Iterated integrals, diagonal cycles and rational points on elliptic curves

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Classical Heegner points

Let $E_{/\mathbb{Q}}$ be an elliptic curve and

$$f = \sum_{n \geq 1} a_n q^n \in S_2(N) \text{ with } L(E,s) = L(f,s).$$

The modular parametrization is

$$\begin{array}{rccc} \varphi : & X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathfrak{H}^* & \longrightarrow & E(\mathbb{C}) \\ & \tau & \mapsto & P_\tau := 2\pi i \int_\infty^\tau f(z)dz \\ & & & = \sum_{n\geq 1} \frac{a_n}{n} e^{2\pi i n\cdot\tau} \end{array}$$

If $\tau \in \mathbb{P}^1(\mathbb{Q})$ is a cusp: $P_\tau \in E(\mathbb{Q})_{tors}$.

If $\tau \in \mathcal{H} \cap K$, where K is imaginary quadratic: $P_\tau \in E(K^{ab})$.

Stark-Heegner points

- Bertolini, Darmon, Greenberg replaced $\mathfrak{H}^*$ by the p-adic upper half-plane, using Coleman p-adic path integrals.

- For $E_{/\mathbb{Q}}$, K real quadratic where p is inert and H/K ring class field, Darmon constructs points on $E(\mathbb{C}_p)$ which should be H-rational.
  Bertolini, Dasgupta, Greenberg, Longo, R., Seveso, Vigni complete the conjectural picture.

- For $E_{/F}$ modular over a totally real F, Darmon and Logan use a similar cohomological formalism to construct points on ring class fields H/K of ATR quadratic extensions K/F.
  Gartner generalizes to any K/F provided the signs of the functional equations match, but is not effective.

The modular parametrization revisited

- The universal covering of $X_0(N)$ is

$$\mathbf{P}(X_0(N);\infty) = \{\gamma : [0,1] \longrightarrow X_0(N),\ \gamma(0) = \infty\}/\text{homotopy}.$$

- The modular parametrization factors through

$$\begin{array}{rccc} \varphi : & X_0(N) = \pi_1(X_0(N))\backslash \mathbf{P}(X_0(N)) & \longrightarrow & J_0(N) \to E \\ & \gamma : \infty \rightsquigarrow \tau & \mapsto & P_\tau := \int_\gamma \omega_f, \end{array}$$

as $\pi_1(X_0(N)) \to \mathbb{C}$, $\gamma \mapsto \int_\gamma \omega_f$ factors through $H_1(X_0(N),\mathbb{Z})$.

- Chen's *iterated integrals* may give rise to *anabelian* modular parametrizations of points in $E(\mathbb{C})$.

Chen's iterated path integrals

- Y smooth quasi-projective curve, $o \in Y$ base point, $\tilde{Y}$ universal covering.

- The *iterated integral* attached to a tuple of smooth 1-forms $(\omega_1, \ldots, \omega_n)$ on Y is the functional

$$\gamma \mapsto \int_\gamma \omega_1 \cdot \omega_2 \cdot \ldots \cdot \omega_n := \int_\Delta (\gamma^*\omega_1)(t_1)(\gamma^*\omega_2)(t_2) \cdots (\gamma^*\omega_n)(t_n),$$

where $\Delta = \{0 \leq t_n \leq t_{n-1} \leq \cdots \leq t_1 \leq 1\}$.

- When $n = 2$: $\int_\gamma \omega \cdot \eta = \int_{\tilde{\gamma}} \omega F_\eta, \quad$ for F_η primitive of η on $\tilde{Y}$.

- A linear combination of iterated integrals which is *homotopy invariant* yields $J : \mathbf{P}(Y; o) \longrightarrow \mathbb{C}$.

