$$

$$Y = -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x} =$$

$$= -r\sin\theta\cos\phi\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) + r\cos\theta\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) =$$

$$= \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}$$

$$Z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$= -r\sin\theta\sin\phi\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$+ r\sin\theta\cos\phi\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\theta}{r\sin\theta}\frac{\partial}{\partial\phi}\right) =$$

$$= \frac{\partial}{\partial\phi}$$

Observació: Aquestes expressions no contenen $\frac{\partial}{\partial r}$. Això és lògic, perquè aquests camps vectorials són tangents a l'esfera $S: x^2+y^2+z^2=R^2$, i les coordenades esfèriques són adaptades a S.

$$
\begin{aligned}
dx \otimes dx = {} & \sin^2\theta \cos^2\phi \; dr \otimes dr + r^2 \cos^2\theta \cos^2\phi \; d\theta \otimes d\theta + r^2 \sin^2\theta \sin^2\phi \; d\phi \otimes d\phi \\
& + r \sin\theta \cos\theta \cos^2\phi \; (dr \otimes d\theta + d\theta \otimes dr) \\
& - r \sin^2\theta \cos\phi \sin\phi \; (dr \otimes d\phi + d\phi \otimes dr) \\
& - r^2 \cos\theta \sin\theta \cos\phi \sin\phi \; (d\theta \otimes d\phi + d\phi \otimes d\theta)
\end{aligned}
$$

$$
\begin{aligned}
dy \otimes dy = {} & \sin^2\theta \sin^2\phi \; dr \otimes dr + r^2 \cos^2\theta \sin^2\phi \; d\theta \otimes d\theta + r^2 \sin^2\theta \cos^2\phi \; d\phi \otimes d\phi \\
& + r \sin\theta \cos\theta \sin^2\phi \; (dr \otimes d\theta + d\theta \otimes dr) \\
& + r \sin^2\theta \cos\phi \sin\phi \; (dr \otimes d\phi + d\phi \otimes dr) \\
& + r^2 \cos\theta \sin\theta \cos\phi \sin\phi \; (d\theta \otimes d\phi + d\phi \otimes d\theta)
\end{aligned}
$$

$$
\begin{aligned}
dz \otimes dz = {} & \cos^2\theta \; dr \otimes dr + r^2 \sin^2\theta \; d\theta \otimes d\theta \\
& - r \cos\theta \sin\theta \; (dr \otimes d\theta + d\theta \otimes dr)
\end{aligned}
$$

$$
g = dx \otimes dx + dy \otimes dy + dz \otimes dz = dr \otimes dr + r^2 \, d\theta \otimes d\theta + r^2 \sin^2\theta \; d\phi \otimes d\phi
$$

$$
\omega = dx \wedge dy \wedge dz = \det\left(\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} \right) dr \wedge d\theta \wedge d\phi = r^2 \sin\theta \; dr \wedge d\theta \wedge d\phi
$$