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ABELIAN VARIETIES WITH QUATERNIONIC MULTIPLICATION AND THEIR MODULI

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Introduction

In this monograph, we explore several arithmetical, geometrical and diophantine questions concerning *quaternion algebras*, *abelian varieties* and *Shimura varieties* and the rich relationships existing between them. Specifically, these notes are focused on

- Abelian varieties whose ring of endomorphisms is a maximal order in a totally indefinite quaternion algebra B over a totally real number field F, and on the
- Shimura varieties $X_B/\mathbb{Q}$ that naturally occur as their moduli spaces.

As we aim to make apparent, many of the arithmetical and geometrical properties of these abelian varieties are encoded either in the quaternion algebra B or in the Shimura variety X_B. In turn, the nature of these Shimura varieties cannot be handled without an understanding of the objects parametrized by them.

Due to Albert's classification of involuting division algebras (cf. [Mu70]) and the work of Shimura [Sh63], there is a limited number of rings that can occur as the endomorphism ring of an abelian variety. Namely, if A is a simple abelian variety over an algebraically closed field, $\mathrm{End}(A)$ is an order in either a totally real number field, a quaternion algebra over a totally real number field or a division algebra over a complex multiplication field. We stress that abelian varieties with distinct endomorphism rings tend to exhibit very different behaviour in many aspects.

There is a considerable body of work on abelian varieties with complex multiplication due to Shimura and Taniyama [ShTa61], Mumford [Mu70], Lang [Lan83] and others. The impact of these contributions on class field theory and the Birch and Swinnerton-Dyer conjecture is enormous.

There is also abundant literature on abelian varieties with real multiplication due to Humbert [Hu93], Shimura [Sh63], Ribet [Ri80], [Ri94], Lange

[La88], Wilson [Wi02] and others. These works are highly relevant to the generalized Shimura-Taniyama-Weil conjectures due to the fact that modular abelian varieties $A_f/\mathbb{Q}$ attached to cusp forms $f \in S_2(\Gamma_0(N))$ have totally real multiplication over $\mathbb{Q}$. We refer the reader to [HaHaMo99] and [Ri90] for more details.

However, few authors have studied quaternionic multiplication (QM) on abelian varieties. In this case, the arithmetic of their endomorphism ring is considerably more involved than in the commutative cases, forcing their Néron-Severi groups to be less accessible. We refer the reader to [No01], [JoMo94], [HaMu95], [HaHaMo99] and [Oh74] for some recent contributions.

Shimura [Sh63], [Sh67] was the first to consider the coarse moduli spaces of abelian varieties with quaternion multiplication, which admit a canonical model $X_B/\mathbb{Q}$ over the field $\mathbb{Q}$ of rational numbers. As complex manifolds, the varieties $X_B(\mathbb{C})$ can be described as compact quotients of certain bounded symmetric domains by arithmetic groups acting on them. Shimura explored their arithmetic, showing that the coordinates of the so-called *Heegner points* on X_B generate class fields whose Galois action on them can be described by explicit reciprocity laws.

Interest in Shimura varieties has increased in recent years. They have been crucial in several studies of major questions in number theory. Let us quote some of them.

With regard to modular conjectures, Shimura curves play a fundamental role in Ribet's proof of the Epsilon conjecture which, in turn, implies that Fermat's Last Theorem follows from the Shimura-Taniyama-Weil conjecture (cf. [Ri89], [Ri90] and [Pr95]).

In connection to the Birch and Swinnerton-Dyer conjecture, Vatsal [Va02] and Cornut [Cor02] have recently proved, independently, several conjectures of Mazur on higher Heegner points on elliptic curves by means of modular curves, Gross and Shimura curves and Ratner's ergodic theory. Moreover, Bertolini and Darmon [BeDa96], [BeDa98], [BeDa99] have exploited the Čerednik-Drinfeld theory on the bad reduction of Shimura curves to prove anti-cyclotomic versions of conjectures of Mazur, Tate and Teitelbaum on p-adic variants of the Birch and Swinnerton-Dyer conjecture.

Concerning the finiteness and squareness conjectures of the Shafarevich-Tate group of an abelian variety over a number field, Poonen and Stoll [PoSt99] have recently made a careful study of the Cassels-Tate pairing and have given explicit criteria for the squareness of the torsion part of the Shafarevich-Tate group of the Jacobian variety of a curve. On the basis of these results, Jordan and Livné [JoLi99] have exhibited Atkin-Lehner quotients of Shimura curves such that the cardinality of the finite part of the Shafarevich-Tate group of their Jacobian varieties is not a perfect square but twice a perfect square (cf. also [Bab01]). In a recent work, Stein [St02] provides explicit examples of abelian varieties $A/\mathbb{Q}$ such that $\sharp\mathrm{Sha}(A/\mathbb{Q}) = p{\cdot}n^2$, $n \in \mathbb{Z}$, for every odd prime $p < 10000$, $p \neq 37$.

In addition to the many applications of Shimura curves and varieties, many authors have also been interested in their geometrical and diophantine properties themselves. Indeed, integral models of Shimura curves and their special fibres have been considered by Morita [Mo81], Boutot and Carayol [BoCa91], Buzzard [Bu96], Čerednik [Ce76], Drinfeld [Dr76] and Zink [Zi81] among many others. Also of great interest are the series of papers by Kudla and Kudla-Rapoport on height pairings on Shimura curves, intersection numbers of special 0-cycles and the values at the centre of their symmetry of the derivatives of certain Eisenstein series, along the pattern initiated by the classical papers of Hirzebruch-Zagier and Gross-Zagier. See [Kud97] and [KudRa02], for instance.

Effective results and computations on Shimura curves have also been worked out by Kurihara [Ku79], Elkies [El98], Alsina [Al99] and Bayer [Ba02] among others. These are particularly valuable since the absence of cusps on these curves make these approaches more difficult than in the classical modular case.

A notable and very recent result has been obtained by Edixhoven and Yafaev in [EdYa02] and its sequels, proving part of the André-Oort Conjecture on the distribution of special points on Shimura varieties.

In a different direction, Ihara [Ih], Jordan and Livné [JoLi85], [Jo86], [JoLi86], Ogg [Ogg83], [Ogg84], Milne [Mi79] and Kamienny [Ka90] have studied the sets of rational points on Shimura curves, their Atkin-Lehner quotients and their Jacobian varieties over global, local and finite fields. Finally, we refer the reader to [Gr02] for an approach to Shimura surfaces.

This report is organized as follows. In Chapter 1, we fix some notation and review some facts on number fields, quaternion algebras and abelian varieties that we will need throughout the monograph.

In Chapter 2, we look into several questions on the arithmetic of quaternion algebras and orders which arise naturally in our subsequent study of abelian varieties and Shimura varieties. Firstly, in Section 2.1, we consider a problem on the existence of suitable integral bases of quaternion algebras which is related to several papers by Chinburg and Friedman in relation to arithmetic 3-orbifolds (cf. [ChFr86], [ChFr99], [ChFr00]). Secondly, in Section 2.2, we study the set of conjugation classes of pure quaternions of given reduced norm in a quaternion algebra B under an action introduced by O'Connor and Pall [CoPa39] in the 1930s and further studied by Pollack [Po60] in the 1960s. We express the number of conjugation classes in terms of class numbers of quadratic extensions embedded in B and groups of units.

In Chapter 3 we study abelian varieties A with quaternionic multiplication and give an arithmetic criterion for the existence of principal polarizations on them. In particular, we prove that abelian varieties with quaternionic multiplication over a totally real number field F of narrow class number $h_+(F) = 1$ are always principally polarizable. Moreover, we give an expression for the number of isomorphism classes of principal polarizations on A in terms of relative class numbers of orders in CM-fields by means of Eichler's theory of optimal embeddings. As a consequence, we exhibit simple abelian varieties of any even dimension admitting arbitrarily many nonisomorphic principal polarizations. In turn, we obtain that there exist arbitrarily large sets of pairwise non isomorphic curves of genus 2 sharing isomorphic unpolarized Jacobian varieties.

In Chapter 4 we consider the moduli spaces of polarized abelian varieties with multiplication by a maximal order in a totally indefinite quaternion algebra B over a totally real number field F. Shimura constructed canonical models $X_B/\mathbb{Q}$ of these moduli spaces which are proper schemes over $\mathbb{Q}$ of dimension $[F : \mathbb{Q}]$.

We introduce several subgroups $V_0 \subseteq W_0 \subseteq W^1 \subseteq \mathrm{Aut}_{\mathbb{Q}}(X_B)$ of the group of automorphisms of these Shimura varieties and we study their modular interpretation by extending some results due to Jordan [Jo81]. We then consider several maps that occur naturally between the Shimura varieties X_B, Hilbert modular varieties $\mathcal{H}_S$ and the moduli spaces of polarized abelian varieties $\mathcal{A}_g$ which appear when we forget certain endomorphism structures. We prove that, up to birational equivalences, these forgetful maps coincide with the natural projection by suitable groups of Atkin-Lehner involutions.

We then derive several applications of our main result in this chapter. In Section 4.6, we study the quaternionic locus in the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties and we describe it as a union of Atkin-Lehner quotients of Shimura varieties. Subsequently, in Section 4.7, we study the field of moduli of the quaternionic multiplication on a principally polarized abelian variety.

In Chapter 5, we explore the diophantine properties of abelian surfaces A/K with quaternionic multiplication over a number field. In Section 5.1, based on our results in Chapter 3, we look into the representations of the absolute Galois group G_K in the ring of endomorphisms $\mathrm{End}_{\bar{K}}(A)$ and the Néron-Severi group $\mathrm{NS}_{\bar{K}}(A)$, respectively. This allows us to derive a substantial amount of information on the minimal field of definition of the endomorphisms of $A \otimes \mathrm{Spec}\bar{K}$ and the intermediate endomorphism ring $\mathrm{End}_K(A)$.

In Section 5.2, we compare the field of moduli and the field of definition of the quaternionic multiplication on the Jacobian variety of a curve of genus two. Finally, in Section 5.3, we illustrate our results with several explicit examples in dimension 2 based on explicit computations due to Hashimoto, Murabayashi and Tsunogai [HaMu95], [HaTs99].

In Chapter 6 we look into several diophantine questions on Shimura curves. Our main tool in this part is provided by the theory of Čerednik-Drinfeld on the special fibres of the integral models of Shimura curves at the ramified primes. Let $X_D/\mathbb{Q}$ be the Shimura curve over $\mathbb{Q}$ attached to the indefinite rational quaternion algebra B of discriminant D. In Section 6.1 we investigate the group of automorphisms of X_D and prove that, in many cases, $\mathrm{Aut}(X_D \otimes \mathrm{Spec}\bar{\mathbb{Q}}) = \mathrm{Aut}(X_D) = W^1$ is the positive Atkin-Lehner group. In Section 6.2.4, we determine the family of bielliptic Shimura curves and we use it in Section 6.3 to study the set of points on X_D rational over quadratic fields. In this way we answer a question posed by Kamienny [Ka90]. Finally, we obtain explicit equations of *all* elliptic quotients of X_D of degree 2.

Design of the cover. The design of the cover is due to the illustrator Maria Vidal. It is inspired from a drawing that may be found in the PhD. Thesis of Jordan [Jo81] and it makes use of the fundamental domain of the Shimura curve X_6 due to Alsina [Al99].

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Contents

Chapter 1

Background

Introduction

In this chapter we establish some notations and review some well known facts on number fields, quaternion algebras and abelian varieties that we will use in this monograph.

1.1 Basic facts on number fields

Let $\mathbb{Q}$ denote the field of rational numbers and let $\bar{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ in the field $\mathbb{C}$ of complex numbers.

Let F be a number field, that is, a finite field extension of $\mathbb{Q}$. The ring of integers R_F of F is a Dedekind domain: the set of fractional ideals of R_F is a free abelian group generated by the prime ideals of R_F.

Let R be an order in F over $\mathbb{Z}$: a subring of R_F such that $F = R \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $\mathrm{Pic}(R)$ denote the group of fractional projective ideals of R modulo principal ideals and let $h(R) = |\mathrm{Pic}(R)|$ denote the class number of R. We will simply write $\mathrm{Pic}(F)$ instead of $\mathrm{Pic}(R_F)$ and $h(F)$ instead of $h(R_F)$.

The class number of an order R in F is related to the class number $h(F)$ as follows:

$$h(R) = \frac{h(F)}{(R_F^* : R^*)} \frac{|(R_F/\mathfrak{f}_R \cdot R_F)^*|}{|(R/\mathfrak{f}_R)^*|},$$

where we let $\mathfrak{f}_R$ denote the conductor of R.

For any place v of F, we will let F_v denote the completion of F at v. We freely identify the set of non archimedean places of F with the set of prime ideals of F and the finite set of archimedean places of F with the set of immersions $\sigma : F \hookrightarrow \mathbb{C}$ up to complex conjugation.

A totally real number field is a number field F all whose archimedean places σ factor through the field $\mathbb{R}$ of real numbers. A number field F is totally imaginary if none of its archimedean places factor through $\mathbb{R}$. A complex muliplication (CM) field is a totally imaginary quadratic extension of a totally real number field.

An element $a \in F^*$ is called totally positive if $\sigma(a) > 0$ for any real archimedean place of F. We let F_+^* denote the subgroup of totally positive elements of F^*. For any subset S of F^*, we let $S_+ = S \cap F_+^*$. In particular, we let $R_{F+}^* = R_F^* \cap F_+^*$.

A principal ideal of F is called totally positive if it can be generated by a totally positive element of F. We let $\mathrm{Pic}_+(F)$ stand for the narrow class group of fractional ideals of F up to totally positive principal ideals.

More generally, for any subset $\infty = \{\sigma_1, ..., \sigma_r\}$ of real archimedean places on a number field F, we let F_∞^* denote the subgroup of elements $a \in F^*$ such that $\sigma(a) > 0$ for all $\sigma \in \infty$. We let $\mathrm{Pic}_\infty(F)$ denote the class group of fractional ideals of F up to principal ideals which can be generated by an element $a \in F_\infty^*$. We similarly let $h_\infty(F) = |\mathrm{Pic}_\infty(F)|$.

For any subfield K of F, we let $R_{F/K}^\sharp = \{a \in F : \mathrm{tr}_{F/K}(aR_F) \subseteq R_K\}$ be the codifferent of R_F over R_K. The inverse ideal of the codifferent of F over F is the different $\vartheta_{F/K} = R_{F/K}^{\sharp -1}$.

1.2 Basic facts on quaternion algebras

1.2.1 Quaternion algebras, orders and ideals

Let F be either a number field or the completion of a number field at a local place. Unless $F = \mathbb{R}$ or $\mathbb{C}$, let R_F denote the ring of integers of F.

Definition 1.2.1. A quaternion algebra B over F is a central simple algebra over F of $\mathrm{rank}_F(B) = 4$.

Quaternion algebras can be classically described and constructed as follows. Let L be a quadratic separable algebra over the field F, let σ denote the non trivial involution on L over F and let $m \in F^*$ be any non zero element. Then, the algebra

$$B = L + Le$$

with

$$e^2 = m \text{ and } e\beta = \beta^\sigma e \text{ for any } \beta \in L,$$

is a quaternion algebra over F. The classical notation for it is $B = (L, m)$. As it is shown in [Vi80], any quaternion algebra over F is of this form.

A second and alternative construction of quaternion algebras is the following. Let a, $b \in F^*$ be non zero elements. Then, the algebra

$$B = \left(\frac{a, b}{F}\right) = F + Fi + Fj + Fij,$$

with

$$i^2 = a, j^2 = b \text{ and } ij = -ji,$$

is again a quaternion algebra over F and again any quaternion algebra admits such a description. Note that the two constructions are related since $B = (\frac{a,b}{F}) = (F(i), b)$.

Let B be a quaternion algebra over F. The algebra B comes equipped with an anti-involuting conjugation map $\beta \mapsto \bar{\beta}$ such that, when restricted to a quadratic extension $F(\beta)$, $\beta \in B^* \setminus F^*$, is the nontrivial automorphism of $F(\beta)/F$. More explicitly, if $\beta = a + bi + cj + dij$, then $\overline{\beta} = a - bi - cj - dij$. Elements $\beta \in B$ are roots of the quadratic polynomial $x^2 - \mathrm{tr}(\beta)x + \mathrm{n}(\beta)$, where

$$\mathrm{tr}(\beta) = \beta + \overline{\beta}$$

and

$$\mathrm{n}(\beta) = \beta\overline{\beta}$$

denote the *reduced trace* and the *reduced norm* of $\beta \in B$, respectively.

For any subset S of B, we denote by $S_0 = \{\beta \in S : \mathrm{tr}(\beta) = 0\}$ the subgroup of *pure* quaternions of S. We let B^*_+ and B^*_- denote the subgroup of elements of B^* of totally positive and totally negative reduced norm, respectively. For any subset S of B^*, we let $S_+ = B^*_+ \cap S$ and $S_- = B^*_- \cap S$. More generally, for any subset $\infty = \{\sigma_1, ..., \sigma_r\}$ of real archimedean places on F, we let B^*_∞ denote the subgroup of elements $\beta \in B^*$ such that $\sigma(\mathrm{n}(\beta)) > 0$ for all $\sigma \in \infty$.

Let v be a place of F and let F_v denote the completion of F respect to v. If v is a complex archimedean place, necessarily $B \otimes F_v \simeq \mathrm{M}_2(\mathbb{C})$ since this is the only quaternion algebra over $\mathbb{C}$ up to isomorphism. If v is a finite or a real archimedean place, then there are two isomorphism classes of quaternion algebras over the local field F_v: the split algebra $\mathrm{M}_2(F_v)$ and a division algebra that we will denote by $\mathbb{H}_v$. In the real case $F_v = \mathbb{R}$, $\mathbb{H}_v = (\frac{-1,-1}{\mathbb{R}})$ is the classical skew-field of Hamilton's quaternions.

A place v of F *ramifies* in B if $B \otimes F_v \simeq \mathbb{H}_v$ is the nonsplit algebra over F_v. In particular, a real archimedean place v ramifies in B if $B \otimes F_v = (\frac{-1,-1}{\mathbb{R}})$. It is a classical theorem of Hasse that there is a finite and even number of places v on F that ramify in B. We will say that B is *totally indefinite* over F if no real place ramifies in B.

Let us assume from now on that F is either a number field or the completion of a number field with respect to a finite place.

