What do heating your living room, financial investments, and image processing have in common?

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There is a striking resemblance on the modeling of

- heat &
- option prices in Finance

In both cases the basic object is the same: "the Laplacian" after Pierre-Simon, marquis de Laplace (1749-1827)

It is responsible for many phenomena in our lives
A first example:

what is the **temperature** of a certain tile in your living room's floor, long after you turn on the wall radiators at **30°C** while the remaining of the walls are always kept at **0°C**?
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*iCrea*
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A second question (on images):
which color (red level) would you give to the missing pixels?
Robert Brown (1773-1858), biologist

Looking through a microscope at pollen grains in water, he noted that the grains moved randomly through the water.

**BROWNIAN MOTION**

Think also on a large plastic beach ball on the stands of a stadium totally full of people.
A third question (of finance type):

what is your **expected gain** when,

starting always from the same given tile in your living room, you walk randomly and you get **30€** only when you hit a radiator on the first time that you hit your living room's walls (otherwise you get **0€**)?
• A third question (of **finance** type)

**ANSWER:** at every point one has

expected gain = temperature !!
How to solve the problem:

- make a **squared lattice** of very small step-size $h$
- Move from a point to either East, West, North, or South, each one with **probability** 1/4
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$C = \text{starting point of the walk}$

$u(C) = \text{expected gain starting from } C$

\[
u(C) = \frac{1}{4} \{u(E) + u(W) + u(N) + u(S)\}
\]

(average)
Some math:

\[ \Delta u (x, y) = (\partial_{xx} u + \partial_{yy} u) (x, y) = 0 \]

The LAPLACIAN of \( u = 0 \)
Harmonic functions are characterized by the mean value property:

The value of the function at the center of any circle =
the average of the values of the function on the circle

OK with HEAT, and with EXPECTED GAIN!
\[ \Delta u = \partial_{xx} u + \partial_{yy} u = 0 \]

is called the [Laplace equation](#).

It is a **Partial Differential Equation** (a PDE) (also called the equations of Mathematical Physics).

Its solutions are called “harmonic functions”. Together with solutions of the heat or diffusion equation

\[ \partial_t u - \Delta u = 0 \]

(and other equations of the same type), they model:

- **heat**  (Fourier and Einstein)
- **option prices** in Finance
- gravitational and electric potentials  (Laplace)
- densities of biological or chemical species
Partial Differential Equations. Types:

1. Elliptic: Laplace equation: \( \Delta u = \partial_{xx} u + \partial_{yy} u = 0 \)

2. Parabolic:
   - **Heat** or diffusion equation: \( \partial_t u - \Delta u = 0 \)
   - Navier-Stokes (or 1 million $) equations
     (incompressible viscous **fluids**)
     \[ \begin{align*}
     \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} &= - \nabla p \\
     \text{div } \mathbf{u} &= 0
     \end{align*} \]

3. Hyperbolic:
   - **Wave** equation
     (acoustics, sound-waves)
     \( \partial_{tt} u - \Delta u = 0 \)
   - Schrödinger equation
     (quantum mechanics)
     \( i \partial_t \mathbf{u} + \Delta \mathbf{u} = 0 \)
   - Euler's equations
     (incompressible **fluids**)
     \[ \begin{align*}
     \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} &= - \nabla p \\
     \text{div } \mathbf{u} &= 0
     \end{align*} \]
Some other important PDEs:

a. Linear equations.
1. Laplace’s equations: \( \Delta u = 0 \)
2. Helmholtz’s equation (involves eigenvalues): \( -\Delta u = \lambda u \)
3. First-order linear transport equation: \( u_t + cu_x = 0 \)
4. Heat or diffusion equation: \( u_t - \Delta u = 0 \)
5. Schrödinger’s equation: \( iu_t + \Delta u = 0 \)
6. Wave equation: \( u_{tt} - c^2 \Delta u = 0 \)
7. Telegraph equation: \( u_{tt} + d u_t - u_{xx} = 0 \)

b. Nonlinear equations.
1. Eikonal equation: \( |Du| = 1 \)
2. Nonlinear Poisson equation: \( -\Delta u = f(u) \)
3. Burgers’ equation: \( u_t + u u_x = 0 \)
4. Minimal surface equation: \( \text{div} \left( \frac{Du}{(1 + |Du|^2)^{1/2}} \right) = 0 \)
5. Monge-Ampère equation: \( \text{det}(D^2 u) = f \)
6. Korteweg-deVries equation (KdV): \( u_t + u u_x + u_{xxx} = 0 \)
7. Reaction-diffusion equation: \( u_t - \Delta u = f(u) \)

1. Evolution equation of linear elasticity: \( u_{tt} - \mu \Delta u - (\lambda + \mu) D(\text{div} u) = 0 \)
2. System of conservation laws:
   \( u_t + \text{div} F(u) = 0 \)
   \( \{ \begin{align*}
   \text{curl} E &= -B_t \\
   \text{curl} B &= \mu_0 \varepsilon_0 E_t \\
   \text{div} B &= \text{div} E &= 0
   \end{align*} \)
3. Maxwell’s equations in vacuum:
4. Reaction-diffusion system:
5. Euler’s equations for incompressible, inviscid fluid:
6. Navier-Stokes equations for incompressible viscous fluid: