p-ADIC L-FUNCTIONS AND EULER SYSTEMS:
A TALE IN TWO TRILOGIES

MASSIMO BERTOLINI, FRANCESC CASTELLA, HENRI DARMON
SAMIT DASGUPTA, KARTIK PRASANNA, VICTOR ROTGER

ABSTRACT. This article surveys six different special value formulae for p-adic L-functions, stressing their common features and their eventual arithmetic applications via the theory of "Euler systems".

CONTENTS

INTRODUCTION

This article surveys six different special value formulae for p-adic L-functions, stressing their common features and their eventual arithmetic applications via the theory of "Euler systems". The most classical instances are:

(1) Leopoldt's formula for the value at $s = 1$ of the Kubota-Leopoldt p-adic L-function in terms of p-adic logarithms of *circular units*;

(2) Katz's p-adic Kronecker limit formula for values of the two variable p-adic L-function of a quadratic imaginary field at finite order characters in terms of p-adic logarithms of associated *elliptic units*.

They are reviewed in Sections 1.1 and 1.2 respectively. Section 1.3 describes the more recent formula of [BDP] and explains why it is a direct generalisation of the formulae of Leopoldt and Katz in the setting where special units are replaced by *Heegner points*. The three parallel treatments in Chapter 1 suggest that both elliptic and circular units might be viewed as *degenerate cases* of the Euler system of Heegner points, obtained by

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successively replacing the cusp forms and ordinary CM points that arise in the latter setting by Eisenstein series and cusps.

The second part of this survey attempts to view the Euler system introduced by Kato [Kato] in a similar way, as the "most degenerate instance" of a broader class of examples. Referred to as "Euler systems of Garrett-Rankin-Selberg type" because of the role played by the formulae of Rankin-Selberg and Garrett in relating them to special values of L-functions, these examples consist of

(1) Kato's original Euler system of (p-adic families of) Beilinson elements in the second K-group of modular curves, whose global objects are indexed by pairs $\{u_1, u_2\}$ of modular units. Their connection to L-values follows from Rankin's method applied to a cusp form and the pair of weight two Eisenstein series corresponding to the logarithmic derivatives of u_1 and u_2.

(2) The Euler system of *Beilinson-Flach* elements in the first K-group of a product of two modular curves.

(3) The Euler system of *generalised Gross-Kudla-Schoen* diagonal cycles, whose connection with L-values arises from the formula of Garrett for the central critical value of the convolution L-series attached to a triple of newforms.

The global cohomology classes in (3) are indexed by triples (f, g, h) of cusp forms and take values in the tensor product of the three p-adic representations attached to f, g and h. Example (1) (resp. (2)) can in some sense be viewed as a degenerate instance of (3) in which g and h (resp. h only) are replaced by Eisenstein series.

Ever since the seminal work of Kolyvagin [Ko], there have been many proposals for axiomatizing and classifying the Euler systems that should arise in nature (cf. [Ru], [PR3], [Cz1], [Cz2], [MR], ...) with the goal of understanding them more conceptually and systematizing the process whereby arithmetic information is coaxed from their behaviour. The present survey is less ambitious, focussing instead on six settings where the associated global objects have been constructed unconditionally, attempting to organise them coherently, and suggesting that they arise from two rather than six fundamentally distinct classes of examples. Like the ten plagues of Egypt in the Jewish Passover Haggadah, Euler Systems can surely be counted in many ways. The authors believe (and certainly hope) that their "two trilogies" are but the first instalments of a richer story in which higher-dimensional cycles on Shimura varieties and p-adic families of automorphic forms are destined to play an important role.

1. CLASSICAL EXAMPLES

1.1. Circular units.

Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ be a primitive, non-trivial even Dirichlet character of conductor N. The values of the Dirichlet L-function $L(s, \chi)$ at the negative odd integers belong to the field $\mathbb{Q}_\chi \subset \bar{\mathbb{Q}}$ generated by the values of χ. This can be seen by realising $L(1 - k, \chi)$ for even $k \geq 2$ as the constant term of the holomorphic Eisenstein series

$$
(1) \qquad E_{k,\chi}(q) := L(1 - k, \chi) + 2 \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n) q^n, \qquad \sigma_{k-1,\chi}(n) := \sum_{d|n} \chi(d) d^{k-1}
$$

of weight k, level N and character χ, and invoking the q-expansion principle to argue that the constant term in (1) inherits the rationality properties of the coefficients $\sigma_{k-1,\chi}(n)$.

If p is any prime (possibly dividing N), the *ordinary p-stabilisation*

$$
(2) \qquad E_{k,\chi}^{(p)}(q) := E_{k,\chi}(q) - \chi(p) p^{k-1} E_{k,\chi}(q^p)
$$

has Fourier expansion given by

$$
(3) \qquad E_{k,\chi}^{(p)}(q) = L_p(1 - k, \chi) + 2 \sum_{n=1}^{\infty} \sigma_{k-1,\chi}^{(p)}(n) q^n,
$$

where

$$
(4) \qquad L_p(1 - k, \chi) = (1 - \chi(p) p^{k-1}) L(1 - k, \chi), \qquad \sigma_{k-1,\chi}^{(p)}(n) = \sum_{p \nmid d|n} \chi(d) d^{k-1}.
$$

For each $n \geq 1$, the function on $\mathbb{Z}^{\geq 1}$ sending k to the n-th Fourier coefficient $\sigma_{k-1,\chi}^{(p)}(n)$ extends to a p-adic analytic function of $k \in (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p$. The article [Se] explains why the constant term of (3) inherits the same property. The resulting extension to $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$ of $L_p(s, \chi)$, defined *a priori* as a function on the negative odd integers, is the *Kubota-Leopoldt p-adic L-function* attached to χ. The elegant construction of $L_p(s, \chi)$ arising from this circle of ideas (and its subsequent extension to totally real fields) was one of the original motivations for the theory of p-adic modular forms initiated in [Se].

The collection of eigenforms in (3) is a prototypical example of a *p-adic family of modular forms*, whose specialisations at even integers $k \leq 0$, while not classical, continue to admit a geometric interpretation as *p-adic modular forms*[1] of weight k and level N_0, the prime-to-p part of N. When $k = 0$, these are just rigid analytic functions on the *ordinary locus* $\mathcal{A} \subset X_1(N_0)(\mathbb{C}_p)$ obtained by deleting from $X_1(N_0)(\mathbb{C}_p)$ all the residue discs attached to supersingular elliptic curves in characteristic p. In particular, the special value $L_p(1, \chi)$ can be interpreted as the the value at the cusp ∞ of such a rigid analytic function, namely the weight 0 Eisenstein series

$$
(5) \qquad E_{0,\chi}^{(p)}(q) = L_p(1, \chi) + 2 \sum_{n=1}^{\infty} \left(\sum_{p \nmid d|n} \chi(d) d^{-1} \right) q^n.
$$

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[1]they are even overconvergent, but this stronger property will not be exploited here.

An independent expression for this function can be derived in terms of the Siegel units $g_a \in \mathcal{O}_{Y_1(N)}^\times$ attached to a fixed choice of primitive N-th root of unity ζ and a parameter $1 \le a \le N - 1$, whose q-expansions are given by

$$(6) \qquad g_a(q) = q^{1/12}(1 - \zeta^a) \prod_{n>0}(1 - q^n\zeta^a)(1 - q^n\zeta^{-a}).$$

More precisely, let Φ be the *canonical lift of Frobenius* on $\mathcal{A}$ which sends the point corresponding to the moduli of the pair $(E, t) \in \mathcal{A}$ to the moduli of the pair $(E/C, t+C)$ where $C \subset E(\mathbb{C}_p)$ is the *canonical subgroup* of order p in E. The rigid analytic function

$$(7) \qquad g_a^{(p)} := \Phi^*(g_a)g_a^{-p} = g_{pa}(q^p)g_a(q)^{-p}$$

maps the ordinary locus $\mathcal{A}$ to the residue disc of 1 in $\mathbb{C}_p$, and therefore its p-adic logarithm $\log_p g_a^{(p)}$ is a rigid analytic function on $\mathcal{A}$, with q-expansion given by

$$\log_p g_a^{(p)} = \log_p\left(\frac{1 - \zeta^{ap}}{(1 - \zeta^a)^p}\right) + p\sum_{n=1}^\infty\left(\sum_{p\nmid d|n}\frac{\zeta^{ad} + \zeta^{-ad}}{d}\right)q^n.$$

Letting

$$(8) \qquad \mathfrak{g}(\chi) = \sum_{a=1}^{N-1}\chi(a)\zeta^a$$

denote the Gauss sum attached to χ, a direct computation shows that the rigid analytic function on $\mathcal{A}$ given by

$$(9) \qquad h_\chi^{(p)} := \frac{1}{p\mathfrak{g}(\chi^{-1})} \times \sum_{a=1}^{N-1}\chi^{-1}(a)\log_p g_a^{(p)}$$

has q-expansion equal to

$$(10) \qquad h_\chi^{(p)}(q) = -\frac{(1 - \chi(p)p^{-1})}{\mathfrak{g}(\chi^{-1})}\sum_{a=1}^{N-1}\chi^{-1}(a)\log_p(1 - \zeta^a) + 2\sum_{n=1}^\infty\left(\sum_{p\nmid d|n}\chi(d)d^{-1}\right)q^n.$$

Theorem 1.1 (Leopoldt). *Let χ be a non-trivial even primitive Dirichlet character of conductor N. Then*

$$L_p(1, \chi) = -\frac{(1 - \chi(p)p^{-1})}{\mathfrak{g}(\chi^{-1})}\sum_{a=1}^{N-1}\chi^{-1}(a)\log_p(1 - \zeta^a).$$

Proof. Comparing q-expansions in (5) and (10) shows that the difference $E_{0,\chi}^{(p)} - h_\chi^{(p)}$ is constant on the residue disc of a cusp, and hence on all of $\mathcal{A}$ since it is rigid analytic on this domain. In fact,

$$(11) \qquad E_{0,\chi}^{(p)} = h_\chi^{(p)},$$

since both these p-adic modular functions have nebentype character $\chi \ne 1$. Leopoldt's formula follows by equating the constant terms in the q-expansions in (5) and (10). For more details on this "modular" proof of Leopoldt's formula, see [Katz, §10.2]. $\qquad\square$

Remark: Recall the customary notations in which $E_k(\psi, \chi)$ denotes the Eisenstein series attached to a pair (ψ, χ) of Dirichlet characters, having the q-expansion

$$E_k(\psi, \chi) = \delta_{\psi=1} L(1-k, \psi\chi) + 2 \sum_{n=1}^{\infty} \left(\sum_{d|n} \psi(n/d)\chi(d)d^{k-1} \right) q^n,$$

and let χ_p be the Dirichlet character of modulus Np which agrees with χ on $(\mathbb{Z}/Np\mathbb{Z})^{\times}$, so that $E_k(1, \chi) = E_{k,\chi}$ and $E_k(1, \chi_p) = E_{k,\chi}^{(p)}$. The rigid analytic function $E_{0,\chi}^{(p)} = h_{\chi}^{(p)}$ which is a key actor in the proof of Leopoldt's formula above is a *Coleman primitive* of the weight two Eisenstein series $E_2(\chi_p, 1)$, a non-ordinary modular form of *critical slope* since its p-th Fourier coefficient is equal to p. The pattern whereby special values of p-adic L-series outside the range of classical interpolation arise as values of Coleman primitives of p-adic modular forms at distinguished points of the modular curve (namely cusps, or ordinary CM points) will recur in Sections 1.2 and 1.3.

The expressions of the form $(1 - \zeta^a)$ (when N is composite) and $\frac{1-\zeta^a}{1-\zeta^b}$ (when N is prime) that occur in Leopoldt's formula are called *circular units*. These explicit units play an important role in the arithmetic of the cyclotomic field $\mathbb{Q}(\zeta)$. Letting F_{χ} denote the field cut out by χ viewed as a Galois character, and $\mathbb{Z}_{\chi}$ the ring generated by its values, the expression

$$u_{\chi} := \prod_{a=1}^{N-1} (1 - \zeta^a)^{\chi^{-1}(a)} \in (\mathcal{O}_{F_{\chi}}^{\times} \otimes \mathbb{Z}_{\chi})^{\chi}$$

is a distinguished unit in F_{χ} (or rather, a $\mathbb{Z}_{\chi}$-linear combination of such) which lies in the χ-eigenspace for the natural action of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ (in which the second factor $\mathbb{Z}_{\chi}$ in the tensor product is fixed by this group).

A notable feature of the unit u_{χ} is that it is essentially a "universal norm" over the tower of cyclotomic fields whose n-th layer is $F_{\chi,n} = F_{\chi}(\mu_{p^n})$. More precisely, after fixing a sequence $(\zeta = \zeta_N, \zeta_{Np}, \zeta_{Np^2}, \dots, \zeta_{Np^n}, \dots)$ of primitive Np^n-th roots of unity which are compatible under the p-power maps, and setting

$$u_{\chi,n} = \prod_{a=1}^{N-1} (1 - \zeta_{Np^n}^a)^{\chi^{-1}(a)} \quad \in (\mathcal{O}_{F_{\chi,n}}^{\times} \otimes \mathbb{Z}_{\chi})^{\chi},$$

we find that

$$\mathrm{Norm}_{F_{\chi,n}}^{F_{\chi,n+1}}(u_{\chi,n+1}) = \begin{cases} u_{\chi,n} & \text{if } n \geq 1, \\ u_{\chi} \otimes (\chi(p) - 1) & \text{if } n = 0. \end{cases}$$

After viewing χ as a $\mathbb{C}_p$-valued character, let $\mathbb{Z}_{p,\chi}$ be the ring generated over $\mathbb{Z}_p$ by the values of χ (endowed with the trivial $G_{\mathbb{Q}}$-action) and let $\mathbb{Z}_{p,\chi}(\chi)$ be the free module of rank one over $\mathbb{Z}_{p,\chi}$ on which $G_{\mathbb{Q}}$ acts via the character χ. The symbols $\mathbb{Q}_{p,\chi}$ and $\mathbb{Q}_{p,\chi}(\chi)$ are likewise given the obvious meaning. The images

$$\kappa_{\chi,n} := \delta u_{\chi,n} \in H^1(F_{\chi,n}, \mathbb{Z}_{p,\chi}(1))^{\chi} = H^1(F_n, \mathbb{Z}_{p,\chi}(1)(\chi^{-1}))$$

under the connecting homomorphism $\delta : (F_{\chi,n}^\times \otimes \mathbb{Z}_\chi)^\chi \longrightarrow H^1(F_{\chi,n}, \mathbb{Z}_{p,\chi}(1))^\chi$ of Kummer theory can be patched together in a canonical element $\kappa_{\chi,\infty} := (\kappa_{\chi,n})_{n \geq 0}$ belonging to

$$(12) \qquad \varprojlim_n H^1(F_n, \mathbb{Z}_{p,\chi}(1)(\chi^{-1})) \;=\; \varprojlim_n H^1(\mathbb{Q}, \mathbb{Z}_p[G_n] \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p,\chi}(1)(\chi^{-1}))$$

$$= \; H^1(\mathbb{Q}, \Lambda_{\mathrm{cyc}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p,\chi}(1)(\chi^{-1})),$$

where

- $\mathbb{Z}_p[G_n]$ is the group ring of $G_n := \mathrm{Gal}(F_n/\mathbb{Q}) = (\mathbb{Z}/p^n\mathbb{Z})^\times$, equipped with the tautological action of $G_{\mathbb{Q}}$ in which $\sigma \in G_{\mathbb{Q}}$ acts via multiplication by its image in G_n, and the identification (12) follows from Shapiro's lemma;
- $\Lambda_{\mathrm{cyc}} = \varprojlim_n \mathbb{Z}_p[G_n] = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ is the completed group ring of $\mathbb{Z}_p^\times$ equipped with the similar "tautological" action of $G_{\mathbb{Q}}$.

The Galois module Λ_{cyc} can be viewed as a p-adic interpolation of the Tate twists $\mathbb{Z}_p(k)$ for all $k \in \mathbb{Z}$. More precisely, given $k \in \mathbb{Z}$ and a Dirichlet character ξ of p-power conductor, let $\nu_{k,\xi} : \Lambda \longrightarrow \mathbb{Z}_{p,\xi}$ be the ring homomorphism sending the group-like element $a \in \mathbb{Z}_p^\times$ to $a^{k-1}\xi^{-1}(a)$. It induces a $G_{\mathbb{Q}}$-equivariant specialisation map

$$\nu_{k,\xi} : \Lambda_{\mathrm{cyc}} \longrightarrow \mathbb{Q}_{p,\xi}(k-1)(\xi^{-1}),$$

giving rise to a collection of global cohomology classes

$$\kappa_{k,\chi\xi} := \nu_{k,\xi}(\kappa_{\chi,\infty}) \quad \in \quad H^1(\mathbb{Q}, \mathbb{Q}_{p,\chi\xi}(k)((\chi\xi)^{-1})).$$

These classes can be viewed as the "arithmetic specialisations" of the p-adic family $\kappa_{\chi,\infty}$ of cohomology classes.

Given any Dirichlet character η with $\eta(p) \neq 1$, let $F_{p,\eta}$ be the finite extension of $\mathbb{Q}_p$ cut out by the corresponding Galois character and denote by $G_\eta = \mathrm{Gal}(F_{p,\eta}/\mathbb{Q}_p)$ its Galois group. Restriction to $F_{p,\eta}$ composed with δ^{-1} leads to the identifications

$$H^1(\mathbb{Q}_p, \mathbb{Q}_{p,\eta}(1)(\eta)) \;=\; H^1(F_{p,\eta}, \mathbb{Q}_{p,\eta}(1)(\eta))^{G_\eta} = \left(H^1(F_{p,\eta}, \mathbb{Q}_p(1)) \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p,\eta}(\eta)\right)^{G_\eta}$$

$$= \; \left(\mathcal{O}_{F_{p,\eta}}^\times \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p,\eta}(\eta)\right)^{G_\eta}.$$

Applying the p-adic logarithm $\log_p : \mathcal{O}_{F_{p,\eta}}^\times \to F_{p,\eta}$ to this last module leads to the map

$$(13) \qquad \log_\eta : H^1(\mathbb{Q}_p, \mathbb{Q}_{p,\eta}(1)(\eta)) \longrightarrow (F_{p,\eta} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p,\eta}(\eta))^{G_\eta} = (\mathbb{C}_p \otimes_{\mathbb{Q}_{p,\eta}} \mathbb{Q}_{p,\eta}(\eta))^{G_{\mathbb{Q}_p}},$$

where the Tate-Sen isomorphism $F_{p,\eta} = \mathbb{C}_p^{\mathrm{Gal}(\bar{\mathbb{Q}}_p/F_{p,\eta})}$ has been used to make the last identification.