Definition 1.2.2. An element $\beta \in B$ is *integral* if $\mathrm{tr}(\beta)$, $\mathrm{n}(\beta) \in R_F$.

Unlike number fields or local fields, the set of integral elements of B is not a ring anymore.

Definition 1.2.3. An *order* $\mathcal{O} \subset B$ over R_F is a ring of integral elements in B which is finitely generated as R_F-module and such that $\mathcal{O} \otimes_{R_F} F = B$. It is a *maximal order* if it is not properly contained in any other order.

Let us agree to say that two orders $\mathcal{O}$ and $\mathcal{O}'$ of B are *conjugate* if $\mathcal{O} = \gamma^{-1}\mathcal{O}'\gamma$ for some $\gamma \in B^*$. The conjugation class $\{\gamma^{-1}\mathcal{O}\gamma : \gamma \in B^*\}$ is also known as the *type* of $\mathcal{O}$.

Maximal orders in quaternion algebras are in general not unique, often not even up to conjugation by elements of B^*. The finite number of conjugation classes of maximal orders in B is called the *type number* of B and it is denoted by $t(B)$. In general, we have the following

Definition 1.2.4. Let $\mathcal{O}$ be an order in B. For any finite place $\wp$ of F, let $\mathcal{O}_\wp = \mathcal{O} \otimes_{R_F} R_{F_\wp}$. The *type number* $t(\mathcal{O})$ of $\mathcal{O}$ is the number of conjugation classes of orders $\mathcal{O}'$ of B such that $\mathcal{O}'_\wp \simeq \mathcal{O}_\wp$ for all prime ideals $\wp$ of F.

It follows from [Vi80] that the type number $t(\mathcal{O})$ of an arbitrary order $\mathcal{O}$ of B is always a finite number.

The following result is known as the Hasse-Schilling-Maass Norms Theorem for quaternion algebras (cf. [HaSc36], [Ei37], [Ei38], [Vi80]).

Proposition 1.2.5. *Let $\mathcal{O}$ be a maximal order in B. Let $\infty = Ram_\infty(B)$ be the set of archimedean places of F that ramify in B. Then*

$$\mathrm{n}(B^*) = F^*_\infty$$

and

$$\mathrm{n}(\mathcal{O}^*) = R^*_{F\infty}.$$

Note that the set $\mathrm{Ram}_\infty(B)$ may be empty.

Definition 1.2.6. Let $\mathcal{O}$ be an order in B. A *left ideal* (respectively *right ideal*) of $\mathcal{O}$ is a finitely generated R_F-module I with $I \otimes_{R_F} F = B$ and such that $\mathcal{O}I \subseteq I$ (respectively $I\mathcal{O} \subseteq I$).

There are several ideals related to a left ideal I of a maximal order $\mathcal{O}$. Firstly, the *inverse ideal* of a $\mathcal{O}$-left ideal I is defined to be $I^{-1} = \{\beta \in B : I\beta I \subseteq I\}$. It is a right $\mathcal{O}$-ideal such that $II^{-1} = \mathcal{O}$.

The conjugate ideal of I is the right ideal $\overline{I} = \{\overline{\alpha} : \alpha \in I\}$. The product ideal $I \cdot \overline{I} = \{\sum \alpha_i \overline{\alpha_i} : \alpha_i \in I\}$ is a two-sided ideal of $\mathcal{O}$. If $\mathrm{n}(I) = \{\mathrm{n}(\alpha) : \alpha \in I\} \subset F$ denotes the norm ideal of I, we have that $I \cdot \overline{I} = \mathrm{n}(I) \cdot \mathcal{O}$ (cf. [Sh63J]). We introduce the following

Definition 1.2.7. Let I be a left ideal of a maximal order $\mathcal{O}$ in B. Then, we define

$$\mathcal{N}(I) = \mathrm{n}(I)\mathcal{O} = I\overline{I}$$

to be the two-sided ideal of $\mathcal{O}$ generated by the ideal $\mathrm{n}(I)$ of F.

The set of classes of left ideals of a maximal order $\mathcal{O}$ is

$$\mathrm{Pic}_\ell(\mathcal{O}) = \{I \subset B : \mathcal{O}I \subseteq I\}/\{\mathcal{O}\beta : \beta \in B^*\}.$$

By its adelic description and approximation theorems, $\mathrm{Pic}_\ell(\mathcal{O})$ is a finite set (cf. [Vi80]) and the map $I \mapsto I^{-1}$ induces a bijection between $\mathrm{Pic}_\ell(\mathcal{O})$ and the set $\mathrm{Pic}_r(\mathcal{O})$ of classes of right ideals of $\mathcal{O}$. In fact, its cardinality does not depend on the choice of the maximal order $\mathcal{O}$ and it is denoted by $h(B)$. The following can be found in [Ei55].

Proposition 1.2.8. *Let $\mathcal{O}$ be a maximal order in B. If there is an archimedean place v on F which does not ramify in B, then the class number of B*

coincides with that of the field F: $h(B) = h(F)$. More precisely, there is a bijection of sets

$$n : \mathrm{Pic}_\ell(\mathcal{O}) \xrightarrow{\sim} \mathrm{Pic}(F)$$

induced by the reduced norm of B over F.

The existence of an unramified archimedean place on B is called *Eichler's condition* in the literature (cf. [Vi80]). We refer the reader to [Joh], [Re75] and [Vi80] for more details on quaternion algebras and orders.

1.2.2 Codifferent and discriminant

As above, let F be either a number field or the completion of a number field with respect a non archimedean place, let R_F denote the ring of integers of F and let B be a quaternion algebra over F.

Definition 1.2.9. Let $\mathcal{O}$ be an order of B and let I be a left ideal of $\mathcal{O}$. Let $K \subset F$ be a subfield of F and let $R_K \subset R_F$ denote its ring of integers. Then, the *codifferent* of I over R_K is defined by $I^\sharp_{B/K} = \{\beta \in B : \mathrm{tr}_{B/K}(I\beta) \subseteq R_K\}$. It is a right ideal of $\mathcal{O}$ over R_F.

The following facts are well known, but we include here a proof of them due to the lack of a suitable reference.

Proposition 1.2.10. *Let $\mathcal{O}$ be a maximal order of B and let I be a left ideal of $\mathcal{O}$.*

1. $I^\sharp_{B/K} = I^{-1} \cdot \mathcal{O}^\sharp_{B/K}$.
2. $\mathcal{O}^\sharp_{B/K} = \mathcal{O}^\sharp_{B/F} \cdot R^\sharp_{F/K}$.
3. $\mathcal{O}^\sharp_{B/F} = \prod_\wp D_\wp^{-1}$, *where the product runs over the finite set of prime ideals $\wp$ of F that ramify in B and $D_\wp$ are two-sided integral ideals of $\mathcal{O}$ such that $D_\wp^2 = \wp$.*

Proof. 1. For any prime $\wp$ of F, let $I_\wp = I \otimes R_{F_\wp}$. Since it is a local statement, it suffices to prove that $(I_\wp)^\sharp_{B_\wp/K_\wp} = I_\wp^{-1} \cdot (\mathcal{O}_\wp)^\sharp_{B_\wp/K_\wp}$ for any prime $\wp$, where $K_\wp$ denotes the completion of K respect to the prime below $\wp$. As it is shown in [Vi80], $I_\wp$ must be a principal ideal and hence $I_\wp = \mathcal{O}_\wp\beta$ for some $\beta \in B_\wp$. Then obviously $I^\sharp_\wp = \beta^{-1} \cdot \mathcal{O}^\sharp_\wp$.

2. The proof translates step by step the one given for number fields in [Se68], Chapter III, Proposition 8.

3. This fact is stated, without proof, in [Sh63J]. By Proposition 1.2.8 and Proposition 1.2.5, for any ramifying prime ideal $\wp$ there is an integral $\mathcal{O}$-left ideal $D_\wp$ such that $\mathrm{n}(D_\wp) = \wp$. At all finite prime ideals $\mathfrak{q} \neq \wp$, $D_\wp \otimes R_{F_\mathfrak{q}} \simeq \mathcal{O} \otimes R_{F_\mathfrak{q}}$ and $D_\wp \otimes R_{F_\wp}$ is the (unique, two-sided) maximal ideal of $\mathcal{O} \otimes R_{F_\wp}$. Since so it is locally, $D_\wp$ is a two-sided ideal.

We show now that $\mathcal{O}^\sharp_{B/F} = \prod_\wp D_\wp^{-1}$ locally at any finite prime ideal of F. If $\wp$ is a ramified prime in B, then $B_\wp$ is discrete valuation division algebra over $F_\wp$ (cf. [Vi80]). Let π be an uniformizer of $B_\wp$ such that $\pi^2 = \pi_{F_\wp}$ is an uniformizer of the completion of F at $\wp$. In this case, necessarily $I_\wp = \mathcal{O}_\wp \pi^n$ for some $n \in \mathbb{Z}$ (cf. [Vi80]) and $I^\sharp_\wp = \pi^{(-n-1)}\mathcal{O}_\wp = I^{-1} \cdot \mathcal{O}^\sharp_\wp$. If $\mathfrak{q}$ does not ramify, $\mathcal{O}_\mathfrak{q} \simeq \mathrm{M}_2(R_{F_\mathfrak{q}})$ and $I_\mathfrak{q} = \mathcal{O}_\mathfrak{q} \cdot \beta$, $\beta \in B_\mathfrak{q}$, is a principal ideal. Then $I^\sharp_\mathfrak{q} = \beta^{-1} \cdot \mathcal{O}_\mathfrak{q} = I_\mathfrak{q}^{-1} \cdot \mathcal{O}^\sharp_\mathfrak{q}$. □

Definition 1.2.11. Let $\mathcal{O}$ be an order of B and let I be a left ideal of $\mathcal{O}$. Let $K \subset F$ be a subfield of F and let R_K denote its ring of integers. The discriminant of I over R_K is

$$\mathrm{disc}_{B/K}(I) := \mathrm{n}_{B/K}(I^\sharp_{B/K})^{-1} \cdot \mathrm{n}_{B/K}(I).$$

In particular, the discriminant of B is defined to be

$$\mathrm{disc}(B) = \mathrm{disc}_{B/F}(\mathcal{O}),$$

for any maximal order $\mathcal{O}$ of B. The following result ensures that the definition of $\mathrm{disc}(B)$ does not depend of the choice of the maximal order $\mathcal{O}$.

Proposition 1.2.12. *Let $\mathcal{O}$ be a maximal order of B and let I be a left ideal of $\mathcal{O}$.*

1. $\mathrm{disc}_{B/K}(I) = \mathrm{n}_{B/K}(I)^2 \cdot \mathrm{disc}_{B/K}(\mathcal{O})$.
2. $\mathrm{disc}_{B/K}(\mathcal{O}) = N_{F/K}(\mathrm{disc}_{B/F}(\mathcal{O})) \cdot \mathrm{disc}_{F/K}(R_F)^2$.
3. $\mathrm{disc}(B) = \mathrm{disc}_{B/F}(\mathcal{O}) = \wp_1 \cdot \ldots \cdot \wp_r$, *where $\wp_i$ are the prime ideals of F which ramify in B.*
4. $\mathrm{disc}_{B/K}(I)^2 = \langle \det(\mathrm{tr}_{B/K}(\alpha_i \cdot \alpha_j))\rangle_{R_F} = \langle det\ (\mathrm{tr}_{B/K}(\alpha_i \cdot \overline{\alpha_j}))\rangle_{R_F}$, *where $\{\alpha_i\}$ run on all K-bases of B that are contained in I.*

Proof. The first three statements are consequence of Proposition 1.2.10. The first equality of 4. is proved in [Vi80], p. 25. For the last equality, we may suppose by 2. that $F = K$ and it suffices to show that, for any prime ideal $\wp$ of F, $\langle \det(\mathrm{tr}_{B_\wp/F_\wp}(\alpha_i \cdot \alpha_j))\rangle_{R_{F_\wp}} = \langle \det(\mathrm{tr}_{B_\wp/F_\wp}(\alpha_i \cdot \overline{\alpha_j}))\rangle_{R_{F_\wp}}$, where $B_\wp = B \otimes F_\wp$ and $\{\alpha_i\}$ run on all $F_\wp$-bases of $B_\wp$ that are contained in $I_\wp = I \otimes R_{F_\wp}$. Now all left ideals of $\mathcal{O}_\wp$ are principal and, for any α, $\alpha_{ij} \in B_\wp$, we have that $\det(\mathrm{tr}(\alpha\alpha_{ij})) = \mathrm{n}(\alpha)^2 \det(\mathrm{tr}(\alpha_{ij}))$. Hence we may suppose that $I_\wp = \mathcal{O}_\wp$. If $\wp \nmid \mathrm{disc}(B)$, then $\mathcal{O}_\wp \simeq \mathrm{M}_2(R_{F_\wp}) = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$ and a simple computation shows the claim.

Analogously, if $\wp | \mathrm{disc}(B)$,

$$\mathcal{O}_\wp \simeq \left\{ \begin{pmatrix} a & b \\ \pi b^\sigma & a^\sigma \end{pmatrix} : a, b \in R_{F_\wp^2} \right\},$$

where $R_{F_\wp^2}$ denotes the ring of integers of the quadratic unramified extension $F_{\wp^2}$ of $F_\wp$, π is a local uniformizer and σ is the nontrivial element in the Galois group $\mathrm{Gal}(F_\wp^2/F_\wp)$. As in the case $\wp \nmid \mathrm{disc}(B)$, the consideration of an explicit basis for this order allows us to conclude. □

Definition 1.2.13. An order $\mathcal{O}$ of B is an *Eichler order* if it is the intersection of two maximal orders. It is an *hereditary order* if all left ideals I of $\mathcal{O}$ are projective over R_F.

The proof of the following facts can be found in [Re75] and [Vi80].

Proposition 1.2.14. *1. Let $\mathcal{O}$ be an order of B. Then* $\mathrm{disc}(B)|\mathrm{disc}(\mathcal{O})$ *and $\mathcal{O}$ is maximal if and only if* $\mathrm{disc}(B) = \mathrm{disc}(\mathcal{O})$.

2. *Let $\mathcal{O}$ be an Eichler order of B. Then, there exists an ideal $\mathcal{N}$ of F,* $(\mathcal{N}, \mathrm{disc}(B)) = 1$, *such that* $\mathrm{disc}(\mathcal{O}) = \mathrm{disc}(B) \cdot \mathcal{N}$. *The ideal $\mathcal{N}$ is called the level of $\mathcal{O}$.*

3. *An order $\mathcal{O}$ of B is hereditary if and only* $\mathrm{disc}(\mathcal{O})$ *is a square-free ideal and if and only if $\mathcal{O}$ is an Eichler order of square-free level.*

1.2.3 Eichler theory on optimal embeddings

Let B be a quaternion algebra over a field F. The following two statements are equivalent versions of the Skolem-Noether Theorem.

Theorem 1.2.15. *1. Any automorphism $\varphi : B \xrightarrow{\sim} B$ of F-algebras is inner: there exists $\gamma \in B^*$ such that $\varphi(\beta) = \gamma^{-1}\beta\gamma$ for all $\beta \in B$.*

2. Let L be a quadratic separable algebra of F and let $i : L \hookrightarrow B$ and $j : L \hookrightarrow B$ be two embeddings of L into B. Then there exists $\gamma \in B^*$ such that $i(L) = \gamma^{-1}j(L)\gamma$.

Let us assume that F is a number field and let $\mathcal{O}$ be an order in B. Let us define the groups $\mathcal{O}^* \supset \mathcal{O}^*_+ \supseteq \mathcal{O}^1$ of units in $\mathcal{O}$, units in $\mathcal{O}$ of totally positive reduced norm and units in $\mathcal{O}$ of reduced norm 1 respectively. We also let $\mathrm{Norm}_{B^*}(\mathcal{O}) = \{\gamma \in B^* : \gamma^{-1}\mathcal{O}\gamma \subseteq \mathcal{O}\}$ be the normalizer of $\mathcal{O}$ in B^*. Notice that the Skolem-Noether Theorem 1.2.15 can also be rephrased by saying that $\mathrm{Aut}_F(B) \simeq B^*/F^*$ and that the automorphism group of $\mathcal{O}$ is $\mathrm{Aut}_{R_F}(\mathcal{O}) \simeq \mathrm{Norm}_{B^*}(\mathcal{O})/F^*$.

A positive anti-involution ϱ on B is a map $\varrho : B \to B$ such that $(\beta_1 + \beta_2)^\varrho = \beta_1^\varrho + \beta_2^\varrho$ and $(\beta_1 \cdot \beta_2)^\varrho = \beta_2^\varrho \cdot \beta_1^\varrho$ for any β_1, $\beta_2 \in B$, and such that $\mathrm{tr}(\beta \cdot \beta^\varrho) \in F^*_+$ for any $\beta \in B^*$. By the Skolem-Noether Theorem, if ϱ is a positive anti-involution, there exists $\mu \in B^*$ such that $\beta^\varrho = \mu^{-1}\overline{\beta}\mu$. Further, it is easily shown that the positiveness of ϱ implies that $\mathrm{tr}(\mu) = 0$ and $\mathrm{n}(\mu) \in F^*_+$ (cf. [Mu70], [LaBi92]). The element μ is determined by ϱ up to multiplication by elements of F^* and we will sometimes use the notation $\varrho = \varrho_\mu$.

Definition 1.2.16. Let $\mathcal{O}$ be an order in B and let S be an order over R_F in a quadratic algebra L over F.

1. An embedding $i : S \hookrightarrow \mathcal{O}$ is *optimal* if $i(S) = i(L) \cap \mathcal{O}$.
2. Two optimal embeddings $i : S \hookrightarrow \mathcal{O}$ and $j : S \hookrightarrow \mathcal{O}$ are Eichler conjugate over $\mathcal{O}^*$ if there exists $\gamma \in \mathcal{O}^*$ such that $i(L) = \gamma^{-1}j(L)\gamma$. We will denote it by $i \sim_e j$.
3. We let $E(S, \mathcal{O})$ denote the set of Eichler conjugation classes of optimal embeddings of S in $\mathcal{O}$ and $e(S, \mathcal{O}) = |E(S, \mathcal{O})|$.

As we quote in the theorem below, the set $E(S, \mathcal{O})$ is indeed finite and it makes sense to consider its cardinality.

