Stark-Heegner Points

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Second Lecture

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Arizona Winter School

Tucson, Arizona

March 2011

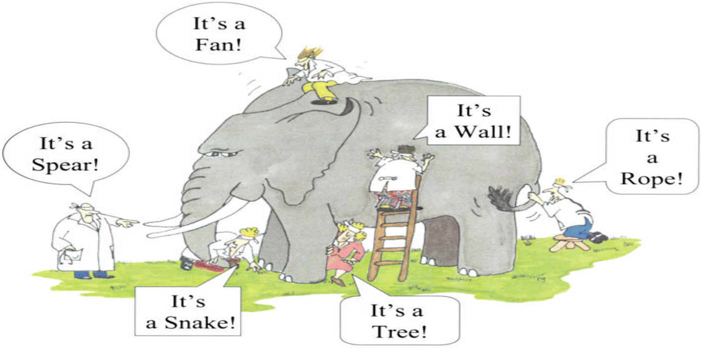
Summary of the first lecture

Victor described variants of the Heegner point construction based on higher dimensional algebraic cycles: the so-called *Chow-Heegner points*.

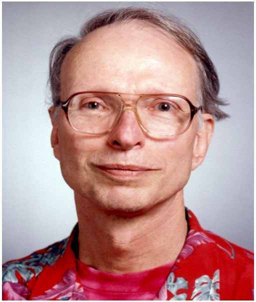
Our last two lectures, and the student projects will focus *exclusively* on Chow-Heegner points attached to *diagonal cycles* on triple products of modular curves and Kuga-Sato varieties.

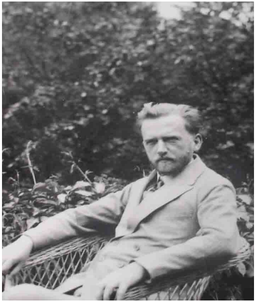
Goal of this morning's lecture: indicate how these ostensibly very special constructions fit into the "broader landscape" of *Stark-Heegner points.*

... otherwise the les experienced participants might feel like the protagonists in the tale of the elephant and the six blind men!



What is a Stark-Heegner point?





Executive summary: Stark-Heegner points are points on elliptic curves arising from (*not necessarily algebraic*) cycles on modular varieties.

A prototypical example: points arising from ATR cycles

Motivation. Thanks to Heegner points, we know:

$$ord_{s=1}L(E,s) \leq 1 \implies \mathrm{rank}(E(\mathbb{Q})) = \mathrm{ord}_{s=1} L(E,s),$$

for all elliptic curves $E/\mathbb{Q}$.

By work of Zhang and his school, exploiting Heegner points on Shimura curves, similar results are known for *many* elliptic curves over totally real fields...

but not for all of them!!

The mysterious elliptic curves

F = a real quadratic field;

E = elliptic curve of conductor 1 over F;

$\chi : \mathrm{Gal}(K/F) \longrightarrow \pm 1$ = quadratic character of F.

Question: Show that

$$\mathrm{ord}_{s=1} L(E/F, \chi, s) \leq 1 \implies \mathrm{rank}(E^{\chi}(F)) = \mathrm{ord}_{s=1} L(E/F, \chi, s).$$

The mysterious elliptic curves

Theorem (Matteo Longo)

$$L(E/F,\chi,1)\neq 0 \implies \#E^{\chi}(F)<\infty.$$

Yu Zhao's PhD thesis (defended March 10, 2011):

Theorem (Rotger, Zhao, D)

If E is a $\mathbb{Q}$-curve, i.e., is isogenous to its Galois conjugate, then

$$\mathrm{ord}_{s=1} L(E/F,\chi,s)=1 \implies \mathrm{rank}(E^{\chi}(F))=1.$$

We have no idea how to prove the existence of a point in $E^{\chi}(F)$ in general!

Logan, D, (2003): we can nonetheless propose a *conjectural formula* to compute it in practice, via *ATR cycles*.

ATR cycles

Let Y be the (open) *Hilbert modular surface* attached to E/F:

$$Y(\mathbb{C}) = \mathbf{SL}_2(\mathcal{O}_F)\backslash(\mathcal{H}_1 \times \mathcal{H}_2).$$

Let $\gamma \in \mathbf{SL}_2(\mathcal{O}_F)$, with a (unique) fixed point $\tau_1 \in \mathcal{H}_1$.

Then the field K generated by the eigenvalues of γ is an ATR extension of F.

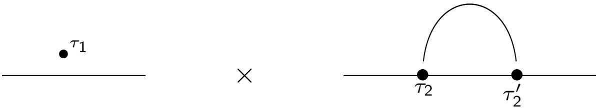
To each γ, we will attach a cycle $\Delta_\gamma \subset Y(\mathbb{C})$ of real dimension one which "behaves like a Heegner point".

ATR cycles

$\tau_1 :=$ fixed point of $\gamma \circlearrowleft \mathcal{H}_1$;

$\tau_2, \tau_2' :=$ fixed points of $\gamma \circlearrowleft (\mathcal{H}_2 \cup \mathbb{R})$;

$\Upsilon_\gamma = \{\tau_1\} \times \text{geodesic}(\tau_2 \to \tau_2')$.



$$\Delta_\gamma = \Upsilon_\gamma / \langle \gamma \rangle \subset Y(\mathbb{C}).$$

Key fact: The cycles Δ_γ are *null-homologous*.