Iterated integrals of modular forms

- $X = X_0(N)$, $Y = X \setminus \{\infty\}$, cusp 0 as base point.
- Let $\omega \in \Omega^1(X)$ and $\eta \in \Omega^1(Y)$, with a pole at ∞.
- Let $\alpha = \alpha_{\omega,\eta} \in \Omega^1(Y)$ such that $\omega F_\eta - \alpha_{\omega,\eta}$ on $\tilde{Y}$ has log poles over ∞.
- $J_{\omega,\eta} := \int \omega \cdot \eta - \alpha_{\omega,\eta}$ is homotopy-invariant.

Iterated integrals of modular forms

- Let $E_{/\mathbb{Q}}$ be an elliptic curve and $f = f_E \in S_2(N_E)$.
- Let $g \in S_2(M)$ be a newform of some level M, with $[\mathbb{Q}(\{a_n(g)\}) : \mathbb{Q}] = t \geq 1$. Put $N = \mathrm{lcm}(M, N_E)$.
- $\gamma_f \in H_1(X, \mathbb{C})$ Poincaré dual of ω_f.
- Let $\{\omega_{g,i}, \eta_{g,i}\}_{i=1,\dots,t}$ be a symplectic basis of $H^1(X)[g]$.
- **Define** $P_{g,f} := \sum_{i=1}^{t} \int_{\gamma_f} \omega_{g,i} \cdot \eta_{g,i} - \eta_{g,i} \cdot \omega_{g,i} - 2\alpha_i \in E(\mathbb{C})$.
- The point is independent of the choice of base point 0, path γ_f or basis of $H^1(X)[g]$.

Numerical computation

- With Michael Daub, Henri Darmon and Sam Lichtenstein we have an algorithm to compute $P_{g,f}$:

- Given $N \geq 1$, define c_N the smallest integer for which there are
$$\gamma_j = \left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right) \in \Gamma_0(N), \quad c \leq \mathrm{c}_N$$
such that $H_1(X, \mathbb{Z}) = \langle ..., [\gamma_j], ... \rangle_{\mathbb{Z}}$.

- The number n_D of Fourier coefficients required to compute $P_{g,f}$ to a given number D of digits of accuracy is
$$n_D = O(\max\{N \cdot \mathrm{c}_N \cdot (D + N^{11\sigma_0(N)+2}), \mathrm{c}_N^2 \cdot N^{2\sigma_0(N)+2}\}).$$

- We represent the 1-forms $\eta_{g,i}$ as differentials of the *2nd kind*: $\sum u_i \cdot \omega_{g,i}$ where u_i are modular units given as eta products.

Some points on curves of rank 1 and conductor < 100

E	P_{gen}	g	n	$P_{g,f,n}$
37a1	$(0,-1)$	1	1	$-6P$
43a1	$(0,-1)$	1	1	$4P$
53a1	$(0,-1)$	1	1	$-2P$
57a1	$(2,1)$	1	1	$\frac{4}{3}P$
		2	1	$-\frac{16}{3}P$
		3	1	$-4P$
58a1	$(0,-1)$	1	1	$4P$
		2	1	0
			2	$4P$
77a1	$(2,3)$	1	1	$\frac{12}{5}P$
		2	1	$-\frac{4}{3}P$
		3	1	$\frac{4}{3}P$
		4	1	$-\frac{12}{5}P$
79a1	$(0,0)$	1	1	$-4P$
82a1	$(0,0)$	1	1	0
			3	$2P$
		2	1	$2P$
83a1	$(0,0)$	1	1	0
			2	$2P$
88a1	$(2,-2)$	1	1	0
		2	1	0
			2	$8P$
		3	1	0
			2	$8P$
91a1	$(0,0)$	1	1	$2P$
		2	1	$2P$
		3	1	$4P$
91b1	$(-1,3)$	0	1	0
		2	1	0
		3	1	0
92b1	$(1,1)$	1	1	0
		2	1	0
99a1	$(2,0)$	1	1	$-\frac{2}{3}P$
		2	1	0
		3	1	$\frac{2}{3}P$

Connection with diagonal cycles

- **Theorem 1** (Darmon-R.-Sols) The points $P_{f,g}$ are $\mathbb{Q}$-rational.