Definition 1.2.17. Let L be a quadratic separable extension of F and let S be an order in L over R_F of conductor $\mathfrak{f}_S$. Let $\wp$ be a prime ideal of F. The *Eichler symbol* of S and $\wp$ is

$$(\frac{S}{\wp}) = \begin{cases} 1 & \text{if } \wp | \mathfrak{f}_S \text{ or } \wp \text{ decomposes in } L, \\ -1 & \text{if } \wp \nmid \mathfrak{f}_S \text{ and } \wp \text{ remains inert in } L, \\ 0 & \text{otherwise.} \end{cases}$$

A proof of next statement can be found e. g. in [Vi80, p. 96-97].

Theorem 1.2.18. *Let B be a division quaternion algebra over a number field F and let ∞ be the set of real archimedean places of F which are ramified in B. Assume that there is at least one archimedan place of F which does not belong to ∞. Let $\mathcal{O}$ be an Eichler order of square-free level $\mathcal{N}$ in B and let S be an order in a quadratic algebra L over F. Then, the number of Eichler conjugation classes of optimal embeddings $i : S \hookrightarrow \mathcal{O}$ is*

$$e(S, \mathcal{O}) = \frac{h(S)}{h_\infty(F)} \prod_{\wp | \mathrm{disc}(B)} (1 - (\frac{S}{\wp})) \prod_{\wp | \mathcal{N}} (1 + (\frac{S}{\wp})).$$

Notice that, in particular, Eichler's Theorem 1.2.18 establishes a criterion for the embeddability of a quadratic order S into an Eichler order $\mathcal{O}$.

Since it will be of use later in Chapter 3, we consider in Proposition 1.2.19 below a stronger form of a particular case of Eichler's Theorem. Let B be a totally indefinite division quaternion algebra over a totally real number field F. Let $\mathcal{O}$ be an Eichler order of level $\mathcal{N}$ in B and let S be an order over R_F in a quadratic field extension L/F.

Assume that all prime ideals $\wp | \mathrm{disc}(\mathcal{O})$ ramify in L but do not divide the conductor $\mathfrak{f}_S$ of S. This is a strong restriction on S which we will naturally encounter in Chapter 3. Let H_S be the *ring class field* of S over L. The Galois group $\mathrm{Gal}(H_S/L)$ is isomorphic, via the Artin reciprocity map, to the Picard group $\mathrm{Pic}(S)$ of classes of locally invertible ideals of S. In the particular case that S is the ring of integers of L, then H_S is the *Hilbert class field* of L. It follows from our assumption on S that L and H_S are linearly disjoint over F, that is, $F = L \cap H_S$. The norm induces a map $N_{L/F} : \mathrm{Pic}(S) \to \mathrm{Pic}(R_F)$ that, by the reciprocity isomorphism can be interpreted as the restriction map $\mathrm{Gal}(H_S/L) \to \mathrm{Gal}(L \cdot H_F/L) \simeq \mathrm{Gal}(H_F/F)$ (cf. [Ne99], Chapter VI, Section 5). In particular, we have an exact sequence

$$0 \to \Delta \to \mathrm{Pic}(S) \stackrel{N_{L/F}}{\to} \mathrm{Pic}(R_F) \to 0.$$

Here, $\Delta = \mathrm{Ker}\ (N_{L/F})$ can be viewed as the Galois group of H_S over the fixed field $L_\triangle$ of H_S by Δ. The group $\Delta = \mathrm{Gal}(H_S/L_\triangle)$ acts on $E(S,\mathcal{O})$ by a reciprocity law as follows: let $i : S \hookrightarrow \mathcal{O}$ be an optimal embedding and let $\tau \in \mathrm{Gal}(H_S/L_\triangle)$. Then let $\mathfrak{b} = [\tau, H_S/L]$ be the locally invertible ideal in S corresponding to τ by the Artin's reciprocity map. Since the reduced norm on B induces a bijection of sets $\mathrm{n} : \mathrm{Pic}_\ell(\mathcal{O}) \simeq \mathrm{Pic}(F)$ and $N_{L/K}(\mathfrak{b})$ is a principal ideal in F, it follows that $i(\mathfrak{b})\mathcal{O} = \beta\mathcal{O}$ is a principal right ideal of $\mathcal{O}$ and we can choose a generator $\beta \in \mathcal{O}$. Then τ acts on $i \in E(S,\mathcal{O})$ as

$$i^\tau = \beta^{-1} i \beta.$$

It can be checked that this action does not depend on the choice of the ideal $\mathfrak{b}$ in its class in $\mathrm{Pic}(S)$ nor on the choice of the element $\beta \in \mathcal{O}$. Moreover, a local argument shows that this action is free. Since $|\Delta| = |E(S,\mathcal{O})|$, we obtain

Proposition 1.2.19. *The action of* Δ *on the set of Eichler conjugacy classes of optimal embeddings* $E(S,\mathcal{O})$ *is free and transitive.*

The above action acquires a real arithmetic meaning and coincides with Shimura's reciprocity law in the particular case that L is a CM-field over F. In this situation, $E(S,\mathcal{O})$ can also be interpreted as the set of *Heegner points* on a Shimura variety $\mathfrak{X}$ on which the Galois group Δ is acting (cf. [Sh67], Section 9.10).

Finally, several manuscripts deal with the computation of the numbers $e(S,\mathcal{O})$. See [HiPiSh89] and [Br90] for Gorenstein and Bass orders.

1.3 Basic facts on abelian varieties

Let k be a subfield of the field $\mathbb{C}$ of complex numbers and let $\bar{k}$ denote an algebraic closure of k in $\mathbb{C}$. Let A be an abelian variety of dimension $g \geq 1$ defined over k.

For any field extension K/k, let $G_K = \mathrm{Gal}(\bar{k}/K)$. We let $A_K = A \otimes_{\mathrm{Spec}k} \mathrm{Spec}K$ and we let $K(A)$ denote the function field of A_K.

Let $\mathrm{Div}(A_{\bar{k}})$ denote the group of Weil divisors of $A_{\bar{k}}$. For any field extension K/k in $\bar{k}$, a Weil divisor Θ is rational over K, that is, $\Theta \in \mathrm{Div}(A_K) = H^0(G_K, \mathrm{Div}(A_{\bar{k}}))$, if it is stable by the action of the absolute Galois group of K.

Let $\mathrm{Pic}(A_{\bar{k}})$ denote the group of invertible sheaves on $A_{\bar{k}}$. There is a natural exact sequence

$$0 \to \bar{k}(A)^*/\bar{k}^* \to \mathrm{Div}(A_{\bar{k}}) \to \mathrm{Pic}(A_{\bar{k}}) \to 0$$

which induces a long exact sequence of Galois cohomology groups and an isomorphism $\mathrm{Div}(A_K)/K(A)^* \simeq \mathrm{Pic}(A_{\bar{k}})$ for any K/k, because $A(K) \neq \emptyset$ (cf. [PoSt99]).

Let $\mathrm{Pic}^0_{\bar{k}}(A)$ denote the subgroup of $\mathrm{Pic}_{\bar{k}}(A)$ of invertible sheaves on A which are algebraically equivalent to 0 and let $\mathrm{Pic}^0_K(A) = \mathrm{Pic}_K(A) \cap \mathrm{Pic}^0_{\bar{k}}(A)$ for any field extension K/k.

The Néron-Severi group $\mathrm{NS}(A)$ of algebraic equivalence classes of invertible sheaves on A is the group $\mathrm{NS}(A) = \mathrm{Pic}(A)/\mathrm{Pic}^0(A)$. We note that not all elements in $\mathrm{NS}(A_{\bar{k}})^{G_K} = H^0(G_K, \mathrm{NS}_{\bar{k}}(A))$ are represented by an invertible sheaf $\mathcal{L}$ on A defined over K. This translates into the fact that $\mathrm{NS}(A_K) \subseteq \mathrm{NS}(A_{\bar{k}})^{G_K}$ but these two groups do not need to be equal.

By the Néron Basis Theorem, $\mathrm{NS}(A_{\bar{k}})^{G_K}$ is a finitely generated and torsion-free abelian group and we will agree to define the Picard number of A_K to be

$$\rho(A_K) = \mathrm{rank}_{\mathbb{Z}}\, \mathrm{NS}(A_{\bar{k}})^{G_K}.$$

For any closed point $P \in A(\bar{k})$, let $t_P : A_{\bar{k}} \to A_{\bar{k}}$ denote the translation-by-P map. As is well known, an invertible sheaf $\mathcal{L} \in \mathrm{NS}(A_{\bar{k}})^{G_k}$ induces a morphism

$$\begin{array}{rccc} \varphi_{\mathcal{L}} : & A & \to & \hat{A} \\ & P & \mapsto & t_P^*(\mathcal{L}) \otimes \mathcal{L}^{-1} \end{array}$$

defined over k.

We let $K(\mathcal{L})$ denote the kernel of $\varphi_{\mathcal{L}}$. An invertible sheaf $\mathcal{L} \in \mathrm{NS}(A_{\bar{k}})$ is *nondegenerate* if $K(\mathcal{L})$ is a finite group. Since the image and kernel of the morphism $\varphi_{\mathcal{L}}$ attached to a degenerate invertible sheaf $\mathcal{L} \in \mathrm{NS}(A_{\bar{k}})^{G_k}$ is a proper subabelian variety of A_k, it follows that all non algebraically equivalent to zero invertible sheaves $\mathcal{L} \in \mathrm{NS}(A_{\bar{k}})^{G_k}$ on a simple abelian variety A over k are nondegenerate.

Definition 1.3.1. The degree of a nondegenerate invertible sheaf $\mathcal{L}$ on A is

$$\deg(\mathcal{L}) = |K(\mathcal{L})|^{1/2}.$$

An invertible sheaf $\mathcal{L}$ is *principal* if and only if $K(\mathcal{L})$ is trivial, that is, if $\varphi_{\mathcal{L}} : A \to \hat{A}$ is an isomorphism.

Let us agree to say that two invertible sheaves $\mathcal{L}$ and $\mathcal{L}' \in \mathrm{NS}(A_{\bar{k}})^{G_k}$ are isomorphic, denoted by $\mathcal{L} \simeq \mathcal{L}'$, if $(A, \mathcal{L}) \simeq (A, \mathcal{L}')$ as polarized abelian varieties, that is, there exists an automorphism $\alpha \in \mathrm{Aut}(A)$ such that $\mathcal{L} = \alpha^*(\mathcal{L}')$.

Definition 1.3.2. We let

$$\Pi(A) = \{\mathcal{L} \in \mathrm{NS}(A_{\bar{k}})^{G_k} : \deg(\mathcal{L}) = 1\}/_{\simeq}$$

denote the set of isomorphism classes of principal invertible sheaves on A.

The following theorem follows from a result of Narasimhan and Nori [NaNo81]. See also [GoGuRo02].

Theorem 1.3.3. *The set* $\Pi(A)$ *is a finite set.*

We shall let $\pi(A) = |\Pi(A)|$ denote the cardinality of $\Pi(A)$.

Let $\mathcal{L} \in \mathrm{NS}(A_{\bar{k}})^{G_k}$ be a nondegenerate invertible sheaf on an abelian variety A over k. By Mumford's Vanishing Theorem (cf. [Mu70], §16), there is a unique integer $i(\mathcal{L})$ such that $H^{i(\mathcal{L})}(A, \mathcal{L}) \neq 0$ and $H^j(A, \mathcal{L}) = 0$ for all $j \neq i(\mathcal{L})$. The integer $i(\mathcal{L})$ is called the *index of* $\mathcal{L}$ and it only depends on the class of $\mathcal{L}$ in $\mathrm{NS}(A_{\bar{k}})$. We have $0 \leq i(\mathcal{L}) \leq g = \dim(A)$. The class of algebraic equivalence $\mathcal{L} \in \mathrm{NS}(A_{\bar{k}})^{G_k}$ of an invertible sheaf on A is a *polarization over* k if $i(\mathcal{L}) = 0$ or, equivalently, if $\mathcal{L}$ is ample. By the Riemann-Roch Theorem, $|K(\mathcal{L})| = |\mathrm{Ker}\ \varphi_{\mathcal{L}} : A \to \hat{A}| = \dim_k(H^{i(\mathcal{L})}(A, \mathcal{L}))$.

In particular, $\mathcal{L}$ is *principal* if and only if $\dim_{\bar{k}}(H^{i(\mathcal{L})}(A_{\bar{k}}, \mathcal{L})) = 1$. In consequence, $\mathcal{L} \in \mathrm{NS}(A_{\bar{k}})^{G_k}$ is a *principal polarization* if and only if $H^0(A_{\bar{k}}, \mathcal{L}) = \bar{k} \cdot \vartheta$, for a certain nonzero automorphic form ϑ.

Definition 1.3.4. For any nonnegative integer $0 \leq i \leq g$, let

$$\Pi_i(A) = \{\mathcal{L} \in \mathrm{NS}(A_{\bar{k}})^{G_k} : i(\mathcal{L}) = i, \deg(\mathcal{L}) = 1\}/_{\simeq}$$

denote the set of isomorphism classes of principal invertible sheaves on A of index i.

The set $\Pi(A)$ can be naturally written as the disjoint union $\Pi(A) = \Pi_0(A) \cup ... \cup \Pi_g(A)$. If we let $\pi_i(A) = |\Pi_i(A)|$, it then holds that $\pi(A) =$

$\pi_0(A)+...+\pi_g(A)$. Moreover, as is shown in [Mu70], it holds for any invertible sheaf $\mathcal{L}$ on $A_{\bar{k}}$ that $i(\mathcal{L}) + i(\mathcal{L}^{-1}) = g$. Consequently, the map $\mathcal{L} \mapsto \mathcal{L}^{-1}$ induces a one-to-one correspondence between $\Pi_i(A)$ and $\Pi_{g-i}(A)$ and therefore $\pi_i(A) = \pi_{g-i}(A)$.

Let now $A(\mathbb{C})$ be the set of complex-valued points of the abelian variety $A_{\mathbb{C}}$. There exists a complex vector space V of dimension g and a lattice $\Lambda \subset V$ of rank $2g$ over $\mathbb{Z}$ such that $V/\Lambda \simeq A(\mathbb{C})$.

A Riemann form on Λ is an $\mathbb{R}$-alternating bilinear form $E : V \times V \to \mathbb{R}$ such that $E(\Lambda \times \Lambda) \subset \mathbb{Z}$ and $E(\sqrt{-1}u, \sqrt{-1}v) = E(u, v)$ for all u, $v \in V$.

By the Appell-Humbert Theorem (cf. [Mu70], [LaBi92]), the first Chern class induces an isomorphism of additive groups between the Néron-Severi group $\mathrm{NS}(A_{\mathbb{C}})$ of $A_{\mathbb{C}}$ and the group of Riemann forms on Λ. An invertible sheaf $\mathcal{L}$ on A is a polarization if and only if the corresponding Riemann form $E_{\mathcal{L}}$ attached to $\mathcal{L}$ by the Appell-Humbert Theorem satisfies that $E(\sqrt{-1}u, u) > 0$ for all $u \in V \setminus \{0\}$.

Let

$$\begin{array}{rccc} H: & V \times V & \to & \mathbb{C} \\ & (u,v) & \mapsto & E(\sqrt{-1}u, v) + \sqrt{-1}E(u,v) \end{array}$$

be the hermitian form attached to a Riemann form E on Λ. If H is the hermitian form associated to an invertible sheaf $\mathcal{L}$, the index $i(\mathcal{L})$ agrees with the number of negative eigen-values of H (cf. [Mu70], §16).

Let $\mathcal{L}$ be a polarization on $A_{\mathbb{C}}$ and $E_{\mathcal{L}}$ the corresponding Riemann form. There exists a suitable basis of Λ such that the matrix expression of $E_{\mathcal{L}}$ is

$$\begin{pmatrix} & & & d_1 & & \\ & 0 & & & ... & \\ & & & & & d_g \\ -d_1 & & & & & \\ & ... & & & 0 & \\ & & -d_g & & & \end{pmatrix}$$

for some positive integers $d_1|d_2|...|d_g$. The sequence $(d_1, ..., d_g)$ is called the *type* of the polarization $\mathcal{L}$. A polarization $\mathcal{L} \in \mathrm{NS}(A)$ is called *primitive* if $\mathcal{L} \notin d \cdot \mathrm{NS}(A)$ for any $d \in \mathbb{Z}, d \geq 2$. Its type is then $(1, d_2, ..., d_g)$ for $d_i|d_{i+1}$, $i = 2, ..., g-1$. The polarization $\mathcal{L}$ is principal if and only if $d_1 = ... = d_g = 1$.

For any field extension K/k in $\mathbb{C}$, let $\mathrm{End}_K(A)$ denote the ring of endomorphisms of the abelian variety A_K. This is a possibly non commutative $\mathbb{Z}$-algebra of finite rank.

Let $\mathrm{End}^0_K(A) = \mathbb{Q} \otimes \mathrm{End}_K(A)$. It is well known that this is a semisimple algebra of finite rank over $\mathbb{Q}$. Moreover, $\mathrm{End}^0_K(A)$ is a simple division algebra if and only if A_K is a simple abelian variety, i. e., A contains no proper subabelian varieties defined over K. The following theorem follows from Albert's classification of anti-involuting division algebras.

Theorem 1.3.5. *Let A/k be a simple abelian variety over k of dimension g. Let $B = \mathrm{End}^0(A)$, F be the centre of B and F_0 be the maximal totally real subfield of F. Let $e = [F : \mathbb{Q}]$, $e_0 = [F_0 : \mathbb{Q}]$ and $d^2 = [B : F]$. Then one of the following possibilities must hold:*

1. *$B = F = F_0$ and $e|g$.*
2. *$F = F_0$, B is a totally indefinite quaternion algebra over F and $2e|g$.*
3. *$F = F_0$, B is a totally definite quaternion algebra over F and $2e|g$.*
4. *F is a CM-field over F_0, B is a division algebra over F and $e_0d^2|g$.*

Chapter 2

Arithmetic of quaternion algebras

Introduction

In this chapter, we study two questions on the arithmetic of quaternion algebras and their orders which arise from our work on abelian varieties with quaternionic multiplication and the Shimura varieties which occur as their moduli spaces.

Let B be a quaternion algebra over either a number field or the completion of a number field with respect to a finite place and let $\mathcal{O}$ be an arbitrary order in B. In Section 2.1, we consider the problem of finding suitable integral bases of B lying in $\mathcal{O}$. This question is related to several papers by Chinburg and Friedman in relation to arithmetic 3-orbifolds (cf. [ChFr86], [ChFr99], [ChFr00]) and also arises naturally in Chapter 4 in the study of forgetful maps between Shimura varieties and the field of moduli of the quaternionic multiplication on an abelian variety.

