When the L-value vanishes

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January 1, 2016

- Choose odd Artin representations
  $\varrho_1, \varrho_2 : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(L), \quad \det(\varrho_1) = \det(\varrho_2)^{-1}$
- Define $\varrho := \varrho_1 \otimes \varrho_2 : G_{\mathbb{Q}} \to \mathrm{Gal}(H/\mathbb{Q}) \hookrightarrow \mathrm{GL}_4(L)$
- There are $g \in M_1(N, \chi)$,
  $h \in M_1(N, \bar{\chi}) : \quad \varrho_g \simeq \varrho_1 \otimes L_p, \quad \varrho_h \simeq \varrho_2 \otimes L_p$.
- **Example:** $\quad g = \bar{h} \Rightarrow \varrho = 1 \oplus \mathrm{ad}(g)$
- **Example:** Let K be a real or imaginary quadratic field.
  - $\psi_1, \psi_2 : G_K \to L^\times$ finite order characters
  - $\varrho_1 = \mathrm{Ind}_{\mathbb{Q}}^K(\psi_1)$, $\varrho_2 = \mathrm{Ind}_{\mathbb{Q}}^K(\psi_2)$
  - Assume $\det(\varrho_1) = \det(\varrho_2)^{-1}$. Then $\psi_1\psi_2$ and $\psi_1\psi_2'$ are self-dual.
  - $g = \theta(\psi_1),\ h = \theta(\psi_2)$
  - $\varrho = \varrho_1 \otimes \varrho_2 \simeq \mathrm{Ind}_{\mathbb{Q}}^K(\psi_1\psi_2) \oplus \mathrm{Ind}_{\mathbb{Q}}^K(\psi_1\psi_2')$.

- Let $E/\mathbb{Q}$ be an elliptic curve and fix $p \nmid \mathrm{cond}(E) \cdot \mathrm{cond}(\varrho)$.
- Choose ordinary p-stabilizations g_α, h_α in level Np.
- Let $\mathbf{g} = \{g_x\}$ be a Hida family passing through g_α
- Let $\mathbf{h} = \{h_x\}$ be a Hida family passing through h_α.
- It is possible to construct a p-adic family of global cohomology classes

$$\{ \quad \kappa(E, g_x, h_x) \quad \}$$

which for $k(x) = 2$ are the image under $\mathrm{reg}_{\mathrm{et}}$ of elements in

$$\begin{cases} \mathrm{CH}^2(X_1(Np^s)^3)_0 & \text{if } \mathbf{g}, \mathbf{h} \text{ are cuspidal} \\ \mathrm{K}_1(X_1(Np^s)^2) & \text{if } \mathbf{g} \text{ cuspidal}, \mathbf{h} \text{ Eisenstein} \\ \mathrm{K}_2(X_1(Np^s)) & \text{if } \mathbf{g}, \mathbf{h} \text{ are Eisenstein} \end{cases}$$

- For $k(x) \geq 2$, $\kappa(E, g_x, h_x)$ lies in Bloch-Kato's $H^1_f(\mathbb{Q}, -)$.
- At $k(x) = 1$ such that $g_x = g_\alpha$ and $h_x = h_\alpha$,

  $\kappa(E, g_\alpha, h_\alpha) \in H^1(\mathbb{Q}, V_f \otimes V_g \otimes V_h) = H^1(\mathbb{Q}, V_p(E) \otimes V_p(\varrho))$

- But $\kappa(E, g_\alpha, h_\alpha)$ does not necessarily lie in

  $\mathrm{Sel}_p(E_{/H})^\varrho = \mathrm{Hom}(V_p(\varrho), H^1_f(H, V_p(E)))$,

  which in turn contains

  $E(H)^\varrho = \mathrm{Hom}(V_p(\varrho), E(H) \otimes L_p)$

- **Theorem:** $L(E,\varrho,1)=0 \Rightarrow \kappa(E,g_\alpha,h_\alpha)\in \mathrm{Sel}_p(E_{/H})^\varrho$
- This reciprocity law is proved by showing that the Bloch-Kato logarithms of $\kappa(E,g_x,h_x)$ for $k(x)\geq 2$ recover the p-adic L-function $L_p(f,\mathbf{g},\mathbf{h})$.
- Assume $L(E,\varrho,1)=0$ so $\kappa_{\alpha\alpha}=\kappa(E,g_\alpha,h_\alpha)\in \mathrm{Sel}_p(E_{/H})^\varrho$.
- One checks that $\mathrm{Sign}(E,\varrho)=+1$, hence $L'(E,\varrho,1)=0$.
- **Conjecture:** $L''(E,\varrho,1)\neq 0 \overset{?}{\Leftrightarrow} \kappa_{\alpha\alpha}\neq 0$ in $\mathrm{Sel}_p(E_{/H})^\varrho$