- $P_{g,f}$ is the *Chow-Heegner* point associated with the $[g,g,f]$-isotypical component of Gross-Kudla-Schoen's diagonal cycle

$$\Delta = \{(x,x,x), x \in X\} -$$

$$-\{(x,x,0)\} - \{(x,0,x)\} - \{(0,x,x)\} +$$

$$+\{(0,0,x)\} + \{(0,x,0)\} + \{(x,0,0)\} \subset X^3,$$

a null-homologous cycle of codimension two in X^3:

- Putting
$\Pi = \{(x,x,y,y)\} \subset X^4$, $P_{g,f} = \pi_{f,*}(\Pi \cdot \pi_{123}^*(\Delta[g,g,f]))$.

Connection with diagonal cycles

In fact, for any divisor

$$T \in \frac{\mathrm{Pic}(X \times X)}{\pi_1^*\mathrm{Pic}(X) \oplus \pi_2^*\mathrm{Pic}(X)} \simeq \mathrm{End}(J_0(N))$$

we obtain a point

$$P_T = \Pi \cdot \pi_{123}^*(\Delta_T) \quad \text{for a suitable } \Delta_T \in \mathrm{CH}^2(X^3)_0.$$

This gives rise to a new *modular parametrization* of points

$$\mathrm{End}(J_0(N)) \to \mathrm{Hodge}(X_0(N)^2) \to J_0(N)(\mathbb{Q}) \stackrel{\pi_{f,*}}{\to} E(\mathbb{Q}), \quad T \mapsto P_T,$$

which is $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariant for its natural extension to $\bar{\mathbb{Q}}$.

Connection with L-functions

The triple L-function of $f \in S_k(N_f)$, $g \in S_\ell(N_g)$, $h \in S_m(N_h)$ is

$$L(f,g,h;s) = L(V_f \otimes V_g \otimes V_h; s) = \prod_p L^{(p)}(f,g,h;p^{-s})^{-1},$$

For $p \nmid N = \mathrm{lcm}(N_f, N_g, N_h)$, the Euler factor $L^{(p)}(f,g,h;T)$ is

$$(1-\alpha_f\alpha_g\alpha_h T)\cdot(1-\alpha_f\alpha_g\beta_h T)\cdot \ldots \cdot (1-\beta_f\beta_g\beta_h T).$$

- The completed L-function satisfies

$$\Lambda(f,g,h;s) = \prod_{p|N\infty} \varepsilon_p(f,g,h)\cdot\Lambda(f,g,h;k+\ell+m-2-s).$$

- $\varepsilon_\infty(f,g,h) = \begin{cases} -1 & \text{if } (k,\ell,m) \text{ are balanced.} \\ +1 & \text{if } (k,\ell,m) \text{ are unbalanced.} \end{cases}$

Combining our result with Yuan-Zhang-Zhang

Theorem 2. Let $E_{/\mathbb{Q}}$ be an elliptic curve of conductor N_E and g a newform of level M.

Assume $\varepsilon_p(g, g, f) = +1$ at the primes $p \mid N = \mathrm{lcm}(M, N_E)$.

Then the module of points

$$\underline{P}_{g,f} := \sum_{d \mid \frac{N}{N_E}} \pi_{f(d)} \{P_T, T \in \mathrm{End}^0(J_0(N))[g]\} \subseteq E(\mathbb{Q})$$

is nonzero if and only if:

i. $L(f, 1) = 0$,
ii. $L'(f, 1) \neq 0$, and
iii. $L(f \otimes \mathrm{Sym}^2(g^\sigma), 2) \neq 0$ for all $\sigma : K_g \longrightarrow \mathbb{C}$.

Examples

- $E = 37a$, $g = 37b$. $\underline{P}_{g,f} = \langle P_{g,f} \rangle$ is not torsion. $\varepsilon_{37}(g,g,f) = +1$ and $L(f \otimes \mathrm{Sym}^2(g), 2) \neq 0$.

