

Stark-Heegner points Arizona Winter School 2011

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Classical Heegner points

Let E/\mathbb{Q} be an elliptic curve and

$$f = f_E = \sum_{n \geq 1} a_n q^n \in S_2(\Gamma_0(N)) \text{ with } L(E, s) = L(f, s).$$

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The modular parametrization is

$$\begin{array}{ccc} \varphi : X_0(N) & \longrightarrow & E \\ \infty & \longmapsto & 0 \\ \tau & \longmapsto & P_\tau := 2\pi i \int_\infty^\tau f(z) dz \\ & & = \sum_{n \geq 1} \frac{a_n}{n} e^{2\pi i n \cdot \tau} \end{array}$$

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Put

$$\mathcal{O}_\tau = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : N \mid c, \gamma \cdot \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} \tau \\ 1 \end{pmatrix} \right\} \subset M_0(N) \subseteq M_2(\mathbb{Z}).$$

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\mathcal{O}_τ is an order in K in which all $p \mid N$ split or ramify, and

$$P_\tau \in E(H_{\mathcal{O}_\tau}),$$

where $\text{Gal}(H_{\mathcal{O}_\tau}/K) \simeq \text{Pic}(\mathcal{O}_\tau)$.

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Theorem (Gross-Zagier)

$L'(E/K, 1)/\Omega_E \doteq \text{height}(P_K)$ where $P_K = \text{Tr}_{H_{\mathcal{O}_T}/K}(P_T)$.

Corollary

$P_K \in E(K)$ has infinite order if and only if $L'(E/K, 1) \neq 0$.

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Theorem (+Kolyvagin)

If $r_{an}(E/K) = 1$, BSD holds true for E/K .

Corollary

If $r_{an}(E/\mathbb{Q}) \leq 1$, BSD holds true for E/\mathbb{Q} .

Heegner points on Shimura curves

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We *still* have $\varphi : X_0^{N^-}(N^+) \dashrightarrow E, [\tau] \mapsto P_\tau \in E(H_{\mathcal{O}_\tau})$. All works nicely thanks to Zhang.

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And generalizes well to modular elliptic curves E/F over a totally real number field F and totally imaginary quadratic K/F *provided* $[F : \mathbb{Q}]$ **is odd or** $\exists \wp \mid \mathfrak{N}$.

What can we say if any of these fails? How do we construct points on E over other fields?

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and

$$\begin{aligned} \text{Pic}_0(X)(\mathbb{C}) &\xrightarrow{\sim} (H^{1,0})^\vee / H_1(X, \mathbb{Z}) \simeq \mathbb{C}^g / \Lambda \\ D &\mapsto \int_D \mapsto (\int_D \omega_1, \dots, \int_D \omega_g) \end{aligned}$$

$$H^{1,0} := H^0(X_{\mathbb{C}}, \Omega^1).$$

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For $X = X_0(N)$ the modular parametrization factors as:

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Over the complex numbers, via AJ, this looks

$$\begin{array}{ccccc} \varphi_{\mathbb{C}} : \Gamma_0(N) \backslash \mathcal{H}^* & \xhookrightarrow{i} & (H^{1,0})^{\vee} / H_1(X, \mathbb{Z}) & \xrightarrow{\pi_f} & \mathbb{C} / \Lambda_f \\ [\tau] & \mapsto & (\int_{\infty}^{\tau} f \frac{dq}{q}, \dots, \int_{\infty}^{\tau} f_g \frac{dq}{q}) & \mapsto & \int_{\infty}^{\tau} f(q) dq / q \end{array}$$

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For non-split Shimura curves $X_0^{N^-}(N^+)$ there is no choice of a base point $\infty \in X(\mathbb{Q})$ and it is more natural to simply consider

$$\mathrm{Pic}_0(X) \xrightarrow{\pi_f} E.$$

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Cohomology in higher dimension

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$$H_{dR}^n(V), \quad 0 \leq n \leq 2d$$

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For curves: $\text{Fil}^0 = H_{dR}^1(X) = \Omega^1(X)/dF(X) \supset \text{Fil}^1 = \Omega^1(X)$.

For any prime p , the p -adic étale cohomology groups

$$H_{\text{ét}}^n(V_{\bar{F}}, \mathbb{Q}_p), \quad 0 \leq n \leq 2d,$$

are finite dim'l \mathbb{Q}_p -vector spaces with an action of $\text{Gal}(\bar{F}/F)$.