The module $\mathbb{D}_{\mathbb{C}_p}(\mathbb{Q}_{p,\eta}(\eta)) := (\mathbb{C}_p \otimes_{\mathbb{Q}_{p,\eta}} \mathbb{Q}_{p,\eta}(\eta))^{G_{\mathbb{Q}_p}}$ appearing as the target in (13) is the Dieudonné module (with $\mathbb{C}_p$ as "period ring") attached to the Artin representation $\mathbb{Q}_{p,\eta}(\eta)$. It is a one-dimensional $\mathbb{Q}_{p,\eta}$-vector space generated by the "Gauss sum"

$$\mathfrak{g}(\eta) = \sum_{a=1}^{m-1} \zeta_m^a \otimes \eta(a)$$

which can thus be viewed as a "p-adic period" attached to the Galois representation $\mathbb{Q}_{p,\eta}(\eta)$. Theorem 1.1 can be re-phrased as the following relationship between the classes $\kappa_{1,\chi\xi}$ and

the values of the Kubota Leopoldt L-function at $s = 1$, twisted eventually by *finite order characters*:

$$(14) \qquad L_p(1, \chi\xi) = -\frac{(1 - \chi\xi(p)p^{-1})}{(1 - (\chi\xi)^{-1}(p))} \times \frac{\log_{\chi\xi}(\kappa_{1,\chi\xi})}{\mathfrak{g}(\chi\xi)}.$$

Note in particular that the global classes $\kappa_{1,\chi\xi}$ determine the Kubota-Leopoldt L-function completely, since an element of the Iwasawa algebra has finitely many zeroes.

For all $k \geq 1$ and characters η of conductor prime to p, Bloch and Kato have defined a generalisation of the map $\log_\eta$ of (13) for the representation $\mathbb{Q}_{p,\eta}(k)(\eta)$, in which $\mathbb{C}_p$ is replaced by the larger Fontaine period ring B_{dR}:

$$(15) \qquad \log_{k,\eta} : H^1(\mathbb{Q}_p, \mathbb{Q}_{p,\eta}(k)(\eta)) \longrightarrow \mathbb{D}_{\mathrm{dR}}(\mathbb{Q}_{p,\eta}(k)(\eta)) := (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p,\eta}(k)(\eta))^{G_{\mathbb{Q}_p}}.$$

(See for instance [Kato], [Cz1, §2.6]; we have implicitly used the fact that for $k \geq 1$, all extensions of $\mathbb{Q}_p$ by $\mathbb{Q}_p(k)$ are cristalline, and that $\mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(\mathbb{Q}_{p,\eta}(k)(\eta)) = 0$.) The target of the logarithm map is a one-dimensional $\mathbb{Q}_{p,\eta}$-vector space with a canonical generator $t^{-k}\mathfrak{g}(\eta)$, where $t \in B_{\mathrm{dR}}$ is Fontaine's p-adic analogue of $2\pi i$ on which $G_{\mathbb{Q}_p}$ acts as $\sigma t = \chi_{\mathrm{cyc}}(\sigma)t$. The Bloch-Kato logarithm $\log_{k,\eta}$ with $k = 1$ is related to the map $\log_\eta$ of (13) by the formula

$$\log_\eta = t \log_{1,\eta},$$

and (14) admits the following extension for all $k \geq 1$ (cf. [PR4, 3.2.3]):

$$(16) \qquad L_p(k, \chi) = \frac{(1 - \chi(p)p^{-k})}{(1 - \chi^{-1}(p)p^{k-1})} \times \frac{(-t)^k}{(k-1)!\mathfrak{g}(\chi)} \times \log_{k,\chi}(\kappa_{k,\chi}).$$

When $k \leq 0$, the source and target of the logarithm map are both zero, and (16) does not extend to the negative integers. An interpretation of $L_p(k, \chi)$ can be given in terms of the *dual exponential map* of Bloch-Kato,

$$(17) \qquad \exp^*_{k,\eta} : H^1(\mathbb{Q}_p, \mathbb{Q}_{p,\eta}(k)(\eta)) \longrightarrow \mathbb{D}_{\mathrm{dR}}(\mathbb{Q}_{p,\eta}(k)(\eta)),$$

obtained by dualising the map $\exp_{1-k,\eta^{-1}} := \log^{-1}_{1-k,\eta^{-1}}$ and combining local Tate duality with the natural duality between $\mathbb{D}_{\mathrm{dR}}(\mathbb{Q}_{p,\eta}(1-k)(\eta^{-1}))$ and $\mathbb{D}_{\mathrm{dR}}(\mathbb{Q}_{p,\eta}(k)(\eta))$. The kernel of the map $\exp^*_{k,\eta}$ consists precisely of the extensions of $\mathbb{Q}_p$ by $\mathbb{Q}_{p,\eta}(k)(\eta)$ which are cristalline, and $\exp^*_{k,\eta}$ is an isomorphism of one-dimensional $\mathbb{Q}_{p,\eta}$-vector spaces for all $k \leq 0$. The following theorem for $k \leq 0$ [PR4, 3.2.2] is one of the simplest instances of so-called *reciprocity laws* relating L-values to distinguished global cohomology classes with values in the associated p-adic representation:

$$(18) \qquad L_p(k, \chi) = \frac{(1 - \chi(p)p^{-k})}{(1 - \chi^{-1}(p)p^{k-1})} \times \frac{(-k)!t^k}{\mathfrak{g}(\chi)} \times \exp^*_{k,\chi}(\kappa_{k,\chi}).$$

Equations (16) and (18) give a satisfying intepretation of $L_p(k, \chi)$ at *all* integers $k \in \mathbb{Z}$ in terms of the global classes $\kappa_{k,\chi}$. In particular the global classes $\kappa_{k,\chi}$ are non-trivial, and in fact non-cristalline, whenever $L_p(k, \chi) \neq 0$ and $k < 0$. Since k is then in the region of classical interpolation defining $L_p(s, \chi)$, the non-vanishing of $L_p(k, \chi)$ is directly related to the behaviour of the corresponding classical L-function. In this region of classical interpolation, the complex L-value attached to the p-adic representation $\mathbb{Q}_{p,\chi}(k)(\eta)$ is thus

given new meaning, as the *obstruction* to the global class $\kappa_{k,\chi} \in H^1(\mathbb{Q}, \mathbb{Q}_{p,\chi}(k)(\chi))$ being cristalline at p.

1.2. **Elliptic units.** In addition to the cusps, modular curves are endowed with a second distinguished class of algebraic points: the *CM points* attached to the moduli of elliptic curves with complex multiplication by an order in a quadratic imaginary field K. The values of modular units at such points give rise to units in abelian extensions of K, the so-called *elliptic units*, which play the same role for abelian extensions of K as circular units in the study of cyclotomic fields. The resulting p-adic families of global cohomology classes are the main ingredient in the seminal work of Coates and Wiles on the arithmetic of elliptic curves with complex multiplication.

The Eisenstein series $E_{k,\chi}$ of (1), viewed as a function of a variable τ in the complex upper half-plane $\mathcal{H}$ by setting $q = e^{2\pi i \tau}$, is given by the well-known formula

$$(19) \qquad E_{k,\chi}(\tau) := N^k \mathfrak{g}(\bar{\chi})^{-1} \frac{(k-1)!}{(2\pi i)^k} {\sum_{(m,n) \in N\mathbb{Z} \times \mathbb{Z}}}' \frac{\bar{\chi}(n)}{(m\tau + n)^k},$$

where $\mathfrak{g}(\bar{\chi})$ is the Gauss sum defined in (8), and the superscript $'$ indicates that the sum is to be taken over the non-zero lattice vectors in $N\mathbb{Z} \times \mathbb{Z}$.

Assume for simplicity that K has class number one, trivial unit group $\mathcal{O}_K^\times = \pm 1$, and odd discriminant $D < 0$. Assume also that there is an integral ideal–and hence, an element $\mathfrak{n}$ of $\mathcal{O}_K$–satisfying

$$(20) \qquad \mathcal{O}_K/\mathfrak{n} = \mathbb{Z}/N\mathbb{Z}.$$

One then says that the Eisenstein series $E_{k,\chi}$ satisfies the *Heegner hypothesis* relative to K. Under this hypothesis, the even character χ gives rise to a finite order character $\chi_\mathfrak{n}$ of conductor $\mathfrak{n}$ on the ideals of K by the rule

$$(21) \qquad \chi_\mathfrak{n}((\alpha)) := \bar{\chi}(\alpha \bmod \mathfrak{n}).$$

After writing

$$(22) \qquad \tau_\mathfrak{n} = \frac{b + \sqrt{D}}{2N}, \qquad \text{where} \quad \mathfrak{n} = \mathbb{Z}N + \mathbb{Z}\frac{b + \sqrt{D}}{2},$$

a direct calculation using (19) shows that

$$(23) \qquad E_{k,\chi}(\tau_\mathfrak{n}) = N^k \mathfrak{g}(\bar{\chi})^{-1} \frac{(k-1)!}{(2\pi i)^k} L(K, \chi_\mathfrak{n}, k, 0),$$

where for all $k_1, k_2 \in \mathbb{Z}$ with $k_1 + k_2 > 2$,

$$L(K, \chi_\mathfrak{n}, k_1, k_2) := {\sum_{\alpha \in \mathcal{O}_K}}' \chi_\mathfrak{n}(\alpha) \alpha^{-k_1} \bar{\alpha}^{-k_2}.$$

Note that $L(K, \chi_\mathfrak{n}, s) := \frac{1}{2} L(K, \chi_\mathfrak{n}, s, s)$, viewed as a function of a complex variable s, is the usual Hecke L-function attached to the finite order character $\chi_\mathfrak{n}$. The relation (23) between values of $E_{k,\chi}$ at CM points and L-series of quadratic imaginary fields is the direct counterpart of (1) relating the value at the cusps with Dirichlet L-series. For the

purpose of p-adic interpolation it will be convenient, at least initially, to consider the values $L(K, \chi_{\mathfrak{n}}, k, 0)$ rather than the values $L(K, \chi_{\mathfrak{n}}, k, k)$.

Under the moduli interpretation of $\Gamma_1(N) \backslash \mathcal{H} = Y_1(N)(\mathbb{C})$, the point $\tau \in \mathcal{H}$ corresponds to the pair $(\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau, 1/N)$ consisting of an elliptic curve over $\mathbb{C}$ with a marked point of order N. The point $\tau_{\mathfrak{n}}$ corresponds in this way to the pair $(\mathbb{C}/\bar{\mathfrak{n}}^{-1}, 1/N)$. Since $\bar{\mathfrak{n}}^{-1}$ is a fractional ideal of $\mathcal{O}_K$, the elliptic curve $A = \mathbb{C}/\bar{\mathfrak{n}}^{-1}$ has complex multiplication by $\mathcal{O}_K$ and hence has a model defined over K and even over $\mathcal{O}_K$, while $1/N$ represents an $\mathfrak{n}$-torsion point of A, denoted $t_{\mathfrak{n}}$, and hence is defined over the ray class field $K_{\mathfrak{n}}$ of K of conductor $\mathfrak{n}$. In particular the point $P_{\mathfrak{n}}$ of $X_1(N)$ attached to the pair $(A, t_{\mathfrak{n}})$ is rational over $K_{\mathfrak{n}}$.

Choose a Néron differential $\omega_A \in \Omega^1(A/\mathcal{O}_K)$ and let $\Omega_K \cdot \mathcal{O}_K \subset \mathbb{C}$ be the associated period lattice. This determines the complex number Ω_K uniquely up to sign, once ω_A has been chosen. Following Katz, an algebraic modular form of weight k on $\Gamma_1(N)$ can be viewed as a function f on the isomorphism classes of triples (E, t, ω_E) consisting of an elliptic curve (E, t) with $\Gamma_1(N)$-level structure and a choice of regular differential ω_E on E, satisfying a weight $-k$-homogeneity condition under scaling of ω_E. The convention relating both points of view is that $f(\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau, 1/N, dz) := f(\tau)$ for $\tau \in \mathcal{H}$, where dz is the standard differential on $\mathbb{C}/\Lambda$ whose period lattice is equal to Λ. In particular,

$$(24) \qquad E_{k,\chi}(A, t_{\mathfrak{n}}, \omega_A) = E_{k,\chi}\left(\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau_{\mathfrak{n}}, 1/N, \Omega_K \bar{\mathfrak{n}} dz\right) = \frac{E_{k,\chi}(\tau_{\mathfrak{n}})}{(\Omega_K \bar{\mathfrak{n}})^k},$$

where by an abuse of notation $\bar{\mathfrak{n}}$ is identified with one of its generators.

Because $E_{k,\chi}$ is defined over the field $\mathbb{Q}_\chi$ generated by the values of χ and the triple $(A, t_{\mathfrak{n}}, \omega_A)$ is defined over $K_{\mathfrak{n}}$, the quantities in (24) are algebraic, and in fact belong to the compositum $K_{\mathfrak{n},\chi}$ of $K_{\mathfrak{n}}$ and $\mathbb{Q}_\chi$. Therefore by (23), the normalised L-value

$$(25) \qquad \frac{E_{k,\chi}(\tau_{\mathfrak{n}})}{(\Omega_K \bar{\mathfrak{n}})^k} = \mathfrak{n}^k \mathfrak{g}(\bar\chi)^{-1} \frac{(k-1)!}{(2\pi i \cdot \Omega_K)^k} L(K, \chi_{\mathfrak{n}}, k, 0)$$

also belongs to $K_{\mathfrak{n},\chi}$.

Fix a prime $p \nmid N$, and let $X_{01}(p, N)$ be the modular curve attached to $\Gamma_0(p) \cap \Gamma_1(N)$. Recall the affinoid region $\mathcal{A}$ of Section 1.1 obtained by deleting the supersingular residue discs from $X_1(N)$. The natural algebraic projection $X_{01}(p, N) \longrightarrow X_1(N)$ admits a rigid analytic section $s : \mathcal{A} \longrightarrow X_{01}(p, N)(\mathbb{C}_p)$ over this ordinary locus, which sends the point attached to an ordinary pair (E, t) with $\Gamma_1(N)$-level structure to the point attached to the triple (E, C, t) where C is the *canonical subgroup* of E of order p. The ordinary p-stabilisation $E_{k,\chi}^{(p)}$ of $E_{k,\chi}$ defined in (2) can thus be viewed either as a classical modular form on $X_{01}(p, N)$ or as a p-adic modular form on $X_1(N)$ by pulling back to $\mathcal{A}$ via s.

Assume further that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K, let $K \hookrightarrow \mathbb{Q}_p$ be the embedding of K into its completion at $\mathfrak{p}$, and extend this to an embedding $\imath_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. The elliptic curve $A/\mathbb{C}_p$ deduced from A via $\imath_p$ is ordinary at p, and its canonical subgroup of order p is equal to the group scheme of its $\mathfrak{p}$-division points. In particular, this group is defined over K. It follows that the image $s(P_{\mathfrak{n}})$ of $P_{\mathfrak{n}} = (A, t_{\mathfrak{n}})$ belongs to $X_{01}(p, N)(K_{\mathfrak{n}})$. A direct calculation shows that it is represented by the pair $(\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau_{\mathfrak{p}\mathfrak{n}}, \frac{\bar{\mathfrak{p}}_{\mathfrak{n}}^{-1}}{pN})$, where $\tau_{\mathfrak{p}\mathfrak{n}} \in \Gamma_{01}(p, N) \backslash \mathcal{H}$ is defined by

the same equation as (22) but with $\mathfrak{n}$ replaced by $\mathfrak{pn}$:

$$(26) \qquad \tau_{\mathfrak{pn}} = \frac{b+\sqrt{D}}{2Np}, \qquad \text{where} \quad \mathfrak{pn} = \mathbb{Z}pN + \mathbb{Z}\frac{b+\sqrt{D}}{2},$$

and $\bar{\mathfrak{p}}_{\mathfrak{n}}^{-1}$ is any element of $\mathcal{O}_K/\mathfrak{n}\mathcal{O}_K$ which is congruent to $\bar{\mathfrak{p}}^{-1}$. The value of the p-adic modular form $E_{k,\chi}^{(p)}$ on the ordinary triple $(A, t_{\mathfrak{n}}, \omega_A)$ can therefore be calculated as

$$
E_{k,\chi}^{(p)}(A, t_{\mathfrak{n}}, \omega_A) \;=\; E_{k,\chi}^{(p)}\left(\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau_{\mathfrak{pn}}, \; \frac{\bar{\mathfrak{p}}_{\mathfrak{n}}^{-1}}{pN}, \; \bar{\mathfrak{p}}\bar{\mathfrak{n}}\Omega_K dz\right) \;=\; \frac{\chi_{\mathfrak{n}}(\bar{\mathfrak{p}})E_{k,\chi}^{(p)}(\tau_{\mathfrak{pn}})}{(\bar{\mathfrak{p}}\bar{\mathfrak{n}}\Omega_K)^k}
$$

$$(27) \qquad\qquad\qquad = \;(1 - \chi_{\mathfrak{n}}^{-1}(\mathfrak{p})\mathfrak{p}^k/p) \times \mathfrak{n}^k \mathfrak{g}(\bar{\chi})^{-1} \frac{(k-1)!}{(2\pi i \cdot \Omega_K)^k} L(K, \chi_{\mathfrak{n}}, k, 0),$$

where the second and third occurrence of $E_{k,\chi}^{(p)}$ are treated as classical modular forms of level pN, and (27) follows from a direct calculation based on (19) and (26). The fact that all the coefficients in the Fourier expansion (3) of $E_{k,\chi}^{(p)}$ extend to p-adic analytic functions of k on weight space $W = \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$ and that the pair $(A, t_{\mathfrak{n}})$ belongs to $\mathcal{A}$ suggests that the right-hand side of (27), normalised by a suitable p-adic period, should admit a similar prolongation. More precisely, the formal completion $\hat{A}$ of $A_{\mathcal{O}_{\mathbb{C}_p}}$ along its identity section is isomorphic to the formal multiplicative group $\hat{\mathbf{G}}_m$, and upon choosing an isomorphism $\imath : \hat{A} \longrightarrow \hat{\mathbf{G}}_m$ we may define a p-adic period $\Omega_p \in \mathbb{C}_p^\times$ by the rule

$$\omega_A = \Omega_p \cdot \omega_{\mathrm{can}},$$

where $\omega_{\mathrm{can}} := \imath^* \frac{dt}{t}$, with $\frac{dt}{t}$ the canonical differential on $\hat{\mathbf{G}}_m$, plays the role of the complex differential dz in the p-adic setting. The function $L_p(K, \chi_{\mathfrak{n}}, k)$ of k defined by

$$(28) \qquad L_p(K, \chi_n, k) = E_{k,\chi}^{(p)}(A, t_{\mathfrak{n}}, \omega_{\mathrm{can}}) = \Omega_p^k \cdot E_{k,\chi}^{(p)}(A, t_{\mathfrak{n}}, \omega_A),$$

extends to a p-adic analytic function of $k \in W$ and is equal (up to the p-adic period Ω_p^k) to the right-hand side of (27) for all integers $k \geq 2$. It is called the *Katz one-variable p-adic L-function* attached to K and to the character $\chi_{\mathfrak{n}}$.

