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Variational principles in mechanics: geometric aspects

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Abstract

A geometric description of variational principles in mechanics is given, with special attention to constrained systems. The equations of motion for the case of nonholonomic constraints are obtained in both the vakonomic and nonholonomic formalisms from a variational principle. Both formalisms are compared and it is proved that they coincide when the constraints are integrable. As an example of vakonomic mechanics, a formulation of the theory of optimal control is described.

1 Introduction

To describe the motion of mechanical systems there is a variety of mathematical models which are based on different principles. Most of the physical models are obtained using an appropriate variational principle in a certain evolution space. But variational principles are not only important in physics, but also in many branches of engineering [11] or economics [8], where one is interested in optimizing a given functional, possibly subject to some restrictions. In fact, constraints are ubiquitous in many mechanical systems and much more different situations.

In this paper we are going to study Lagrangian systems, i.e., dynamical systems in which the equations of motion are obtained by finding the critical paths of a functional

$$\int_{t_1}^{t_2} L \,\mathrm{d}t \,,$$

where L is a function defined on the tangent bundle $\mathrm{T}Q$ of a given differentiable manifold Q, the configuration space. We will not consider arbitrary variations, but only variations satisfying certain conditions. These conditions arise from some given constraints on the dynamics of the system. We will analize the case when the constraints are defined by a certain submanifold C of the tangent bundle $\mathrm{T}Q$. Such constraints are usually called nonholonomic.

There are two different approaches when dealing with constraints. The first one is based on the idea of understanding the constraints as constraint forces. This point of view, that seems very natural in a physical context, gives rise to the classical d'Alembert-Lagrange principle. Mechanics of Lagrangian systems with nonholonomic constraints based on this principle is called nonholonomic mechanics [2], [7], [10]. But there is another different point of view, that seems more natural when one is interested in optimizing a given functional defined as above when there are constraints. For example, if we wish to change the state of a given system minimizing a cost functional (a typical problem in engineering or economics), it is not natural to understand the constraints as forces acting on the system. In this case one is interested in minimizing the functional considering only the variations allowed by the constraints. Mechanics of Lagrangian systems with nonholonomic constraints based on this idea is often called vakonomic mechanics (mechanics of **v**ariational **a**xiomatic **k**ind [2]). For example, optimal control theory is a typical example of vakonomic mechanics. It is interesting to notice that both mechanics do not coincide in general, but they agree when the constraints are integrable. Several references on these topics are [2], [3], [6] and [9].

The paper is organized as follows. In section 2 we give a generalized notion of a variational problem. Section 3 is devoted to study vakonomic mechanics from a geometric point of view, with special attention to admissible variations and the equations of motion. In section 4 we do the same for nonholonomic mechanics. Both mechanics are understood as generalized variational problems. In section 5 it is proved that, when the constraints are integrable, vakonomic and nonholonomic mechanics coincide. Finally, in section 6, as an example of vakonomic mechanics, a geometric formulation of optimal control theory is studied.

Basic knowledge of differential geometric structures is assumed. The interested reader may consult the bibliography for more specific topics as the vertical lift, the fibre derivative and the canonical involution [1], or the Euler-Lagrange operator [5].

2 Variational problems

2.1 Elements of a variational problem

First, we are going to define what we mean by a variational problem. A variational problem consists of the data $(Q, L, C, \mathcal{C}, \mathcal{W})$ where:

- Q is a n-dimensional differentiable manifold, the *configuration space*.
- L is the *lagrangian function* defined on the tangent bundle, $L\colon \mathrm{T}Q \to \mathcal{R}$.
- C is the *constraint submanifold*, and it is a submanifold of $\mathrm{T}Q$.
- $\mathcal{C}$ is the family of *admissible paths*. Given two points $q_1, q_2 \in Q$ and a compact interval $I = [t_1, t_2]$, we will say that a path $\gamma\colon I \to Q$ of class C^2 is admissible if $\gamma(t_1) = q_1$, $\gamma(t_2) = q_2$ and $\dot\gamma(t) \in C$, for all $t \in I$.
- $\mathcal{W}$ are the *admissible variation fields* (or infinitesimal variations). For a given admissible path γ, $\mathcal{W}_\gamma$ consists on a certain set of vector fields along γ.

Now, we are ready to define the variational problem associated to the data $(Q, L, C, \mathcal{C}, \mathcal{W})$. The *action* of L along a path γ is the functional $S\colon \mathcal{C} \longrightarrow \mathcal{R}$ given by the integral

$$S[\gamma] = \int_I L(\dot\gamma(t))\,\mathrm{d}t\,. \tag{2.1}$$

A variational problem consists in finding the critical admissible paths of the functional S, in a sense that will be precised later.

If $\gamma\colon I \to Q$, $\gamma(t_1) = q_1$, $\gamma(t_2) = q_2$ is an admissible path, a *variation* of γ is a C^2 function $\Gamma\colon (-\delta, \delta) \times I \longrightarrow Q$ such that:

1. $\Gamma_\varepsilon = \Gamma(\varepsilon, \cdot)$ is a one-parameter family of paths defined on I with fixed end-points, $\Gamma(\varepsilon, t_i) = q_i$, $\forall \varepsilon \in (-\delta, \delta)$, $i = 1, 2$, and

2. $\Gamma(0,t) = \gamma(t)$, $\forall t \in I$ (if there is no variation, $\varepsilon = 0$, we obtain the original path γ).

Being $\Gamma(\varepsilon, t)$ a function of two real variables, we will denote its derivatives with respect to ε and t as Γ' and $\dot{\Gamma}$, respectively. Both Γ' and $\dot{\Gamma}$ are vector fields along Γ.

Given a variation $\Gamma(\varepsilon, t)$ of an admissible path γ, the *variation field* of Γ is the vector field $\mathbf{w}$ along γ defined by

$$\mathbf{w}(t) = \Gamma'(0,t) \ .$$

Notice that $\mathbf{w}(t_1) = 0$ and $\mathbf{w}(t_2) = 0$ since the Γ_ε have fixed end-points.

Therefore, given a family of variations of γ, we can associate to them a family of variation vector fields along γ. We will say that a variation Γ of γ is *admissible* if its associated variation vector field along γ, $\mathbf{w}(t) = \Gamma'(0,t)$, is an admissible variation field of γ, i.e., $\mathbf{w} \in \mathcal{W}_\gamma$.

For any $\mathbf{w}$ vector field along γ and any function $\lambda\colon I \to \mathcal{R}$, we have the following property, whose proof is straighforward in local coordinates.

$$\mathsf{s} \circ (\lambda \mathbf{w})^{\cdot} = (\mathrm{D}\lambda)\, \mathrm{vl}(\dot{\gamma}, \mathbf{w}) + \lambda\, \mathsf{s} \circ \dot{\mathbf{w}} \ . \tag{2.2}$$

Notice that, if $\mathbf{w}$ is a vector field along γ, then $\dot{\mathbf{w}}$ is a vector field along $\mathbf{w}$ and $\mathsf{s} \circ \dot{\mathbf{w}}$ is a vector field along $\dot{\gamma}$. The function $\lambda\colon I \to \mathcal{R}$ denotes a function of time, and it is clear that $\lambda\mathbf{w}$ is another vector field along γ. D is the usual derivative with respect to the time. The map vl denotes the vertical lift $\mathrm{vl}\colon \mathrm{T}Q \times_Q \mathrm{T}Q \to \mathrm{T}(\mathrm{T}Q)$. Its local expression is $\mathrm{vl}(q,v,u) = (q,v;0,u)$. Finally, $\mathsf{s}\colon \mathrm{T}(\mathrm{T}Q) \to \mathrm{T}(\mathrm{T}Q)$ denotes the canonical involution, which is an isomorphism between the two vector bundle structures of $\mathrm{T}(\mathrm{T}Q)$. Its local expression is $\mathsf{s}(q,v;u,a) = (q,u;v,a)$.

2.2 Critical admissible paths

An admissible path γ is said to be *critical* if, for each admissible variation Γ_ε, the first variation of $S[\Gamma_\varepsilon]$ is zero, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} S[\Gamma_\varepsilon]\bigg|_{\varepsilon=0} = 0 \ .$$

Now, suppose that $\Gamma(\varepsilon, t)$ is a variation of a path γ. Let us consider $\dot{\Gamma}\colon (-\delta, \delta) \times I \to \mathrm{T}Q$. Derivation of $\dot{\Gamma}$ with respect to ε and t yields $(\dot{\Gamma})'$ and

$(\dot{\Gamma})^{\cdot}$, which are now vector fields along $\dot{\Gamma}$. Taking $\varepsilon = 0$ yields two vector fields along $\dot{\gamma}$, which are $\boldsymbol{s} \circ \dot{\mathbf{w}}$ and $\ddot{\gamma}$. Then, if $f\colon \mathrm{T}Q \to \mathcal{R}$ is a function, we have

$$\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} f(\dot{\Gamma}(\varepsilon, t)) = \langle \mathrm{d}f(\dot{\gamma}(t)), \boldsymbol{s}(\dot{\mathbf{w}}(t)) \rangle \ . \tag{2.3}$$

(Notice that $\dot{\mathbf{w}}$ is a vector field along $\mathbf{w}$, and $\boldsymbol{s} \circ \dot{\mathbf{w}}$ is a vector field along $\dot{\gamma}$, so the contraction makes sense.)