In Section 2.2, we reconsider a quaternionic equation introduced by Pall and O'Connor in [Pa37] and [CoPa39] and further explored by Pollack in [Po60]. Motivated by their work, we introduce a conjugation relation in the set of pure quaternions in $\mathcal{O}$ of given reduced norm and we compute the number of orbits in terms of class numbers of quadratic orders embedded in $\mathcal{O}$ by making use of Eichler's theory of optimal embeddings. Our results in this section are crucial to our study of the set of isomorphism classes of line bundles on abelian varieties with quaternionic multiplication in Chapter 3.

The results of this chapter are contained in [Ro2] and [Ro4].

2.1 Integral quaternion bases and distance ideals

In this section we focus on the following questions on quaternion algebras and orders which naturally arise from our results in Chapter 4.

Question 2.1.1. Let F be either a number field or the completion of a number field with respect to a finite place and let B be a quaternion algebra over F. Let $\mathcal{O}$ be an arbitrary order in B.

1. If $B \simeq (\frac{a,b}{F})$ for some $a, b \in R_F$, can one find integral elements ι, $\eta \in \mathcal{O}$ such that $\iota^2 = a$, $\eta^2 = b$, $\iota\eta = -\eta\iota$?
2. If $B \simeq (L, m)$ for a quadratic separable algebra over F and $m \in R_F$, can one find $\chi \in \mathcal{O}$ such that $\chi^2 = m$, $\chi\beta = \overline{\beta}\chi$ for any $\beta \in L$?

We note that the second question may be considered as a refinement of the first. Indeed, let $\mathcal{O}$ be an order in $B = (\frac{a,b}{F})$ and fix an arbitrary element $j \in \mathcal{O}$ such that $i^2 = a$. Then, while our first question asks whether there exist arbitrary elements ι, $\eta \in \mathcal{O}$ such that $\iota^2 = a$, $\eta^2 = b$ and $\iota\eta = -\eta\iota$, the second wonders whether such an integral basis exists with $\iota = i$.

If $B = \left(\frac{a,b}{F}\right) = F + Fi + Fj + Fij$, let $\mathcal{O}_0 = R_F[i, j]$. Obviously, the first part of the question is answered positively whenever $\gamma^{-1}\mathcal{O}\gamma \supseteq \mathcal{O}_0$ for some $\gamma \in B^*$. The following proposition asserts that this is actually a necessary condition. Although it is not stated in this form in [ChFr00F], it is due to Chinburg and Friedman, and follows from the ideas therein. It is a consequence of Hilbert's Satz 90.

Proposition 2.1.2. *Let $B = F + Fi + Fj + Fij = \left(\frac{a,b}{F}\right)$ with a, $b \in R_F$. Let $\mathcal{O}_0 = R_F[i, j]$.*

An order $\mathcal{O}$ in B contains a basis ι, $\eta \in \mathcal{O}$, $\iota^2 = a$, $\eta^2 = b$, $\iota\eta = -\eta\iota$ of B if and only if the type of $\mathcal{O}_0$ is contained in the type of $\mathcal{O}$.

Proof. Assume that there exist ι, $\eta \in \mathcal{O}$ satisfying the above relations. By the Skolem-Noether Theorem (cf. [Vi80]), j and η are conjugate (by, say, $\alpha \in B^*$). Thus, by replacing i by $\alpha^{-1}i\alpha$ and $\mathcal{O}_0$ by $\alpha^{-1}\mathcal{O}_0\alpha$, we may assume that $j = \eta \in \mathcal{O}$. We then need to show the existence of an element $\gamma \in F(j) = F(\eta)$ such that $\gamma^{-1}\iota\gamma = i$.

We have $i\eta = -\eta i$ and thus $\eta = -i^{-1}\eta i$. In addition, since $\iota\eta = -\eta\iota$, $\iota i^{-1}\eta i = \eta\iota$. Hence, $(\iota i^{-1})\eta = \eta(\iota i^{-1})$ and we deduce that $\iota i^{-1} \in F(\eta)$ is an element of norm $\mathrm{Norm}_{F(\eta)/F}(\iota i^{-1}) = 1$.

By Hilbert's Satz 90, there exists $\omega \in F(\eta)$ such that $\iota i^{-1} = \omega\overline{\omega}^{-1}$, that is, $\iota = \omega\overline{\omega}^{-1}i$. Stated in this form, we need to find an element $\gamma \in F(\eta)$ with $\gamma^{-1}\omega\overline{\omega}^{-1}i\gamma = i$. Since $\gamma i = i\overline{\gamma}$, we can choose $\gamma = \omega$. □

Corollary 2.1.3. *Assume that F is the completion of a number field with respect to a finite place. Let $\mathcal{O}$ be an Eichler order of level $\mathcal{N}$ in $B = (\frac{a,b}{F})$, $a, b \in R_F$. Then, there exist $\iota, \eta \in \mathcal{O}$, $\iota^2 = a$, $\eta^2 = b$, $\iota\eta = -\eta\iota$ if and only if $\mathcal{N}|4ab$.*

Proof. By [Vi80], §2, there is only one type of Eichler orders of fixed level $\mathcal{N}$ in B. Remark that, if B is division, necessarily $\mathcal{N} = 1$. Let $\mathcal{O}_0 = R_F[i, j]$. Since $\mathrm{disc}(\mathcal{O}_0) = 4ab$, as one can check, a necessary and sufficient condition on $\mathcal{O}$ to contain a conjugate order of $\mathcal{O}_0$ is that $\mathcal{N}|4ab$. The corollary follows from Proposition 2.1.2. □

In the global case, the approach to question 2.1.1 §1 can be made more effective under the assumption that B satisfies the *Eichler condition*. Namely, a quaternion algebra B over F is said to satisfy the Eichler condition if some archimedean place v of F does not ramify in B, that is, $B \otimes_F F_v \simeq \mathrm{M}_2(F_v)$. Here, we let $F_v \simeq \mathbb{R}$ or $\mathbb{C}$ denote the completion of F at v.

The following theorem of Eichler describes the set $\mathcal{T}(\mathcal{N})$ of types of Eichler orders of given level $\mathcal{N}$ purely in terms of the arithmetic of F. Let $\mathrm{Pic}_\infty(F)$ be the narrow class group of F of fractional ideals up to principal fractional ideals (a) generated by elements $a \in F^*$ such that $a > 0$ at any real archimedean place v that ramifies in B and let $h_\infty(F) = |\mathrm{Pic}_\infty(F)|$.

Definition 2.1.4. The group $\overline{\mathrm{Pic}}_\infty^{\mathcal{N}}(F)$ is the quotient of $\mathrm{Pic}_\infty(F)$ by the subgroup generated by the squares of fractional ideals of F, the prime ideals $\wp$ that ramify in B and the prime ideals $\mathfrak{q}$ such that $\mathcal{N}$ has odd $\mathfrak{q}$-valuation.

The group $\overline{\mathrm{Pic}}_\infty^{\mathcal{N}}(F)$ is a 2-torsion finite abelian group. Therefore, if $h_\infty(F)$ is odd, then $\overline{\mathrm{Pic}}_\infty^{\mathcal{N}}(F)$ is trivial.

The following can be found in [Ei37], [Ei38] and [Vi80], p. 89.

Proposition 2.1.5. *The reduced norm* n *induces a bijection of sets*

$$\mathcal{T}(\mathcal{N}) \xrightarrow{\sim} \overline{\mathrm{Pic}}^{\mathcal{N}}_{\infty}(F).$$

The bijection is not canonical in the sense that it depends on the choice of an arbitrary Eichler order $\mathcal{O}$ in B. For $\mathcal{N}=1$, the bijection is explicitly described as follows. For any two maximal orders $\mathcal{O}$, $\mathcal{O}'$ of B over R_F, define the *distance ideal* $\rho(\mathcal{O},\mathcal{O}')$ to be the order-ideal of the finite R_F-module $\mathcal{O}/\mathcal{O}\cap\mathcal{O}'$ (cf. [Re75], p. 49). Alternatively, $\rho(\mathcal{O},\mathcal{O}')$ can also be defined locally in terms of the local distances between $\mathcal{O}\otimes_{R_F} R_{F_\wp}$ and $\mathcal{O}'\otimes_{R_F} R_{F_\wp}$ in the Bruhat-Tits tree $\mathcal{T}_\wp$ for any (nonarchimedean) prime ideal $\wp$ of F that does not ramify in B (cf. [ChFr99]). Finally, $\rho(\mathcal{O},\mathcal{O}')$ is also the *level* of the Eichler order $\mathcal{O}\cap\mathcal{O}'$. This notion of distance proves to be suitable to classify the set of types of maximal orders of B, as the assignation $\mathcal{O}'\mapsto\rho(\mathcal{O},\mathcal{O}')$ induces the bijection claimed in Proposition 2.1.5.

Corollary 2.1.6. *Let $B=\left(\frac{a,b}{F}\right)$, a, $b\in R_F$ be a quaternion algebra over a global field F. If B satisfies Eichler's condition and $h_\infty(F)$ is odd then, for any Eichler order $\mathcal{O}$ in B, there is an integral basis ι, $\eta\in\mathcal{O}$, $\iota^2=a$, $\eta^2=b$, $\iota\eta=-\eta\iota$ of B.*

As for question 2.1.1 §2, let $B=F+Fi+Fj+Fij=\left(\frac{a,b}{F}\right)=(L,b)$ with a, $b\in R_F$ and $L=F(\sqrt{a})$. Choose an arbitrary order $\mathcal{O}$ of B. For given $\eta\in\mathcal{O}$, $\eta^2=a$, we ask whether there exists $\chi\in\mathcal{O}$, $\chi^2=b$, such that $\eta\chi=-\eta\chi$. By Proposition 2.1.2, a necessary condition is that $\mathcal{O}_0=R_F[i,j]\subseteq\mathcal{O}$ up to conjugation by elements of B^* and, without loss of generality, we assume that this is the case. With these notations, we have

Definition 2.1.7. Let $\mathcal{O}\supseteq\mathcal{O}'$ be two arbitrary orders in B. The transportator of $\mathcal{O}'$ into $\mathcal{O}$ over B^* is $(\mathcal{O}:\mathcal{O}'):=\{\gamma\in B^*:\gamma^{-1}\mathcal{O}'\gamma\subset\mathcal{O}\}$.

Note that $\mathrm{Norm}_{B^*}(\mathcal{O})$ is a subgroup of finite index of $(\mathcal{O}:\mathcal{O}')$.

Proposition 2.1.8. *Let $\mathcal{O}\supseteq\mathcal{O}_0$ be an order in B and let $\eta\in\mathcal{O}$, $\eta^2=a$. Then, there exists $\chi\in\mathcal{O}$ such that $\chi^2=b$ and $\eta\chi=-\chi\eta$ if and only if $\eta=\gamma^{-1}i\gamma$ for $\gamma\in(\mathcal{O}:\mathcal{O}_0)$.*

Let $f=|(\mathcal{O}:\mathcal{O}_0):\mathrm{Norm}_{B^*}(\mathcal{O})|$ be the index of the normalizer group $\mathrm{Norm}_{B^*}(\mathcal{O})$ in $(\mathcal{O}:\mathcal{O}_0)$. Consider η_1, ... η_h representatives of elements in $\mathcal{O}$ such that $\eta_i^2=a$ up to conjugation by elements in $\mathrm{Norm}_{B^*}(\mathcal{O})$. Then, it follows from the above proposition that there exist η_{i_1}, ..., η_{i_f} such that, for a given element $\eta \in \mathcal{O}$, there exists $\chi \in \mathcal{O}$, $\chi^2 = b$, $\eta\chi = -\chi\eta$ if and only if η lies in one of the f $\mathrm{Norm}_{B^*}(\mathcal{O})$-conjugation classes generated by η_{i_1}, ..., η_{i_f}.

Again, the number of $\mathrm{Norm}_{B^*}(\mathcal{O})$-conjugation classes of elements $\eta \in \mathcal{O}$ such that $\eta^2 = a$ can be explicitly computed in many cases in terms of class numbers by means of the theory of Eichler of optimal embeddings. We refer the reader to [Vi80] for details.

2.2 Pollack conjugation

Let F be a number field and let B be a division quaternion algebra over F. Let $\mathcal{O}$ be an order in B.

Definition 2.2.1. Two quaternions μ_1, $\mu_2 \in B$ are *Pollack conjugate* over $\mathcal{O}$ if $\mu_2 = \overline{\alpha}\mu_1\alpha$ for some unit $\alpha \in \mathcal{O}^*$. We will denote it by $\mu_1 \sim_p \mu_2$.

The motivation for the above definition comes from Pollack's work [Po60], in which he studied the obstruction for two pure quaternions μ_1 and $\mu_2 \in B$ with the same reduced norm to be conjugate over B^* in the above sense, that is, $\mu_1 = \overline{\alpha}\mu_2\alpha$ with $\alpha \in B^*$. He expressed this obstruction in terms of the 2-torsion subgroup $\mathrm{Br}_2(F)$ of the Brauer group $\mathrm{Br}(F)$ of F. Furthermore, he investigated the solvability of the equation $\mu_2 = \overline{\alpha}\mu_1\alpha$ over $\mathcal{O}^*$ for quaternions μ_1 and μ_2 in a maximal order $\mathcal{O}$ of B.

As a refinement of his considerations, it is natural to consider the set of orbits of pure quaternions $\mu \in \mathcal{O}$ of fixed reduced norm $\mathrm{n}(\mu) = d \in F^*$ under the action of the group of units $\mathcal{O}^*$ by Pollack conjugation. We drop the restriction on $\mathcal{O}$ to be maximal in our statements.

Definition 2.2.2. For any integral element $d \in R_F$, we let

$$\mathcal{P}(d, \mathcal{O}) = \{\mu \in \mathcal{O} : \mathrm{tr}(\mu) = 0, \mathrm{n}(\mu) = d\}.$$

Let

$$P(d, \mathcal{O}) = \mathcal{P}(d, \mathcal{O})/ \sim_p$$

and $p(d, \mathcal{O}) = |P(d, \mathcal{O})|$.

In order to avoid trivialities, we assume for the rest of this section that $-d$ is not a perfect square in R_F. Then, any quaternion $\mu \in \mathcal{P}(d, \mathcal{O})$ induces an embedding

$$\begin{array}{rcl} i_\mu : F(\sqrt{-d}) & \hookrightarrow & B \\ a + b\sqrt{-d} & \mapsto & a + b\mu \end{array}$$

for which $i_\mu(R_F[\sqrt{-d}]) \subset \mathcal{O}$.

Let S be an order over R_F in a quadratic algebra L over F. Recall from Section 1.2.3 that an embedding $i : S \hookrightarrow \mathcal{O}$ is *optimal* if $i(S) = i(L) \cap \mathcal{O}$. For any $\mu \in \mathcal{P}(d, \mathcal{O})$ there is a uniquely determined order $S_\mu \supseteq R_F[\sqrt{-d}]$ such that i_μ is optimal at S_μ.

Moreover, two equivalent quaternions μ_1, $\mu_2 \in \mathcal{P}(d, \mathcal{O})$, $\mu_1 \sim_p \mu_2$, are optimal at the same order S. Indeed, if $\alpha \in \mathcal{O}^*$ is such that $\mu_1 = \overline{\alpha}\mu_2\alpha$, then α is forced to have reduced norm $\mathrm{n}(\alpha) = \pm 1$. Hence $\overline{\alpha} = \pm\alpha^{-1} \in \mathcal{O}^*$ and the observation follows since α normalizes $\mathcal{O}$. Conversely, any optimal embedding $i : S \hookrightarrow \mathcal{O}$, $S \supseteq R_F[\sqrt{-d}]$ determines a quaternion $\mu = i(\sqrt{-d}) \in \mathcal{P}(d, \mathcal{O})$.

Hence, a necessary condition for $\mu_1 \sim_p \mu_2$ over $\mathcal{O}^*$ is that μ_1 and μ_2 induce an optimal embedding at the same quadratic order $F(\sqrt{-d}) \supset S \supseteq R_F[\sqrt{-d}]$.

Definition 2.2.3. For any quadratic order S over R_F, we let

$$\mathcal{P}(S, \mathcal{O}) = \{ \text{ Optimal embeddings } i : S \hookrightarrow \mathcal{O}\}.$$

Let

$$P(S, \mathcal{O}) = \mathcal{P}(S, \mathcal{O}) / \sim_p$$

and $p(S, \mathcal{O}) = |P(S, \mathcal{O})|$.

In contrast to Pollack conjugation, we recall from Section 1.2.3 that two optimal embeddings $i, j : S \hookrightarrow \mathcal{O}$ lie on the same conjugation class in the sense of Eichler, written $i \sim_e j$, if there exists $\alpha \in \mathcal{O}^*$ such that $i = \alpha^{-1} j \alpha$. We also recall that, according to our above definition, we let $E(S, \mathcal{O}) = \mathcal{P}(S, \mathcal{O})/_{\sim_e}$ and $e(S, \mathcal{O}) = |E(S, \mathcal{O})|$.

Proposition 2.2.4. *Let S be an order in a quadratic algebra L over F and let $\mathcal{O}$ be an order in a division quaternion algebra B over F. Then, the number of Pollack conjugation classes of optimal embeddings of S into $\mathcal{O}$ is*

$$p(S, \mathcal{O}) = |\mathrm{n}(\mathcal{O}^*)/N_{L/F}(S^*)| \cdot \frac{e(S, \mathcal{O})}{2}.$$

Proof. Let us agree to say that two pure quaternions μ_1 and $\mu_2 \in \mathcal{O}$ lie in the same $\pm$Eichler conjugation class if there exists $\alpha \in \mathcal{O}^*$ such that $\mu_1 = \pm\alpha^{-1}\mu_2\alpha$. We shall denote it by $\mu_1 \sim_{\pm e} \mu_2$ and $E_\pm(S,\mathcal{O}) = \mathcal{P}(S,\mathcal{O})/_{\sim_{\pm e}}$. The identity map $\mu \mapsto \mu$ descends to a natural surjective map

$$\rho : P(S,\mathcal{O}) \quad \to \quad E_\pm(S,\mathcal{O})$$

and the proposition now follows from the following lemma.