Oda's conjecture on periods

For any closed 2-form $\omega_G \in \Omega_G$, let Λ_G denote its set of periods, as in Kartik's lectures:

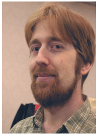
$$\Lambda_G = \{ \int_\gamma \omega_G, \quad \gamma \in H_2(X(\mathbb{C}), \mathbb{Z}). \}$$

Conjecture (Oda, 1982)

For a suitable choice of ω_G, we have $\mathbb{C}/\Lambda_G \sim E(\mathbb{C})$.

Points attached to ATR cycles

$$P_\gamma^?(G) := \mathrm{AJ}(\Delta_\gamma)(\omega_G) := \int_{\partial^{-1}\Delta_\gamma} \omega_G \quad \in \quad \mathbb{C}/\Lambda_G = E(\mathbb{C}).$$



$$\Gamma_{\mathrm{trace}=t} = \Gamma\gamma_1\Gamma^{-1} \cup \cdots \cup \Gamma\gamma_h\Gamma^{-1}.$$

Conjecture (Logan, D, 2003)

The points $P_{\gamma_j}^?(G)$ belongs to $E(H) \otimes \mathbb{Q}$, where H is a specific ring class field of K. The points $P_{\gamma_1}^?(G), \ldots, P_{\gamma_h}^?(G)$ are conjugate to each other under $\mathrm{Gal}(H/K)$. Finally, the point $P_K^?(G) := P_{\gamma_1}^?(G) + \cdots + P_{\gamma_h}^?(G)$ is of infinite order iff $L'(E/K, 1) \neq 0$.

Stark-Heegner points attached to real quadratic fields

ATR points are defined over abelian extensions of a quadratic ATR extension K of a real quadratic field F.

There is a second setting, equally fraught with mystery, involving an elliptic curve $E/\mathbb{Q}$ over $\mathbb{Q}$ and class fields of *real quadratic fields*.

Simplest case: $E/\mathbb{Q}$ is of prime conductor p, and K is a real quadratic field in which p is inert.

$$\mathcal{H}_p = \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$$

A dictionary between the two settings

ATR cycles	Real quadratic points
F real quadratic	$\mathbb{Q}$
∞_0, ∞_1	p, ∞
E/F of conductor 1	$E/\mathbb{Q}$ of conductor p
$\mathbf{SL}_2(\mathcal{O}_F)\backslash(\mathcal{H}\times\mathcal{H})$	$\mathbf{SL}_2(\mathbb{Z}[1/p])\backslash(\mathcal{H}_p\times\mathcal{H})$
K/F ATR	$K/\mathbb{Q}$ real quadratic, with p inert
ATR cycles	Cycles in $\mathbf{SL}_2(\mathbb{Z}[1/p])\backslash(\mathcal{H}_p\times\mathcal{H})$.

From ATR extensions to real quadratic fields

One can develop the notions in the right-hand column to the extent of

1. Attaching to $f \in S_2(\Gamma_0(p))$ a "Hilbert modular form" G on $\mathbf{SL}_2(\mathbb{Z}[1/p])\backslash(\mathcal{H}_p \times \mathcal{H})$.
2. Making sense of the expression

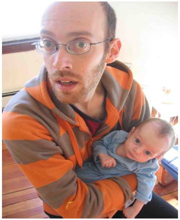
$$\int_{\partial^{-1}\Delta_\gamma} \omega_G \quad \in \quad K_p^\times / q^{\mathbb{Z}} = E(K_p)$$

   for any "p-adic ATR cycle" Δ_γ.

The resulting local points are defined (*conjecturally*) over ring class fields of K. They are prototypical "Stark-Heegner points" ...

Computing Stark-Heegner points attached to real quadratic fields

There are fantastically efficient *polynomial-time* algorithms for calculating Stark-Heegner points, based on the ideas of Glenn Stevens and Rob Pollack. (Cf. their AWS lectures.)



Drawback of Stark-Heegner vs Chow-Heegner points

They are completely mysterious and the mecanisms underlying their algebraicity are poorly understood.

Advantage of Stark-Heegner vs Chow-Heegner points

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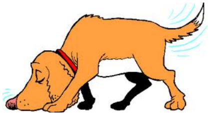

New cases of the Birch and Swinnerton-Dyer conjecture

Theorem (Bertolini, Dasgupta, D)

Assume the conjectures on Stark-Heegner points attached to real quadratic fields (in the stronger, more precise form given in Samit Dasgupta's PhD thesis). Then

$$L(E/K, \chi, 1) \neq 0 \implies (E(H) \otimes \mathbb{C})^{\chi} = 0,$$

for all $\chi : \mathrm{Gal}(H/K) \longrightarrow \mathbb{C}^{\times}$ *with* H *a ring class field of the real quadratic field* K

Question. Can we control the arithmetic of E over ring class fields of real quadratic fields *without* invoking Stark-Heegner points?