Using (2.3) in (2.1), we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon} S[\Gamma_\varepsilon]\right|_{\varepsilon=0} = \int_I \langle \mathrm{d}L(\dot{\gamma}(t)), \boldsymbol{s}(\dot{\mathbf{w}}(t)) \rangle \ \mathrm{d}t \ .$$

Therefore, given a variational problem $(Q, L, C, \mathcal{C}, \mathcal{W})$, an admissible path γ is critical if and only if

$$\int_I \langle \mathrm{d}L(\dot{\gamma}(t)), \boldsymbol{s}(\dot{\mathbf{w}}(t)) \rangle \ \mathrm{d}t \ = 0 \ , \tag{2.4}$$

for each admissible vector field $\mathbf{w} \in \mathcal{W}_\gamma$. Observe that this condition does not depend on the full variation $\Gamma(\varepsilon, t)$, but only on its variation field (see also [4]).

To obtain a more manageable condition of criticity, it is convenient to perform an integration by parts. First, we can define the *Euler-Lagrange operator* associated with a function $L : \mathrm{T}Q \longrightarrow \mathcal{R}$ as a mapping $\mathcal{E}_L\colon \mathrm{T}^2Q \to \mathrm{T}^*Q$ characterized by the relation

$$\langle \mathcal{E}_L \circ \ddot{\gamma}, \mathbf{w} \rangle = \langle \mathrm{d}L \circ \dot{\gamma}, \boldsymbol{s} \circ \dot{\mathbf{w}} \rangle - \mathrm{D}\langle \mathcal{D}L \circ \dot{\gamma}, \mathbf{w} \rangle \ , \tag{2.5}$$

for any path $\gamma\colon I \to Q$ and vector field $\mathbf{w}$ along γ.

Here, the map $\mathcal{D}L\colon \mathrm{T}Q \to \mathrm{T}^*Q$ is the fibre derivative of L. Recall that, given a vector bundle $E \to B$, if $f\colon E \to \mathcal{R}$ is a function, then the derivatives of the restrictions of f to the fibres define the fibre derivative of f, which is a map $\mathcal{D}f\colon E \to E^*$. Its local expression is $\mathcal{D}f(b; a) = (b; \partial f/\partial a)$.

It is easy to check (in coordinates) that the Euler-Lagrange operator is well-defined by this relation. The expression in local coordinates of $\mathcal{E}_L$ is the usual one,

$$\mathcal{E}_L = \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v}\right)\right) \mathrm{d}q \ ,$$

where $\mathrm{d}/\mathrm{d}t$ is the total time-derivative operator. The Euler-Lagrange operator can be extended in the same way to a time dependent Lagrangian.

Using (2.5), we obtain that

$$\int_I \langle \mathrm{d}L(\dot{\gamma}(t)), \boldsymbol{s}(\dot{\mathbf{w}}(t)) \rangle \ \mathrm{d}t = \int_I \langle \mathcal{E}_L(\ddot{\gamma}(t)), \mathbf{w}(t) \rangle \ \mathrm{d}t + \Big[\langle \mathcal{D}L(\dot{\gamma}(t)), \mathbf{w}(t) \rangle \Big]_{t_1}^{t_2} \ .$$

Since $\mathbf{w}$ is a variation field, the last term vanishes ($\mathbf{w}(t_1) = 0$, $\mathbf{w}(t_2) = 0$). Therefore, using (2.4), we conclude that an admissible path γ is critical if and only if

$$\int_I \langle \mathcal{E}_L(\ddot{\gamma}(t)), \mathbf{w}(t) \rangle \, \mathrm{d}t = 0 \,, \tag{2.6}$$

for each admissible variation field $\mathbf{w} \in \mathcal{W}_\gamma$.

For the case when there are no constraints, $C = \mathrm{T}Q$ and $\mathcal{W}_\gamma$ is the set of all the vector fields along γ, we obtain the well-known *Euler-Lagrange equation*,

$$\mathcal{E}_L \circ \ddot{\gamma} = 0 \,.$$

In this paper we will be interested in variational problems when $C \subseteq \mathrm{T}Q$, $C \neq \mathrm{T}Q$. Hence, given a set of admissible paths, it is necessary to select a set of admissible variation fields (or infinitesimal variations) along the admissible paths. We will consider two different approaches to this problem. The first one is vakononomic mechanics, which can be considered as a strictly variational approach. The second one is nonholonomic mechanics, which is variational in our generalized sense, but not in the classical one. Nonholonomic mechanics is the usual way to describe the dynamics of a mechanical system with constraints. In the next two sections we will describe the dynamical equations obtained in each case. It is interesting to remark that both approaches are equivalent when the constraints are integrable (holonomic constraints).

We finish this section with a useful property of the Euler-Lagrange operator that will be used in many calculations in the following. For any $f\colon \mathrm{T}Q \to \mathcal{R}$ and $\mu\colon I \to \mathcal{R}$ (a function of time),

$$\mathcal{E}_{\mu f} = \mu \, \mathcal{E}_f - (\mathrm{D}\mu)\mathcal{D}f \circ \tau^2_1 \,, \tag{2.7}$$

where $\tau^2_1\colon \mathrm{T}^2Q \to \mathrm{T}Q$ is the canonical projection.

3 Vakonomic mechanics

Roughly speaking, vakonomic mechanics is the result of variational calculus when the variations are restricted by some constraints on the positions and also the velocities.

Our initial setting is therefore a submanifold $C \subset \mathrm{T}Q$ of codimension $m < n$; let us denote by j the inclusion of C in $\mathrm{T}Q$. A *constraint* is any function ϕ vanishing on C. Locally C is defined by the vanishing of some constraints $\phi^i\colon \mathrm{T}Q \to \mathcal{R}$ ($i = 1, \ldots, m$) whose differentials $\mathrm{d}\phi^i$ are linearly independent at each point of C.

We will assume that the composition $\tau_Q \circ j: C \longrightarrow Q$ is a submersion. This is equivalent to say that the constraints ϕ^i can be chosen such that their fibre derivatives $\mathcal{D}\phi^i$ are linearly independent at every point of C. In local coordinates, this means that $\partial\phi^i/\partial v^k$ has maximal rank. That is, the constraints restrict the velocities, not the positions.

With the assumptions above, the image $(\tau_Q \circ j)(C) \subset Q$ is open, so we may assume that the projection $C \to Q$ is a *surjective* submersion. Then there exists the vertical subbundle $\mathrm{V}(C) \subset \mathrm{T}(C)$, which has rank $n - m$ (the dimension of the fibres of the submersion). Indeed, at each $v_q \in C$ we have $\mathrm{V}_{v_q}(C) = \mathrm{T}_{v_q}(C) \cap \mathrm{V}_{v_q}(\mathrm{T}Q)$.

To obtain the equations of motion of vakonomic mechanics, we need first to describe which are the admissible variations.

We remember that an admissible path is a mapping $\gamma: I \to Q$ such that $\dot\gamma$ takes its values in the submanifold $C \subset \mathrm{T}Q$. Let Γ be a variation of an admissible path γ. The variation Γ is called a *strongly admissible variation* of γ if every path Γ_ε is admissible.

If Γ is a strongly admissible variation, then $\phi(\dot\Gamma(\varepsilon, t)) = 0$, for any constraint ϕ. Taking the derivative with respect to ε at $\varepsilon = 0$ and using (2.3) we have

$$\langle \mathrm{d}\phi \circ \dot\gamma, \boldsymbol{s} \circ \dot{\mathbf{w}} \rangle = 0 \tag{3.1}$$

for every constraint ϕ. This can also be expressed as

$$\boldsymbol{s}(\dot{\mathbf{w}}(t)) \in \mathrm{T}_{\dot\gamma(t)}(C) \tag{3.2}$$

for each $t \in I$.

A variation field $\mathbf{w}$ of an admissible path γ satisfying (3.1) or (3.2) is called an *admissible variation field* (for vakonomic mechanics).

It is important to remark that an admissible path may not have any nontrivial strongly admissible variation, and so an admissible variation field may not arise from a strongly admissible variation. One may say that the variations having admissible variation fields are the variations that preserve the constraints up to first order in ε. These variations may be called *weakly admissible variations*, and can always be constructed from an admissible variation field.