A conjectural description of $\kappa_{\alpha,\alpha} := \kappa(E, g_\alpha, h_\alpha)$

- Assume $L(E,\varrho,1) = 0$, $L'(E,\varrho,1) = 0$, $L''(E,\varrho,1) \neq 0$.
- Fix two independent $P, Q : V_p(\varrho) \to E(H)_{L_p}$
- Fix a vector $v = v_{\alpha\alpha} \in V_p(\varrho)$, $\mathrm{Frob}_p(v) = \alpha_g \alpha_h \cdot v$
- Set $\quad P_{\alpha\alpha} = P(v_{\alpha\alpha}), \quad Q_{\alpha\alpha} = Q(v_{\alpha\alpha}) \in E(H)_{L_p}$
- Define likewise $\quad v_{\alpha\beta}, \quad P_{\alpha\beta}, \quad Q_{\alpha\beta}$ ...
- **Conjecture:** The class $\kappa_{\alpha,\alpha} \in \mathrm{Hom}(V_p(\varrho), E(H) \otimes L_p)$ is

$$\kappa_{\alpha,\alpha} = \Omega_p \cdot \left(\log(Q_{\beta\beta}) \cdot P - \log(P_{\beta\beta}) \cdot Q \right).$$

Example 1

- $g \in S_1(N, \chi), \quad h = g \otimes \chi^{-1}, \quad L(f, g, h, s) = L(E, s) \cdot L(E, \mathrm{Ad}(g), s)$.
- Assume $r_{\mathrm{an}}\, E(\mathbb{Q}) = 2,\ r_{\mathrm{an}}(E, \mathrm{Ad}(g)) = 0$.
- Fix a basis $E(\mathbb{Q}) \ = \ \langle\, P, Q\, \rangle$ up to torsion.
- $\alpha_g \beta_h = \alpha_g \beta_g \chi^{-1}(p) = 1$
- $\kappa(f, g_\alpha, h_\beta)(v_{\beta\alpha}) \ \doteq\ \log(Q) \cdot P \ - \ \log(P) \cdot Q \in E(\mathbb{Q}) \otimes \mathbb{Q}_p$

Example 2

- $g = \theta(\psi_1), h = \theta(\psi_2)$ for $\psi_1, \psi_2 : G_K \to L^\times$.
- $L(f, g, h, s) = L(E/K, \psi_1\psi_2, s) \cdot L(E/K, \psi_1\psi_2', s)$
- Assume $\mathrm{ord}_{s=1} L(f, g, h, s) = 2$.
- Assume in fact $r\,(E_{/K}, \psi_1\psi_2) = r\,(E_{/K}, \psi_1\psi_2') = 1$.
- Fix

$$\begin{array}{llllll} P: & V_p(\varrho) & \twoheadrightarrow & \mathrm{Ind}(\psi_1\psi_2) & \subseteq & E(H)_{L_p} \\ Q: & V_p(\varrho) & \twoheadrightarrow & \mathrm{Ind}(\psi_1\psi_2') & \subseteq & E(H)_{L_p} \end{array}$$

- $\kappa_{\alpha\alpha}(v_{\beta\beta}) \stackrel{?}{=} \log(Q_{\beta\beta}) \cdot P_{\beta\beta} - \log(P_{\beta\beta}) \cdot Q_{\beta\beta} \in E(H) \otimes L_p$.
- $\kappa_{\alpha\alpha}(v_{\alpha\beta}) \stackrel{?}{=} \log(Q_{\alpha\alpha}) \cdot P_{\alpha\beta} - \log(P_{\alpha\alpha}) \cdot Q_{\alpha\beta} \in E(H) \otimes L_p$.