Recall the Siegel units g_a and $g_a^{(p)}$ defined in equations (6) and (7) of Section 1.1. Evaluating these functions at the pair $(A, t_{\mathfrak{n}})$ gives rise to the *elliptic units*

$$u_{a,\mathfrak{n}} := g_a(A, t_{\mathfrak{n}}) = g_a(\tau_{\mathfrak{n}}), \qquad u_{a,\mathfrak{n}}^{(p)} := g_a^{(p)}(A, t_{\mathfrak{n}}) = g_a^{(p)}(\tau_{\mathfrak{pn}}) = u_{a,\mathfrak{n}}^{\sigma_{\mathfrak{p}}-p}$$

in $\mathcal{O}_{K_{\mathfrak{n}}}^\times$, where $\sigma_{\mathfrak{p}} \in \mathrm{Gal}(K_{\mathfrak{n}}/K)$ denotes the Frobenius element at $\mathfrak{p}$.

The following result of Katz (as well as its proof) is the direct counterpart of Leopoldt's formula (Theorem 1.1) in which cusps are replaced by CM points and circular units by elliptic units.

Theorem 1.2 (Katz). *Let χ be a non-trivial even primitive Dirichlet character of conductor N and let K be a quadratic imaginary field equipped with an ideal $\mathfrak{n}$ satisfying $\mathcal{O}_K/\mathfrak{n} = \mathbb{Z}/N\mathbb{Z}$. Let $\chi_{\mathfrak{n}}$ be the ideal character of K associated to the pair $(\chi, \mathfrak{n})$ as in (21).*

Then

$$L_p(K, \chi_{\mathfrak{n}}, 0) = -\frac{(1 - \chi_{\mathfrak{n}}(\mathfrak{p})p^{-1})}{\mathfrak{g}(\bar{\chi})} \times \sum_{a=1}^{N-1} \chi^{-1}(a) \log_p u_{a,\mathfrak{n}}.$$

Proof. Setting $k = 0$ in (28) gives

$$L_p(K, \chi_{\mathfrak{n}}, 0) = E_{0,\chi}^{(p)}(A, t_{\mathfrak{n}}),$$

where $E_{0,\chi}^{(p)}(A, t_{\mathfrak{n}})$ refers to $E_{0,\chi}^{(p)}(A, t_{\mathfrak{n}}, \omega)$ for *any* choice of regular differential ω on A, since $E_{0,\chi}$ is of weight zero and this value is therefore independent of ω. But by equations (11) and (9) in the proof of Theorem 1.1,

$$E_{0,\chi}^{(p)}(A, t_{\mathfrak{n}}) = h_\chi^{(p)}(A, t_{\mathfrak{n}}) = \frac{1}{p\mathfrak{g}(\bar{\chi})} \times \sum_{a=1}^{N-1} \chi^{-1}(a) \log_p g_a^{(p)}(A, t_{\mathfrak{n}})$$

$$= \frac{1}{p\mathfrak{g}(\bar{\chi})} \times \sum_{a=1}^{N-1} \chi^{-1}(a) \log_p u_{a,\mathfrak{n}}^{\sigma_{\mathfrak{p}}-p} = \frac{\chi_{\mathfrak{n}}(\mathfrak{p}) - p}{p\mathfrak{g}(\bar{\chi})} \times \sum_{a=1}^{N-1} \chi^{-1}(a) \log_p u_{a,\mathfrak{n}}.$$

The theorem follows. □

The calculations above can be extended by introducing the Katz two-variable p-adic L-function $L_p(K, \chi_{\mathfrak{n}}, k_1, k_2)$ which interpolates the values of $L(K, \chi_{\mathfrak{n}}, k_1, k_2)$ as k_1 and k_2 both vary over weight space. These more general L-values are related to the values at CM points of the non-holomorphic Eisenstein series

$$E_{k_1,k_2,\chi}(\tau) = N^{k_1+k_2}\mathfrak{g}(\bar{\chi})^{-1}\frac{(k_1-1)!}{(2\pi i)^{k_1}}(\tau - \bar{\tau})^{k_2} \sideset{}{'}\sum_{(m,n)\in N\mathbb{Z}\times\mathbb{Z}} \frac{\bar{\chi}(n)}{(m\tau+n)^{k_1}(m\bar{\tau}+n)^{k_2}}$$

by the formula generalising (23)

$$(29) \qquad E_{k_1,k_2,\chi}(\tau_{\mathfrak{n}}) = N^{k_1}\mathfrak{g}(\bar{\chi})^{-1}\frac{(k_1-1)!}{(2\pi i)^{k_1}}\sqrt{D}^{k_2}L(K, \chi_{\mathfrak{n}}, k_1, k_2).$$

The function $E_{k_1,k_2,\chi}$ is a real analytic function on $\mathcal{H}$ which transforms like a modular form of weight $k_1 - k_2$ and character χ under the action of $\Gamma_0(N)$. Although it is non-holomorphic in general, it can sometimes be expressed as the image of holomorphic modular forms under iterates of the *Shimura-Maass* derivative operator

$$(30) \qquad \delta_k = \frac{1}{2\pi i}\left(\frac{d}{d\tau} + \frac{k}{\tau - \bar{\tau}}\right)$$

sending real analytic modular forms of weight k to real analytic modular forms of weight $k + 2$. More precisely, after setting $\delta_k^r = \delta_{k+2r-2} \circ \cdots \circ \delta_{k+2} \circ \delta_k$, a direct calculation reveals that

$$\delta_k^r E_{k,\chi} = E_{k+r,-r,\chi}.$$

A *nearly holomorphic modular form of weight k on $\Gamma_1(N)$* is a linear combination

$$f = \sum_{i=1}^{t} \delta_{k-2j_i}^{j_i} f_i, \qquad f_i \in M_{k-2j_i}(\Gamma_1(N)),$$

where the f_i are classical modular forms of weight $k - 2j_i$ on $\Gamma_1(N)$. If the Fourier expansions of all the f_i at all the cusps have Fourier coefficients in a field L, then the nearly holomorphic modular form f is said to be *defined over L*.

Shimura proved that nearly holomorphic modular forms of weight k which are defined over $\bar{\mathbb{Q}}$ take algebraic values at CM triples like $(A, t_{\mathfrak{n}}, \omega_A)$. More precisely, if f is defined over $L \subset \bar{\mathbb{Q}}$, then

$$(31) \qquad f(A, t_{\mathfrak{n}}, \omega_A) := \frac{f(\tau_{\mathfrak{n}})}{(\bar{\mathfrak{n}}\Omega_K)^k} \text{ belongs to } LK_{\mathfrak{n}}.$$

A conceptual explanation for this striking algebraicity result rests on the relationship between nearly holomorphic modular forms of weight k and global sections of an algebraic vector bundle arising from the relative de Rham cohomology of the universal elliptic curve over $Y_1(N)$, and on the resulting interpretation of the Shimura-Maass derivative in terms of the Gauss-Manin connection on this vector bundle. See for instance Section 1.5 of [BDP] or Section 2.4 of [DR1] for a brief account of this circle of ideas, and Section 10.1 of [Hi2] for a more elementary treatment.

Specialising (31) to the setting where f is the Eisenstein series $E_{k+r,-r,\chi}$ of weight $k + 2r$ with $k \geq 2$ and $r \geq 0$, and invoking (29) leads to the conclusion that the special values

$$(32) \qquad E_{k+r,-r,\chi}(A, t_{\mathfrak{n}}, \omega_A) = \mathfrak{n}^{k+r}\bar{\mathfrak{n}}^{-r}\mathfrak{g}(\bar{\chi})^{-1}\frac{(k+r-1)!}{(2\pi i)^{k+r}\Omega_K^{k+2r}}\sqrt{D}^{-r}L(K, \chi_{\mathfrak{n}}, k+r, -r)$$

belong to the compositum $K_{\mathfrak{n},\chi}$ of $K_{\mathfrak{n}}$ and $\mathbb{Q}_\chi$, just as in (25), for all $k \geq 2$ and $r \geq 0$.

In light of this algebraicity result, it is natural to attempt to interpolate these values p-adically as a function of both k and r in weight space. This p-adic interpolation rests on the fact that the Shimura-Maass derivative admits a counterpart in the realm of p-adic modular forms: the Atkin-Serre operator d which raises the weight by two and acts as the differential operator $d = q\frac{d}{dq}$ on q-expansions. If f is a classical modular form of weight k with rational Fourier coefficients, the nearly holomorphic modular form $\delta_k^r f$ and the p-adic modular form $d^r f$ are objects of a very different nature, but nonetheless their values *agree* on ordinary CM triples (where it makes sense to compare them) so that in particular

$$(33) \qquad \delta_k^r f(A, t_{\mathfrak{n}}, \omega_A) = d^r f(A, t_{\mathfrak{n}}, \omega_A), \qquad \text{for all } r \geq 0.$$

The reason, which is explained for instance in Section 1.5 of [BDP], is that d admits *the same algebraic description* as δ_k in terms of the Gauss-Manin connection on a relative de Rham cohomology sheaf, with the sole difference that the (non-holomorphically varying) Hodge decomposition on the complex de Rham cohomology of the fibers is replaced in the p-adic setting by the *Frobenius decomposition* of the de Rham cohomology of the universal (ordinary) elliptic curve over $\mathcal{A}$. But the functorial action of the endomorphism algebra on algebraic de Rham cohomology causes these two decompositions to agree for ordinary CM elliptic curves.

Equation (33) means that, as far as values at ordinary CM triples are concerned, the p-adic modular form $d^r E_{k,\chi}$ is a perfect substitute for $E_{k+r,-r,\chi} = \delta_k^r E_{k,\chi}$. The Fourier

expansion of this p-adic avatar is given, for $r \geq 1$, by

$$(34) \qquad d^r E_{k,\chi} = \sum_{n=1}^{\infty} n^r \sigma_{k-1,\chi}(n) q^n.$$

The coefficients of q^n when $p|n$ do not extend to a p-adic analytic function of $(k, r) \in W^2$, and this difficulty persists after replacing $E_{k,\chi}$ by its ordinary p-stabilisation $E_{k,\chi}^{(p)}$. One is therefore led to consider instead the so-called p-*depletion* of $E_{k,\chi}$, defined by

$$(35) \qquad E_{k,\chi}^{[p]}(\tau) = E_{k,\chi}(\tau) - (1 + \chi(p)p^{k-1})E_{k,\chi}(p\tau) + \chi(p)p^{k-1}E_{k,\chi}(p^2\tau),$$

so that $E_{k+r,-r,\chi}^{[p]} := \delta_k^r E_{k,\chi}^{[p]}$–a nearly holomorphic modular form of level Np^2–is given by

$$(36) \qquad E_{k+r,-r,\chi}(\tau) - (p^r + \chi(p)p^{k+r-1})E_{k+r,-r,\chi}(p\tau) + \chi(p)p^{k+2r-1}E_{k+r,-r,\chi}(p^2\tau),$$

while its p-adic avatar $d^r E_{k,\chi}^{[p]}$–a p-adic modular form of level N–has Fourier expansion

$$d^r E_{k,\chi}^{[p]} = \sum_{p \nmid n}^{\infty} n^r \sigma_{k-1,\chi}(n) q^n.$$

The coefficients in this expansion do extend to p-adic analytic functions of k and r on weight space $W = \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$. Just as in the construction of $L_p(K, \chi_{\mathfrak{n}}, k)$, this suggests that the function $L_p(K, \chi_{\mathfrak{n}}, k_1, k_2)$ defined (for $k \geq 2$ and $r \geq 0$) by the rule

$$
\begin{aligned}
\frac{L_p(K, \chi_{\mathfrak{n}}, k+r, -r)}{\Omega_p^{k+2r}} &:= d^r E_{k,\chi}^{[p]}(A, t_{\mathfrak{n}}, \omega_A) = E_{k+r,-r,\chi}^{[p]}\left(\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau_{\mathfrak{p}^2\mathfrak{n}}, \frac{\bar{\mathfrak{p}}_{\mathfrak{n}}^{-2}}{p^2 N}, \ \bar{\mathfrak{p}}^2\bar{\mathfrak{n}}\Omega_K dz\right) \\
(37) \qquad\qquad &= (1 - \chi_{\mathfrak{n}}^{-1}(\mathfrak{p})\mathfrak{p}^{k+r}\bar{\mathfrak{p}}^{-r}/p) \times (1 - \chi_{\mathfrak{n}}(\bar{\mathfrak{p}})\mathfrak{p}^r\bar{\mathfrak{p}}^{-k-r}) \ \times \\
&\qquad \mathfrak{n}^{k+r}\bar{\mathfrak{n}}^{-r}\mathfrak{g}(\bar{\chi})^{-1}\frac{(k+r-1)!}{(2\pi i)^{k+r}\Omega_K^{k+2r}}\sqrt{D}^{-r}L(K, \chi_{\mathfrak{n}}, k+r, -r)
\end{aligned}
$$

extends to an analytic function of $(k, r) \in W^2$. The function $L_p(K, \chi_{\mathfrak{n}}, k_1, k_2)$ is called the Katz *two-variable p-adic L-function* attached to the character $\chi_{\mathfrak{n}}$. Note that the restriction of $L_p(K, \chi_{\mathfrak{n}}, k_1, k_2)$ to the line $k_2 = 0$ is related to the Katz one-variable p-adic L-function by the rule

$$L_p(K, \chi_{\mathfrak{n}}, k, 0) = (1 - \chi_{\mathfrak{n}}(\bar{\mathfrak{p}})\bar{\mathfrak{p}}^{-k})L_p(K, \chi_{\mathfrak{n}}, k).$$

The ratio of the two sides (which can be seen to be a p-adic analytic function of k, since $\bar{\mathfrak{p}}$ belongs to $\mathcal{O}_{K_{\mathfrak{p}}}^{\times}$) reflects the difference between working with the ordinary p-stabilisation $E_{k,\chi}^{(p)}$ and the p-depletion $E_{k,\chi}^{[p]}$.

The following variant of Katz's Theorem 1.2 expresses the special value $L_p(K, \chi_{\mathfrak{n}}^{-1}, 1, 1)$ in terms of the elliptic units $u_{a,\mathfrak{n}}^* := g_a(w(A, t_{\mathfrak{n}}))$, where w is the *Atkin-Lehner involution* such that $g_a(w(A, t_{\mathfrak{n}})) = g_a(-1/N\tau_{\mathfrak{n}})$.

Theorem 1.3 (Katz). *With notations as in Theorem 1.2,*

$$L_p(K, \chi_{\mathfrak{n}}^{-1}, 1, 1) = (1 - \chi_{\mathfrak{n}}^{-1}(\bar{\mathfrak{p}}))(1 - \chi_{\mathfrak{n}}(\mathfrak{p})/p) \times \sum_{a=1}^{N-1} \chi^{-1}(a) \log_p u_{a,\mathfrak{n}}^*.$$

Proof. By setting $k = 2$ and $r = -1$ in equation (37) (and replacing $\chi_{\mathfrak{n}}$ by $\chi_{\mathfrak{n}}^{-1}$), we obtain

$$L_p(K, \chi_{\mathfrak{n}}^{-1}, 1, 1) = d^{-1} E_{2,\chi^{-1}}^{[p]}(A, t_{\mathfrak{n}}, \omega_A).$$

The p-adic modular form of weight 0 which appears on the right hand side of this identity has the same Fourier expansion as the p-depletion of the Eisenstein series $E_0(\chi^{-1}, 1)$ introduced in the remark following Theorem 1.1 of Section 1. Combined with [Hi1, Lemma 5.3], we thus see that

$$L_p(K, \chi_{\mathfrak{n}}^{-1}, 1, 1) = \mathfrak{g}(\bar{\chi}) w E_{0,\chi}^{[p]}(A, t_{\mathfrak{n}}, \omega_A),$$

where w is the *Atkin-Lehner involution* at N introduced above. The result then follows from an almost identical calculation to the proof of Theorem 1.2. $\square$

In stark analogy with the case of circular units, the expressions

$$u_{\chi_{\mathfrak{n}}} := \prod_{a=1}^{N-1} u_{a,\mathfrak{n}}^{\chi^{-1}(a)} = \prod_{\sigma \in \mathrm{Gal}(K_{\mathfrak{n}}/K)} (\sigma u_{1,\mathfrak{n}})^{\chi_{\mathfrak{n}}(\sigma)}$$

are (formal $\mathbb{Q}_\chi$-linear combination of) special units in $K_{\mathfrak{n}}$ lying in the $\chi_{\mathfrak{n}}^{-1}$-eigenspace for the natural action of G_K.

These units arise as the "bottom layer" of a norm-coherent family of elliptic units over the two-variable $\mathbb{Z}_p$-extension K_∞ of K. The same construction as in equation (12) of Section 1.1 leads to a global cohomology class

$$\kappa_{\chi_{\mathfrak{n}},\infty} \in H^1(K, \Lambda_K(\chi_{\mathfrak{n}})),$$

where $\Lambda_K = \mathbb{Z}_p[\![\mathrm{Gal}(K_\infty/K)]\!]$ is the *two-variable* Iwasawa algebra attached to K, equipped with its tautological G_K-action. The Galois module $\Lambda_K(\chi_{\mathfrak{n}})$ gives a p-adic interpolation of the Hecke characters of the form $\chi_{\mathfrak{n}}\phi$ where ϕ is a Hecke character of p-power conductor.

In particular, if ψ is a Hecke character of infinity type (k_1, k_2) arising as a specialisation of $\Lambda_K(\chi_{\mathfrak{n}})$, the global class

$$\kappa_\psi \in H^1(K, \mathbb{Q}_p(\psi))$$

obtained by specialising $\kappa_{\chi_{\mathfrak{n}},\infty}$ at ψ, although it arises from elliptic units, encodes arithmetic information about a Galois representation V_ψ of K attached to a Hecke character of possible infinite order.

The reciprocity law of Coates and Wiles expresses the Katz two-variable p-adic L-function $L_p(K, \chi_{\mathfrak{n}}, k_1, k_2)$ as the image under a "big exponential map" of the global class $\kappa_{\mathfrak{n},\infty}$. By an analogue of (18) of Section 1.1, the special value $L(\psi^{-1}, 0)$ can then be interpreted as the obstruction to the class κ_ψ being cristalline at $\mathfrak{p}$, whenever ψ lies in the range of classical interpolation for the Katz p-adic L-function.