Next, we are going to give a more detailed description of admissible variation fields. Among all the vector fields $\mathbf{w}$ along γ, we consider a particular submodule. Take the subbundle

$$\boldsymbol{L}^C_\gamma \subset \gamma^*\mathrm{T}(Q) = I \times_\gamma \mathrm{T}(Q)$$

whose sections are the vector fields $\mathbf{w}$ along γ of class C^1 whose vertical lifts $\mathrm{vl}(\dot\gamma, \mathbf{w})$ are tangent to C. Using this subbundle we can express the admissible variation fields in a more manageable way. First, notice that, since I is an interval, both L^C_γ and $\gamma^*\mathrm{T}(Q)$ are trivializable. Therefore there exists a global frame for $\gamma^*\mathrm{T}(Q)$, $(\mathbf{w}_k)$ $(k = 1, \ldots, n)$. Since L^C_γ is a subbundle of rank $n-m$, we can assume that the last $n-m$ of the $\mathbf{w}_k$ span this subbundle.

Any vector field along γ can be thus uniquely written $\mathbf{w} = \sum_{k=1}^n \lambda^k \mathbf{w}_k$, where λ^k are functions of time. Then $\mathbf{w}$ is a variation field if and only if the coefficients λ^k vanish at the end-points of I. Moreover, according to (3.1), $\mathbf{w}$ is an admissible variation field if it is a variation field and $\langle \mathrm{d}\phi^i \circ \dot\gamma, \mathsf{s} \circ \dot{\mathbf{w}} \rangle = 0$, for $i = 1, \ldots, m$. Taking into account (2.2), this condition can be written

$$\sum_{k=1}^n \langle \mathcal{D}\phi^i \circ \dot\gamma, \mathbf{w}_k \rangle \mathrm{D}\lambda^k + \sum_{k=1}^n \langle \mathrm{d}\phi^i \circ \dot\gamma, \mathsf{s} \circ \dot{\mathbf{w}}_k \rangle \lambda^k = 0 \,. \tag{3.3}$$

Notice that, since the fibre derivatives $\mathcal{D}\phi^i$ are linearly independent at each point, the matrix with entries $\langle \mathcal{D}\phi^i \circ \dot\gamma, \mathbf{w}_k \rangle$ has maximal rank, m. By the special choice of the $\mathbf{w}_k$, the last $n-m$ of them vanish under the $\mathcal{D}\phi^i$, and hence the square matrix $A = (\langle \mathcal{D}\phi^i \circ \dot\gamma, \mathbf{w}_j \rangle)_{i,j=1,\ldots,m}$ is invertible. So, writing the equation as

$$\sum_{j=1}^m A^i_j \, \mathrm{D}\lambda^j + \sum_{k=1}^n B^i_k \, \lambda^k = 0 \,, \tag{3.4}$$

we can isolate the $\mathrm{D}\lambda^j$ $(j = 1, \ldots, m)$ linearly in terms of all the λ^k $(k = 1, \ldots, n)$. This determines uniquely λ^j due to the initial condition $\lambda^j(t_1) = 0$. In conclusion, if the coefficients λ^k satisfy the relations given by (3.4) leaving the $\lambda^{m+1}, \ldots, \lambda^n$ free, then $\mathbf{w} = \sum_{k=1}^n \lambda^k \mathbf{w}_k$ is an admissible variation field. We will suppose that variation fields exist and are of this kind.

Now, we are going to obtain the equation of motion of vakonomic mechanics. As we have shown, a critical path of the action with constraints is an admissible path γ such that $\displaystyle\int_I \langle \mathcal{E}_L(\ddot\gamma(t)), \mathbf{w}(t) \rangle \, \mathrm{d}t$ vanishes for each *admissible* variation field $\mathbf{w}$ (2.6).

If we consider the action $\int_{\dot\gamma} L \mathrm{d}t$ along γ, whatever the functions $\mu_i(t)$ are, the action is also

$$\int_{\dot\gamma} (L - \sum_{i=1}^m \mu_i \phi^i) \, \mathrm{d}t \,. \tag{3.5}$$

In principle, since the variations may not be strongly admissible, it is not clear that the variations of both actions yield the same result. However, using

(2.6) and (2.5), the difference of the first-order variations of the actions is

$$\int_I \Big\langle \mathcal{E}_{\sum_{i=1}^m \mu_i \phi^i} \circ \ddot{\gamma}, \mathbf{w} \Big\rangle \,\mathrm{d}t = \sum_{i=1}^m \int_I \mu_i \left\langle \mathrm{d}\phi^i \circ \dot{\gamma}, s \circ \dot{\mathbf{w}} \right\rangle \mathrm{d}t - \sum_{i=1}^m \int_I \mathrm{D} \left\langle \mu_i (\mathcal{D}\phi^i \circ \dot{\gamma}), \mathbf{w} \right\rangle \mathrm{d}t,$$

and both terms vanish whenever $\mathbf{w}$ is an admissible variation field. Therefore, the variations of the two actions coincide when $\mathbf{w}$ is an admissible variation field.