- **Theorem** (Darmon-R):

  If $L_p^g(f,\mathbf{g},\mathbf{h})(1,1) \neq 0$ then $\kappa_{\alpha,\alpha}$ and $\kappa_{\alpha,\beta}$ are linearly independent in $\mathrm{Sel}_p(E_{/H})^{\varrho}$

  It suffices to show that

  $$\begin{pmatrix} \log \kappa_{\alpha\alpha}(v_{\beta\beta}) & \log \kappa_{\alpha\alpha}(v_{\beta\alpha}) \\ \log \kappa_{\alpha\beta}(v_{\beta\beta}) & \log \kappa_{\alpha\beta}(v_{\beta\alpha}) \end{pmatrix} \doteq \begin{pmatrix} 0 & L_p^g(f,\mathbf{g},\mathbf{h})(1,1) \\ L_p^g(f,\mathbf{g},\mathbf{h})(1,1) & 0 \end{pmatrix}$$

  Sketck of proof of the formula for the off-diagonal terms:

  $L_p^g(f,\mathbf{g},\mathbf{h})(1,1) = \lim_{x\to 1} L_p(f,\mathbf{g},\mathbf{h})(x,x)$

  $= \lim_{x\to 1} \mathrm{AJ}_p(\Delta[f,g_x,h_x])(\omega_f \eta_{g_x} \omega_{h_x})$

  $= \lim_{x\to 1, k(x)=2} \log_p \mathrm{AJ}_{\mathrm{et}}(\Delta[f,g_x,h_x])(\omega_f \eta_{g_x} \omega_{h_x})$

  $= \Omega_p \cdot \log_{\omega_f} \kappa_{\alpha\alpha}(v_{\beta\alpha})$ for some period Ω_p.

Numerical verification of the conjecture: we can not compute the class $\kappa_{\alpha\alpha}$ itself, but we can compute its logarithm. Namely, with A. Lauder we can check to high *p*-adic precision that:

$$\log \kappa_{\alpha\alpha}(v_{\beta\alpha}) = \frac{1}{\log_p(u)} \cdot \det \begin{pmatrix} \log_p(P_{\beta\beta}) & \log_p(P_{\beta\alpha}) \\ \log_p(Q_{\beta\beta}) & \log_p(Q_{\beta\alpha}) \end{pmatrix}$$

for a specific Stark unit u.

Our class in the computer

- $E_{57b}: y^2 + xy + y = x^3 - 7x + 5 \qquad \rightsquigarrow f \in S_2(57)$.
- $K = \mathbb{Q}(\sqrt{-23}),\ \chi = (\frac{\cdot}{23}),\ \mathrm{Gal}(H/K) = \{1, \sigma, \sigma^2\}$
- $S_1(23, \chi)$ spanned by $g = \theta(\psi), \quad \psi : \mathrm{Gal}(H/K) \to L^\times$.
- $V_g \otimes V_g = V_\psi \oplus L \oplus L(\chi)$.
- $\mathrm{rank}\, E(H)^\psi = 2, \quad \mathrm{rank}\, E(\mathbb{Q}) = \mathrm{rank}\, E(K)^\chi = 0$
- Heegner points are useless here! Using descent:

  $P = (9a^2 - 4a + 17, -45a^2 + 20a - 80),\ Q = (a^2 + 3, 2a^2 + 2)$,

  where $a \in H$ satisfies $a^3 - a^2 + 2a - 1 = 0$.

Our class in the computer

- Points

$$P^+ := 2P - P^\sigma - P^{\sigma^2}, \qquad P^- := P^\sigma - P^{\sigma^2}$$
$$Q^+ := 2Q - Q^\sigma - Q^{\sigma^2}, \qquad Q^- := Q^\sigma - Q^{\sigma^2}$$

span the two copies of V_ψ in $E(H)_L$.

- Take $p = 19$ and $H \hookrightarrow \mathbb{C}_{19}$ so that $P, Q \in E(\mathbb{Q}_{19})$.
- Take $g_\alpha \in S_1(3 \cdot 23, \chi)$ with U_3-eigenvalue ζ_3.

$$\log_p \kappa = \int_{g_\alpha} f \cdot g = \frac{\log(P^+)\log(Q^-) - \log(P^-)\log(Q^+)}{\lambda \cdot \log u}$$

$$= -25103076413984358720047537708218 \bmod 19^{25}.$$

Stark-Heegner points for real quadratic fields

- $E : y^2 + xy + y = x^3 - x^2 - x - 14 \qquad \rightsquigarrow f \in S_2(17)$.
- $\psi_o =$ quartic char of $K = \mathbb{Q}(\sqrt{5})$ ramified at λ, $\mathbf{N}(\lambda) = 29$.
- $g = \theta(\psi_o) \in S_1(5 \cdot 29, \chi),\ h = \theta(\bar{\psi}_o) \in S_1(5 \cdot 29, \bar{\chi})$,

  with fourier coefficients in $L = \mathbb{Q}(i)$.
- $V_g \otimes V_h = L \oplus L(\chi_5) \oplus V_\psi$,

  $\psi = \psi_o / \psi_o'$ is a dihedral character of K of conductor 29.