A classical result of Deuring asserts that the p-adic Tate module of a CM elliptic curve, viewed as a representation of G_K, is always of the form V_ψ for a suitable Hecke character ψ of K of infinity type $(1, 0)$. The global class κ_ψ for such a ψ acquires a special interest in relation to the Birch and Swinnerton-Dyer conjecture for the elliptic curve A_ψ attached to it. This connection between $L(A_\psi, 1)$ and the singular parts of global classes κ_ψ was used

by Coates-Wiles to give what historically was the first broad piece of convincing supporting evidence for the conjecture of Birch and Swinnerton-Dyer, notably the implication

$$L(A, 1) \neq 0 \quad \Longrightarrow \quad A(K) \otimes \mathbb{Q} = \{0\},$$

for any elliptic curve $A/\mathbb{Q}$ with complex multiplication by K.

1.3. **Heegner points.** This section replaces the Eisenstein series $E_{k,\chi}$ of Section 1.2 by a *cusp form* of weight k. The argument used in the proof of Theorem 1.3 then leads naturally to the p-adic Gross-Zagier formula of [BDP] expressing the special values of certain *p-adic Rankin L-series* attached to f and K in terms of the p-adic Abel-Jacobi images of so-called *generalised Heegner cycles*.

For simplicity, let $f \in S_k(N)$ be a normalized cuspidal eigenform of *even weight k* on $\Gamma_1(N)$ with rational Fourier coefficients and trivial nebentypus character. (When $k = 2$, the form f is therefore associated to an elliptic curve $E/\mathbb{Q}$.) Let K be a quadratic imaginary field satisfying all the hypotheses of Section 1.2, including the Heegner assumption (20). We will also assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ is, as before, a rational prime which splits in K and does not divide N.

By analogy with the construction of the Katz two-variable p-adic L-function, it is natural to consider the quantities

(38) $$\delta_k^r f(A, t_\mathfrak{n}, \omega_A) = d^r f(A, t_\mathfrak{n}, \omega_A),$$

which belong to $K_\mathfrak{n}$ for all $r \geq 0$. The role of the relatively elementary formula (32) of Section 1.2 relating such quantities to L-values when f is an Eisenstein series is played in this context by a seminal formula of Waldspurger, whose importance for the arithmetic study of generalised Heegner cycles and points would be hard to overstate, even though its proof lies beyond the scope of this survey.

Waldspurger's formula relates (38) to the L-function of f twisted (over K) by certain unramified Hecke characters of K. If ϕ is such a character (viewed as a mutiplicative function on fractional ideals of K in the usual way) then the L-series $L(f, K, \phi, s)$ of f/K twisted by ϕ is defined (for $s \in \mathbb{C}$ in some right-half plane) by the Euler product

$$L(f, K, \phi, s) = \prod_\mathfrak{l} \left[(1 - \alpha_{\mathbf{N}\mathfrak{l}}(f) \cdot \phi(\mathfrak{l}) \mathbf{N}\mathfrak{l}^{-s})(1 - \beta_{\mathbf{N}\mathfrak{l}}(f) \cdot \phi(\mathfrak{l}) \mathbf{N}\mathfrak{l}^{-s}) \right]^{-1}$$

taken over the prime ideals $\mathfrak{l}$ in $\mathcal{O}_K$, where $\alpha_\ell(f)$ and $\beta_\ell(f)$ are the roots of the Hecke polynomial $x^2 - a_\ell(f)x + \ell^{k-1}$ for f at ℓ, and we set $\alpha_{\mathbf{N}\mathfrak{l}} := \alpha_\ell(f)^t$ and $\beta_{\mathbf{N}\mathfrak{l}} := \beta_\ell(f)^t$ if $\mathbf{N}\mathfrak{l} = \ell^t$. Rankin's method can be used to show that $L(f, K, \phi, s)$ admits an analytic continuation to the entire complex plane.

If k_1 and k_2 are integers with the same parity, let ϕ_{k_1,k_2} be the unramified Hecke character of K of infinity type (k_1, k_2) defined on fractional ideals by the rule

(39) $$\phi_{k_1,k_2}((\alpha)) = \alpha^{k_1} \bar{\alpha}^{k_2},$$

and set

$$L(f, K, k_1, k_2) := L(f, K, \phi_{k_1,k_2}^{-1}, 0).$$

Because

$$L(f, K, k_1 + s, k_2 + s) = L(f, K, \phi_{k_1,k_2}^{-1}, s),$$

we may view $L(f, K, k_1, k_2)$ as a function on pairs $(k_1, k_2) \in \mathbb{C}^2$ with $k_1 - k_2 \in 2\mathbb{Z}$. The functional equation relates the values $L(f, K, k_1, k_2)$ and $L(f, K, k - k_2, k - k_1)$ and hence corresponds to the reflection about the line $k_1 + k_2 = k$ in the (k_1, k_2)-plane, preserving the perpendicular lines $k_1 - k_2 = k + 2r$ for any $r \in \mathbb{Z}$. The restriction of $L(f, K, k_1, k_2)$ to such a line therefore admits a functional equation whose sign depends on r in an interesting way. More precisely, it turns out that assumption (20) forces this sign to be -1 when $1 - k \leq r \leq -1$, and to be $+1$ for other integer values of r. In particular, the central critical value $L(f, K, k + r, -r)$ vanishes for reasons of sign when $1 - k \leq r \leq -1$, but is expected to be non-zero for infinitely many $r \geq 0$. The quantities $L(f, K, k + r, -r)$ are precisely the special values that arise in the formula of Waldspurger, which asserts (cf. [BDP, Thm. 5.4]) that

$$(40) \qquad (\delta_k^r f(A, t_\mathfrak{n}, \omega_A))^2 = 1/2 \cdot (2\pi/\sqrt{D})^{k+2r-1} r!(k+r-1)! \cdot \frac{L(f, K, k + r, -r)}{(2\pi i \cdot \bar{\mathfrak{n}}\Omega_K)^{2(k+2r)}}.$$

Note the square that appears on the left-hand side. It is an enlightening exercise to recover equation (32) of Section 1.2 (up to an elementary constant) by setting $f = E_{k,\chi}$ and replacing $\phi_{k+r,-r}$ by $\chi_\mathfrak{n}\phi_{k+r,-r}$ in equation (40). This suggests that Waldspurger's formula should be viewed as the natural extension of (32) to the setting where Eisenstein series are replaced by cusp forms. An important difference with the setting of Section 1.2 is that the quantities $\delta_k^r f(A, t_\mathfrak{n}, \omega_A)$ only encode the *central critical values* $L(f, K, k + r, -r)$. In the setting of Eisenstein series the analogous value $L(E_{k,\chi}, K, (\chi_\mathfrak{n}\phi_{k+r,-r})^{-1}, 0)$ is also central critical, but breaks up as a product

$$L(E_{k,\chi}, K, (\chi_\mathfrak{n}\phi_{k+r,-r})^{-1}, 0) = L(K, (\chi_\mathfrak{n}\phi_{k+r,-r})^{-1}, 0) \times L(K, (\chi_\mathfrak{n}^{-1}\phi_{1-k-r,1+r})^{-1}, 0)$$

whose two factors, which are values of L-functions of Hecke characters, are interchanged by the functional equation and are non-self dual in general.

In order to interpolate the values $d^r f(A, t_\mathfrak{n}, \omega_A)$ p-adically, we replace, exactly as in (35), the modular form f by its p-depletion

$$(41) \qquad\qquad f^{[p]}(\tau) := f(\tau) - a_p(f)f(p\tau) + p^{k-1}f(p^2\tau).$$

A direct calculation using (41) shows that for all $r \geq 0$,

$$(42) \qquad d^r f^{[p]}(A, t_\mathfrak{n}, \omega_A) = (1 - a_p\mathfrak{p}^r\bar{\mathfrak{p}}^{-k-r} + \mathfrak{p}^{k+2r-1}\bar{\mathfrak{p}}^{-k-2r-1})d^r f(A, t_\mathfrak{n}, \omega_A).$$

Since the collection of modular forms indexed by the parameter r

$$(43) \qquad\qquad d^r f^{[p]}(q) = \sum_{(n,p)=1} n^r a_n(f) q^n$$

is a p-adic family of modular forms in the sense defined in Section 1.1, it follows, just as in Section 1.2, that the product of (42) by the p-adic period Ω_p^{k+2r} extends to a p-adic analytic function of $r \in W = (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p$. In light of (38) and (40), the quantity defined by

$$(44) \qquad\qquad L_p(f, K, k + r, -r) := \Omega_p^{2(k+2r)} \times d^r f^{[p]}(A, t_\mathfrak{n}, \omega_A)^2$$

is referred to as the *anticyclotomic p-adic L-function* attached to f and K.

The value $L_p(f, K, k + r, -r)$ of this p-adic L-function for $r < 0$ is defined by p-adic continuity rather than by the direct interpolation of classical L-values. For $r \in [1 - k, -1]$, the quantity $L_p(f, K, k + r, -r)$ should be regarded as a genuinely p-adic avatar of the special value $L(f/K, \phi_{k+r,-r}^{-1}, 0)$—or rather, of the first derivative $L'(f/K, \phi_{k+r,-r}^{-1}, 0)$ since the value vanishes for parity reasons. The main result of [BDP] relates it to the image under the p-adic Abel-Jacobi map of the *generalised Heegner cycles* introduced in loc.cit.

We will now state the main result of [BDP] in the illustrative special case where $k = 2$ and f is attached to an elliptic curve E of conductor N. Let $P_K \in J_0(N)(K)$ be the class of the degree 0 divisor $(A, t_\mathfrak{n}) - (\infty)$ in the Jacobian variety $J_0(N)$ of $X_0(N)$, let $P_{f,K}$ denote its image in $E(K)$ under the modular parametrisation

$$\varphi_E : J_0(N) \longrightarrow E$$

arising from the modular form f, and let ω_E be the regular differential on E satisfying $\varphi_E^*(\omega_E) = \omega_f := (2\pi i)f(\tau)d\tau$.

Theorem 1.4. *Let $f \in S_2(N)$ be a normalized cuspidal eigenform of level $\Gamma_0(N)$ with N prime to p and let K be a quadratic imaginary field equipped with an integral ideal $\mathfrak{n}$ satisfying $\mathcal{O}_K/\mathfrak{n} = \mathbb{Z}/N\mathbb{Z}$. Then*

$$L_p(f, K, 1, 1) = \left(\frac{1 - a_p(f) + p}{p} \right)^2 \log_p(P_{K,f})^2,$$

where $\log_p$ is the formal group logarithm on E associated with the regular differential ω_E.

Sketch of proof. By (44) with $k = 2$ and $r = -1$,

$$(45) \qquad L_p(f, K, 1, 1) = d^{-1} f^{[p]}(A, t_\mathfrak{n}, \omega_A)^2.$$

As in the proof of Theorems 1.1, 1.2 and 1.3, the result will be obtained by interpreting the p-adic modular function $F^{[p]} := d^{-1} f^{[p]}$ as the rigid analytic primitive of the differential $\omega_{f^{[p]}}$ on $\mathcal{A}$ which vanishes at the cusp ∞, and relating this rigid analytic function to the *Coleman primitive* of ω_f.

Recall the lift of Frobenius Φ on the ordinary locus $\mathcal{A} \subset X_0(N)(\mathbb{C}_p)$. A direct computation shows that

$$(46) \qquad \Phi\omega_f = p\omega_{Vf}$$

as sections of $\Omega^1_{X_0(N)}$ over $\mathcal{A}$, where V is the operator on p-adic modular forms acting as $f(q) \mapsto f(q^p)$ on q-expansions.

Let F denote the Coleman primitive of ω_f on $\mathcal{A}$. It is a *locally analytic* function on $\mathcal{A}$, which is well-defined up to a constant and satisfies $dF = \omega_f$ on $\mathcal{W}_\epsilon$. By definition of the Coleman primitive,

$$F(A, t_\mathfrak{n}) = \log_{\omega_f}(P_K) = \log_p(P_{K,f}),$$

where $\log_{\omega_f}$ is the p-adic logarithm on $J_0(N)$ attached to the differential ω_f, and we have used the fact that $\varphi_E(P_K) = P_{K,f}$ and $\varphi_E^*(\omega_E) = \omega_f$ to derive the second equality. The result now follows from noting that

$$F^{[p]}(A, t_\mathfrak{n}) = (1 - a_p(f)/p + 1/p)F(A, t_\mathfrak{n}).$$

$\square$

Like circular units and elliptic units, the Heegner point $P_{K,f} \in E(K)$ arises naturally as a *universal norm* of a compatible system of points defined over the so-called *anti-cyclotomic* $\mathbb{Z}_p$-*extension* of K. This extension, denoted K_∞^-, is contained in the two-variable $\mathbb{Z}_p$-extension K_∞ of Section 1.2 and is the largest subextension which is Galois over $\mathbb{Q}$ and for which $\mathrm{Gal}(K/\mathbb{Q})$ acts as -1 on $\mathrm{Gal}(K_\infty^-/K)$ via conjugation. After letting $\Lambda_K^- = \mathbb{Z}_p[\![\mathrm{Gal}(K_\infty^-/K)]\!]$ be the Iwasawa algebra attached to this extension, equipped with its tautological action of G_K, this norm-compatible collection of Heegner points can be parlayed into the construction of a global cohomology class

$$\kappa_{f,K,\infty} \in H^1(K, V_p(E) \otimes \Lambda_K^-),$$

where $V_p(E)$ is the Galois representation attached to the p-adic Tate module of E.

The module $\Lambda_K^- \otimes_{\mathbb{Z}_p} V_p(E)$ is a deformation of $V_p(E)$ which p-adically interpolates the twists of $V_p(E)$ by the *anticyclotomic* Hecke characters $\phi_{r,-r}$, and hence the class $\kappa_{f,K,\infty}$ admits specialisations

$$\kappa_{f,K,r} \in H^1(K, V_p(E) \otimes \mathbb{Q}_p(\phi_{r,-r})).$$

When $r = 0$ this class arises from the image of Heegner points under the Kummer map, and hence is cristalline at p. Theorem 1.4 asserts that its p-adic logarithm is related to the values of the anti-cyclotomic p-adic L-function $L_p(f, K, 1+r, 1-r)$ at the point $(1,1)$ that lies outside the range of classical interpolation.

In contrast, the classes $\kappa_{f,K,r}$ need not be cristalline at p when $r > 0$, and the formalism described in the previous sections suggests that the image of $\kappa_{f,K,r}$ under the dual exponential map at p should be related to the value $L_p(f, K, 1+r, 1-r)$, a simple non-zero multiple of $L(f, K, \phi_{r,-r}^{-1}, 1)$ since $(1+r, 1-r)$ lies in the range of classical interpolation defining the anticyclotomic p-adic L-function. One thus expects, just as in Sections 1.1 and 1.2, a direct relation between the images of the "higher weight" specializations $\kappa_{f,K,r}$ under the Bloch-Kato dual exponential map and the central critical values $L(f, \phi_{-r,r}, 1)$. Current ongoing work of the second author aims to exploit the classes $\kappa_{f,K,r}$ and Kolyvagin's "method of Euler systems", as summarised in [Kato, §13], for example, to derive new cases of the Bloch-Kato conjecture of the form

(47) $L(f, \phi_{r,-r}, 1) \neq 0 \implies \mathrm{Sel}_K(V_p(E)(\phi_{r,-r})) = \{0\}, \qquad (r > 0).$

Using Hida families and a p-adic deformation along them of the Euler system of Heegner points due to Howard [How], one can more generally hope to establish the analogue of (47) for p-ordinary cuspidal eigenforms of even weight $k \geq 2$. This program is being carried out in [Cas] based in part on a suitable extension of the circle of ideas described in this section.

2. EULER SYSTEMS OF GARRETT-RANKIN-SELBERG TYPE

The three formulae described in Chapter 1 relate the p-adic logarithms of circular units, elliptic units and Heegner points to values of associated p-adic L-functions at points that lie outside their range of classical interpolation. The construction of the p-adic L-function in all three cases rests on formulae for critical L-values in terms of the *values* of modular forms at distinguished points of modular curves, namely cusps or CM points.

There is a different but equally useful class of special value formulae arising from the Rankin-Selberg method. The prototypical such formula is concerned with a triple (f, g, h) of eigenforms of weights k, ℓ, and m respectively with $k = \ell + m + 2r$ and $r \geq 0$. It involves the Petersson scalar product

$$I(f, g, h) := \langle f, g \times \delta_m^r h \rangle,$$

and relates the square of this quantity to the *central critical value $L(f \otimes g \otimes h, \frac{k+\ell+m-2}{2})$* of the convolution L-function attached to f, g and h. An overarching theme of Chapter 2 is that the quantity $I(f, g, h)$ can be p-adically interpolated as f, g and h are made to vary over a suitable set of classical specialisations of Hida families. In particular, when f, g and h are of weight two, forcing r to tend p-adically to -1 in weight space, the resulting p-adic limit of $I(f, g, h)$, denoted $I_p(f, g, h)$—a p-adic L-value—acquires an interpretation as the Bloch-Kato p-adic logarithm of a global cohomology class arising from a suitable geometric construction, thereby motivating the study of the following Euler systems:

(1) When g and h are Eisenstein series, the invariant $I_p(f, g, h)$ is related in Section 2.1 to the *p-adic regulator*

$$\mathbf{reg}_p\{u_g, u_h\}(\eta_f),$$

were u_g and u_h are the modular units whose logarithmic derivatives are equal to g and h respectively, $\{u_g, u_h\} \in K_2(X_1(N))$ is the *Beilinson element* in the second K-group of $X_1(N)$ formed essentially by taking the cup-product of these two units, and η_f is a suitable class in $H^1_{\mathrm{dR}}(X_1(N))$ attached to f. The p-adic regulator has a counterpart in p-adic étale cohomology and the images of the Beilinson elements under this map lead to a system of global cohomology classes which underlie Kato's study of the Mazur-Swinnerton-Dyer p-adic L-function attached to classical modular forms via the theory of Euler systems.

(2) When only h is an Eisenstein series and f and g are cuspidal, the invariant $I_p(f, g, h)$ is again related, in Section 2.2, to a p-adic regulator of the form

$$\mathbf{reg}_p(\Delta_{u_h})(\eta_f \wedge \omega_g),$$

where Δ_{u_h} is a *Beilinson-Flach element* in $K_1(X_1(N) \times X_1(N))$ attached to the modular unit u_h viewed as a function on a diagonally embedded copy of $X_1(N) \subset X_1(N)^2$. The Euler system of *Beilinson-Flach elements* is obtained by replacing the p-adic regulator by its p-adic étale counterpart; some of its possible arithmetic applications are discussed in Section 2.2.

(3) When f, g and h are all cusp forms, the invariant $I_p(f, g, h)$ is related in Section 2.3 to

$$\mathrm{AJ}_p(\Delta)(\eta_f \wedge \omega_g \wedge \omega_h),$$

where
$$\mathrm{AJ}_p : \mathrm{CH}^2(X_1(N)^3)_0 \longrightarrow \mathrm{Fil}^2(H^3_{\mathrm{dR}}(X_1(N)^3))^\vee$$
is the p-adic Abel-Jacobi map, and Δ is the *Gross-Kudla-Schoen* cycle obtained by a simple modification of the diagonal cycle in $X_1(N)^3$. The resulting Euler system of *Gross-Kudla-Schoen diagonal cycles* and some of its eventual arithmetic applications are described in Section 2.3.

2.1. **Beilinson-Kato elements.** Let $f \in S_2(N)$ be a cuspidal eigenform on $\Gamma_0(N)$ (not necessarily new of level N), and let p be an odd prime not dividing N. Assume that p is ordinary for f, relative to a fixed embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$. Denote by $L_p(f, s)$ the Mazur-Swinnerton-Dyer p-adic L-function attached to f. This p-adic L-function is defined as in [MTT] via the p-adic interpolation of the complex central critical values $L(f, \xi, 1)$, with ξ varying among the Dirichlet characters of p-power conductor.

One goal of this section is to explain the connection between the value of $L_p(f, s)$ at the point $s = 2$, lying outside the range of classical interpolation for $L_p(f, s)$, and the image of so-called Beilinson-Kato elements by the p-adic syntomic regulator on K_2 of the modular curve of level N. The resulting formula is a p-adic analogue of Beilinson's theorem [Bei2] relating $L(f, 2)$ to the complex regulator of the Beilinson-Kato elements considered above.

Before stating the main theorem of this section, we recall the definition of the p-adic regulator, following [Bes3] and [BD1] (to which we refer for more detailed explanations). The reader should notice that this definition builds upon the techniques of p-adic integration that played a crucial role in the preceding sections.

Write Y for the open modular curve $Y_1(N)$ over $\mathbb{Q}$, and X for its canonical compactification $X_1(N)$; furthermore, denote by $\bar{Y}$ and $\bar{X}$ the extension to $\bar{\mathbb{Q}}$ of Y and X, respectively. Let F be a field of characteristic 0, and let $\mathrm{Eis}_2(\Gamma_1(N), F)$ denote the F-vector space of weight 2 Eisenstein series on $\Gamma_1(N)$ with Fourier coefficients in F. There is a surjective homomorphism

(48)
$$\mathcal{O}^\times_{\bar{Y}} \otimes F \xrightarrow{\mathrm{dlog}} \mathrm{Eis}_2(\Gamma_1(N), F),$$

sending a modular unit u (or rather, an F-linear combination of such) to the Eisenstein series $\frac{1}{2\pi i} \frac{u'(z)}{u(z)}$. Given $u_1, u_2 \in \mathcal{O}^\times_{\bar{Y}}$, write

$$\{u_1, u_2\} \in K_2(\bar{Y}) \otimes \mathbb{Q}$$

for their Steinberg symbol in the second K-group of $\bar{Y}$. As in Section 1.1, let $\mathcal{A}$ be the ordinary locus of Y (viewed here as a rigid analytic curve over $\mathbb{C}_p$), obtained by removing both the supersingular and the cuspidal residue discs. It is equipped with a system of *wide open neighborhoods* $\mathcal{W}_\epsilon$ in the terminology of Coleman, as described for example in [DR1] and [BD1]. Let Φ be the canonical lift of Frobenius on $\mathcal{A}$, and let $\Phi \times \Phi$ be the corresponding lift of Frobenius on $\mathcal{A} \times \mathcal{A}$. Choose a polynomial $P(t) \in \mathbb{Q}[t]$ satisfying

- $P(\Phi \times \Phi)$ annihilates the class of $\frac{du_1}{u_1} \otimes \frac{du_2}{u_2}$ in $H^2_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon)$,
- $P(\Phi)$ annihilates the target of the residue map on $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ at the supersingular annuli,
- $P(\Phi)$ is invertible on $H^1_{\mathrm{dR}}(X)$.

Such a P exists because the eigenvalues of Φ on $H^1_{\mathrm{dR}}(X)$ have complex absolute value $\sqrt{p}$, while Φ acts on the target of the residue map, and on $H^2_{\mathrm{rig}}(\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon)$, with eigenvalues of strictly larger modulus. The first condition on P implies the existence of a rigid analytic one-form ρ_P in $\Omega^1(\mathcal{W}_\epsilon^2)$ such that

$$(49) \qquad d\rho_P = P(\Phi \times \Phi)\left(\frac{du_1}{u_1} \otimes \frac{du_2}{u_2}\right).$$

The form ρ_P is well-defined up to closed rigid one-forms on $\mathcal{W}_\epsilon \times \mathcal{W}_\epsilon$. Fix a base point $x \in \mathcal{W}_\epsilon$, and set

$$\tilde{\xi}_{P,x} := (\delta^* - h_x^* - v_x^*)\rho_P \in \Omega^1(\mathcal{W}_\epsilon),$$

where $\delta(w) := (w, w)$, $h_x(w) := (w, x)$ and $v_x(w) := (x, w)$ are the diagonal, horizontal and vertical inclusions of $\mathcal{W}_\epsilon$ in $\mathcal{W}_\epsilon^2$, respectively. One checks that the image $\xi_{P,x}$ of $\tilde{\xi}_{P,x}$ in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ does not depend on the choice of a form ρ_P satisfying equation (49), and vanishes at the supersingular annuli by the second condition imposed on P. It follows that $\xi_{P,x}$ can be identified via restriction with an element of the algebraic de Rham cohomology group $H^1_{\mathrm{dR}}(X)$. Moreover, setting $\xi_x := P(\Phi)^{-1}\xi_{P,x}$ in view of the third condition on P, we find that the class ξ_x is independent of the choice of P. The p-adic regulator of $\{u_1, u_2\}$ is defined as

$$(50) \qquad \mathbf{reg}_p\{u_1, u_2\} := \xi_x \in H^1_{\mathrm{dR}}(X).$$

By Poincaré duality, the p-adic regulator $\mathbf{reg}_p\{u_1, u_2\}$ can and will be identified with a linear functional on $H^1_{\mathrm{dR}}(X)$.

We are now ready to state the main theorem of this section. Let χ be a primitive, even Dirichlet character of conductor N. Recall the Eisenstein series $E_{2,\chi}$ appearing in equation (1) of Section 1.1, and let u_χ be a modular unit satisfying

$$(51) \qquad \mathrm{dlog}(u_\chi) = E_{2,\chi}.$$

Normalize the Petersson scalar product on real analytic modular forms of weight k and character ψ on $\Gamma_0(N)$ by setting

$$(52) \qquad \langle f_1, f_2 \rangle_{k,N} := \int_{\Gamma_0(N)\backslash\mathcal{H}} y^k \overline{f_1(z)} f_2(z) \frac{dxdy}{y^2}.$$

Write $\alpha_p(f)$, resp. $\beta_p(f)$ for the unit, resp. non-unit root of the Frobenius polynomial $x^2 - a_p(f)x + p$ associated to f. Consider the unit root subspace

$$H^1_{\mathrm{dR}}(X)^{f,\mathrm{ur}} \subset H^1_{\mathrm{dR}}(X)^f$$

of the f-isotypic part of $H^1_{\mathrm{dR}}(X)$, on which Frobenius acts as multiplication by $\alpha_p(f)$. We attach to f a canonical element η_f^{ur} of $H^1_{\mathrm{dR}}(X)^{f,\mathrm{ur}}$ in the following way (cf. [BD1], Sections 2.5 and 3.1 for more details). First, we define an anti-holomorphic differential

$$(53) \qquad \eta_f^{\mathrm{ah}} := \langle f, f \rangle_{2,N}^{-1} \cdot \bar{f}(z) d\bar{z}.$$

The differential η_f^{ah} gives rise to a class in $H^1_{\mathrm{dR}}(X_{\mathbb{C}})$, whose natural image in $H^1(X_{\mathbb{C}}, \mathcal{O}_X)$ is in fact defined over $\bar{\mathbb{Q}}$ via our fixed embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}$. Using now the embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$ we obtain a class η_f in $H^1(X, \mathcal{O}_X)$, and a lift η_f^{ur} of η_f to $H^1_{\mathrm{dR}}(X)^{f,\mathrm{ur}}$.

Theorem 2.1. *The equality*

$$L_p(f, 2) \cdot \frac{L(f, \chi, 1)}{\Omega_f^+} = (2iN^{-2}\mathfrak{g}(\chi))(1 - \beta_p(f)p^{-2})(1 - \beta_p(f)) \cdot \mathbf{reg}_p\{u_{\chi^{-1}}, u_\chi\}(\eta_f^{\mathrm{ur}})$$

holds, where Ω_f^+ is a real period attached to f as in Section 2.3 of [BD1], *and $\mathfrak{g}(\chi)$ is the Gauss sum defined in equation* (8).

Remark 2.2. A version of Theorem 2.1 has been obtained by Brunault [Br], as a consequence of Kato's reciprocity law [Kato]. A different proof, proposed by Bannai-Kings [BK] and Niklas [Nik], relies on the Eisenstein measures introduced by Panchiskine. The approach sketched here is based on the methods of [BD1], depending crucially on Hida's p-adic deformation of f.

The key steps in this approach consist in:
- The p-adic approximation of the value $L_p(f, 2)$ (lying outside the range of classical interpolation of $L_p(f, s)$) by means of values in the range of classical interpolation of the so-called Mazur-Kitagawa p-adic L-function.
- The description of the Mazur-Kitagawa p-adic L-function as a factor of a p-adic Rankin L-series $L_p(\mathbf{f}, \mathbf{E}_\chi)$ associated to the convolution of the Hida families $\mathbf{f}$ and $\mathbf{E}_\chi$ interpolating in weight 2 the ordinary forms f and $E_{2,\chi}$, respectively. (This factorisation follows from a corresponding factorisation of complex special values.)
- The explicit evaluation of $L_p(\mathbf{f}, \mathbf{E}_\chi)$ at the weights $(2, 2)$ (lying outside the range of classical interpolation for this p-adic L-function), which yields an expression directly related to the p-adic regulator $\mathbf{reg}_p\{u_{\chi^{-1}}, u_\chi\}(\eta_f^{\mathrm{ur}})$ described above.

Write $U_{\mathbf{f}} \subset (\mathbb{Z}/(p - 1)\mathbb{Z}) \times \mathbb{Z}_p$ for the weight space attached to $\mathbf{f}$, and denote by $f_k \in S_k(N)$ the classical eigenform whose ordinary p-stabilisation is equal to the weight k specialisation of $\mathbf{f}$, for all k in the space of classical weights $U_{\mathbf{f},\mathrm{cl}} := U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}$. (In particular, $f_2 = f$.) Let $L_p(f_k, \rho, s)$ be the Mazur-Swinnerton-Dyer p-adic L-function [MTT] associated to f_k and to a Dirichlet character ρ (equal to $\mathbf{1}$ or to χ in our study). Thus $L_p(f_k, \rho, s)$ interpolates the special values $L_p(f_k \otimes \xi\rho, j)$, for $1 \leq j \leq k - 1$ and ξ in the set of Dirichlet characters of p-power conductor. As k varies, the p-adic L-functions $L_p(f_k, \rho, s)$ can be patched together to yield the Mazur-Kitagawa two-variable p-adic L-function $L_p(\mathbf{f}, \rho)(k, s)$, defined on the domain $U_{\mathbf{f}} \times \mathbb{Z}_p$. For $k \in U_{\mathbf{f},\mathrm{cl}}$, one has the identity

$$L_p(\mathbf{f}, \rho)(k, s) = \lambda(k) \cdot L_p(f_k, \rho, s),$$

where $\lambda(k)$ is a p-adic period equal to 1 at $k = 2$ and non-vanishing in a neighborhood of $k = 2$. Note that $L_p(f, 2)$ can be described as the p-adic limit, as $(k, \ell) \in U_{\mathbf{f},\mathrm{cl}} \times [1, k/2]$ tends to $(2, 2)$, of the values $L_p(\mathbf{f}, \mathbf{1})(k, k/2 + \ell - 1)$ occurring in the range of classical interpolation for $L_p(\mathbf{f}, \mathbf{1})$.

Let $\mathbf{E}_\chi$ be the Hida family of Eisenstein series whose weight $\ell \in \mathbb{Z}^{\geq 2}$ specialisation is equal to the ordinary p-stabilisation of the Eisenstein series $E_{\ell,\chi}$. We recall the definition [BD1] of the two-variable Rankin p-adic L-function $L_p(\mathbf{f}, \mathbf{E}_\chi)(k, \ell)$, where (k, ℓ) belongs to $U_{\mathbf{f}} \times \mathbb{Z}_p$. It is defined via the p-adic interpolation of the critical values

$$L(f_k \otimes E_{\ell,\chi}, k/2 + \ell - 1), \qquad \ell \in [1, k/2].$$

Set $t := k/2 - \ell$ (so that $t \geq 0$). Let δ_ℓ be the Shimura-Maass differential operator of equation (30), mapping the space $M_\ell^{\mathrm{nh}}(N, \phi)$ of weight ℓ nearly holomorphic modular forms on $\Gamma_0(N)$ and character ϕ to $M_{\ell+2}^{\mathrm{nh}}(N, \phi)$, and let δ_ℓ^t denote its t-fold iterate $\delta_{\ell+2t-2} \circ \ldots \circ \delta_\ell$. Theorem 2 of [Sh1] yields the special value formula

$$(54) \qquad L(f_k \otimes E_{\ell,\chi}, k/2 + \ell - 1) = B_{N,k,\ell,\chi} \cdot \langle f_k(z), (\delta_\ell^t E_{\ell,\chi^{-1}}) \cdot E_{\ell,\chi} \rangle_{k,N},$$

where $B_{k,\ell,\chi}$ is an explicit non-zero algebraic constant depending on N, k, ℓ and χ. Set $\Xi_{k,\ell} := (\delta_\ell^t E_{\ell,\chi^{-1}}) \cdot E_{\ell,\chi}$, and denote by $\Xi_{k,\ell}^{\mathrm{hol}}$ its holomorphic projection. Hence

$$\langle f_k(z), \Xi_{k,\ell} \rangle_{k,N} = \langle f_k(z), \Xi_{k,\ell}^{\mathrm{hol}} \rangle_{k,N},$$

and (54) implies that the ratio

$$\frac{L(f_k \otimes E_{\ell,\chi}, k/2 + \ell - 1)}{\langle f_k, f_k \rangle_{k,N}}$$

is algebraic. (More precisely, it belongs to the extension of $\mathbb{Q}$ generated by the Fourier coefficients of f_k and the values of χ.) Viewing $\Xi_{k,\ell}^{\mathrm{hol}}$ as a p-adic modular form, the calculations of Section 3.1 of [BD1] – see in particular equation (46)– show that the ratios

$$\frac{\langle f_k(z), \Xi_{k,\ell}^{\mathrm{hol}} \rangle_{k,N}}{\langle f_k, f_k \rangle_{k,N}},$$

normalised by multiplying them by suitable Euler factors, are interpolated p-adically by a p-adic L-function $L_p(\mathbf{f}, \mathbf{E}_\chi)(k, \ell)$. As a by-product of this construction, one obtains the formula (cf. equation (50) of loc. cit.)

$$(55) \qquad L_p(\mathbf{f}, \mathbf{E}_\chi)(2, 2) = \frac{1}{1 - \beta_p(f)^2 p^{-1}} \langle \eta_f^{\mathrm{ur}}, e_{\mathrm{ord}}(d^{-1} E_{2,\chi^{-1}}^{[p]} \cdot E_{2,\chi}) \rangle_Y,$$

where e_{ord} denotes Hida's ordinary projector, $E_{2,\chi^{-1}}^{[p]}$ is the p-depletion of $E_{2,\chi^{-1}}$ defined in equation (35), $d = q\frac{d}{dq}$ is the Atkin-Serre derivative operator, and $\langle\ ,\ \rangle_Y$ is the natural Poincaré pairing on Y. The right-hand side of (55) is equal to $\mathbf{reg}_p\{u_{\chi^{-1}}, u_\chi\}(\eta_f^{\mathrm{ur}})$ up to an Euler factor (whose precise form is given in equation (60) of loc. cit.).

Remark 2.3. Note that the rigid-analytic function $d^{-1} E_{2,\chi^{-1}}^{[p]}$ is the Coleman primitive of $E_{2,\chi^{-1}}^{[p]}$, so that equation (55) expresses values of p-adic L-functions in terms of the theory of p-adic integration. This feature is in common with the formulae presented in the previous sections, with the notable difference that in this case (and in the cases described in the following sections) it is a so-called "iterated Coleman integral" that makes its appearance.

As a final step, we remark that the factorisation of complex L-functions

$$L(f_k \otimes E_{\ell,\chi}, k/2 + \ell - 1) = L(f_k, k/2 + \ell - 1) \cdot L(f_k, \chi, k/2)$$

implies directly the factorisation of p-adic L-functions

$$(56) \qquad L_p(\mathbf{f}, \mathbf{E}_\chi)(k, \ell) = \eta(k) \cdot L_p(\mathbf{f}, \mathbf{1})(k, k/2 + \ell - 1) \cdot L_p(\mathbf{f}, \chi)(k, k/2),$$

where $\eta(k)$ is a p-adic analytic function whose exact value at 2 is determined in Theorem 3.4 of loc. cit. Theorem 2.1 follows by combining equations (55) and (56), given the

interpretation of the right-hand side of (55) as a p-adic regulator. This concludes our outline of the proof of Theorem 2.1.