Notice that the μ_i can be chosen such that

$$\langle \mathcal{E}_{L - \sum_{i=1}^m \mu_i \phi^i} \circ \ddot{\gamma}, \mathbf{w}_j \rangle = 0 \tag{3.6}$$

for $j = 1, \ldots, m$. Indeed, by relation (2.7) this equation can be written as

$$\langle \mathcal{E}_L \circ \ddot{\gamma}, \mathbf{w}_j \rangle - \sum_{i=1}^m \mu_i \langle \mathcal{E}_{\phi^i} \circ \ddot{\gamma}, \mathbf{w}_j \rangle + \sum_{i=1}^m \mathrm{D}\mu_i \langle \mathcal{D}\phi^i \circ \dot{\gamma}, \mathbf{w}_j \rangle = 0$$

for each $j = 1, \ldots, m$. That is, we have $(\mathrm{D}\mu_i)\, A^i_j - \mu_i\, C^i_j + D_j = 0$, where A is the invertible matrix we have used previously. So again we have a linear differential equation that determines the functions μ_i (up to initial conditions).

Now, let us apply the variational principle: $\int_I \langle \mathcal{E}_L \circ \ddot{\gamma}, \mathbf{w} \rangle \,\mathrm{d}t$ must vanish for each admissible variation field $\mathbf{w} = \sum_{k=1}^n \lambda^k \mathbf{w}_k$. As we have said, instead of L we can use $L - \sum_{i=1}^m \mu_i \phi^i$.

So we have

$$\sum_{k=1}^n \int_I \langle \mathcal{E}_{L - \sum_{i=1}^m \mu_i \phi^i} \circ \ddot{\gamma}, \mathbf{w}_k \rangle \lambda^k \,\mathrm{d}t = 0$$

for each set of functions λ^k yielding an admissible variation field.

If we choose the functions μ_i satisfying (3.6), then the sum is only from $m+1$ to n:

$$\sum_{j=m+1}^n \int_I \langle \mathcal{E}_{L - \sum_{i=1}^m \mu_i \phi^i} \circ \ddot{\gamma}, \mathbf{w}_j \rangle \lambda^j \,\mathrm{d}t = 0 \,.$$

This must hold for any choice of the functions $\lambda^{m+1}, \ldots, \lambda^n$, which vanish at the end-points of I but are otherwise arbitrary. So we conclude that

$$\langle \mathcal{E}_{L - \sum_{i=1}^m \mu_i \phi^i} \circ \ddot{\gamma}, \mathbf{w}_j \rangle = 0$$

for each $j = m+1, \ldots, n$. Combining this with (3.6) we conclude that

$$\mathcal{E}_{L - \sum_{i=1}^m \mu_i \phi^i} \circ \ddot{\gamma} = 0 \,, \tag{3.7}$$

which is the *equation of motion of vakonomic mechanics*.

Conversely, if γ satisfies (3.7) for a certain set of $\mu_i(t)$, then it is straightforward to show that $\displaystyle\int_I \langle \mathcal{E}_L \circ \ddot{\gamma}, \mathbf{w} \rangle$ vanishes for each admissible variation field, and thus γ is a critical path of the action with constraints.

Notice that, using (2.7), the equation of motion may also be written as

$$\mathcal{E}_L \circ \ddot{\gamma} = \sum_{i=1}^{m} \left(\mu_i \, \mathcal{E}_{\phi^i} \circ \ddot{\gamma} - (\mathrm{D}\mu_i) \, \mathcal{D}\phi^i \circ \dot{\gamma} \right) . \tag{3.8}$$

4 The variations and equations of motion of nonholonomic mechanics

In this section we are going to show that nonholonomic mechanics may be understood as a variational problem.