Lemma 2.2.5. *Let* $e_S = \dim_{\mathbb{F}_2}(\mathrm{n}(\mathcal{O}^*)/N_{L/F}(S^*))$*. Let* $\mu \in \mathcal{P}(S,\mathcal{O})$ *and let* $\varepsilon_\mu = 1$ *if* $\mu \sim_e -\mu$ *and* $\varepsilon_\mu = 2$ *otherwise. Then, in the* $\pm$*Eichler conjugation class* $\{\pm\alpha^{-1}\mu\alpha : \alpha \in \mathcal{O}^*\}$ *of* μ*, there are exactly* $\varepsilon_\mu 2^{e_S-1}$ *Pollack conjugation classes of pure quaternions.*

Proof of Lemma 2.2.5. Suppose first that $\varepsilon_\mu = 1$. Then, the $\pm$Eichler conjugation class of $\mu \in \mathcal{P}(S,\mathcal{O})$ is $\{\alpha^{-1}\mu\alpha : \alpha \in \mathcal{O}^*\}$. Let $\gamma \in \mathcal{O}^*$ be such that $-\mu = \gamma\mu\gamma^{-1} = \gamma^{-1}\mu\gamma$. We claim that, for any given $\alpha \in \mathcal{O}^*$, it holds that $\mu \sim_p \alpha^{-1}\mu\alpha$ if and only if $\mathrm{n}(\alpha) \in N_{L/F}(S^*) \cup (-\mathrm{n}(\gamma)N_{L/F}(S^*))$. Indeed, if $\mu \sim_p \alpha^{-1}\mu\alpha$, let $\beta \in \mathcal{O}^*$ with $\mathrm{n}(\beta) = \pm 1$ be such that $\overline{\beta}\alpha^{-1}\mu\alpha\beta = \mu$. If $\mathrm{n}(\beta) = 1$, then $\alpha\beta\mu = \mu\alpha\beta$ and hence $\alpha\beta \in L \cap \mathcal{O}^* = S^*$. Thus $\mathrm{n}(\alpha\beta) = \mathrm{n}(\alpha) \in N_{L/F}(S^*)$. If $\mathrm{n}(\beta) = -1$, a similar argument shows that $\mathrm{n}(\alpha) \in -\mathrm{n}(\gamma) \cdot N_{L/F}(S^*)$. Conversely, let $\mathrm{n}(\alpha) = v \in N_{L/F}(S^*) \cup (-\mathrm{n}(\gamma)N_{L/F}(S^*))$ and let $s \in S^*$ be such that $N_{L/F}(s) = v$ or $-v\mathrm{n}(\gamma)^{-1}$. Since μ induces an embedding $S \hookrightarrow \mathcal{O}$, we can regard s as an element in $\mathcal{O}^*$ such that $\mathrm{n}(s) = v$ or $-v\mathrm{n}(\gamma)^{-1}$ and $s\mu = \mu s$. Hence $\alpha^{-1}\mu\alpha = \alpha^{-1}s\mu s^{-1}\alpha = (\alpha^{-1}s)\mu\overline{(\alpha^{-1}s)}$ or $(\alpha^{-1}s\gamma^{-1})\mu\overline{(\alpha^{-1}s\gamma^{-1})}$. This proves the claim.

Since B is division, Pollack's Theorem on Pall's Conjecture ([Po60], Theorem 4) applies to show that $-\mathrm{n}(\gamma) \notin N_{L/F}(S^*)$. We then conclude that the distinct Pollack conjugation orbits in $\{\alpha^{-1}\mu\alpha : \alpha \in \mathcal{O}^*\}$ are exactly the classes

$$\mathcal{C}_u = \{\alpha^{-1}\mu\alpha : \mathrm{n}(\alpha) \in uN(S^*) \cup (-\mathrm{n}(\gamma)uN(S^*))\}$$

as $u \in \mathrm{n}(\mathcal{O}^*)$ runs through a set of representatives in $\mathrm{n}(\mathcal{O}^*)/\langle -\mathrm{n}(\gamma), N(S^*)\rangle$. There are 2^{e_S-1} of them.

Assume that $\varepsilon_\mu = 2$, that is, $\mu \not\sim_e -\mu$. Then, the $\pm$Eichler conjugation class of $\mu \in \mathcal{P}(S,\mathcal{O})$ is $\{\alpha^{-1}\mu\alpha : \alpha \in \mathcal{O}^*\} \cup \{-\alpha^{-1}\mu\alpha : \alpha \in \mathcal{O}^*\}$. As in the previous case, it is shown that $\mu \sim_p \alpha^{-1}\mu\alpha$ if and only if $\mathrm{n}(\alpha) \in N_{L/F}(S^*)$ and $\mu \sim_p -\alpha^{-1}\mu\alpha$ if and only if $\mathrm{n}(\alpha) \in -N_{L/F}(S^*)$. We obtain that, as $u \in \mathrm{n}(\mathcal{O}^*)$ runs through a set of representatives in $\mathrm{n}(\mathcal{O}^*)/N(S^*)$, the 2^{e_S}

distinct Pollack conjugation classes in the ±Eichler conjugation class of the quaternion $\mu \in \mathcal{P}(S,\mathcal{O})$ are

$$\mathcal{C}'_u = \{\alpha^{-1}\mu\alpha : \mathrm{n}(\alpha) \in uN(S^*)\} \cup \{-\alpha^{-1}\mu\alpha : \mathrm{n}(\alpha) \in -uN(S^*)\}. \quad \square$$

Remark 2.2.6. In view of Proposition 2.2.4, the effective computation of the number of Pollack conjugation classes $p(S,\mathcal{O})$ for arbitrary orders lies on the computability of the groups $N_{L/F}(S^*)$ and $\mathrm{n}(\mathcal{O}^*)$ and the number $e(S,\mathcal{O})$. The study of the former depends on the knowledge of the group of units S^* and there is abundant literature on the subject. If $\mathcal{O}$ is an Eichler order, the Hasse-Schilling-Maass Theorem in its integral version describes $\mathrm{n}(\mathcal{O}^*)$ in terms of the archimedean ramified places of B (cf. [Vi80]). Finally, we refer the reader to Section 1.2.3 for details on the computation of the number of Eichler conjugation classes of optimal embeddings.

Corollary 2.2.7. *Let $\mathcal{O}$ be an order in a division quaternion algebra B over F and let $-d \in R_F$ be not a perfect square. Then, the number of Pollack conjugation classes of pure quaternions $\mu \in \mathcal{O}_0$ of reduced norm $\mathrm{n}(\mu) = d$ is*

$$p(d,\mathcal{O}) = \sum_{S \supseteq R_F[\sqrt{-d}]} |\mathrm{n}(\mathcal{O}^*)/N_{L/F}(S^*)| \cdot \frac{e(S,\mathcal{O})}{2},$$

where S runs through the finite set of quadratic orders in $F(\sqrt{-d})$ which contain $R_F[\sqrt{-d}]$.

Chapter 3

Abelian varieties with quaternionic multiplication

Introduction

It is well known that an elliptic curve E over an arbitrary algebraically closed field always admits a unique principal polarization up to translations. This is in general no longer shared by higher dimensional abelian varieties. In fact, it is a delicate question to decide whether a given abelian variety A is principally polarizable. Even, if this is the case, it is an interesting problem to investigate the set $\Pi_0(A)$ of isomorphism classes of principal polarizations on A. By a theorem of Narasimhan and Nori [NaNo81], $\Pi_0(A)$ is a finite set. We shall denote its cardinality by $\pi_0(A)$.

The aim of this chapter is to study these questions on abelian varieties with quaternionic multiplication. It will be made apparent that the geometrical properties of these abelian varieties are encoded in the arithmetic of their ring of endomorphisms. As we will see in Chapter 4, our results shed some light on the geometry and arithmetic of the Shimura varieties that occur as moduli spaces of abelian varieties with quaternionic multiplication. Also, these are the basis of our study of the diophantine properties of abelian varieties with quaternion multiplication over number fields carried in Chapter 5. The results of this chapter are an expanded version of [Ro2].

3.1 Main results

A generic principally polarizable abelian variety admits a single class of principal polarizations. Humbert [Hu93] was the first to exhibit simple complex abelian surfaces with two nonisomorphic principal polarizations on them. Later, Hayashida and Nishi (cf. [HaNi65] and [Ha68]) computed $\pi_0(E_1 \times E_2)$ for isogenous elliptic curves $E_1/\mathbb{C}$ and $E_2/\mathbb{C}$ with complex multiplication. In positive characteristic, Ibukiyama, Katsura and Oort [IbKaOo86] related the number of principal polarizations on the power E^n of a supersingular elliptic curve to the class number of certain hermitian forms. Lange [La88] translated this problem into a number-theoretical one involving the arithmetic of the ring $\mathrm{End}(A)$ and produced examples of simple abelian varieties of high dimension with several principal polarizations. However, he showed that for an abelian variety with endomorphism algebra $\mathrm{End}(A) \otimes \mathbb{Q} = F$, a totally real number field, the number $\pi_0(A)$ is uniformly bounded in terms of the dimension of A: $\pi_0(A) \leq 2^{\dim(A)-1}$. That is, abelian varieties with real multiplication may admit several but not *arbitrarily many* principal polarizations.

It could be expected that Lange's or some other bound for $\pi_0(A)$ held for any simple abelian variety. Hence the

Question 3.1.1. Given $g \geq 1$, are there simple abelian varieties of dimension g with arbitrarily many nonisomorphic principal polarizations?

As was already observed, this is not the case in dimension 1. In $g = 2$, only simple abelian surfaces with at most $\pi_0(A) = 2$ were known. One of our main results, stated in a particular case, is the following.

Theorem 3.1.2. *Let F be a totally real number field of degree $[F : \mathbb{Q}] = n$, let R_F denote its ring of integers and $\vartheta_{F/\mathbb{Q}}$ the different of F over $\mathbb{Q}$. Let A be a complex abelian variety of dimension $2n$ whose ring of endomorphisms $\mathrm{End}(A) \simeq \mathcal{O}$ is a maximal order in a totally indefinite quaternion division algebra B over F.*

Assume that the narrow class number $h_+(F)$ of F is 1 and that $\vartheta_{F/\mathbb{Q}}$ and $\mathrm{disc}(B)$ are coprime ideals. Then,

1. *A is principally polarizable.*

2. *The number of isomorphism classes of principal polarizations on A is*

$$\pi_0(A) = \frac{1}{2} \sum_S h(S),$$

where S runs through the set of orders in the CM-field $F(\sqrt{-D})$ that contain $R_F[\sqrt{-D}]$, $D \in F_+^*$ is taken to be a totally positive generator of the reduced discriminant ideal $\mathcal{D}$ of B and $h(S)$ denotes its class number.

In particular, if A is an abelian surface,

$$\pi_0(A) = \begin{cases} \dfrac{h(-4D) + h(-D)}{2} & \text{if } D \equiv 3 \mod 4, \\ \dfrac{h(-4D)}{2} & \text{otherwise.} \end{cases}$$

We prove Theorem 3.1.2 in the more general form of Proposition 3.6.5 and our main Theorem 3.7.2. In order to accomplish it, we present an approach to the problem which stems from Shimura's classical work [Sh63] on analytic families of abelian varieties with prescribed endomorphism ring.

Our approach is essentially different to Lange's in [La88] or Ibukiyama-Katsura-Oort's in [IbKaOo86]. Indeed, whereas in [La88] and [IbKaOo86] the (noncanonical) interpretation of line bundles as symmetric endomorphisms is exploited, we translate the questions we are concerned with to Eichler's language of optimal embeddings. In particular, this leads us to solve a problem that has its roots in the work of O'Connor, Pall and Pollack (see [Po60]) and that has its own interest: see Section 2.2 for details.

In regard to the question above, the second main result of this chapter is the following.

Theorem 3.1.3. *Let g be a positive integer. Then*

1. *If g is even, there exist simple abelian varieties A of dimension g such that $\pi_0(A)$ is arbitrarily large.*

2. *If g is odd and square-free, $\pi_0(A) \leq 2^{g-1}$ for any simple abelian variety A of dimension g over $\mathbb{C}$.*

The boundless growth of $\pi_0(A)$ when g is even stems from our main Theorem 3.7.2 combined with analytical results on the asymptotic behaviour and explicit bounds for relative class numbers of CM-fields due to Horie-Horie [HoHo90] and Louboutin [Lo00], [Lo02]. The second part of Theorem 3.1.3 follows from the ideas of Lange in [La88]. The details of the proof are completed in Section 3.9.

The following corollary follows from Theorem 3.1.3 and the fact that any simple principally polarized abelian surface is the Jacobian of a smooth curve of genus 2 which, by Torelli's Theorem, is unique up to isomorphism.

Corollary 3.1.4. *There are arbitrarily large sets C_1, ..., C_N of pair-wise nonisomorphic genus 2 curves with isomorphic simple unpolarized Jacobian varieties $J(C_1) \simeq J(C_2) \simeq ... \simeq J(C_N)$.*

In view of Theorem 3.1.3, it is natural to wonder whether there exist arbitrarily large sets of pairwise nonisomorphic curves of given even genus $g \geq 4$ with isomorphic unpolarized Jacobian varieties. In this direction, Ciliberto and van der Geer [CivdGe92] proved the existence of two nonisomorphic curves of genus 4 whit isomorphic Jacobian varieties. Explicit examples of curves with isomorphic (nonsimple) Jacobians have been constructed by Howe [Ho00], while examples of pairs of distinct modular curves of genus 2 defined over $\mathbb{Q}$ with isomorphic unpolarized absolutely simple Jacobian varieties have been obtained in [GoGuRo02].

Finally, let us note that the statement of Theorem 3.1.3 does not cover odd non square-free dimensions.

Conjecture 3.1.5. *Let g be a non square free positive integer. Then there exist simple abelian varieties of dimension g such that $\pi_0(A)$ is arbitrarily large.*

The conjecture is motivated by the fact that, when g is non square-free, there exist abelian varieties whose ring of endomorphisms is an order in a noncommutative division algebra over a CM-field and there is a strong similitude between the arithmetic of the Néron-Severi groups of these abelian varieties and those in the quaternion case.

3.2 Abelian varieties with quaternionic multiplication

The main object of study of this chapter and the rest of the monograph will be the following class of abelian varieties.

Definition 3.2.1. Let k be a field and let $\bar{k}$ be an algebraic closure of k. An abelian variety A/k over k is an abelian variety with quaternionic multiplication (QM) if the following conditions are fulfilled:

(i) $\mathrm{End}_{\bar{k}}(A) \simeq \mathcal{O}$ is a maximal order in a totally indefinite quaternion algebra B over a totally real number field F.

(ii) $\dim(A) = 2[F : \mathbb{Q}]$.

Remark 3.2.2. In this chapter we focus our attention on abelian varieties with quaternionic multiplication over the field $\mathbb{C}$ of complex numbers. However, we make the above definition in greater generality since in the subsequent chapters we will consider abelian varieties with quaternionic multiplication over number fields.

Remark 3.2.3. We warn the reader that there are several and non coincident notions of quaternionic multiplication in the literature.

According to our definition, there do not exist abelian varieties with quaternionic multiplication over finite fields nor over the algebraic closure of a finite field. Indeed, this follows from the theory of Honda-Tate (cf. [Ta66], [Ta68]).

In particular, if we let A/K be an abelian variety with quaternionic multiplication over a number field K and we let $\wp$ be a prime ideal of K of good reduction of A, let k be the residue field of K at $\wp$ and let $\widetilde{A} = A \otimes \mathrm{Spec}\, k$. Then it holds that $\mathrm{End}_{\bar{K}}(A) \simeq \mathcal{O}$ is a maximal quaternion order, whereas $\mathrm{End}_{\bar{k}}(\widetilde{A}) \supsetneq \mathcal{O}$. Indeed, $\widetilde{A}$ fails to be simple and it isogenous to the product of two abelian varieties of dimension $\dim(A)/2$.

Let thus F be a totally real number field of degree $[F : \mathbb{Q}] = n$ and let R_F be its ring of integers. Let B denote a totally indefinite division quaternion algebra over F and let $\mathcal{D} = \mathrm{disc}(B) = \prod_{i=1}^{2r} \wp_i$, where $\wp_i$ are finite prime ideals of F and $r \geq 1$, be its reduced discriminant ideal.

We may fix an isomorphism of F-algebras

$$(\eta_\sigma) : B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \bigoplus_{\sigma} \mathrm{M}_2(\mathbb{R}^\sigma),$$

where σ runs in $\mathrm{Gal}(F/\mathbb{Q})$ and $\mathbb{R}^\sigma$ denotes $\mathbb{R}$ as a F-vector space via the immersion $\sigma : F \hookrightarrow \mathbb{R}$. For any $\beta \in B$, we will often abbreviate $\beta^\sigma = \eta_\sigma(\beta) \in \mathrm{M}_2(\mathbb{R})$.

Let $A/\mathbb{C}$ be an abelian variety with quaternionic multiplication by $\mathcal{O}$. As a complex manifold, $A(\mathbb{C}) \simeq V/\Lambda$ for V a complex vector space of dimension g and $\Lambda \subset V$ a co-compact lattice that may be identified with the first group of integral singular homology $H_1(A, \mathbb{Z})$. The lattice Λ is naturally a left $\mathcal{O}$-module and $\Lambda \otimes \mathbb{Q}$ is a left B-module of the same rank over $\mathbb{Q}$ as B. Since every left B-module is free (cf. [We67], §9), there is an element $v_0 \in V$ such that $\Lambda \otimes \mathbb{Q} = B \cdot v_0$ and therefore $\Lambda = \mathcal{I} \cdot v_0$ for some left $\mathcal{O}$-ideal $\mathcal{I} \subset B$.

Note that $\mathcal{I}$ is determined by A up to principal ideals and we can choose (and fix) a representative of $\mathcal{I}$ in its class in $\mathrm{Pic}_\ell(\mathcal{O})$ such that $\mathrm{n}(\mathcal{I}) \subset F$ is coprime with the discriminant $\mathcal{D}$. This is indeed possible because B is totally indefinite: it is a consequence of Proposition 1.2.5, Proposition 1.2.8 and the natural epimorphism of ray class groups $\mathrm{Pic}^{\mathcal{D}}(F) \to \mathrm{Pic}(F)$ of ideals of F (cf. [Ne99], §6).

Let $\rho_a : B \hookrightarrow \mathrm{End}(V) \simeq \mathrm{M}_{2n}(\mathbb{C})$ and $\rho_r : \mathcal{O} \hookrightarrow \mathrm{End}(\Lambda) \simeq \mathrm{M}_{4n}(\mathbb{Z})$ denote the analytic and rational representations of B and $\mathcal{O}$ on V and Λ, respectively. It is well known that $\rho_r \sim \rho_a \bigoplus \bar{\rho}_a$.

