Diagonal cycles, triple product L-functions and rational points on elliptic curves

(Séminaire de Théorie des Nombres de Bordeaux)

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(Joint work with Henri Darmon)

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Classical Heegner points

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The modular parametrization is

$$\begin{array}{llcl} \varphi: & X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathfrak{H}^* & \longrightarrow & E(\mathbb{C}) \\ & \tau & \mapsto & P_\tau := 2\pi i \int_\infty^\tau f(z)dz \\ & & & = \sum_{n\geq 1} \frac{a_n}{n} e^{2\pi i n \cdot \tau} \end{array}$$

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If $\tau \in \mathcal{H} \cap K$, where K is imaginary quadratic: $\quad P_\tau \in E(K^{ab})$.

The modular parametrization revisited

- The universal covering of $X_0(N)$ is

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- Chen's *iterated integrals* may give rise to *anabelian* modular parametrizations of points in $E(\mathbb{C})$.

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- A linear combination of iterated integrals which is *homotopy invariant* yields $J : \mathbf{P}(Y; o) \longrightarrow \mathbb{C}$.

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- $J_{\omega,\eta} := \int \omega \cdot \eta - \alpha_{\omega,\eta}$ is homotopy-invariant.

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Examples

E	P_{gen}	N_g	$P_{g,f}$
37a	$(0,-1)$	37	$-6P$
43a	$(0,-1)$	43	$4P$
53a	$(0,-1)$	53	$-2P$
57a	$(2,1)$	57	$\frac{4}{3}P$
		57	$-\frac{16}{3}P$
		19	$-4P$
58a	$(0,-1)$	58	$4P$
		29	0
		29	$4P$
77a	$(2,3)$	77	$\frac{12}{5}P$
		77	$-\frac{4}{3}P$
		11	$\frac{4}{3}P$
79a	$(0,0)$	79	$-4P$
82a	$(0,0)$	82	0
		82	$2P$
		41	$2P$
		41	0

83a	$(0,0)$	83	0
		83	$2P$
88a	$(2,-2)$	88	0
		44 g	0
		44 $g(2)$	$8P$
		11 g	0
		11 $g(2)$	$8P$
91a	$(0,0)$	91	$2P$
		91	$2P$
		91	$4P$
91b	$(-1,3)$	91	0
		91	0
		91	0
92b	$(1,1)$	92	0
		46	0
99a	$(2,0)$	99	$-\frac{2}{3}P$
446d	(1,0),(0,2)	446	0
681a	$(4,4)$	681	$-24P$

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Define $\underline{P}_{g,f} := \langle P_{\sigma_{g(az),f(bz)}} \rangle \subseteq E(\mathbb{Q})$ where

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The module $\underline{P}_{g,f}$ is nonzero if and only if:

i. $L(f,1) = 0$, $L'(f,1) \neq 0$

ii. the local signs at finite primes of $L(g^\sigma \otimes g^\sigma \otimes f, s)$ are all $+1$

iii. $L(\mathrm{Sym}^2(g^\sigma) \otimes f, 2) \neq 0$.

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A complex Gross-Zagier formula for Δ

Theorem (Yuan-Zhang-Zhang)

$$h(\Delta[f,g,h]) = (\text{Explicit non-zero factor}) \times L'(f,g,h,2)$$

where

$$h : \mathrm{CH}^2(X^3)_0 \longrightarrow \mathbb{R}$$

is Beilinson-Bloch's height pairing.

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- Points with $\kappa(x) = 2$ are *not* interpolated: $L(f_x, g, h, 2) = 0$.
- For x_0 with $\kappa(x_0) = 2$ and $\mathbf{f}_{x_0} = f$, regard $\mathcal{L}_p(\mathbf{f}, g, h)(x_0)$ as a p-adic avatar of $L'(f, g, h, 2)$.

A p-adic avatar of the Gross-Zagier formula

Theorem. (Darmon-R.) Assume for simplicity that $N_f = N_g = N_h$. Then

$$\mathcal{L}_p(\mathbf{f}, g, h)(x_0) = \frac{\mathcal{E}(f,g,h)}{\mathcal{E}_0(f)\mathcal{E}_1(f)} \times \mathrm{AJ}_p(\Delta)(\eta_f \otimes \omega_g \otimes \omega_h).$$

where

$$\begin{aligned}
\mathcal{E}(f,g,h) &:= \left(1 - \beta_p(f)\alpha_p(g)\alpha_p(h)p^{-2}\right)\left(1 - \beta_p(f)\alpha_p(g)\beta_p(h)p^{-2}\right) \\
&\quad \left(1 - \beta_p(f)\beta_p(g)\alpha_p(h)p^{-2}\right)\left(1 - \beta_p(f)\beta_p(g)\beta_p(h)p^{-2}\right) \\
\mathcal{E}_0(f) &:= (1 - \beta_p^2(f)\chi_f^{-1}(p)p^{-1}) \\
\mathcal{E}_1(f) &:= (1 - \beta_p^2(f)\chi_f^{-1}(p)p^{-2}).
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where we had set $\rho = d^{-1} P(\Phi)(\omega_g \otimes \omega_h)$.