Stark-Heegner points for real quadratic fields

- $\mathrm{ord}_{s=1} L(E/K, s) = \mathrm{ord}_{s=1} L(E/K, \psi, s) = 1$.
- $E(K) = \langle P_K \rangle, \quad P_K = \left(\frac{379}{20}, \frac{-1995+7218\sqrt{5}}{200}\right)$.
- H/K = extension cut out by ψ, $\mathrm{Gal}(H/\mathbb{Q}) = D_8$.

  $H = M(\sqrt{\delta}), \quad M = \mathbb{Q}(\sqrt{5}, \sqrt{29}), \quad \delta = \frac{-29+3\sqrt{29}}{2}$.
- $E(H)^{\psi}$ is spanned by linear combinations of conjugates of

$$P_H = \left(\frac{-220777-17703\sqrt{145}}{5800}, \frac{214977+17703\sqrt{145}}{11600} + \frac{28584525+3803103\sqrt{5}+1645605\sqrt{29}+2364771\sqrt{145}}{290000}\sqrt{\delta}\right)$$

which was computed using Darmon-Pollack's algorithm.

Stark-Heegner points for real quadratic fields

- By applying the ordinary projection algorithms of A. Lauder to a space of ordinary overconvergent 17-adic modular forms of weight one and level $5 \cdot 29$:

$$\log_p \kappa = \int_{g_\alpha} f \cdot h = \frac{1}{3 \cdot 17} \cdot \frac{\log(P_K)\log(P_H)}{\log(u_K)}$$

$$= 1259389260500681328 \times 17 \bmod 17^{16}$$

- Hence we find a recipe for $P_H \in E(H)$ (mod torsion) as

$$P_H = \exp_{E,p}\left(\frac{3 \cdot 17 \cdot \log_p(u_K)}{\log_{E,p}(P_K)} \cdot \int_{g_\pm} f \cdot h\right)$$

Putting Kato in the computer

- $E : y^2 + xy + y = x^3 + x^2 - 4x + 5 \qquad \rightsquigarrow f \in S_2(42).$
- $\chi =$ even cubic Dirichlet character of conductor 19.
- Field cut out by χ is $H = \mathbb{Q}(\alpha), \quad \alpha^3 - \alpha^2 - 6\alpha + 7 = 0.$
- $E(\mathbb{Q}) = 0, \qquad r(E(H), \chi) = r(E(H), \bar{\chi}) = 1.$
- $E(H)^\chi = \langle P_\chi \rangle, \quad P_\chi = P + \chi(\sigma^2) P^\sigma + \chi(\sigma) P^{\sigma^2}$

$$\text{where } \mathrm{Gal}(H/\mathbb{Q}) = \{1, \sigma, \sigma^2\},$$

$$P = (2\alpha^2 + 3\alpha - 8, 4\alpha^2 + 6\alpha - 10).$$

Putting Kato in the computer

- $\varepsilon = (\frac{-3}{\cdot})$, so $\chi \cdot \varepsilon$ is odd sextic. Take $p = 7$.
- $g = \mathrm{Eis}_1(1, \varepsilon^{-1}\chi^{-1})$, $h = \mathrm{Eis}_1(\chi, \varepsilon) \in M_1(3 \cdot 19)$.
- $\varrho = \varrho_g \otimes \varrho_h = \chi + \varepsilon + \varepsilon^{-1} + \chi^{-1}$

$$\log_p \kappa = \int_{g_\alpha} f \cdot h = \frac{64}{7 \cdot 9} \cdot \frac{\log_{E,7}(P_\chi) \cdot \log_{E,7}(P_{\bar{\chi}})}{\log_7(u_{\chi\varepsilon}) + \log_7(u_{\bar{\chi}\varepsilon})}$$

$$= -12640038280624118214395581 \bmod 7^{36}.$$