- $E = 58a$ and $g = 29a$. $\varepsilon_2(g,g,f) = \varepsilon_{29}(g,g,f) = +1$ and $L(f \otimes \mathrm{Sym}^2(g), 2) \neq 0$. But $P_{g,f}$ is torsion. $\underline{P}_{g,f}$ contains the non-torsion point $P_{g,f,2} := \pi_f(P_{T_g \cdot T_2})$.

- $E = 91b$, $g = 91a$. $\underline{P}_{g,f} = \langle P_{g,f} \rangle$ is torsion, because $\varepsilon_7(g,g,f) = \varepsilon_{13}(g,g,f) = -1$. Wants a Shimura curve.

- $E = 158b$, $g = 158d$. While $\varepsilon_2(g,g,f) = \varepsilon_{79}(g,g,f) = +1$, $\underline{P}_{g,f} = \langle P_{g,f} \rangle$ is torsion, because $L(f \otimes \mathrm{Sym}^2(g), 2) = 0$.

Motivic explanation of the rationality of $P_{g,f}$

- Set $Y = X_0(N) \setminus \{\infty\}$, $\Gamma = \pi_1(Y; 0)$.
- $I = \langle \gamma - 1 \rangle$ augmentation ideal of $\mathbb{Z}[\Gamma]$: $\quad \mathbb{Z}[\Gamma]/I = \mathbb{Z}$.
- $I/I^2 = \Gamma_{ab} = H_1(Y, \mathbb{Z}) = H_1(X, \mathbb{Z})$.
- $I^2/I^3 = (\Gamma_{ab} \otimes \Gamma_{ab}),\ \gamma_1 \otimes \gamma_2 \mapsto (\gamma_1 - 1)(\gamma_2 - 1)$.
- $\{\mathbf{P}(Y; 0) \rightarrow \mathbb{C} \text{ of length } \leq n\} \simeq \mathrm{Hom}(I/I^{n+1}, \mathbb{C})$.

An extension of mixed motives

- The exact sequence $0 \longrightarrow I^2/I^3 \longrightarrow I/I^3 \longrightarrow I/I^2 \longrightarrow 0$

  becomes $0 \longrightarrow H^1_{\mathrm{B}}(Y) \longrightarrow M_{\mathrm{B}} \longrightarrow H^1_{\mathrm{B}}(Y)^{\otimes 2} \longrightarrow 0$.

- The first and third groups are the Betti realizations of a pure motive defined over $\mathbb{Q}$.

- The complexification of both is equipped with a Hodge filtration: both are *pure Hodge structures*.

- $M_{\mathrm{B}} = \mathrm{Hom}(I/I^3, \mathbb{C}) = \{J : \mathbf{P}(Y;0) \rightarrow \mathbb{C}$ of length $\leq 2\}$ underlies a *mixed Hodge structure* and should arise from a *mixed motive* over $\mathbb{Q}$.

Motivic explanation of the rationality of $P_{g,f}$

- M_{B} yields an extension class $\kappa \in \mathrm{Ext}^1_{\mathrm{MHS}}(H^1_{\mathrm{B}}(X)^{\otimes 2}, H^1_{\mathrm{B}}(X))$.

- Any $\xi : \mathbb{Z}(-1) \longrightarrow H^1_{\mathrm{B}}(X)^{\otimes 2}$ yields

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_B(X) & \longrightarrow & M_B(\xi) & \longrightarrow & \mathbb{Z}(-1) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \xi & & \\ 0 & \longrightarrow & H^1_B(X) & \longrightarrow & M_B & \longrightarrow & H^1_B(X)^{\otimes 2} & \longrightarrow & 0. \end{array}$$

- $\varphi : \ \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-1), H^1_{\mathrm{B}}(X)) = \frac{H^1_{\mathrm{dR}}(X/\mathbb{C})}{\Omega^1(X(\mathbb{C})) + H^1_{\mathrm{B}}(X)} \simeq J_0(N)(\mathbb{C})$.