We now turn to a brief discussion of the theory of Euler systems, which in the current context aims to relate the values of $L_p(f, s)$ at integer points to a collection of classes in various continuous Galois cohomology groups associated to f. Let F be a p-adic field containing the values of χ, and let $\delta_{\chi^\pm}$ be the image of $u_{\chi^\pm}$ in $H^1_{\mathrm{et}}(Y, F(1))$ arising from Kummer theory. Define the (p-adic) étale regulator

$$\mathbf{reg}_{\mathrm{et}}\{u_{\chi^{-1}}, u_\chi\} := \delta_{\chi^{-1}} \cup \delta_\chi \in H^2_{\mathrm{et}}(Y, F(2)) = H^1(\mathbb{Q}, H^1_{\mathrm{et}}(\bar{Y}, F(2))),$$

where the last identification is a consequence of the Hochschild-Serre spectral sequence. Consider the isomorphism

$$(57) \qquad \log_{Y,2} : H^1(\mathbb{Q}_p, H^1_{\mathrm{et}}(\bar{Y}, F(2))) \longrightarrow D_{\mathrm{dR}}(H^1_{\mathrm{et}}(\bar{Y}, F(2))) = H^1_{\mathrm{dR}}(Y/F),$$

where the first map is the Bloch-Kato logarithm (which is an isomorphism in our setting), and the second equality follows from the comparison theorem between étale and de Rham cohomology. The map (57) sends the restriction at p of the étale regulator, denoted $\mathbf{reg}_{\mathrm{et},p}$, to the p-adic regulator:

$$(58) \qquad \log_{Y,2}(\mathbf{reg}_{\mathrm{et},p}\{u_{\chi^{-1}}, u_\chi\}) = \mathbf{reg}_p\{u_{\chi^{-1}}, u_\chi\}$$

(cf. Proposition 9.11 and Corollary 9.10 of [Bes2]). Thus Theorem 2.1 can be rephrased as a relation between the value $L_p(f, 2)$ and the Bloch-Kato logarithm of the étale regulator:

$$(59) \quad L_p(f, 2) \cdot \frac{L(f, \chi, 1)}{\Omega_f^+} \;=\; (2iN^{-2}\mathfrak{g}(\chi))(1 - \beta_p(f)p^{-2})(1 - \beta_p(f))$$

$$\times \log_{Y,2}(\mathbf{reg}_{\mathrm{et},p}\{u_{\chi^{-1}}, u_\chi\})(\eta_f^{\mathrm{ur}}).$$

The identity (59) should be viewed as the analogue of equation (14) of Section 1.1. More general versions of (59) can be obtained by replacing u_χ by the modular units corresponding to the Eisenstein series $E_2(\chi_1, \chi_2)$ defined in Section 1.1.

Extend F so that it contains the Fourier coefficients of f, and fix a Galois and Hecke equivariant (for the prime-to-N Hecke operators) projection

$$\pi_f : H^1_{\mathrm{et}}(\bar{Y}, F) \longrightarrow V_f,$$

where V_f is the Galois representation attached to f. Let κ_f be the image in $H^1(\mathbb{Q}, V_f(2))$ of $\mathbf{reg}_{\mathrm{et}}\{u_{\chi^{-1}}, u_\chi\}$ by the natural map induced by π_f. Although κ_f depends on the choice of an auxiliary character χ, this dependency may be eliminated by "stripping off" the scalar $\frac{L(f,\chi,1)}{\Omega_f^+} \in F$, which can assumed to be *non-zero* by a judicious choice of χ.

Kato shows that $\kappa_f = \kappa_{f,0}$ is the bottom element of a norm-compatible system of classes $\kappa_{f,n} \in H^1(\mathbb{Q}(\mu_{p^n}), V_f(2))$, constructed from Eisenstein series of level divisible by p^n (cf. [Kato] and [Br]). The formalism outlined in Section 1.1 identifies the system of classes $(\kappa_{f,n})_{n \geq 0}$ with an element $\kappa_{f,\infty}$ of $H^1(\mathbb{Q}, \Lambda_{\mathrm{cyc}} \otimes V_f(2))$. Setting

$$\kappa_{f,\xi}(k) := \nu_{k,\xi}(\kappa_{f,\infty}) \in H^1(\mathbb{Q}, \mathbb{Q}_{p,\xi} \otimes V_f(1+k)(\xi^{-1}))$$

for the specialisation of $\kappa_{f,\infty}$ under $\nu_{k,\xi}$, it turns out that the image of $\kappa_{f,\xi}(1)$ by the Bloch-Kato logarithm encodes the values $L_p(f, \xi, 2)$, whereby generalising (59) (cf. [Kato], (16.6)).

In this way, $L_p(f, s)$ can be "read-off" from the class $\kappa_{f,\infty}$. The work [BD2] undertakes the task of obtaining such a description of $L_p(f, s)$ by extending the techniques of [BD1] outlined above.

More generally, Perrin-Riou's description of $L_p(f, s)$ as the image of $\kappa_{f,\infty}$ by a "big" log-map (interpolating the Bloch-Kato logarithms) allows one to recover the values $L_p(f, \xi, 1 + k)$ for all $k \geq 1$ from the logarithmic images of the classes $\kappa_{f,\xi}(k)$ (cf. for example [Br], Theorem 23, [PR2], 3.3.10 and [Cz1]).

Consider now the dual exponential map (in the case $k = 0$)

$$\exp_{0,\xi}^* : H^1(\mathbb{Q}_p, \mathbb{Q}_{p,\xi} \otimes V_f(1)(\xi^{-1})) \longrightarrow D_{\mathrm{dR}}(\mathbb{Q}_{p,\xi} \otimes V_f(1)(\xi^{-1})).$$

Kato's explicit reciprocity law describes $L_p(f, \xi, 1)$ in terms of $\exp_{0,\xi}^*(\kappa_{f,\xi}(0))$ (cf. for example [PR2], §2.1). In light of the interpolation formula for $L_p(f, s)$, this gives the chain of equivalences

$$(60) \qquad \exp_{0,\xi}^*(\kappa_{f,\xi}(0)) \neq 0 \qquad \Leftrightarrow \qquad L_p(f, \xi, 1) \neq 0 \qquad \Leftrightarrow \qquad L(f, \xi, 1) \neq 0.$$

When combined with Kolyvagin's theory of Euler systems, (60) implies the following case of the Birch and Swinnerton-Dyer conjecture (where ξ can be assumed to be an arbitrary Dirichlet character):

$$(61) \qquad L(f, \xi, 1) \neq 0 \qquad \Rightarrow \qquad \mathrm{Hom}(\mathbb{C}(\xi), E(\bar{\mathbb{Q}}) \otimes \mathbb{C}) = 0.$$

2.2. **Beilinson-Flach elements.** Let $f \in S_2(\Gamma_1(N), \chi_f)$ and $g \in S_2(\Gamma_1(N), \chi_g)$ be two cusp forms. In this section we discuss the p-adic Beilinson formula of [BDR1] for the value of the p-adic Rankin L-series attached to $f \otimes g$ at the non-critical value $s = 2$. Roughly speaking, this formula is achieved by applying the formalism of the previous section with the p-adic family of Eisenstein series $\mathbf{E}_\chi$ replaced by the Hida family $\mathbf{g}$ interpolating the cusp form g.

The characters χ_f and χ_g are taken to have modulus N, so $\chi_f(p) = 0$ for $p \mid N$. We assume that the forms f and g are normalized eigenforms of level N (not necessarily new), including for the operators U_p for $p \mid N$. Recall the imprimitive Rankin L-series associated to f and g:

$$(62) \qquad \begin{aligned} L^{\mathrm{imp}}(f \otimes g, s) := \prod_p \big[&(1 - \alpha_p(f)\alpha_p(g)p^{-s})(1 - \alpha_p(f)\beta_p(g)p^{-s}) \\ &\times (1 - \beta_p(f)\alpha_p(g)p^{-s})(1 - \beta_p(f)\beta_p(g)p^{-s}) \big]^{-1}, \end{aligned}$$

where $\alpha_p(f), \beta_p(f)$ are the roots of the Hecke polynomial $x^2 - a_p(f)x + \chi_f(p)p$ of f if $p \nmid N$, and $(\alpha_p(f), \beta_p(f)) = (a_p(f), 0)$ if $p \mid N$. We adopt similar notations for g. The Rankin series $L^{\mathrm{imp}}(f \otimes g, s)$ is the imprimitive L-function associated to the tensor product $V_f \otimes V_g$ of the motives attached to f and g, and differs from the full $L(f \otimes g, s)$ only possibly in Euler factors at primes $p \mid N$.

Using the Rankin-Selberg method, Shimura gave an explicit formula for $L^{\mathrm{imp}}(f \otimes g, s)$ in terms of the Petersson inner product (52). Let $\chi = (\chi_f \chi_g)^{-1}$, and let $E_\chi(z, s - 1)$ be

the non-holomorphic Eisenstein series of weight zero and character χ:

$$E_\chi(z, s-1) = \sideset{}{'}\sum_{(m,n) \in N\mathbb{Z} \times Z} \chi^{-1}(n) \operatorname{Im}(z)^s |mz+n|^{-2s}.$$

Shimura's formula reads (cf. for example equation (4) on [BDR1], with $k = \ell = 2$)

$$(63) \qquad L^{\operatorname{imp}}(f \otimes g, s) = \frac{1}{2} \frac{(4\pi)^s}{\Gamma(s)} \langle f^*(z), E_\chi(z, s-1) g(z) \rangle_{2,N},$$

where $f^*(z) = \overline{f(-\bar{z})} \in S_2(N_f, \chi_f^{-1})$ denotes the cusp form obtained by conjugating the Fourier coefficients of f. Since non-holomorphic Eisenstein series satisfy a functional equation relating s and $1-s$, Shimura's formula leads to a functional equation for $L^{\operatorname{imp}}(f \otimes g, s)$ relating the values at s and $3-s$.

Let X denote the modular curve $X_1(N)$. Beilinson gave a geometric interpretation for $L^{\operatorname{imp}}(f \otimes g, s)$ at the near central point $s = 2$ in terms of higher Chow groups on the surface $S = X \times X$. The Rankin L-function $L^{\operatorname{imp}}(f \otimes g, s)$ has no critical points, and in particular Beilinson's formula concerns the non-critical point $s = 2$. The higher Chow group $\operatorname{CH}^2(S, 1) \cong K_1(S)$ is defined to be the homology (in the middle) of the Gersten complex

$$K_2(K(S)) \xrightarrow{\partial} \oplus_{Z \subset S} K(Z)^* \xrightarrow{\operatorname{div}} \oplus_{P \in S} \mathbb{Z}.$$

Here Z ranges over irreducible curves in S, and P ranges over closed points in S. The map denoted div sends a rational function to its divisor. The map ∂ sends a symbol $\{f, g\} \in K_2(K(S))$ associated to pair of functions $x, y \in K(S)^*$ to the tame symbol

$$(64) \qquad \partial(\{x, y\}) = (u_Z)_{Z \subset S}, \qquad u_Z = (-1)^{\nu_Z(x)\nu_Z(y)} \frac{x^{\nu_Z(y)}}{y^{\nu_Z(x)}}.$$

Let $u \in K(X)^*$ be a modular unit as in (48), i.e. a rational function on X whose divisor is supported on the cusps. By viewing u as a rational function on the diagonal $\Delta \subset S$, one can define certain distiguished elements $\Delta_u \in \operatorname{CH}^2(S, 1)$ as follows.

Lemma 2.4. *Given a modular unit u on X, there exists an element of the form*

$$(65) \qquad \Delta_u = (\Delta, u) + \sum a_i(Z_i, u_i) \in \operatorname{CH}^2(S, 1) \otimes \mathbb{Q},$$

where $a_i \in \mathbb{Q}$ and each $Z_i \subset S$ is a horizontal or vertical divisor, i.e. a curve of the form $X \times P$ or $P \times X$ for a point $P \in X$.

Definition 2.5. An element of the form Δ_u as in (65) is called a *Beilinson–Flach element* in $\operatorname{CH}^2(S, 1) \otimes \mathbb{Q}$.

We will be interested in the modular unit u_χ such that $\operatorname{dlog} u_\chi = E_{2,\chi}$ as in (51) and the associated Beilinson–Flach element Δ_{u_χ}.

There is a complex regulator map

$$\mathbf{reg}_\mathbb{C} : \operatorname{CH}^2(S, 1) \to (\operatorname{Fil}^1 H^2_{\operatorname{dR}}(S/\mathbb{C}))^\vee$$

defined by

$$\mathbf{reg}_{\mathbb{C}}\left(\sum(Z_i, u_i)\right)(\omega) = \frac{1}{2\pi i}\int_{Z_i'}\omega\log|u_i|,$$

where ω is a smooth (1,1)-form on S and $Z_i' \subset Z_i$ is the locus on which u_i is regular. We may now state Beilinson's formula

Theorem 2.6 (Beilinson). *We have*

$$\frac{L^{\mathrm{imp}}(f\otimes g, 2)}{\langle f^*, f^*\rangle_{2,N}} = C_\chi\mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})(\omega_g\otimes\eta_f^{\mathrm{ah}})$$

where $C_\chi = (8i)\pi^3[\Gamma_0(N):\Gamma_1(N)(\pm)]^{-1}N^{-2}\mathbf{g}(\chi^{-1})$, $\omega_g = 2\pi i g(z)dz$ *and*

$$\eta_f^{\mathrm{ah}} = \langle f^*, f^*\rangle_{2,N}^{-1}\bar{f}^*(z)d\bar{z}$$

as in (53).

The main theorem of [BDR1] is a p-adic analogue of Beilinson's formula. Before stating this result, it is worth noting that the very definition of the p-adic Rankin L-series associated to $f\otimes g$ is subtle for a reason mentioned before: the classical Rankin L-series $L(f\otimes g, s)$ has no critical values. When f and g are ordinary at a prime $p\nmid N$, Hida showed how to define a p-adic Rankin L-series as follows. Let $\mathbf{f}$ and $\mathbf{g}$ denote the Hida families whose weight two specializations are f and g, respectively. Then for weights k and ℓ such that $2\le\ell\le s\le k-1$, the values $L^{\mathrm{imp}}(f_k\otimes g_\ell, s)$ are critical. Hida proved that there exists a p-adic L-function $L_p(\mathbf{f}, \mathbf{g})(k, \ell, s)$ interpolating the values

(66) $$\frac{L^{\mathrm{imp}}(f_k\otimes g_\ell, s)}{(2\pi i)^{\ell+2s-1}\langle f_k, f_k\rangle_{k,N}}\in\overline{\mathbb{Q}}\qquad\text{for }2\le\ell\le s\le k-1.$$

Note that the roles of $\mathbf{f}$ and $\mathbf{g}$ are not symmetric in this definition, and we therefore obtain two p-adic Rankin L-series associated to f and g, namely:

$$L_p^f(f\otimes g, s) := L_p(\mathbf{f}, \mathbf{g}, 2, 2, s),\qquad L_p^g(f\otimes g, s) := L_p(\mathbf{g}, \mathbf{f}, 2, 2, s).$$

Just as we saw in Section 2.1 regarding the p-adic regulator attached to $K_2(Y_1(N))$, there is a p-adic (or "syntomic") regulator attached to $K_1(S)\cong\mathrm{CH}^2(S, 1)$. This is a map

(67) $$\mathbf{reg}_p : \mathrm{CH}^2(S, 1)\to\mathrm{Fil}^1 H_{\mathrm{dR}}^2(S/\mathbb{Q}_p)^\vee$$

defined by Besser [Bes4] in terms of Coleman's theory of p-adic integration. As for the p-adic regulator discussed in Section 2.1, the map $\mathbf{reg}_p$ of (67) satisfies the property that it factors through the étale regulator via the Bloch–Kato logarithm. The main theorem of [BDR1] is as follows.

Theorem 2.7. *We have*

$$L_p^f(f\otimes g, 2) = \frac{\mathcal{E}(f, g, 2)}{\mathcal{E}(f)\mathcal{E}^*(f)}\mathbf{reg}_p(\Delta_{u_\chi})(\omega_g\otimes\eta_f^{\mathrm{ur}}),$$

where

$$\mathcal{E}(f, g, 2) = (1 - \beta_p(f)\alpha_p(g)p^{-2})(1 - \beta_p(f)\beta_p(g)p^{-2})$$
$$\times (1 - \beta_p(f)\alpha_p(g)\chi(p)p^{-1})(1 - \beta_p(f)\beta_p(g)\chi(p)p^{-1})$$
$$\mathcal{E}(f) = 1 - \beta_p(f)^2 \chi_f^{-1}(p)p^{-2}$$
$$\mathcal{E}^*(f) = 1 - \beta_p(f)^2 \chi_f^{-1}(p)p^{-1}$$

and η_f^{ur} is as in Section 2.1 (see the discussion following (53)).

As indicated earlier, the proof of theorem 2.7 follows along that of Theorem 2.1, with the Hida family $\mathbf{E}_\chi$ replaced by $\mathbf{g}$; we therefore indicate now only a few salient aspects. First, one uses Shimura's generalization of (63) for the critical values of $L(f_k \otimes g_\ell, s)$—essentially (54) with $E_{\ell,\chi}$ replaced by g_ℓ—and interprets the right side of this formula in terms of the Poincaré pairing on algebraic de Rham cohomology. This pairing can be realized in p-adic de Rham cohomology, and one obtains that the algebraic numbers (66) vary p-adic analytically, up to the multiplication of appropriate Euler factors. This defines a p-adic L-function $L_p(\mathbf{f}, \mathbf{g})(k, \ell, s)$, whose value at $k = \ell = s = 2$ is given by the formula

$$(68) \qquad L_p(\mathbf{f}, \mathbf{g})(2, 2, 2) = \frac{1}{\mathcal{E}^*(f)} \left\langle \eta_f^{\mathrm{ur}}, e_{\mathrm{ord}}(d^{-1} E_{2,\chi}^{[p]} \cdot g) \right\rangle_X.$$

Theorem 2.7 is then deduced by using Besser's work [Bes4], which relates the right-hand side of equation (68) to the p-adic regulator.

We conclude this section with a brief discussion of various works in progress regarding the construction of an Euler system of Beilinson–Flach elements along the p-power level tower of self-products of modular curves $X_1(Np^r) \times X_1(Np^r)$, along the lines that were suggested in [DR1] and [BDR1]. This Euler system, which has been developped further by Lei, Loeffler and Zerbes [LLZ] and in [BDR2], holds the promise of several arithmetic applications.

Firstly, such an Euler system would yield a generalization of Theorem 2.7 that varies in p-adic families. This generalization would in particular apply when f and g have level divisible by p, and s is a general arithmetic weight (e.g. $s(x) = x^j\psi(x)$ for a p-power conductor Dirichlet character ψ and integer j). Using this generalization, the fourth author has indicated a proof of a factorization of Hida's p-adic Rankin L-function $L_p(f \otimes f, s)$ of the Rankin square into the Coates-Schmidt p-adic L-function of the symmetric square of f and a Kubota–Leopoldt p-adic L-function [Das]. The approach taken in loc.cit. parallels closely the one in [Gr] to factor the restriction to the cyclotomic line of the Katz p-adic L-function into a product of two Kubota-Leopoldt p-adic L-functions, but with the Katz L-function replaced by $L_p(f \otimes f, s)$, elliptic units by Beilinson-Flach elements, and Katz's p-adic Kronecker limit formula by the p-adic Beilinson formula of [BDR1]. As shown by Citro, a suitable twist of this factorization formula implies Greenberg's trivial zero conjecture for the p-adic L-function of the adjoint of f [Cit].