Our initial setting is also the submanifold $C \subset \mathrm{T}Q$, which may be locally defined by the vanishing of the constraints ϕ^i. An *admissible path* is still a path $\gamma \colon I \to Q$ such that $\dot{\gamma}$ takes its values in C.

A variation field $\mathbf{w}$ of an admissible path γ is called an *admissible variation field* (in nonholonomic mechanics) if it is a section of the subbundle $\mathcal{L}^C_\gamma \subset \gamma^*\mathrm{T}Q$. That is,

$$\mathrm{vl}(\dot{\gamma}(t), \mathbf{w}(t)) \in \mathrm{T}_{\dot{\gamma}(t)}(C) \; . \tag{4.1}$$

Using the constraints, equivalent statements are

$$\langle \mathrm{d}\phi \circ \dot{\gamma}, \mathrm{vl}(\dot{\gamma}, \mathbf{w}) \rangle = 0 \; , \tag{4.2}$$

or

$$\langle \mathcal{D}\phi \circ \dot{\gamma}, \mathbf{w} \rangle = 0 \; , \tag{4.3}$$

for each constraint ϕ.

Notice the key difference with respect to vakonomic mechanics: now the admissibility is a $C^1(I)$-linear condition on $\mathbf{w}$. This linearity makes things easier. Next, we obtain the equation of motion of nonholonomic mechanics.

A *critical path* for nonholonomic mechanics is an admissible path γ such that the first-order variation of the action, $\displaystyle\int_I \langle \mathcal{E}_L(\ddot{\gamma}(t)), \mathbf{w}(t) \rangle \, \mathrm{d}t$, vanishes for each admissible variation field $\mathbf{w}$. By equation (4.3), this means that $\mathcal{E}_L(\ddot{\gamma}(t))$ is a linear combination of the $\mathcal{D}\phi^i \circ \dot{\gamma}$. Therefore, an admissible path γ is critical if and only if there exist functions μ_j, $j = 1, \ldots m$, such that

$$\mathcal{E}_L \circ \ddot{\gamma} = \sum_{i=1}^{m} \mu_i \, \mathcal{D}\phi^i \circ \dot{\gamma} \; . \tag{4.4}$$

5 The case of integrable constraints

Given a differentiable manifold Q and a Lagrangian function $L : \mathrm{T}Q \longrightarrow \mathcal{R}$, a *holonomic problem* is a variational problem where the constraint submanifold is given by a submanifold $P \subset Q$, thus $C = \mathrm{T}P$. Admissible paths are paths $\gamma : I \longrightarrow P \subset Q$. Variation fields $\mathbf{w}$ along γ are *admissible* if they are tangent to P.

Notice that, from any admissible variation field along an admissible path γ, one may construct a variation Γ contained in P. Therefore, it is clear that the problem with holonomic constraints is equivalent to the unconstrained variational problem defined on P by taking the restriction of the Lagrangian L to $\mathrm{T}P \subset \mathrm{T}Q$.

Now, consider the cases of both vakonomic and nonholonomic mechanics when the constraints are defined by an integrable subbundle $C \subset \mathrm{T}Q$. In this situation, both vakonomic and nonholonomic mechanics coincide, and they are equivalent to a holonomic constrained problem on each integral submanifold of C.

To show this, first notice that, in both cases, a path $\gamma : I \longrightarrow Q$ is admissible (i.e., $\dot\gamma$ is in C) if and only if it is contained in an integral submanifold $P \subset Q$ of C. Recall that $\mathrm{T}_q(P) = C_q$, for each point in P.

Let $\mathbf{w}$ be a variation field of γ. We know that $\mathbf{w}$ is admissible in the nonholonomic framework if $\mathrm{vl}(\dot\gamma(t), \mathbf{w}(t)) \in \mathrm{T}_{\dot\gamma(t)}(C)$ (see (4.1)). Since the vertical lift restricts naturally to subbundles, this is equivalent to say that $\mathrm{vl}(\dot\gamma(t), \mathbf{w}(t)) \in \mathrm{V}_{\dot\gamma(t)}C \subset \mathrm{V}_{\dot\gamma(t)}(\mathrm{T}Q)$, that is, $\mathbf{w}(t) \in C_{\gamma(t)}$ or, what is the same, $\mathbf{w}(t) \in \mathrm{T}_{\gamma(t)}P$. And this last condition says that $\mathbf{w}$ is admissible for the holonomic problem. Therefore, we have proved that a nonholonomic problem with integrable constraints is equivalent to a holonomic problem on each integral submanifold of C.

Now, we show the equivalence with the vakonomic problem. Since C is an integrable subbundle of $\mathrm{T}Q$, it is known that the integral submanifolds of C can be locally described as $\psi =$ constant, for some independent functions ψ on Q. This implies that the constraint submanifold C can be locally described by $\phi = \tilde{\mathrm{d}}\psi = 0$. Here, $\tilde{\mathrm{d}}\psi : \mathrm{T}Q \longrightarrow \mathcal{R}$ is the differential $\mathrm{d}\psi$ of $\psi : Q \longrightarrow \mathcal{R}$ considered as a function on the tangent bundle.

Let $\mathbf{w}$ be a variation field of γ, in the sense of vakonomic mechanics. Then, for any $\phi = \tilde{\mathrm{d}}\psi$, we have

$$0 = \langle \mathrm{d}\phi \circ \dot\gamma, \boldsymbol{s} \circ \dot{\mathbf{w}} \rangle = \tilde{\mathrm{d}}\tilde{\mathrm{d}}\psi \circ \boldsymbol{s} \circ \dot{\mathbf{w}}\ .$$

Using the property $\tilde{\mathrm{d}}\tilde{\mathrm{d}}\psi = \tilde{\mathrm{d}}\tilde{\mathrm{d}}\psi \circ \boldsymbol{s}$, we obtain

$$0 = \tilde{\mathrm{d}}\tilde{\mathrm{d}}\psi \circ \dot{\mathbf{w}} = \langle \mathrm{d}\tilde{\mathrm{d}}\psi \circ \mathbf{w}, \dot{\mathbf{w}} \rangle = \langle \mathrm{d}\phi \circ \mathbf{w}, \dot{\mathbf{w}} \rangle = \mathrm{D}(\phi \circ \mathbf{w}) = \mathrm{D}\langle \mathrm{d}\psi \circ \gamma, \mathbf{w} \rangle \ .$$

Thus, $\mathbf{w}$ is an admissible variation field in vakonomic mechanics if and only if $\langle \mathrm{d}\psi \circ \gamma, \mathbf{w} \rangle$ is constant. Since $\mathbf{w}(t_1) = 0$, this constant is zero, $\langle \mathrm{d}\psi \circ \gamma, \mathbf{w} \rangle = 0$, which means that $\mathbf{w}$ is tangent to P. Therefore, $\mathbf{w}$ is admissible for the vakonomic problem if and only if it is admissible for the holonomic problem on P.

6 An example: optimal control and vakonomic mechanics

A problem of optimal control may be given by the following data: a configuration space B describing the state variables, a fibre bundle $\pi: M \longrightarrow B$ whose fibres describe the control variables, a vector field Y along the projection of the bundle, $Y: M \longrightarrow \mathrm{T}B$, and a "Lagrangian function" $L: M \longrightarrow \mathcal{R}$. For a path $\gamma: I \to M$ where $\pi \circ \gamma$ (not γ!) has fixed end-points, the problem is to find an extremum of the action

$$\int_\gamma L(\gamma(t))\, \mathrm{d}t$$

when γ satisfies the differential equation

$$(\pi \circ \gamma)^\cdot = Y \circ \gamma \tag{6.1}$$

that rules the evolution of the state variables.

It is easy to show that this is indeed a vakonomic problem on the manifold M, in which the Lagrangian L is very singular, since it does not depend on the velocities. The constraint submanifold $C \subset \mathrm{T}M$, given by the differential equation above, is

$$C = \{ w_u \in \mathrm{T}M \mid \mathrm{T}\pi(w_u) = Y(u) \} \ . \tag{6.2}$$

In this way, a path γ is admissible if and only if it is a solution of the differential equation (6.1) or, equivalently, if it takes values in the affine subbundle C of $\mathrm{T}M$.

Notice that for this vakonomic problem the constraints are very particular: they express the velocities of the state variables in terms of the state and control variables. In coordinates, if x^i, $i = 1, \ldots m$, are local coordinates in

B and (x^i, u^α), $i = 1, \dots m$, $\alpha = 1, \dots, n-m$ are local coordinates in M the action is given by

$$\int_{t_1}^{t_2} L(x^i, u^\alpha)\, \mathrm{d}t$$

and the constraints are given by a set of first order differential equations

$$\dot{x}^i = f^i(x, u) \ , \quad i = 1, \dots, m,$$

with boundary conditions $x_1^i = x^i(t_1)$ to $x_2^i = x^i(t_2)$ (there are no boundary conditions on the control variables).

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