- $\xi_g : 1 \mapsto \mathrm{cl}(T_g) \in H^2_{\mathrm{B}}(X^2) \overset{Kunneth}{\longrightarrow} H^1_{\mathrm{B}}(X)^{\otimes 2}$,

- $P_g := \varphi(\xi_g) \in J_0(N), \quad P_{g,f} := \pi_f(P_g)$.

$P_{g,f}$ as a Chow-Heegner point

The complex Abel-Jacobi map for curves

$$\mathrm{AJ}_{\mathbb{C}} : \mathrm{CH}^1(X)_0 \longrightarrow \Omega_X^{1,\vee}/H_1(X,\mathbb{Z}), \quad D \mapsto \int_D$$

generalizes to varieties V of higher dimension d and null-homologous cycles of codimension c:

$$\mathrm{AJ}_{\mathbb{C}} : \mathrm{CH}^c(V)_0 \to J^c(V) = \frac{\mathrm{Fil}^{d-c+1} H_{dR}^{2d-2c+1}(V_{\mathbb{C}})^{\vee}}{H_{2d-2c+1}(V,\mathbb{Z})}, \ \Delta \mapsto \int_{\tilde{\Delta}},$$

where $\tilde{\Delta}$ is a $2(d-c)+1$-differentiable chain on the real manifold $V(\mathbb{C})$ with boundary Δ.

$P_{g,f}$ as a complex Chow-Heegner point

$$\begin{array}{ccc}
\mathrm{CH}^2(X^3)_0 & \stackrel{\mathrm{AJ}_{\mathbb{C}}}{\rightarrow} & J^2(X^3) = \frac{\mathrm{Fil}^2 H^3_{dR}(X^3)^\vee}{H_3(X^3,\mathbb{Z})} \\
\Pi^* \downarrow & & \downarrow \Pi^*_{\mathbb{C}} \\
E & \stackrel{\mathrm{AJ}_{\mathbb{C}}}{\rightarrow} & \mathbb{C}/\Lambda_E,
\end{array}$$

$$\Delta \in \mathrm{CH}^2(X^3)_0 \mapsto \pi^*_{123}\Delta \mapsto \pi^*_{123}\Delta \cdot \Pi \mapsto P_\Delta := \pi_{E,*}(\pi^*_{123}\Delta \cdot \Pi) \in E.$$

Theorem. (Darmon-R.-Sols) In $E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$\mathrm{AJ}_{\mathbb{C}}(\Delta_{GKS})(\mathrm{cl}(T_g) \wedge \omega_f) = \int_{\gamma_f} \left(\sum_{i=1}^{t} \omega_{g,i} \cdot \eta_{g,i} - \eta_{g,i}\omega_{g,i} - 2\alpha_i \right).$$

- The p-adic Abel-Jacobi map at a prime $p \nmid N$ is

$$\mathrm{AJ}_p : \mathrm{CH}^{r+2}(X^3)_0(\mathbb{Q}_p) \longrightarrow \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X^3/\mathbb{Q}_p)^\vee.$$

- $\log_{\omega_f}(P_{g,f}) = -2\mathrm{AJ}_p(\Delta_{GKS})(\eta_g \wedge \omega_g \wedge \omega_f)$.

Theorem. (Darmon-R.) Let $(\mathcal{W}, \Phi)$ be a wide open of

$$X_0(N)(\mathbb{C}_p) \setminus \mathrm{red}^{-1}(X(\bar{\mathbb{F}}_p)_{ss})$$

and a lift of Frobenius. Let $\rho \in \Omega^1(\mathcal{W} \times \mathcal{W})$ such that $d\rho = P(\Phi)(\omega_g \otimes \omega_f)$ for a suitable polynomial P. Then

$$\mathrm{AJ}_p(\Delta)(\eta_g \otimes \omega_g \otimes \omega_f) = \langle \eta, P(\Phi)^{-1} \epsilon^* \rho \rangle_X$$

where $\epsilon^* = \epsilon_{12}^* - \epsilon_1^* - \epsilon_2^*$, for $\epsilon_{12}, \epsilon_1, \epsilon_2 : X \hookrightarrow X^2$.