The construction an Euler system of Beilinson–Flach elements is also the subject of independent work in progress by Lei, Loeffler, and Zerbes [LLZ] with the goal of studying the Iwasawa theory of modular forms over quadratic imaginary fields.

Finally, we mention the application of an Euler system of Beilinson–Flach elements toward the rank zero BSD conjecture, which serves as the motivation of [BDR1]; details will appear in [BDR2]. Let $E/\mathbb{Q}$ be an elliptic curve, and let $f \in S_2(\Gamma_0(N))$ denote its associated modular form with rational Fourier coefficients. The image of the Beilinson–Flach element $\Delta_{u_\chi} \in \mathrm{CH}^2(S, 1)$ under the étale regulator yields a class in $H^1(\mathbb{Q}, H^2_{\mathrm{et}}(S \times \overline{\mathbb{Q}}, \mathbb{Z}_p(2)))$. By projecting onto the (f, g)-isotypic component of $H^2_{\mathrm{et}}(S \times \overline{\mathbb{Q}}, \mathbb{Z}_p)$, we obtain a class

$$\kappa_E(g) \in H^1(\mathbb{Q}, V_E \otimes V_g(1))$$

where $V_g = H^1_{\mathrm{et}}(X \times \overline{\mathbb{Q}}, \mathbb{Z}_p)^g$ denotes the Galois representation coming from the g-isotypic component of the first étale cohomology of the modular curve X and $V_E \cong V_f(1)$ denotes the (rational) p-adic Tate-module of E.

The interpolation of the Beilinson–Flach elements Δ_{u_χ} into an Euler system allows for the construction of a p-adic family of classes

$$\kappa_E(\mathbf{g}) \in H^1(\mathbb{Q}, \mathbf{V}_{E,\mathbf{g}})$$

where $\mathbf{V_g}$ is Hida's two-dimensional Λ-adic Galois representation associated to the family $\mathbf{g}$, and

$$\mathbf{V}_{E,\mathbf{g}} := V_E \otimes_{\mathbb{Z}_p} \mathbf{V_g} \otimes_\Lambda \Lambda_{\mathrm{cyc}},$$

where Λ_{cyc} is as in §1.1 and the tensor product $\mathbf{V_g} \otimes_\Lambda \Lambda_{\mathrm{cyc}}$ is taken with respect to a suitable algebra homomorphism $\Lambda \hookrightarrow \Lambda \otimes_{\mathbb{Z}_p} \Lambda$.

An appropriate generalization of Theorem 2.7 relates the image of $\kappa_E(\mathbf{g})$ under the Bloch–Kato logarithm to the p-adic L-function $L_p(\mathbf{f}, \mathbf{g})(k, \ell, s)$ when $k = 2$.

Let us now suppose that the specialization of $\mathbf{g}$ in weight 1 is a classical cusp form, and hence its associated Galois representations $V_{\mathbf{g}_1}$ is an odd 2-dimensional Artin representation (i.e. has finite image), which we denote by ρ. The specialization of the class $\kappa_E(\mathbf{g})$ in weight 1 need not be crystalline at p, since it is defined as the specialization of a p-adic family of classes at a non-classical weight, and is not directly defined via a geometric construction. In fact, one shows that $\kappa_E(\mathbf{g})_1$ is crystalline at p if and only if $L(E, \rho, 1) = 0$. Similarly to (61), Kolyvagin's theory of Euler systems can then be used to deduce the following case of the BSD conjecture for E (cf. [BDR2]):

$$L(E, \rho, 1) \neq 0 \Longrightarrow \mathrm{Hom}(\rho, E(\overline{\mathbb{Q}}) \otimes \mathbb{C}) = 0.$$

2.3. **Gross-Kudla-Schoen cycles.** The setting of this section is obtained from that of Section 2.1 and 2.2 by replacing, in the triple of modular forms one starts with, Eisenstein series with cusp forms. Thus, let $f, g, h \in S_2(N)$ be a triple of normalized *cuspidal* eigenforms of weight 2, level N and nebentypus characters χ_f, χ_g, and χ_h, respectively. Assume $\chi_f \cdot \chi_g \cdot \chi_h$ is the trivial character, so that the tensor product $V_{f,g,h} := V_f \otimes V_g \otimes V_h$ of the compatible system of Galois representations associated by Shimura to f, g and h is self-dual and the Garrett-Rankin L-function $L(f, g, h, s)$ of $V_{f,g,h}$ satisfies a functional equation relating the values s and $4 - s$.

Let $\mathbb{Q}_{f,g,h} = \mathbb{Q}(\{a_n(f), a_n(g), a_n(h)\}_{n \geq 1})$ denote the field generated by the Fourier coefficients of f, g and h. For the sake of simplicity in the exposition, let us also assume that N is square-free, the three eigenforms are new in that level, and $a_\ell(f)a_\ell(g)a_\ell(h) = -1$ for

all primes $\ell \mid N$. The results described in this section hold (suitably adapted) in much greater generality (cf. [DR1]) – a fact that is important to bear in mind for the arithmetic applications we shall discuss at the end of this section.

Fix a prime $p \nmid N$ and an embedding $\mathbb{Q}_{f,g,h} \hookrightarrow \mathbb{C}_p$ for which the three newforms are ordinary, and let $\mathbf{f} : \Omega_f \longrightarrow \mathbb{C}_p[[q]]$ denote the Hida family of overconvergent p-adic modular forms passing though f. The space Ω_f is a finite rigid analytic covering of the weight space $W = \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$.

Write $\Omega_{f,\mathrm{cl}} \subset \Omega_f$ for the subset of points x whose image in $\mathbb{Z}_p^\times$ is an integer $\kappa(x) \geq 2$ and let $f_x \in S_{\kappa(x)}(N)$ denote the eigenform whose p-stabilisation equals the specialization of $\mathbf{f}$ at x. Adopt similar notations for g and h.

Single out one of the three eigenforms, say f. Building on Hida's definition of the p-adic Rankin L-function discussed in Section 2.2, Harris and Tilouine [HaTi] defined a p-adic L-function of three variables, denoted

(69) $$\mathscr{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h}) : \Omega_f \times \Omega_g \times \Omega_h \longrightarrow \mathbb{C}_p.$$

See also [DR1, §4] for a description of (69) which corrects the Euler factor in the p-adic interpolation formula arising in [HaTi].

This p-adic L-function interpolates the square-root of the central critical value of the complex L-function $L(f_x, g_y, h_z, s)$, as (x, y, z) ranges over those triples $(x, y, z) \in \Omega_{f,\mathrm{cl}} \times \Omega_{g,\mathrm{cl}} \times \Omega_{h,\mathrm{cl}}$ such that $\kappa(x) \geq \kappa(y) + \kappa(z)$. To construct $\mathscr{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$, one invokes the work of Garrett [Gar], Harris and Kudla [HaKu], which shows that these central critical values are equal, up to certain explicit periods, to the algebraic number

$$J(f_x, g_y, h_z) := \left(\frac{\langle f_x^*, \delta^t(g_y) \times h_z \rangle_N}{\langle f_x^*, f_x^* \rangle_N} \right)^2 \in \bar{\mathbb{Q}},$$

where recall f_x^* is the eigenform obtained from f_x by complex conjugating its Fourier coefficients, $\delta = \delta_{\kappa(y)}$ denotes the Shimura-Maass operator of (30) and $t := \frac{\kappa(x) - \kappa(y) - \kappa(z)}{2} \geq 0$ (cf. also [DR1, Thm. 4.4]). After multiplying $J(f_x, g_y, h_z)$ by a suitable Euler factor at p, these quantities vary continuously and interpolate to a function on $\Omega_f \times \Omega_g \times \Omega_h$ denoted $\mathscr{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$.

Note that the original triple (f, g, h) corresponds to a point (x_0, y_0, z_0) above the triple of weights $(2, 2, 2)$ which lies *outside* the region of interpolation used to define $\mathscr{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$. By an abuse of notation, let us simply write $\mathscr{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 2, 2)$ for the value of this function at the point (x_0, y_0, z_0).

Similarly to Sections 2.1 and 2.2, our goal here is to report on a formula which describes $\mathscr{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 2, 2)$ as the image of a certain cycle on the cube of the modular curve $X = X_1(N)/\mathbb{Q}$ under the p-adic syntomic Abel-Jacobi map. As before, the resulting formula is a p-adic analogue of a complex one: in this case Yuan-Zhang-Zhang's theorem [YZZ] relating the first derivative $L'(f, g, h, 2)$ to the Beilinson-Bloch's height of Δ.

The cycle in question is essentially the diagonal $X_{123} = \{(x, x, x), x \in X\}$ in X^3, which has to be modified to make it null-homologous. More precisely, fix the cusp $\infty \in X$ at infinity as base point and, following Gross, Kudla and Schoen [GrKu], [GrSc], define Δ to be the class in the Chow group $\mathrm{CH}^2(X^3)$ of codimension 2 cycles in X^3 up to rational

equivalence of the formal sum

$$X_{123} - X_{12} - X_{13} - X_{23} + X_1 + X_2 + X_3,$$

where $X_1 = \{(x, \infty, \infty), x \in X\}$, $X_{12} = \{(x, x, \infty), x \in X\}$ and likewise for the remaining summands. One checks that, for any of the standard cohomology theories (e.g. algebraic de Rham, Betti, p-adic syntomic, p-adic étale), the class of Δ in the cohomology group $H^4(X^3)$ vanishes. Thus Δ belongs to the subgroup $\mathrm{CH}^2(X^3)_0$ of null-homologous cycles in $\mathrm{CH}^2(X^3)$, which is the source of the various Abel-Jacobi maps available, e.g.:

$$(70) \quad \begin{aligned} \mathrm{AJ}_{\mathbb{C}} : & \quad \mathrm{CH}^2(X^3)_0 \longrightarrow \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X^3 \times \mathbb{C})^\vee / H_3(X^3(\mathbb{C}), \mathbb{Z}) \\ \mathrm{AJ}_{\mathrm{syn},p} : & \quad \mathrm{CH}^2(X^3)_0 \longrightarrow \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X^3 \times \mathbb{Q}_p)^\vee \\ \mathrm{AJ}_{\mathrm{et},p} : & \quad \mathrm{CH}^2(X^3)_0 \longrightarrow H^1(G_{\mathbb{Q}}, H^3_{\mathrm{et}}(X^3 \times \bar{\mathbb{Q}}, \mathbb{Z}_p(2))). \end{aligned}$$

Since it is an essential ingredient of the formula that we aim to describe here, we recall Besser's description [Bes1] of the image of Δ under the p-adic syntomic Abel-Jacobi map. Specialized to our setting, [Bes1, Theorem 1.2] shows that $\mathrm{AJ}_{\mathrm{syn},p}(\Delta)$ is a $\mathbb{Q}_p$-valued functional

$$(71) \qquad\qquad \mathrm{AJ}_{\mathrm{syn},p}(\Delta) : \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X^3) \longrightarrow \mathbb{Q}_p$$

which can be described purely in terms of Coleman integration. In loc.cit., Besser recasts Coleman's integration theory of 1-forms on curves in a cohomological guise, and exploits this interpretation to provide a generalization of the notion of Coleman's primitive to forms on higher-dimensional varieties $V/\mathbb{Q}_p$ admitting a smooth model $\mathcal{V}$ over $\mathbb{Z}_p$. The spaces in which Besser's primitives live are called *finite polynomial cohomology groups*, denoted $H^i_{\mathrm{fp}}(\mathcal{V}, n)$ for indices $i, n \geq 0$, and his formalism gives rise to a canonical projection $\mathrm{p} : H^i_{\mathrm{fp}}(\mathcal{V}, n) \to \mathrm{Fil}^n H^i_{\mathrm{dR}}(V)$.

In the case of a curve, like our X equipped with its standard integral model $\mathcal{X}/\mathbb{Z}_p$, taking $i = n = 1$ yields an exact sequence $0 \to \mathbb{Q}_p \to H^1_{\mathrm{fp}}(\mathcal{X}, 1) \xrightarrow{\mathrm{p}} \Omega^1(X) \to 0$ where any pre-image $\tilde{\omega} \in H^1_{\mathrm{fp}}(\mathcal{X}, 1)$ of a regular 1-form ω on X may be identified with a choice of a Coleman primitive of ω. That the kernel of p is $\mathbb{Q}_p$ agrees with the well-known fact that such primitives are well-defined only up to constants.

As it will suffice for our purposes, we content here to describe the restriction of (71) to the Künneth component $\mathrm{Fil}^2(H^1_{\mathrm{dR}}(X)^{\otimes 3})$. Up to permutations of the three variables, the typical element in this space is of the type $\eta_1 \otimes \omega_2 \otimes \omega_3$, where η_1 is a class in $H^1_{\mathrm{dR}}(X)$ and $\omega_2, \omega_3 \in \Omega^1(X)$ are regular differential 1-forms. [Bes1, Theorem 1,2] asserts that

$$(72) \quad \mathrm{AJ}_p(\Delta)(\eta_1 \otimes \omega_2 \otimes \omega_3) = \int_\Delta \eta_1 \otimes \omega_2 \otimes \omega_3 := \sum_{\emptyset \neq I \subseteq \{1,2,3\}} \mathrm{sign}(I) \mathrm{tr}_I(\iota_I^*(\tilde{\eta}_1 \cup \tilde{\omega}_2 \cup \tilde{\omega}_3)),$$

where $\tilde{\eta}_1 \in H^1_{\mathrm{fp}}(\mathcal{X}, 0)$, $\tilde{\omega}_2, \tilde{\omega}_3 \in H^1_{\mathrm{fp}}(\mathcal{X}, 1)$ are choices of primitives of $\eta_1, \omega_2, \omega_3$, respectively, and $\tilde{\eta}_1 \cup \tilde{\omega}_2 \cup \tilde{\omega}_3 \in H^3_{\mathrm{fp}}(\mathcal{X}^3, 2)$ is their cup-product. Moreover, in the above formula we set $\mathrm{sign}(I) = (-1)^{|I|+1}$, $\iota_I : X \hookrightarrow X^3$ denotes the natural inclusion which maps X onto the curve $X_I \subset X^3$ and $\mathrm{tr}_I : H^3_{\mathrm{fp}}(\mathcal{X}, 2) \xrightarrow{\sim} \mathbb{Q}_p$ is the canonical trace isomorphism of [Bes1, Prop. 2.5 (4)]. While each of the terms $\mathrm{tr}_I(\iota_I^*(\tilde{\eta}_1 \cup \tilde{\omega}_2 \cup \tilde{\omega}_3))$ does depend on the choice of primitives, one checks that their sum does not.

We are finally in position to state the main formula alluded to at the beginning.

Theorem 2.8. *For each $\phi \in \{f, g, h\}$, let $\omega_\phi \in \Omega^1(X)$ denote the regular 1-form associated to ϕ, and $\eta_f^{\mathrm{ur}} \in H_{\mathrm{dR}}^1(X)^{f,\mathrm{ur}}$ be the unique class in the unit root subspace of the f-isotypical component of $H_{\mathrm{dR}}^1(X)$ such that $\langle \omega_f, \eta_f^{\mathrm{ur}} \rangle = 1$. Let also $\alpha_p(\phi)$, $\beta_p(\phi)$ be the two roots of the Hecke polynomial $x^2 - a_p(\phi)x + p$, labelled in such a way that $\alpha_p(\phi)$ is a p-adic unit. Then the equality*

$$(73) \qquad \mathscr{L}_p^{\,f}(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 2, 2) = \frac{\mathcal{E}(f, g, h)}{\mathcal{E}_0(f)\mathcal{E}_1(f)} \mathrm{AJ}_p(\Delta)(\eta_f^{\mathrm{ur}} \otimes \omega_g \otimes \omega_h)$$

holds, where

$$\mathcal{E}(f, g, h) := \left(1 - \beta_p(f)\alpha_p(g)\alpha_p(h)p^{-2}\right) \times \left(1 - \beta_p(f)\alpha_p(g)\beta_p(h)p^{-2}\right)$$
$$\times \left(1 - \beta_p(f)\beta_p(g)\alpha_p(h)p^{-2}\right) \times \left(1 - \beta_p(f)\beta_p(g)\beta_p(h)p^{-2}\right),$$
$$\mathcal{E}_0(f) := (1 - \beta_p^2(f)\chi_f^{-1}(p)p^{-1}), \qquad \mathcal{E}_1(f) := (1 - \beta_p^2(f)\chi_f^{-1}(p)p^{-2}).$$

This statement holds in greater generality for eigenforms f, g, h of possibly different primitive levels N_f, N_g, N_h, and different weights k, ℓ, m, provided none of the weights is larger than or equal to the sum of the other two. We refer the reader to [DR1] for the precise formulation; here we limit ourselves to provide an overall description of the proof of Theorem 2.8.

To show the identity (73), one first shows that

$$(74) \qquad \mathrm{AJ}_p(\Delta)(\eta_f^{\mathrm{ur}} \otimes \omega_g \otimes \omega_h) = \langle \eta_f^{\mathrm{ur}}, P(\Phi)^{-1}\xi \rangle$$

where, similarly as in the previous sections, Φ is a lift of Frobenius to the system $\{\mathcal{W}_\epsilon\}_{\epsilon > 0}$ of wide open neighborhoods of the ordinary locus of $X(\mathbb{C}_p)$, $P(t) \in \mathbb{C}_p[t]$ is a polynomial satisfying

- $P(\Phi \times \Phi)$ annihilates the class of $\omega_g \otimes \omega_h$ in $H_{\mathrm{rig}}^2(\mathcal{W}_\epsilon^2)$,
- $P(\Phi)$ acts invertibly on $H_{\mathrm{dR}}^1(X)$,
- $P(\Phi)$ annihilates $H_{\mathrm{dR}}^2(X/\mathbb{Q}_p) \simeq \mathbb{Q}_p(-1)$,

and, given a rigid analytic primitive $\rho \in \Omega_{\mathrm{rig}}^1(\mathcal{W}_\epsilon^2)$ of $P(\Phi \times \Phi)(\omega_g \otimes \omega_h)$ and the choice of the cusp ∞ at infinity as base point, we set

$$\xi = (\delta^* - h_\infty^* - v_\infty^*)\rho \in H_{\mathrm{dR}}^1(X) \subset H_{\mathrm{rig}}^1(\mathcal{W}_\epsilon).$$

This is proved in [DR1, Ch. 3] in two steps: a formal calculation permits first to relate the image under the Abel-Jacobi map of the diagonal cycle Δ on X^3 to the diagonal D on the square X^2 of the curve; this leads to a simplification, which allows to apply Besser's machinery [Bes1] to compute $\mathrm{AJ}_p(\Delta)(\eta_f^{\mathrm{ur}} \otimes \omega_g \otimes \omega_h)$ purely in terms of classes of differential 1-forms on X, yielding (74).