Lemma 3.2.4. *There exists a basis of the complex vector space V such that*

(i) $\rho_a(\beta) = \mathrm{diag}(\eta_{\sigma_i}(\beta)) \in \oplus_{i=1}^n \mathrm{M}_2(\mathbb{R}) \subset \mathrm{M}_{2n}(\mathbb{R})$ *for any* $\beta \in B$, *and*

(ii) The coordinates of v_0 *are* $(\tau_1, 1, ..., \tau_n, 1)$ *for certain* $\tau_i \in \mathbb{C}$, *Im*$(\tau_i) > 0$.

Proof. The existence of an appropiate basis $\{e_1, ..., e_{2n}\}$ of V such that the analytic representation ρ_a of B in V is of the form (i) is shown in [LaBi92], §9, 1.1. Note that the same holds if we replace the above basis for the basis $\{S \cdot e_1, ..., S \cdot e_{2n}\}$, for any invertible matrix $S \in \oplus_{i=1}^n \mathrm{M}_2(\mathbb{R}) \subset \mathrm{M}_{2n}(\mathbb{R})$. The lemma now follows from the fact that S can be chosen such that the coordinates of v_0 in the basis $\{S \cdot e_1, ..., S \cdot e_{2n}\}$ are of the form (ii). Indeed, this follows from linear algebra and the fact that $\mathcal{I} \cdot v_0 \subset V$ is a lattice of maximal rank. □

We note that the choice of the element v_0 fixes an isomorphism of real vector spaces $B \otimes \mathbb{R} \simeq V$.

Reciprocally, for any choice of a left $\mathcal{O}$-ideal $\mathcal{I}$ in B and a vector $v_0 \in V$, $v_0 = (\tau_1, 1, ..., \tau_n, 1)$ with $\mathrm{Im}(\tau_i) > 0$, we may consider the complex torus V/Λ with $\Lambda = \mathcal{I} \cdot v_0$ and B acting on V via the fixed diagonal analytic representation ρ_a. The torus V/Λ admits a polarization and can be embedded in a projective space (cf. [Sh63] and [LaBi92]). In consequence, it is the set of complex points of an abelian variety A such that $\mathrm{End}(A) \supseteq \mathcal{O}$.

Moreover, it holds that for the choice of v_0 in a dense subset of V, we exactly have $\mathrm{End}(A) = \mathcal{O}$. Besides, for v_0 in a subset of measure zero of V, A fails to be simple and it is isogenous to the product of two abelian varieties of dimension n with complex multiplication (cf. [Sh63], [LaBi92]).

3.3 The Néron-Severi group

We fix the following notation for this section. We let $A/\mathbb{C}$ denote an abelian variety with quaternionic multiplication by a maximal order $\mathcal{O}$ in B. In addition, we let $\mathcal{I}$ be a left ideal of $\mathcal{O}$ in B such that $H_1(A, \mathbb{Z}) \simeq \mathcal{I}$ as a left $\mathcal{O}$-module and $(\mathrm{n}(\mathcal{I}), \mathcal{D}) = 1$.

Let $\mathrm{NS}(A) = \mathrm{Pic}(A)/\mathrm{Pic}^0(A)$ be the Néron-Severi group of invertible sheaves on A up to algebraic equivalence. We recall from Section 1.3 that two invertible sheaves $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathrm{NS}(A)$ are isomorphic, denoted by $\mathcal{L}_1 \simeq \mathcal{L}_2$, if there is an automorphism $\alpha \in \mathrm{Aut}(A)$ such that $\mathcal{L}_2 = \alpha^*(\mathcal{L}_1)$. Let us also recall from Section 1.2 that for an arbitrary $\mathcal{O}$-left ideal I, we write $I^\sharp = \{\beta \in B : \mathrm{tr}_{B/\mathbb{Q}}(I\beta) \subseteq \mathbb{Z}\}$ and $\mathcal{N}(I) = \mathrm{n}(I)\mathcal{O} = I\overline{I}$.

The following theorem stems from Shimura's work [Sh63] and describes $\mathrm{NS}(A)$ intrinsically in terms of the arithmetic of B. The theorem establishes when two line bundles on A are isomorphic and translates this into a certain conjugation relation in B. We keep the notations as above.

Theorem 3.3.1. *There is a natural isomorphism of groups*

$$\begin{array}{cccc} c_1: & \mathrm{NS}(A) & \xrightarrow{\sim} & \mathcal{N}(\mathcal{I})^\sharp_0 \\ & \mathcal{L} & \mapsto & \mu = c_1(\mathcal{L}) \end{array}$$

between the Néron-Severi group of A and the group of pure quaternions of the codifferent of the two-sided ideal $\mathcal{N}(\mathcal{I})$.

Moreover, for any two invertible sheaves $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathrm{NS}(A)$, we have that $\mathcal{L}_1 \simeq \mathcal{L}_2$ if and only if there exists $\alpha \in \mathcal{O}^*$ such that $c_1(\mathcal{L}_2) = \overline{\alpha} c_1(\mathcal{L}_1)\alpha$.

Proof. By the Appell-Humbert Theorem (cf. [Mu70], [LaBi92]), the first Chern class allows us to interpret an invertible sheaf $\mathcal{L} \in \mathrm{NS}(A)$ as a Riemann form: an $\mathbb{R}$-alternate bilinear form $E_{\mathcal{L}} : V \times V \to \mathbb{R}$ such that $E_{\mathcal{L}}(\Lambda \times \Lambda) \subset \mathbb{Z}$ and $E_{\mathcal{L}}(\sqrt{-1}u, \sqrt{-1}v) = E(u, v)$ for all u, $v \in V$.

Fix an invertible sheaf $\mathcal{L}$ on A and let $E_{\mathcal{L}}$ be the corresponding Riemann form. The linear map $B \to \mathbb{Q}$, $\beta \mapsto E_{\mathcal{L}}(\beta v_0, v_0)$ is a trace form on B and hence, by the nondegeneracy of $\mathrm{tr}_{B/\mathbb{Q}}$, there is a unique element $\mu \in B$

such that $E_{\mathcal{L}}(\beta v_0, v_0) = \mathrm{tr}_{B/\mathbb{Q}}(\mu\beta)$ for any $\beta \in B$. Since $E_{\mathcal{L}}$ is alternate, $E_{\mathcal{L}}(av_0, av_0) = \mathrm{tr}_{F/\mathbb{Q}}(a^2\mathrm{tr}_{B/F}(\mu)) = 0$ for any $a \in F$. It follows again from the nondegeneracy of $\mathrm{tr}_{F/\mathbb{Q}}$ and the fact that the squares F^{*2} span F as a $\mathbb{Q}$-vector space that $\mathrm{tr}_{B/F}(\mu) = 0$. Thus $\mu^2 + \delta = 0$ for some $\delta \in F$.

The invertible sheaf $\mathcal{L}$ induces an anti-involution ϱ on B called the Rosati involution. It is characterized by the rule $E_{\mathcal{L}}(u, \beta v) = E_{\mathcal{L}}(\beta^{\varrho} u, v)$ for any $\beta \in B$ and u, $v \in V$. It follows that $\beta^{\varrho} = \mu^{-1}\overline{\beta}\mu$ and we conclude that the Riemann form $E_{\mathcal{L}}$ attached to the invertible sheaf $\mathcal{L}$ on A is $E_{\mathcal{L}} = E_{\mu} : V \times V \to \mathbb{R}$, $(u, v) \mapsto \mathrm{tr}_{B\otimes_{\mathbb{Q}}\mathbb{R}/\mathbb{R}}(\mu\beta^{\varrho}\gamma) = \mathrm{tr}_{B\otimes_{\mathbb{Q}}\mathbb{R}/\mathbb{R}}(\overline{\beta}\mu\gamma) = \mathrm{tr}_{B\otimes_{\mathbb{Q}}\mathbb{R}/\mathbb{R}}(\mu\gamma\overline{\beta})$, where $\mu \in B$ is determined as above and γ, β are elements in $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})^n$ such that $u = \gamma v_0$ and $v = \beta v_0$. Since $E_{\mathcal{L}}(\Lambda \times \Lambda) \subset \mathbb{Z}$ and $\mathrm{tr}(\mu) = 0$, we deduce that $\mu \in \mathcal{N}(\mathcal{I})_0^{\sharp}$.

Conversely, one checks that any element $\mu \in \mathcal{N}(\mathcal{I})_0^{\sharp}$ defines a Riemann form E_{μ} which is in turn the first Chern class of an invertible sheaf $\mathcal{L}$ on A. Indeed, since $\mu \in \mathcal{N}(\mathcal{I})^{\sharp}$, E_{μ} is integral over the lattice $\Lambda = \mathcal{I} \cdot v_0$ and E_{μ} is alternate because $\mathrm{tr}(\mu) = 0$. Moreover, let $\iota = \mathrm{diag}(\iota_1, ..., \iota_n) \in \mathrm{GL}_{2n}(\mathbb{R})$, $\iota_i \in \mathrm{GL}_2(\mathbb{R})$, $\iota_i^2 + 1 = 0$, be a matrix such that $\iota \cdot v_0 = \sqrt{-1}v_0$. Then $E_{\mu}(\sqrt{-1}u, \sqrt{-1}v) = E_{\mu}(\gamma\sqrt{-1}v_0, \beta\sqrt{-1}v_0) = \mathrm{tr}(\mu\gamma\iota\overline{\iota}\overline{\beta}) = E_{\mu}(u, v)$ for all u, $v \in V$. This concludes the proof of the first part of the theorem.

As for the second, we note that the first Chern class of the pull-back $\alpha^*\mathcal{L}$ of a line bundle $\mathcal{L}$ on A by an automorphism $\alpha \in \mathrm{Aut}(A) = \mathcal{O}^*$ is represented by the Riemann form $\alpha^* E_{\mathcal{L}} : V \times V \to \mathbb{R}$, $(u, v) \mapsto E_{\mathcal{L}}(\alpha u, \alpha v)$. Hence, if $\mathcal{L}_2 = \alpha^*(\mathcal{L}_1)$, then $\mathrm{tr}(\mu_2\gamma\overline{\beta}) = \mathrm{tr}(\mu_1\alpha\gamma\overline{\beta}\overline{\alpha}) = \mathrm{tr}(\overline{\alpha}\mu_1\alpha\gamma\overline{\beta})$ for all γ, $\beta \in B$ and this is satisfied if and only if $\mu_2 = \overline{\alpha}\mu_1\alpha$, by the nondegeneracy of the trace form. Reciprocally, one checks that if $\mu_2 = \overline{\alpha}\mu_1\alpha$ for some $\alpha \in \mathcal{O}^*$, then $E_{\mu_2} = \alpha^* E_{\mu_1}$ and therefore $\mathcal{L}_2 = \alpha^*\mathcal{L}_1$. $\square$

Remark 3.3.2. As we claimed in Section 3.1, the isomorphism between $\mathrm{NS}(A)$ and $\mathcal{N}(\mathcal{I})_0^{\sharp}$ is canonical in the sense that it does not depend on the choice of a possibly nonexisting principal polarization on A. However, we note that it does depend on the choice of an isomorphism between the quaternionic order $\mathcal{O}$ and the ring of endomorphisms $\mathrm{End}(A)$ of A.

According to Definition 2.2.1, the last statement of Theorem 3.3.1 can be rephrased by saying that two invertible sheaves $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathrm{NS}(A)$ are isomorphic if and only if $c_1(\mathcal{L}_1) \sim_p c_1(\mathcal{L}_2)$.

We now address the question whether the abelian variety A admits a principal bundle. We provide an explicit criterion in terms of the arithmetic of the order $\mathcal{O}$ and the left ideal $\mathcal{I}$. Crucial in the proof of the main result in this direction is the theory of Eichler on optimal embeddings (cf. Section 1.2.3).

Proposition 3.3.3. *Let $\mathcal{L}$ be an invertible sheaf on A and let $c_1(\mathcal{L}) = E_\mu$ be its first Chern class for some element $\mu \in B$, $\mu^2 + \delta = 0$, $\delta \in F$. Then*

$$\deg(\mathcal{L}) = \mathrm{N}_{F/\mathbb{Q}}(\vartheta_{F/\mathbb{Q}}^2 \cdot \mathrm{n}(\mathcal{I})^2 \cdot \mathcal{D} \cdot \delta).$$

Proof. The degree $\deg(\mathcal{L}) = \deg(\varphi_{\mathcal{L}})^{1/2}$ can be computed in terms of the Riemann form as follows:

$$\deg(\varphi_{\mathcal{L}}) = \det(E_\mu(x_i, x_j)) = \det(\mathrm{tr}_{B/\mathbb{Q}}(\mu\beta_i\overline{\beta_j})),$$

where $x_i = \beta_i v_0$ runs through a $\mathbb{Z}$-basis of the lattice Λ. Now, it holds that $\det(\mathrm{tr}_{B/\mathbb{Q}}(\mu\beta_i\overline{\beta_j})) = \mathrm{n}_{B/\mathbb{Q}}(\mu)^2 \det(\mathrm{tr}_{B/\mathbb{Q}}(\beta_i \cdot \overline{\beta_j})) = \mathrm{n}_{B/\mathbb{Q}}(\mu)^2 \mathrm{disc}_{B/\mathbb{Q}}(\mathcal{I})^2 = (\mathrm{N}_{F/\mathbb{Q}}(\delta \cdot \mathrm{n}(\mathcal{I})^2 \cdot \vartheta_{F/\mathbb{Q}}^2 \cdot \mathcal{D}))^2$ by Proposition 1.2.12. □

For the sake of simplicity and unless otherwise stated, we assume for the rest of the chapter the following

Assumption 3.3.4. The ideals $\vartheta_{F/\mathbb{Q}}$ and $\mathcal{D}$ are coprime.

The general case can be dealt by means of the remark below.

Theorem 3.3.5. *The abelian variety A admits a* principal *invertible sheaf if and only if the ideals $\mathcal{D}$ and $\vartheta_{F/\mathbb{Q}} \cdot \mathrm{n}(\mathcal{I})$ of F are* principal*.*

Proof. Let $\mathcal{L}$ be a principal invertible sheaf on A and let $E_\mu = c_1(\mathcal{L})$ be the associated Riemann form for some $\mu \in \mathcal{N}(\mathcal{I})_0^\sharp$, $\mu^2 + \delta = 0$. Since $\mathcal{L}$ is principal, the induced Rosati involution ϱ on $\mathrm{End}(A) \otimes \mathbb{Q} = B$ must also restrict to $\mathrm{End}(A) = \mathcal{O}$ and we already observed that $\beta^\varrho = \mu^{-1}\overline{\beta}\mu$. Therefore μ belongs to the normalizer group $\mathrm{Norm}_{B^*}(\mathcal{O})$ of $\mathcal{O}$ in B. The quotient $\mathrm{Norm}_{B^*}(\mathcal{O})/\mathcal{O}^* F^* \simeq W$ is a finite abelian 2-torsion group and representatives w of W in $\mathcal{O}$ can be chosen so that the reduced norms $\mathrm{n}(w) \in R_F$ are only divisible by the prime ideals $\wp | \mathcal{D}$ (cf. [Vi80], p. 39, 99, [Br90]). Hence, we may express $\mu = u \cdot t \cdot w^{-1}$ for some $u \in \mathcal{O}^*$, $t \in F^*$ and $w \in W$.

Recall that $(\mathrm{n}(\mathcal{I}), \mathcal{D}) = 1$ and $(\vartheta_{F/\mathbb{Q}}, \mathcal{D}) = 1$. Since, from Proposition 3.3.3, $\mathrm{n}(\mathcal{I})^2 \cdot \vartheta_{F/\mathbb{Q}}^2 \cdot \mathcal{D} = (\delta^{-1}) = (t^{-2} \cdot \mathrm{n}(w))$, we conclude that $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}} = (t^{-1})$ and $\mathcal{D} = (\mathrm{n}(w))$ are principal ideals.

Conversely, suppose now that $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}} = (t)$ and $\mathcal{D} = (D)$ are principal ideals, generated by some elements t, $D \in F^*$. Let R_L be the ring of integers in $L = F(\sqrt{-D})$. From Theorem 1.2.18, since any prime ideal $\wp | D$ ramifies in L, Eichler's theory of optimal embeddings guarantees the existence of an embedding $\iota : R_L \hookrightarrow \mathcal{O}$ of R_L into the quaternion order $\mathcal{O}$. Let $w = \iota(\sqrt{-D}) \in \mathcal{O}$ and let $\mu = (t \cdot w)^{-1}$. As one checks locally, $\mu \in \mathrm{Norm}_{B^*}(\mathcal{O}) \cap \mathcal{N}(\mathcal{I})_0^\sharp$ and, by Theorem 3.3.1 and Proposition 3.3.3, μ is the first Chern class of a principal invertible sheaf on A. □

Let us recall that an abelian variety A is self-dual if there is an isomorphism $\varphi : A \xrightarrow{\sim} \hat{A}$. Acoording to Section 1.3, if $\mathcal{L}$ is a principal invertible sheaf on A, then $\varphi_{\mathcal{L}} : A \to \hat{A}$ is an isomorphism. Conversely, it does not need to hold that any isomorphism between A and $\hat{A}$ is of this form. We obtain from the above the following

Corollary 3.3.6. *If $\mathcal{D}$ and $\vartheta_{F/\mathbb{Q}} \cdot \mathrm{n}(\mathcal{I})$ are principal ideals, then A is self-dual.*

Remark 3.3.7. The case when $\vartheta_{F/\mathbb{Q}}$ and $\mathcal{D}$ are non necessarily coprime is reformulated as follows: A admits a principal invertible sheaf if and only if there is an integral ideal $\mathfrak{a} = \wp_1^{e_1} \cdot ... \cdot \wp_{2r}^{e_{2r}} | \vartheta_{F/\mathbb{Q}}$ in F such that both $\mathcal{D} \cdot \mathfrak{a}^2$ and $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}} \cdot \mathfrak{a}^{-1}$ are principal ideals. In this case, A is also self-dual. The proof is *mutatis mutandi* the one given above.