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$$F = \mathbb{C} : \quad H_{\text{dR}}^n(V/\mathbb{C}) = H_{\text{Betti}}^n(V(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{C} \simeq \oplus_{i+j=n} H^{i,j}(V/\mathbb{C})$$

$$\langle \omega_1, \omega_2 \rangle = \frac{1}{(2\pi i)^d} \int_{V(\mathbb{C})} \omega_1 \wedge \omega_2.$$

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Hodge conjecture: cl is surjective.

The complex Abel-Jacobi map

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$\tilde{\Delta} = \partial^{-1}\Delta$ is a $2(d-c)+1$ -differentiable chain on the real manifold $V(\mathbb{C})$ with boundary Δ .

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Tate: there is $\Pi^? \in \text{CH}^{d+1-c}(V \times E)(\mathbb{Q})$ inducing

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$$\Delta \in \text{CH}^c(V)_0 \mapsto \pi_V^* \Delta \mapsto \pi_V^* \Delta \cdot \Pi^? \mapsto P_{\Delta} := \pi_{E,*}(\pi_V^* \Delta \cdot \Pi^?) \in E$$

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Like Heegner divisors $D = ([\tau] - \infty) \in \mathrm{CH}^1(X_0(N))_0(H_{\mathcal{O}_\tau})$.

Shimura varieties associated to a reductive group G/\mathbb{Q} host special cycles.

Example 1: modular and Shimura curves

E/\mathbb{Q} of conductor N and $V = X_0(N)$ or $X_0^{N^-}(N^+)$.

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$$\text{CH}^1(V)_0(\mathbb{C}) \xrightarrow{\text{AJ}_{\mathbb{C}}} \text{Jac}(V)$$

$$\pi \downarrow$$

$$\downarrow \pi_{\mathbb{C}}$$

$$E(\mathbb{C}) \xrightarrow{\text{AJ}_{\mathbb{C}}} \mathbb{C}/\Lambda_E,$$

$$D = ([\tau] - \infty) \in \text{CH}^1(V)_0 \mapsto P_D \in E.$$

Example 2: Kuga-Sato varieties

The universal elliptic curve is

$$\pi : V_1 \twoheadrightarrow X_1(N)$$

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The approach of M. Bertolini, H. Darmon and K. Prasanna:

$$S_{r+2}(\Gamma_1(N)) \simeq \varepsilon H_{par}^{r+1,0}(V_r), \quad f(q) \mapsto f(q) dz_1 \dots dz_r dq/q.$$

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X_r has dimension $2r + 1$ and hosts Heegner cycles of codimension $r + 1$.

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Numerically found that for odd r :

$$P_{r,\mathbb{C}} = \sqrt{-D} \cdot m_r \cdot P_E, \quad m_r^2 = \frac{2r!(2\pi\sqrt{D})^r}{\Omega_E^{2r+1}} L(\psi_E^{2r+1}, r+1) \in \mathbb{Z}.$$

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And proved a p -adic étale version of this.

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It yields

$$\begin{aligned} \pi : \text{CH}^{r+2}(V)_0 &\rightarrow \text{Pic}_0(X) && \xrightarrow{\pi_f} E \\ \Delta &\mapsto P_\Delta = \sum_{(P,P,Q) \in \Delta} \pi_f(Q) \end{aligned}$$

$\dim(V) = 2r + 3$ and there are several natural choices for

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For $r \geq 1$, $\Delta_r := (\epsilon, \epsilon, \mathrm{Id})(\Delta_{\{1,2,3\}} - \Delta_{\{1,2\}}) \in \mathrm{CH}^{r+2}(V)_0$

Theorem (Darmon-R-Sols) $P_r := P_{\Delta_r} \in E(\mathbb{Q})$ satisfies

$$P_r = n_r P_0, \quad n_r \in \mathbb{Z},$$

with $P_0 = \pi_{E,*}(K_X)$, where $K_X \in \text{Pic}(X)$ is the canonical divisor.

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Theorem (Yuan-Zhang-Zhang) $P_g \neq 0$ in $\mathbb{Q} \otimes E(\mathbb{Q}) \Leftrightarrow$

$$\text{ord}_{s=1} L(E, s) = 1 \text{ and } L(E \otimes \text{sym}^2(g), 2) \neq 0.$$