In view of this formula, it is then not difficult to show that, up to an explicit Euler factor at p, (74) is equal to

$$(75) \qquad \langle \eta_f^{\mathrm{ur}}, e_{\mathrm{ord}}(d^{-1}(g^{[p]}) \cdot h) \rangle$$

where, as in §2.1, e_{ord} denotes Hida's ordinary projector, $g^{[p]} := \sum_{p \nmid n} a_n(g)q^n$ is the p-depletion of g and $d := qd/dq$ is Serre's derivative operator. That the left-hand side of

(73) equals (75), again up to an explicit p-multiplier, follows from the explicit calculations involved in the very construction of the p-adic L-function $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$.

We bring (75) to the reader's attention not only because it stands as the basic bridge between the two sides of the equality in (73), but also because this is a quantity which is amenable to effective numerical approximations, thanks to the work of A. Lauder (see [Lau] for more details).

This can be used in turn to design algorithms for computing p-adic numerical approximations to Chow-Heegner and Stark-Heegner points on elliptic curves over various number fields. We refer the reader to [DR2] and the forthcoming works of the third and sixth author, in collaboration with A. Lauder, for some of these constructions. To illustrate the method, we just mention here that the prototypical construction arises when f is taken to be the eigenform associated to an elliptic curve $E/\mathbb{Q}$ and $g = h$. In this setting Zhang [Zh] introduced a rational point $P_{g,f} \in E(\mathbb{Q})$, whose formal group logarithm can be computed as

$$(76) \qquad \log_{\omega_f}(P_{g,f}) = -2 \frac{\mathcal{E}_1(g)}{\mathcal{E}(g, g, f)} \langle \eta_g^{\mathrm{ur}}, e_{\mathrm{ord}}(d^{-1}(g^{[p]}) \cdot f) \rangle,$$

by invoking Theorem 2.8 and the results of [DRS1].

Motivated by the analogies between the description of $P_{g,f}$ given in [DRS1] by means of Chen's complex iterated integrals and the above formula, we refer to (75) as a *p-adic iterated integral*.

We close this section by discussing briefly the Euler system underlying the diagonal cycle Δ, and the arithmetic applications that Theorem 2.8 has in this context.

Assume f has rational Fourier coefficients and trivial nebentypus, and let $E/\mathbb{Q}$ be the (isogeny class of the) elliptic curve associated to it by the Eichler-Shimura construction. If instead of applying the p-adic syntomic Abel-Jacobi map, one considers the image of Δ under the p-adic étale Abel-Jacobi map $\mathrm{AJ}_{\mathrm{et},p}$ recalled in (70), one obtains a global cohomology class with values in $H^3_{\mathrm{et}}(X^3 \times \bar{\mathbb{Q}}, \mathbb{Z}_p(2))$. After projecting to the (f, g, h)-isotypical component of this Galois module, we obtain an element

$$(77) \qquad \kappa_E(g, h) \in H^1(\mathbb{Q}, V_p(E) \otimes V_g \otimes V_h(1)),$$

where, for any of the forms $\phi = f, g, h$, $V_\phi := H^1_{\mathrm{et}}(X \times \bar{\mathbb{Q}}, \mathbb{Z}_p)^\phi$ and $V_p(E) \simeq V_f(1)$ is the Galois representation associated to the p-adic Tate module of E.

Let now Λ_ϕ be the finite extension of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ corresponding to the space Ω_ϕ as considered at the beginning of the section for $\phi = f$. Hida constructed a rank two Galois representation $\mathbf{V}_\phi$ over Λ_ϕ, interpolating the Galois representations associated by Deligne to each of the classical specializations of ϕ.

As in §1.1, let Λ_{cyc} denote the Λ-adic Galois representation whose underlying module is Λ itself, equipped with the Galois action induced by the character

$$\mathrm{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Gal}\,(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \xrightarrow{\chi_{\mathrm{cyc}}} \mathbb{Z}_p^\times \hookrightarrow \Lambda^\times,$$

where χ_{cyc} is the cyclotomic character and the latter inclusion maps an element $z \in \mathbb{Z}_p^\times$ to the corresponding group-like element $[z] \in \Lambda^\times$. Define

$$\mathbf{V}_{f,g,h} := V_p(E) \otimes_{\mathbb{Z}_p} (\mathbf{V}_g \otimes_\Lambda \mathbf{V}_h) \otimes_\Lambda \Lambda_{\mathrm{cyc}}$$

where the tensor products over Λ are taken with respect to algebra homomorphisms

$$\Lambda \hookrightarrow \Lambda \otimes \Lambda \subseteq \Lambda_g \otimes \Lambda_h, \quad \Lambda \hookrightarrow \Lambda \otimes \Lambda \subseteq (\Lambda_g \otimes_\Lambda \Lambda_h) \otimes \Lambda_{\mathrm{cyc}}$$

such that for all classical points (x, y, z) of weights k, ℓ, $m \geq 2$ with $m = k + \ell - 2$, the specialization of $\mathbf{V}_{f,g,h}$ at (x, y, z) is isomorphic as a $G_{\mathbb{Q}}$-module to $V_{f_x} \otimes V_{g_y} \otimes V_{h_z}(k + \ell - 2)$.

One of the main results of [DR2] is the construction of a cohomology class

$$(78) \qquad\qquad \kappa_E(\mathbf{g}, \mathbf{h}) \in H^1(\mathbb{Q}, \mathbf{V}_{f,g,h})$$

which satisfies the following interpolation property. Let $(y, z) \in \Omega_g \times_\Omega \Omega_h$ be a pair of classical points over a weight in Ω of the form $z \mapsto \xi(z)z^2$ for some Dirichlet character ξ of p-power conductor. Then the specialization of $\kappa_E(\mathbf{g}, \mathbf{h})$ at (y, z) satisfies

$$\nu_{y,z}(\kappa_E(\mathbf{g}, \mathbf{h})) = \mathcal{E}_{y,z} \cdot \kappa_E(g_y, h_z).$$

for some explicit Euler factor $\mathcal{E}_{y,z}$.

The most interesting arithmetical applications of the Λ-adic cohomology class (78) arise when we deform it to points (y, z) of weight 1 which are *classical*. Indeed, assume that for such a pair, the specializations g_y and h_z are the q-expansions of classical eigenforms of weight 1; in this case their associated Galois representations are Artin representations, denoted ρ_y and ρ_z respectively. By specializing $\kappa_E(\mathbf{g}, \mathbf{h})$ to this pair, we obtain a cohomology class $\kappa_E(g_y, h_z) \in H^1(\mathbb{Q}, V_p(E) \otimes \rho_y \otimes \rho_z)$. Note that this class needs not be cristalline at p, since it did not arise from a geometric construction, but rather by deforming p-adically a collection of geometric classes. The goal of [DR2] is to show that

$$(79) \qquad \kappa_E(g_y, h_z) \text{ is cristalline at } p \text{ if and only if } L(E, \rho_y \otimes \rho_z, 1) = 0.$$

The main arithmetical application of (79) is the following instance of the Birch and Swinnerton-Dyer conjecture:

Theorem 2.9. *Let E be an elliptic curve over $\mathbb{Q}$ and let*

$$\rho_1, \rho_2 : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

be two continuous odd Galois representations, attached to weight one modular forms g and h respectively. Assume $\det(\rho_1) \cdot \det(\rho_2) = 1$ and there exists $\sigma \in G_{\mathbb{Q}}$ for which $\rho_1 \otimes \rho_2(\sigma)$ has distinct eigenvalues. If $L(E \otimes \rho_1 \otimes \rho_2, 1) \neq 0$, then

$$\dim_{\mathbb{C}}(\hom(\rho_1 \otimes \rho_2, E(\bar{\mathbb{Q}}) \otimes \mathbb{C})) = 0.$$

Conclusion

The contents of this survey are summarized in the table below, whose rows correspond to the six Euler systems covered in each section. The first and second column list the Euler system and its associated *p*-adic *L*-function, while the third gives the *p*-adic special value formula relating the two. The last column indicates the (eventual) application of each Euler system to the Birch and Swinnerton-Dyer conjecture. In this column, $\mathrm{BSD}_r(E, \rho)$ refers to the implication

$$\mathrm{ord}_{s=1} L(E, \rho, s) = r \quad \Rightarrow \quad \dim_{\mathbb{C}} \hom_{G_{\mathbb{Q}}}(\rho, E(K_\rho) \otimes \mathbb{C}) = r,$$

where E is an elliptic curve over $\mathbb{Q}$,

$$\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

is an Artin representation, and $r \geq 0$ is an integer. The letter A alludes to an elliptic curve with complex multiplication, and the letter E to a general (modular) elliptic curve over $\mathbb{Q}$. Likewise, the symbol ρ_ψ refers to the induction from an imaginary quadratic field K to $\mathbb{Q}$ of a *dihedral* character of K, while χ refers to a one-dimensional representation of $G_{\mathbb{Q}}$ (i.e., a Dirichlet character) and ρ, ρ_1 and ρ_2 refer to general odd irreducible two-dimensional Artin representations of $\mathbb{Q}$, with the sole constraint that $\rho_1 \otimes \rho_2$ has trivial determinant.

Euler system	*p*-adic *L*-function	*p*-adic formula	BSD application
Circular units	Kubota-Leopoldt *p*-adic *L*-function	Leopoldt's theorem	None
Elliptic units	Katz two-variable *p*-adic *L*-function	Katz's *p*-adic Kronecker limit formula	Coates-Wiles: $\mathrm{BSD}_0(A, 1)$.
Heegner points	Anticyclotomic *p*-adic *L*-function of [BDP]	*p*-adic Gross-Zagier theorem of [BDP]	Gross-Zagier, Kolyvagin: $\mathrm{BSD}_0(E, \rho_\psi)$ and $\mathrm{BSD}_1(E, \rho_\psi)$.
Beilinson-Kato elements	Mazur-Swinnerton-Dyer-Panciskin	*p*-adic Beilinson formula: [Kato], [Br], [Nik], [BD1]	Kato: $\mathrm{BSD}_0(E, \chi)$.
Beilinson-Flach elements	Hida's *p*-adic Rankin *L*-function	*p*-adic Beilinson formula: [BDR1]	$\mathrm{BSD}_0(E, \rho)$.
Gross-Kudla-Schoen cycles	Hida-Harris-Tilouine triple product *p*-adic *L*-function	*p*-adic Gross-Kudla formula [DR1]	$\mathrm{BSD}_0(E, \rho_1 \otimes \rho_2)$

Before concluding, the following three remarks are in order:

A. Complex formulae. The authors would be remiss if they failed to mention that the p-adic special value formulae described in this survey all have complex counterparts:

1.1. Dirichlet's class number formula relates the complex logarithm of the absolute value of the circular unit u_χ of Section 1.1 to the special value $L(1, \chi)$, or (equivalently, by the functional equation of $L(s, \chi)$) to the first derivative $L'(0, \chi)$ at $s = 0$.

1.2. The Kronecker limit formula relates the complex logarithm of the absolute value of the elliptic unit $u_{\chi_{\mathfrak{n}}}$ of Section 1.2 to the special value $L(K, \chi_{\mathfrak{n}}, 1)$ or (equivalently, by the functional equation) to the first derivative $L'(K, \chi_{\mathfrak{n}}, 0)$. The Kronecker limit formula can also be recast as a simple relation between the *square* of this logarithm and the first derivative at $s = 1$ of the Rankin convolution L-series

$$(80) \qquad L(E_{2,\chi} \otimes \theta_{\chi_{\mathfrak{n}}}, s) = L(\theta_{\chi_{\mathfrak{n}}^{-1}}, s - 1)L(\theta_{\chi_{\mathfrak{n}}}, s),$$

where $\theta_{\chi_{\mathfrak{n}}}$ is the weight one theta series attached to the character $\chi_{\mathfrak{n}}$. Note that the two factors on the right-hand side, at $s = 1$, are interchanged under the functional equation for $L(\theta_{\chi_{\mathfrak{n}}}, s)$.

1.3. The Gross-Zagier formula of [GZ] relates the Néron-Tate canonical height of the Heegner point $P_{K,f}$ of Section 1.3 to the first derivative at $s = 1$ of the Rankin L-series

$$(81) \qquad L(f \otimes \theta_K, s).$$

This L-series is obtained from the L-series in (80) by replacing the Eisenstein series $E_{2,\chi}$ by the weight two cusp form f (and χ by the trivial character).

2.1. Beilison's formula for $K_2(X_1(N))$ relates the square of the complex regulator of the Beilinson element $\{u_\chi, u_{\chi^{-1}}\}$ of Section 2.1, evaluated at the class η_f, (attached, here as in Section 2.1, to a form f with trivial nebentypus character) to the first derivative at the central value $s = 2$ of the triple convolution L-series

$$(82) \qquad L(f \otimes E_{2,\chi} \otimes E_{2,\chi^{-1}}, s) = L(f, s)L(f, s - 2)L(f, \chi, s - 1)L(f, \bar{\chi}, s - 1)$$

(up to a simple elementary fudge factor). When $L(f, \chi, 1) \neq 0$, this triple convolution L-series has a *simple zero* at $s = 2$ arising from the known behaviour of $L(f, s)$ at $s = 0$ and $s = 2$.

2.2. Beilinson's formula for $K_1(X_1(N)^2)$, which is stated in Theorem 2.6, can also be viewed as expressing the *square* of the complex regulator of the element Δ_{u_χ} of Section 2.2 in terms of the first derivative at the central value $s = 2$ of the convolution L-series

$$(83) \qquad L(f \otimes g \otimes E_{2,\chi}, s) = L(f \otimes g, s)L(f \otimes g, \chi, s - 1) = L(f \otimes g, s)L(\bar{f} \otimes \bar{g}, s - 1),$$

(where the last equality follows from the fact that χ is fixed to be the inverse of the product of the nebentypus characters of f and g). Note that the factors on the right of equations (82) and (83), evaluated at $s = 2$, are interchanged under the functional equation, up to simple constants and Gamma factors, and that the L-series on the left admits a simple zero at $s = 2$.

2.3. The Gross-Kudla-Yuan-Zhang-Zhang formula: As described in [GrKu] and [YZZ], it expresses the Arakelov height of the (f, g, h)-isotypic component of the Gross-Kudla-Schoen diagonal cycle $\Delta \in \mathrm{CH}^2(X_1(N)^3)_0$ of Section 2.3 in terms of the first derivative at the central value $s = 2$ of the triple convolution L-series

(84) $$L(f \otimes g \otimes h, s).$$

B. Other p-adic Gross-Zagier formulae. The p-adic formulae of Gross-Zagier type of [BDP] and of [DR1] alluded to in lines 3 and 6 of the table are not the only, or indeed even the most natural, generalisations of the formulae of Gross-Zagier and Gross-Kudla-Zhang to the p-adic setting. Perrin-Riou's p-adic Gross-Zagier formula described in [PR1], which relates the *first derivative* of a suitable p-adic L-function at a point which lies in its range of classical interpolation to the *p-adic height* of a Heegner point, bears a more visible analogy with its original complex counterpart. (The analogue of Perrin-Riou's formula for diagonal cycles has yet to be worked out in the literature, even though it appears to lie within the scope of the powerful techniques developed by Zhang and his school.) In contrast, the p-adic formulae of [BDP] and [DR1] are the direct generalisation of those of Leopoldt and Katz, and are thus better adapted to certain Euler system arguments.

C. Euler systems and central critical zeroes of order one. As will be apparent from the discussion in paragraph A above, all of the Euler systems discussed in this survey are governed by the leading terms of certain L-series at their *central points*, and seem to arise when these L-series admit *simple zeros* at the center, at least generically. The "degenerate instances" described in Sections 1.2, 2.1, and 2.2 correspond to settings where the relevant L-function breaks up into factors that are not central critical but rather are interchanged under the functional equation, as described in (80), (82), and (83). The order of vanishing of an L-series at a non-central point can be read off from the Gamma-factors appearing in its functional equation, and the examples of L-functions admitting simple zeros at such points are essentially exhausted[2] by the examples treated in Sections 1.1, 1.2, 2.1, and 2.2 (along with the simple zeros of Artin L-functions whose leading terms are conjecturally expressed in terms of Stark units). This remark may explain why the Euler systems alluded to in the first, second, fourth and fifth lines of the table, which ultimately rely on properties of modular units, do not generalize readily to other setting (such as totally real base fields), unlike the Euler systems of Heegner points and Gross-Kudla-Schoen cycles which are controlled by "genuine" central critical values.

This survey has taken the view that an Euler system is a collection of *global cohomology classes* which can be *related to L-functions* in a precise way and can be made to *vary in (p-adic) families*. The possibility of p-adic variation is an essential feature because it allows the construction of global classes which do not directly arise, in general, from a geometric construction involving étale Abel-Jacobi images of algebraic cycles or étale regulators of elements in K-theory, but rather from *p-adic limits* of such classes. Frequently, the obstruction to such "p-adic limits of geometric classes" being cristalline at p is encoded in

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[2]The authors are thankful to Benedict Gross for pointing this out to them.

a classical critical L-value, thereby tying this L-value to a global object which can be used to bound the associated Bloch-Kato Selmer group.

One aspect of the picture which has been deliberately *downplayed* is the idea that Euler systems should arise from norm-compatible collections of global elements defined over a varying collection of abelian extensions of a fixed ground field. This feature is clearly present in the first five examples considered in this survey, but not in the sixth, where the only variables of "p-adic deformation" are the *weight variables* arising in Hida theory. Over the years, p-adic families of automorphic forms have been studied for a wide variety of reductive groups. This raises the hope that Gross-Kudla-Schoen diagonal cycles will point the way to further fruitful examples of Euler systems, involving for instance (p-adically varying families of) algebraic cycles on Shimura varieties of unitary or orthogonal type. Examples of this kind would comfort the authors in their belief that Euler systems are far more ubiquitous than would appear from the limited panoply of known instances described in this survey.

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M.B.: Milano, Italy
E-mail address: `massimo.bertolini@unimi.it`

F.C.: Montreal, Canada
E-mail address: `castella@math.mcgill.ca`

H. D.: Montreal, Canada
E-mail address: `darmon@math.mcgill.ca`

S.D.: Santa Cruz, Unites States
E-mail address: `samit_dasgupta@yahoo.com`

K.P.: Ann Arbor, United States
E-mail address: `kartik.prasanna@gmail.com`

V.R.: Barcelona, Spain
E-mail address: `victor.rotger@upc.edu`