Let us recall from Section 1.3 the finite set $\Pi(A)$ of isomorphism classes of principal invertible sheaves on A. Let

$$\mathcal{P}(\mathcal{O}) = \{\mu \in \mathcal{O} : \mathrm{tr}(\mu) = 0, \mathrm{n}(\mu) R_F = \mathcal{D}.\}$$

and let

$$P(\mathcal{O}) = \mathcal{P}(\mathcal{O}) / \sim_p,$$

where, as in Section 2.2, we let $\sim_p$ denote Pollack conjugation over $\mathcal{O}^*$. We note that $\mathcal{P}(\mathcal{O})$ is nonempty if and only if $\mathcal{D}$ is a principal ideal of R_F.

The above proof, together with Theorem 3.3.1, yield the following corollary, which shall serve us in the next section to compute the number of isomorphism classes of principal line bundles on A.

Corollary 3.3.8. *Let A be an abelian variety with quaternion multiplication by a maximal order $\mathcal{O}$. If $\mathcal{D} = (D)$ and $\vartheta_{F/\mathbb{Q}} \cdot \mathrm{n}(\mathcal{I}) = (t)$ are principal ideals, the assignation*

$$\mathcal{L} \mapsto (t \cdot c_1(\mathcal{L}))^{-1}$$

induces a bijection of sets between $\Pi(A)$ *and* $P(\mathcal{O})$.

3.4 Principal invertible sheaves and Eichler optimal embeddings

The aim of this section is to compute the cardinality $\pi(A) = |\Pi(A)|$ of the set of isomorphism classes of principal invertible sheaves on an abelian variety A with quaternionic multiplication by a maximal order $\mathcal{O}$ in a totally indefinite quaternion algebra B over a totally real number field F.

Theorem 3.4.1 below exhibits a close relation between $\pi(A)$ and the class number of F and of certain orders in quadratic extensions L/F that embed in B.

Theorem 3.4.1. *Let* $A/\mathbb{C}$ *be a principally polarizable abelian variety with quaternionic multiplication by a maximal order* $\mathcal{O}$ *in* B *over* F. *Let* $\mathcal{D} = \operatorname{disc}(B) = (D)$ *for some* $D \in F^*$ *and assume that* $(\mathcal{D}, \vartheta_{F/\mathbb{Q}}) = 1$. *Then the number of isomorphism classes of principal invertible sheaves on* A *is*

$$\pi(A) = \frac{1}{2h(F)} \sum_{u} \sum_{S_\mu} 2^{e_{S_\mu}} h(S_\mu),$$

where S_μ *runs through the finite set of orders in the number field* $F(\sqrt{-uD})$ *such that* $R_F[\sqrt{-uD}] \subseteq S_\mu$ *and* $u \in R_F^*/R_F^{*2}$ *runs through a set of representatives of units in* R_F *up to squares. Here,* $2^{e_{S_\mu}} = |R_F^*/N_{F(\sqrt{-uD})/F}(S_\mu^*)|$.

The proof of Theorem 3.4.1 will be completed during the rest of this section. Let us make before several remarks for the sake of its practical application.

Remark 3.4.2. First of all, by Dirichlet's Unit Theorem, $R_F^*/R_F^{*2} \simeq (\mathbb{Z}/2\mathbb{Z})^n$ and thus $e_S \leq n$. The case $F = \mathbb{Q}$ is trivial since $\mathbb{Z}^*/\mathbb{Z}^{*2} = \{\pm 1\}$. In the case of real quadratic fields F, explicit fundamental units $u \in R_F^*$ such that $R_F^*/R_F^{*2} = \{\pm 1, \pm u\}$ are classically well known (cf. [Ne99]). For higher degree totally real number fields there is abundant literature on systems of units. See for instance [La88] for an account.

Remark 3.4.3. Let $2R_F = \mathfrak{q}_1^{e_1} \cdot \ldots \cdot \mathfrak{q}_m^{e_m}$ be the decomposition of 2 into prime ideals in F. For any $u \in R_F^*$, the conductor $\mathfrak{f}$ of $R_F[\sqrt{-uD}]$ over R_F in L is

$$\mathfrak{f} = \prod_{\substack{\mathfrak{q}|2 \\ \mathfrak{q}\nmid D}} \mathfrak{q}^{a_\mathfrak{q}}, \quad 0 \leq a_\mathfrak{q} \leq e_\mathfrak{q}.$$

For a prime ideal $\mathfrak{q}|2$, $\mathfrak{q} \nmid D$, let π be a local uniformizer of the completion of F at $\mathfrak{q}$ and $k = \mathbb{F}_2^f$ be the residue field. Let $e = e_\mathfrak{q} \geq 1$. Since $-uD \in R^*_{F_\mathfrak{q}}$, we have that

$$-uD = x_0 + x_k\pi^k + x_{k+1}\pi^{k+1} + ...,$$

for some $1 \leq k \leq \infty$ and x_i running in a system of representatives of $\mathbb{F}_{2^f}$ in $R_{F_\mathfrak{q}}$ such that $\overline{x_0}$, $\overline{x_k} \neq 0$. Here, we agree to set $k = \infty$ in case that $-uD = x_0$.

It then holds that

$$\min\,([\frac{k}{2}], e) \leq a_\mathfrak{q} \leq e.$$

More precisely, we have

$$a_\mathfrak{q} = \begin{cases} [\frac{k}{2}] & \text{if } k \leq e+1, \\ e & \text{if } [\frac{k}{2}] \geq e. \end{cases}$$

If $[\frac{k}{2}] < e < k-1$, then the determination of $a_\mathfrak{q}$ depends on the choice of the system of representatives of $\mathbb{F}_{2^f}$ in $R_{F_\mathfrak{q}}$ and it deserves a closer inspection. This applies in many cases for deciding whether $R_F[\sqrt{-uD}]$ is the ring of integers of $F(\sqrt{-uD})$, i.e., $\mathfrak{f} = 1$.

In order to show the above formula, we note that the discriminant of $R_F[\sqrt{-uD}]$ over R_F is $-4uD$. Since $u \in R_F^*$ and D has square-free decomposition, $\mathfrak{f}|2$. Further, by [Se68], I, 6, $R_F[\sqrt{-uD}]$ is also maximal at the places $\wp_i|D$. Let thus $\mathfrak{q}|2$, $\mathfrak{q} \nmid D$. The ring of integers of $F_\mathfrak{q}(\sqrt{-uD})$ is $R_{F_\mathfrak{q}}[\alpha]$ for some $\alpha = v\pi^r + w\pi^s\sqrt{-uD}$ with v, $w \in R^*_{F_\mathfrak{q}}$ and r, $s \in \mathbb{Z}$. Then $\mathrm{Tr}\,(\alpha) = 2v\pi^r = v'\pi^{r+e}$, $v' \in R^*_{F_\mathfrak{q}}$, and $\mathrm{N}(\alpha) = v^2\pi^{2r} - Dw^2\pi^{2s}$. Since α must be integral, we obtain that $r = s = -a_\mathfrak{q}$ with $0 \leq a_\mathfrak{q} \leq e$. Computing the local expression of $v^2\pi^{2r} - Dw^2\pi^{2r}$, the rest of our claim follows.

In addition, let us note that the set of orders S in $L = F(\sqrt{-uD})$ that contain $R_F[\sqrt{-uD}]$ can be described as follows. Any order $S \supseteq R_F[\sqrt{-uD}]$ has conductor $\mathfrak{f}_S|\mathfrak{f}$ and for every ideal $\mathfrak{f}'|\mathfrak{f}$ there is a unique order $S \supseteq R_F[\sqrt{-uD}]$ of conductor $\mathfrak{f}'$. Further, $\mathfrak{f}_S|\mathfrak{f}_T$ if and only if $S \supseteq T$.

In order to prove Theorem 3.4.1, we begin by an equivalent formulation of it. As it was pointed out in Corollary 3.3.8, the first Chern class induces a bijection of sets between $\Pi(A)$ and the set of Pollack conjugation classes $P(\mathcal{O})$. As in Section 2.2, for any $u \in R_F^*$, we write $\mathcal{P}(-uD, \mathcal{O}) = \{\mu \in \mathcal{O} : \mu^2 + uD = 0\}$. Observe that $\mathcal{P}(\mathcal{O})$ is the disjoint union of the sets $\mathcal{P}(-u_k D, \mathcal{O})$ as u_k run through units in any set of representatives of R_F^*/R_F^{*2}.

As in Section 2.2, for any quadratic order S over R_F, let $\mathcal{P}(S, \mathcal{O})$ denote the set of optimal embeddings of S in $\mathcal{O}$ and $P(S, \mathcal{O}) = \mathcal{P}(S, \mathcal{O})/_{\sim_p}$. We obtain a natural identification of sets

$$P(\mathcal{O}) = \sqcup_k \sqcup_S P(S, \mathcal{O}),$$

where S runs through the set of quadratic orders $S \supseteq R_F[\sqrt{-u_k D}]$ for any unit u_k in a set of representatives of R_F^*/R_F^{*2}. Hence, in order to prove Theorem 3.4.1, it is enough to show that, for any quadratic order $S \supseteq R_F[\sqrt{-uD}]$, $u \in R_F^*$, it holds that

$$p(S, \mathcal{O}) := |P(S, \mathcal{O})| = \frac{2^{e_S - 1} h(S)}{h(F)}.$$

Since this question is interesting on its own, we have studied it separately in Chapter 2, where we have proved the above statement in greater generality. We are now in position to complete the

Proof of Theorem 3.4.1. Firstly, under the assumptions of Theorem 3.4.1, Proposition 1.2.5 asserts that $\mathrm{n}(\mathcal{O}^*) = R_F^*$. Secondly, we have that

$$e(S, \mathcal{O}) = \frac{h(S)}{h(F)}$$

for any $S \supseteq R_F[\sqrt{-uD}]$, $u \in R_F^*$. This follows from Eichler's Theorem 1.2.18 together with Remark 3.4.3. The combination of these facts together with the preceding discussion and Proposition 2.2.4 yield the theorem. □

3.5 The index of a nondegenerate invertible sheaf

Let $A/\mathbb{C}$ be an abelian variety with quaternionic multiplication by a maximal order $\mathcal{O}$ in B. We have that $A(\mathbb{C}) \simeq V/\Lambda$ for a complex vector space V and a lattice Λ. As in Lemma 3.2.4, we fix for the rest of the section a basis of V such that

(i) The analytic representation of any endomorphism $\beta \in \mathcal{O}$ is

$$\rho_a(\beta) = \text{diag}_{\sigma : F \hookrightarrow \mathbb{R}}(\beta^\sigma), \beta^\sigma \in \mathrm{M}_2(\mathbb{R})$$

and

(ii) $\Lambda = \mathcal{I} \cdot v_0$, where $\mathcal{I}$ is a left ideal of $\mathcal{O}$ such that $(\mathcal{D}, \mathrm{n}(\mathcal{I})) = 1$ and $v_0 = (\tau_1, 1, ..., \tau_n, 1) \in V$ with $\mathrm{Im}\ (\tau_i) > 0$.

We now compute the index of an invertible sheaf $\mathcal{L}$ on an abelian variety A with quaternionic multiplication in terms of the quaternion $\mu = c_1(\mathcal{L})$.

Lemma 3.5.1. *Let $\mu \in B_0$ be a pure quaternion and let $\mathrm{n}(\mu) = \delta \in F^*$. Then, for any immersion $\sigma : F \hookrightarrow \mathbb{R}$, there exists*

$$\nu_\sigma = \nu_\sigma(\mu) \in \mathrm{GL}_2(\mathbb{R})$$

such that $\nu_\sigma \mu^\sigma \nu_\sigma^{-1} = \omega_\sigma$, where

$$\omega_\sigma = \begin{cases} \begin{pmatrix} 0 & \sqrt{\sigma(\delta)} \\ -\sqrt{\sigma(\delta)} & 0 \end{pmatrix} & \text{if } \sigma(\delta) > 0, \\ \begin{pmatrix} \sqrt{\sigma(-\delta)} & 0 \\ 0 & -\sqrt{\sigma(-\delta)} \end{pmatrix} & \text{otherwise.} \end{cases}$$

Moreover, although ν_σ is not uniquely determined by μ, $\mathrm{sign}(\det(\nu_\sigma))$ *is.*

Proof. Since $\mu \in B \hookrightarrow B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R}) \oplus ... \oplus \mathrm{M}_2(\mathbb{R})$ satisfies that $\mu^2 + \delta = 0$, the lemma follows from linear algebra. □

Motivated by the following theorem, we have

Definition 3.5.2. (i) The signature of a nonzero pure quaternion $\mu \in B_0$ such that $\mu^2 + \delta = 0$ for $\delta \in F^*$ is

$$\mathrm{sgn}\ (\mu) = (\mathrm{sgn}\ (\det\ (\nu_\sigma))) \in \{\pm 1\}^n.$$

(ii) We say that a pure quaternion $\mu \in B^*$ is *ample* if $\mathrm{sgn}(\mu) = (1, ..., 1)$.

(iii) For any real immersion $\sigma : F \hookrightarrow \mathbb{R}$, we define the local archimedean index $i_\sigma(\mu)$ of μ by

$$i_\sigma(\mu) = \begin{cases} 0 & \text{if } \sigma(\delta) > 0 \text{ and } \det(\nu_\sigma) > 0, \\ 1 & \text{if } \sigma(\delta) < 0, \\ 2 & \text{if } \sigma(\delta) > 0 \text{ and } \det(\nu_\sigma) < 0. \end{cases}$$

Theorem 3.5.3. *Let $A/\mathbb{C}$ be an abelian variety with quaternionic multiplication by a maximal order $\mathcal{O}$.*

Let $\mathcal{L} \in \mathrm{NS}(A)$ be an invertible sheaf on A and let $\mu = c_1(\mathcal{L})$. Then, the index of $\mathcal{L}$ is

$$i(\mathcal{L}) = \sum_{\sigma : F \hookrightarrow \mathbb{R}} i_\sigma(\mu).$$

Proof. The index of $i(\mathcal{L})$ coincides with the number of negative eigenvalues of the hermitian form H_μ associated to the invertible sheaf $\mathcal{L}$. If we regard $\mathrm{M}_2(\mathbb{R}) \times \overset{n}{\dots} \times \mathrm{M}_2(\mathbb{R})$ embedded diagonally in $\mathrm{M}_{2n}(\mathbb{R})$, there is an isomorphism of real vector spaces between $B \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathrm{M}_2(\mathbb{R}) \times \overset{n}{\dots} \times \mathrm{M}_2(\mathbb{R})$ explicitly given by the map $\beta \mapsto \beta \cdot v_0$. The complex structure that $\mathrm{M}_2(\mathbb{R})^n$ inherits from that of V is such that $\{0\} \times \dots \times \mathrm{M}_2(\mathbb{R}) \times \dots \times \{0\}$ are complex vector subspaces of $\mathrm{M}_2(\mathbb{R})^n$ and we may choose a $\mathbb{C}$-basis of V of the form $\{\mathrm{diag}\,(\beta_1, 0, \dots, 0) \cdot v_0, \mathrm{diag}\,(\gamma_1, 0, \dots, 0) \cdot v_0, \dots, \mathrm{diag}\,(0, \dots, 0, \beta_n) \cdot v_0, \mathrm{diag}\,(0, \dots, \gamma_n) \cdot v_0\}$ for $\beta_i, \gamma_i \in \mathrm{M}_2(\mathbb{R})$.

Let $\iota = \mathrm{diag}(\iota_\sigma) \in \mathrm{GL}_{2n}(\mathbb{R})$ be such that $\iota \cdot v_0 = \sqrt{-1} v_0$. For any $\beta = \mathrm{diag}\,_\sigma(\beta_\sigma)$, we have that $\gamma = \mathrm{diag}\,_\sigma(\gamma_\sigma) \in \mathrm{M}_{2n}(\mathbb{R})$ and

$$H_\mu(\beta v_0, \gamma v_0) = \sum_\sigma \mathrm{tr}\,(\mu^\sigma \beta_\sigma \iota_\sigma \overline{\gamma_\sigma}) + \sqrt{-1} \sum_\sigma \mathrm{tr}\,(\mu^\sigma \beta_\sigma \overline{\gamma_\sigma}).$$

Thus, if we let $H_\sigma \in \mathrm{GL}_2(\mathbb{C})$ denote the restriction of H_μ to $V_\sigma = \mathrm{M}_2(\mathbb{R}) \cdot \begin{pmatrix} \tau_\sigma, & 1 \end{pmatrix}^t$, the matrix of H_μ respect to the chosen basis has diagonal form $H_\mu = \mathrm{diag}\,(H_\sigma)$.

In order to prove Theorem 3.5.3, it suffices to show that the hermitian form H_σ has $i_\sigma(\mu)$ negative eigenvalues. Take $\beta \in \mathrm{M}_2(\mathbb{R})$ and let $v = \beta \cdot \begin{pmatrix} \tau_\sigma, & 1 \end{pmatrix}^t \in V_\sigma$. Then, $H_\sigma(v, v) = \mathrm{tr}(\mu^\sigma \beta_\sigma \iota_\sigma \overline{\beta_\sigma}) = \mathrm{tr}(\omega_\sigma \beta'_\sigma \iota'_\sigma \overline{\beta'_\sigma})$, where $\beta'_\sigma = \nu_\sigma \beta_\sigma \nu_\sigma^{-1}$ and $\iota'_\sigma = \nu_\sigma \iota_\sigma \nu_\sigma^{-1}$. Denote $w_\sigma = \begin{pmatrix} w_1, & w_2 \end{pmatrix}^t = \nu_\sigma \beta_\sigma \cdot \begin{pmatrix} \tau_\sigma, & 1 \end{pmatrix}^t \in \mathbb{C}^2$ and $\|w_\sigma\|^2 = w_1 \overline{w_1} + w_2 \overline{w_2}$.

Some computation yields that

$$H_\sigma(v,v) = \sum_\sigma \frac{C_\sigma\sqrt{|\sigma(\delta)|}}{\det(\nu_\sigma)\mathrm{Im}(\tau_\sigma)},$$

where $C_\sigma = \|w\|^2$ if $\sigma(\delta) > 0$ and $C_\sigma = 2\mathrm{Re}(w_1\overline{w_2})$ if $\sigma(\delta) < 0$. From this, the result follows. □

Remark 3.5.4. From the above formula, the well known relation $i(\mathcal{L}) + i(\mathcal{L}^{-1}) = \dim A$ ([Mu70], Chapter III, Section 16, p. 150) is reobtained.

3.6 Principal polarizations and self-duality

We devote this section to consider the following

Question 3.6.1. Let A be an abelian variety and assume that it is self-dual, that is, $A \simeq \hat{A}$. Then, is A principally polarizable?