Diagonal cycles on triple products of Kuga-Sato varieties

Let $r_1 \geq r_2 \geq r_3$ be integers, with $r_1 \leq r_2 + r_3$.

$$r = \frac{r_1 + r_2 + r_3}{2}$$

$$V = \mathcal{E}^{r_1} \times \mathcal{E}^{r_2} \times \mathcal{E}^{r_3}, \quad \dim V = 2r + 3.$$

$$\Delta = \mathcal{E}^{r} \subset V.$$

$$\Delta \in CH^{r+2}(V).$$

$$\mathrm{cl}(\Delta) = 0 \text{ in } H^{2r+4}_{\mathrm{et}}(V_{\bar{Q}}, \mathbb{Q}_\ell)(r+2)^{G_{\mathbb{Q}}}.$$

$$\mathrm{AJ}_{\mathrm{et}}(\Delta) \in H^1(\mathbb{Q}, H^{2r+3}_{\mathrm{et}}(V_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)(r+2)).$$

Diagonal cycles and L-series

Let f, g, h be modular forms of weights $r_1 + 2$, $r_2 + 2$ and $r_3 + 2$.

By taking the (f, g, h)-isotypic component of the class $\mathrm{AJ}_{\mathrm{et}}(\Delta)$, we obtain a cohomology class

$$\kappa(f, g, h) \in H^1(\mathbb{Q}, V_f \otimes V_g \otimes V_h(r+2))$$

Its behaviour is related to the central critical derivative

$$L'(f \otimes g \otimes h, r+2).$$

We don't "really care" about these rather recundite L-series with Euler factors of degree 8...

From Rankin triple products to Stark-Heegner points

The position of the Stark-Heegner points are controlled by the central critical values $L(E/F, \chi, 1)$, as χ ranges over *ring class characters* of the real quadratic field F.

Write $\chi = \chi_1\chi_2$, where χ_1 and χ_2 are characters of signature $(1, -1)$, so that

$$V_1 = \mathrm{Ind}_F^{\mathbb{Q}} \chi_1, \qquad V_2 = \mathrm{Ind}_F^{\mathbb{Q}} \chi_2$$

are *odd* two-dimensional representations of $\mathbb{Q}$.

Hecke: There exists modular forms g and h of weight one, such that

$$L(g, s) = L(V_1, s), \qquad L(h, s) = L(V_2, s).$$

Furthermore,

$$L(f \otimes g \otimes h, 1) = L(E/F, \chi, 1)L(E/F, \chi_1\chi_2^{\rho}, 1).$$

Hida families

A slight extension of what we learned in Rob's lecture:

Theorem (Hida)

There exist q-series with coefficients in $\mathcal{A}(U)$,

$$\mathbf{g} = \sum_{n=1}^{\infty} \mathbf{b}_n(k) q^n, \qquad \mathbf{h} = \sum_{n=1}^{\infty} \mathbf{c}_n(k) q^n,$$

such that

$$\mathbf{g}(1) = g, \qquad \mathbf{h}(1) = h,$$

and $g_k := \mathbf{g}(k)$ *and* $h_k := \mathbf{h}(k)$ *are (normalised) eigenforms for almost all* $k \in \mathbb{Z}^{\geq 1}$.

The theme of p-adic variation

Philosophy: The natural p-adic invariants attached to (classical) modular forms varying in p-adic families should also vary in p-adic families.

Example: The Serre-Deligne representation V_g of $G_{\mathbb{Q}}$ attached to a classical eigenform g.

Theorem

There exists a Λ-adic representation $\mathbf{V_g}$ of $G_{\mathbb{Q}}$ satisfying

$$\mathbf{V_g} \otimes_{ev_k} \mathbb{Q}_p = V_{g_k}, \qquad \text{for almost all } k \in \mathbb{Z}^{\geq 2}.$$

Diagonal cycles and their p-adic deformations

For each $k \in \mathbb{Z}^{>1}$, consider the cohomology classes

$$\kappa_k := \kappa(f, g_k, h_k) \in H^1(\mathbb{Q}, V_f \otimes V_{g_k} \otimes V_{h_k}(1)).$$

Conjecture

There exists a "big" cohomology class $\kappa \in H^1(\mathbb{Q}, V_f \otimes \mathbf{V_g} \otimes \mathbf{V_h}(1))$ such that $\kappa(k) = \kappa_k$ for almost all $k \in \mathbb{Z}^{\geq 2}$.

Remark: This is in the spirit of work of Ben Howard on the "big" cohomology classes attached to Heegner points.

Question

What relation (if any!) is there between the class

$$\kappa(1) \in H^1(K, V_p(E)(\chi))$$

and Stark-Heegner points attached to $(E/K, \chi)$?

The goal for the AWS

Before seriously attacking the study of p-adic deformations of diagonal cycles and their (eventual) connection with Stark-Heegner points, it is natural to make a careful study of diagonal cycles and their arithmetic properties.

This will be the focus of the next two lectures by Victor and me in this AWS, and of the student projects.