Let us note that in the generic case in which A is an abelian variety whose ring of endomorphisms is $\mathrm{End}(A) = \mathbb{Z}$, it holds that A is principally polarizable if and only if it is self-dual.

As in the previous sections, we let $\mathcal{O}$ denote a maximal order in a totally indefinite quaternion algebra B over a totally real field F and let $\mathcal{D} = \mathrm{disc}(B)$. Let $\mathcal{I}$ be an $\mathcal{O}$-left ideal such that $(\mathrm{n}(\mathcal{I}), \mathcal{D}) = 1$.

Let $\mathcal{O}^*_+$ be the subgroup of units in $\mathcal{O}$ of totally positive reduced norm. By Proposition 1.2.5, we have $\mathrm{n}(\mathcal{O}^*) = R^*_F$ and we let $\Sigma = \Sigma(R^*_F) \subseteq \{\pm 1\}^n$ be the $\mathbb{F}_2$-subspace of signatures of units in R^*_F. As $\mathbb{F}_2$-vector spaces, $\Sigma \simeq R^*_F/R^*_{F+}$ and, by Dirichlet's Unit Theorem, $|\Sigma| = \dfrac{2^n h(F)}{h_+(F)}$. Note that $h_+(F) = h(F)$ if and only if $\Sigma(R^*_F) = \{\pm 1\}^n$. By Proposition 1.2.5, the group Σ fits in the exact sequence

$$1 \to \mathcal{O}^*_+ \to \mathcal{O}^* \overset{\mathrm{sgn}\cdot\mathrm{n}}{\to} \Sigma \to 1$$

Proposition 3.6.2. *There exist principally polarizable abelian varieties $A/\mathbb{C}$ with quaternionic multiplication by $\mathcal{O}$ and $H_1(A,\mathbb{Z}) \simeq \mathcal{I}$ if and only if $\mathcal{D}$ and $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}}$ are principal ideals and $\mathcal{D} = (D)$ is generated by a totally positive element $D \in F^*_+$.*

Proof. Assume first that there exists a principally polarizable abelian variety $A/\mathbb{C}$ with quaternionic multiplication by $\mathcal{O}$ and $H_1(A,\mathbb{Z}) \simeq \mathcal{I}$. Then, by Theorem 3.3.5, $\mathcal{D}$ and $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}}$ are principal ideals. Let $D \in F^*$ be any generator of $(\mathcal{D})$. Let $\mathcal{L}$ be a principal polarization on A and let $\mu = c_1(\mathcal{L})$. By Corollary 3.3.8, $\mathrm{n}(\mu) = uD \in F^*/F^{*2}$ for some $u \in R_F^*$. Since $i(\mathcal{L}) = 0$, it follows from Theorem 3.5.3 that $uD \in F_+^*$.

Conversely, we assume that $\mathcal{D} = (D)$ and $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}} = (t)$ for some $D \in F_+^*$ and $t \in F^*$. Since, from Theorem 3.3.5, A admits a principal line bundle, it follows from Corollary 3.3.8 that the set $\mathcal{P}(\mathcal{O})$ is nonempty. Let $\mu \in \mathcal{P}(\mathcal{O})$ and let $S = \mathrm{diag}\,_{\sigma:F\hookrightarrow\mathbb{R}}(\begin{pmatrix}\mathrm{sgn}\,(\mu^\sigma) & 0\\ 0 & 1\end{pmatrix})$.

Let us fix an immersion $B \hookrightarrow B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R}) \times \overset{n}{\cdots} \times \mathrm{M}_2(\mathbb{R}) \hookrightarrow \mathrm{M}_{2n}(\mathbb{R})$, $\beta \mapsto \mathrm{diag}\,_{\sigma:F\hookrightarrow\mathbb{R}}(\beta^\sigma)$, by embedding diagonally $\mathrm{M}_2(\mathbb{R}) \times \overset{n}{\cdots} \times \mathrm{M}_2(\mathbb{R})$ in $\mathrm{M}_{2n}(\mathbb{R})$. Upon conjugating the embedding $B \hookrightarrow \mathrm{M}_{2n}(\mathbb{R})$ by S, we can assume that $\mathrm{sgn}(\mu) = (1, ..., 1)$. Let V be a complex vector space of dimension $2n$ and let B act on V through the above embedding. Let $\mathfrak{H}$ denote Poincaré's upper half plane. For any choice of $(\tau_1, ..., \tau_n)$ in a dense subset of $\mathfrak{H} \times ...\mathfrak{H}$, the complex torus V/Λ, $\Lambda = \mathcal{I} \cdot v_0$, $v_0 = (\tau_1, 1, ..., \tau_n, 1)$, is the set of complex points of an abelian variety $A/\mathbb{C}$ with quaternionic multiplication by $\mathcal{O}$ such that $H_1(A,\mathbb{Z}) \simeq \mathcal{I}$ as left $\mathcal{O}$-modules. Since conditions (i) and (ii) of Lemma 3.2.4 hold for A, we deduce from Theorem 3.5.3 that A is principally polarizable. □

Remark 3.6.3. As a consequence of the above proposition, we obtain that there exist *self-dual but non principally polarizable abelian varieties*. Indeed, let $F = \mathbb{Q}(\sqrt{3})$ and let $\mathcal{O}$ be a maximal order in the quaternion algebra B of discriminant $\mathcal{D} = 2\sqrt{3}$. Let $\mathcal{I} = \mathcal{O}$. Since $\mathcal{D}$ and $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}}$ are principal ideals but the former can be generated by no totally real elements, it follows from Corollary 3.3.6 and Proposition 3.6.2 that any abelian variety with quaternionic multiplication by $\mathcal{O}$ and such that $H_1(A,\mathbb{Z}) \simeq \mathcal{O}$ is self-dual but non principally polarizable. These abelian varieties indeed exist, as we have shown at the end of Section 3.2.

Let now $A/\mathbb{C}$ be an abelian variety with quaternionic multiplication by $\mathcal{O}$. We choose $\mathcal{I}$ to be a $\mathcal{O}$-left ideal such that $H_1(A,\mathbb{Z}) \simeq \mathcal{I}$ as $\mathcal{O}$-left modules and $(\mathrm{n}(\mathcal{I}), \mathcal{D}) = 1$. We fix a complex vector space V and a vector $v_0 \in V$ such that $A(\mathbb{C}) = V/\Lambda$ with $\Lambda = \mathcal{I} \cdot v_0$ under the same conditions of Lemma 3.2.4.

In regard to the Question 3.6.1, note that, from Corollary 3.3.6, a sufficient condition for A to be self-dual is that $\mathcal{D}$ and $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}}$ are principal ideals. By Corollary 3.3.8 and Theorem 3.5.3, a necessary condition for A to be principally polarizable is that $\mathcal{D}$ be generated by a totally positive element D in F. However, in general these conditions are not sufficient for the existence of a principal polarization on A.

Definition 3.6.4. We denote by $\Omega \subseteq \{\pm 1\}^n$ the set of signatures

$$\Omega = \{(\mathrm{sgn}\ (\det\ \nu_\sigma(\mu))\}_{\mu \in \mathcal{P}(\mathcal{O})}.$$

The set Ω can be identified with a subset of the set of connected components of $\mathbb{R}^n \setminus \cup_{i=1}^n \{x_i = 0\}$. With the notations as above, we obtain the following corollary of Theorems 3.3.5 and 3.5.3.

Proposition 3.6.5. *A complex abelian variety $A/\mathbb{C}$ with quaternionic multiplication by $\mathcal{O}$ and $H_1(A, \mathbb{Z}) \simeq \mathcal{I}$ admits a principal polarization if and only if $\mathcal{D}$ and $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}}$ are principal ideals, $\mathcal{D} = (D)$ is generated by a totally positive element $D \in F_+^*$ and $(1, ..., 1) \in \Omega$.*

Signature questions on number fields are delicate. In order to have a better understanding of Proposition 3.6.5, we describe Ω as the union (as sets) of linear varieties in the affine space $\mathbb{A}^n_{\mathbb{F}_2} = \{\pm 1\}^n$ as follows. Let $\{u_k\}$ be a set of representatives of units in R_F^*/R_F^{*2} and, for any order $S \supseteq R_F[\sqrt{-u_k D}]$ in $L = F(\sqrt{-u_k D})$, choose $\mu_S \in \mathcal{P}(S, \mathcal{O})$. We considered in Section 2.2 the Galois group $\Delta = \mathrm{Ker}(N : \mathrm{Pic}(S) \to \mathrm{Pic}(F))$. Naturally associated to it there is a sub-space of signatures $\Sigma(\Delta)$ in the quotient space $\mathbb{A}^n_{\mathbb{F}_2}/\Sigma(R_F^*)$ as follows: if $\mathfrak{b}$ is an ideal of S such that $N_{L/F}(\mathfrak{b}) = (b)$ for some $b \in F^*$, the signature of b does not depend on the class of $\mathfrak{b}$ in $\mathrm{Pic}(S)$ and only depends on the choice of the generator b up to signatures in $\Sigma(R_F^*)$. By an abuse of notation, we still denote by $\Sigma(\Delta)$ the sub-space of $\mathbb{A}^n_{\mathbb{F}_2}$ generated by $\Sigma(R_F^*)$ and the signatures of the norms of ideals in Δ. Then, from Proposition 1.2.19 we obtain that

$$\Omega = \bigcup_{k,S} \Sigma(\Delta) \cdot \mathrm{sgn}(\mu_S).$$

This allows us to compute Ω in many explicit examples and to show that, in many cases, the set Ω coincides with the whole space of signatures $\{\pm 1\}^n$. The following corollary, which remains valid even if we remove the assumption $(\vartheta_{F/\mathbb{Q}}, \mathcal{D}) = 1$, illustrates this fact.

Corollary 3.6.6. *Assume that*

(i) $\mathcal{D} = (D)$ *for some* $D \in F_+^*$ *and that* $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}} = (t)$ *for some* $t \in F^*$.

(ii) $h_+(F) = h(F)$.

Then, any abelian variety $A/\mathbb{C}$ *with quaternionic multiplication by* $\mathcal{O}$ *and* $H_1(A, \mathbb{Z}) \simeq \mathcal{I}$ *is principally polarizable.*

In particular, if $h_+(F) = 1$*, then the above two conditions on* $\mathcal{O}$ *and* $\mathcal{I}$ *are always accomplished.*

Proof. Since $\Sigma(R_F^*) = \{\pm 1\}^n$, we have that $\Omega = \{\pm 1\}^n$. The result follows from Proposition 3.6.5. □

As we shall see in Chapter 4, this is highly relevant in the study of certain *Shimura varieties*. As was already known to the specialists in dimension 2, we obtain that *any* abelian surface with quaternionic multiplication by a maximal order in an indefinite quaternion algebra $B/\mathbb{Q}$ admits a principal polarization. It can actually be shown that any abelian surface with multiplication by an hereditary quaternion order admits a principal polarization (cf. [Ro2]). This extends our result to a wider class of quaternion orders.

3.7 The number of isomorphism classes of principal polarizations

Let A be a complex abelian variety with quaternion multiplication by a maximal order $\mathcal{O}$. Let V be a complex vector space of dimension $2n$ and let $B \hookrightarrow \oplus_{i=1}^n \mathrm{M}_2(\mathbb{R}) \subset \mathrm{M}_{2n}(\mathbb{R})$ act on V diagonally in 2×2 boxes as in Lemma 3.2.4 in Section 3.2. Let $v_0 = (\tau_1, 1, ..., \tau_n, 1) \in V$, $\mathrm{Im}\,(\tau_i) > 0$, be such that $A(\mathbb{C}) = V/\Lambda$ for $\Lambda = \mathcal{I} \cdot v_0$. As they were introduced in Section 1.3, for any integer $0 \leq i \leq g$, the set $\Pi_i(A)$ denotes the set of isomorphism classes of principal invertible sheaves $\mathcal{L} \in \mathrm{NS}(A)$ of index $i(\mathcal{L}) = i$. The set $\Pi(A)$ naturally splits as the disjoint union $\Pi(A) = \sqcup \Pi_i(A)$. Moreover, we recall that due to the relation $i(\mathcal{L}) + i(\mathcal{L}^{-1}) = g$, the map $\mathcal{L} \mapsto \mathcal{L}^{-1}$ induces a one-to-one correspondence between $\Pi_i(A)$ and $\Pi_{g-i}(A)$ and therefore $\pi_i(A) = \pi_{g-i}(A)$.

Formulas for $\pi_i(A)$, $0 \leq i \leq g$, analogous to that of Theorem 3.4.1 can be derived. Due to its significance, we will only concentrate on the number $\pi_0(A)$ of classes of principal polarizations. The Galois action on the sets

$E(S,\mathcal{O})$ of Eichler classes of optimal embeddings and its behaviour respect to the index of the associated invertible sheaves will play an important role.

Assume then that $\Pi_0(A) \neq \emptyset$. For simplicity, recall that we also assume that $(\vartheta_{F/\mathbb{Q}}, \mathcal{D}) = 1$. By Proposition 3.6.5, we may choose $D \in F_+^*$ and $t \in F^*$ such that $\mathcal{D} = (D)$ and $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}} = (t)$. With these notations, we introduce the following

Definition 3.7.1. Let $u \in R_{F+}^*$ be a totally positive unit. An order $S \supseteq R_F[\sqrt{-uD}]$ is *ample* respect to A if there exists an optimal embedding $i : S \hookrightarrow \mathcal{O}$ such that $\mu = i(\sqrt{-uD})$ is ample. We define $\mathcal{S}_u$ to be the set of ample orders $S \supseteq R_F[\sqrt{-uD}]$ in $F(\sqrt{-uD})$.

The existence of a principal polarization $\mathcal{L}$ on A implies that there is at least some $\mathcal{S}_u$ nonempty. With these notations, we obtain the following expression for $\pi_0(A)$ in terms of the narrow class number of F and the class numbers of certain CM-fields that embed in B.

Theorem 3.7.2. *The number of isomorphism classes of principal polarizations on A is*

$$\pi_0(A) = \frac{1}{2h_+(F)} \sum_{u \in R_{F_+}/R_F^{*2}} \sum_{S \in \mathcal{S}_u} 2^{e_S^+} h(S),$$

where $2^{e_S^+} = |R_{F+}^*/N(S^*)|$.

Proof. By the existing duality between $\Pi_0(A)$ and $\Pi_g(A)$, it is equivalent to show that $\pi_0(A) + \pi_g(A) = \sum_u \sum_{S \in \mathcal{S}_u} 2^{e_S^+} h(S)/h_+(F)$.

Let us introduce the set $\mathcal{P}_{0,g}(\mathcal{O}) = \{\mu \in \mathcal{O} : \mathrm{n}(\mu) \in R_{F+}^* \cdot D, \mathrm{sgn}(\mu) = \pm(1, ..., 1)\}$. By Theorems 3.3.1 and 3.5.3, the set $P_{0,g}(\mathcal{O}) = \mathcal{P}_{0,g}(\mathcal{O})^{\pm}/_{\sim_p}$ is in one-to-one correspondence with $\Pi_0(A) \cup \Pi_g(A)$ and we have a natural decomposition $P_{0,g}(\mathcal{O}) = \sqcup P_{0,g}(S,\mathcal{O})$ as S runs through ample orders in $\mathcal{S}_u$ and $u \in R_{F+}^*/R_F^{*2}$.

Fix $u \in R_{F+}^*$ and S in $\mathcal{S}_u$. In order to compute the cardinality of $P_{0,g}(S,\mathcal{O})$, we relate it to the set $E_{0,g}(S,\mathcal{O}) = \mathcal{P}_{0,g}(S,\mathcal{O})/_{\sim_e}$ of $\mathcal{O}_\pm^*$-Eichler conjugacy classes of optimal embeddings $i_\mu : S \hookrightarrow \mathcal{O}$. Here, we agree to say that two quaternions μ_1 and $\mu_2 \in \mathcal{P}_{0,g}(S,\mathcal{O})$ are Eichler conjugate by $\mathcal{O}_\pm^*$ if there is a unit $\alpha \in \mathcal{O}_\pm^*$ of either totally positive or totally negative reduced norm such that $\mu_2 = \alpha^{-1}\mu_1\alpha$. Note that, by Theorem 3.5.3, the action of $\mathcal{O}_\pm^*$-conjugation on the invertible sheaf $\mathcal{L}$ associated to an element in $\mathcal{P}(S,\mathcal{O})$

either preserves the index $i(\mathcal{L})$ or switches it to $g - i(\mathcal{L})$. This makes sense of the quotient $E_{0,g}(S, \mathcal{O})$.

We have the following exact diagram:

$$\begin{array}{ccccccccc} 0 & \to & \Delta_+ & \to & \mathrm{Pic}(S) & \stackrel{N_{L/F}}{\to} & \mathrm{Pic}_+(F) & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \to & \Delta & \to & \mathrm{Pic}(S) & \stackrel{N_{L/F}}{\to} & \mathrm{Pic}(F) & \to & 0. \end{array}$$

Indeed, there is a natural map $\mathrm{Pic}(S) \stackrel{N_{L/F}}{\to} \mathrm{Pic}_+(F)$, since the norm of an element $a + b\sqrt{-uD} \in L$ for a, $b \in F$ is $a^2 + ub^2D \in F_+^*$. The surjectivity of the map $\mathrm{Pic}(S) \to \mathrm{Pic}_+(F)$ is argued as in Section 2.2 by replacing the Hilbert class field H_F of F by the big Hilbert class field H_F^+, whose Galois group over F is $\mathrm{Gal}(H_F^+/F) = \mathrm{Pic}_+(F)$. By Proposition 1.2.19, Δ acts freely and transitively on $E(S, \mathcal{O})$. Therefore, by Theorem 3.5.3, there is also a free action of Δ_+ on $E_{0,g}(S, \mathcal{O})$. Up to sign, the $\mathcal{O}_\pm^*$-Eichler conjugation class of an element $\mu \in \mathcal{P}(S, \mathcal{O})$ has a well defined signature $\pm\mathrm{sgn}(\mu)$. Note also that two inequivalent $\mathcal{O}_\pm^*$-Eichler classes that fall in the same $\mathcal{O}^*$-conjugation class never have the same signature, even not up to sign. Taken together, this shows that Δ_+ also acts transitively on $E_{0,g}(S, \mathcal{O})$. This means that

$$|E_{0,g}(S, \mathcal{O})| = \frac{h(S)}{h_+(F)}.$$

Arguing as in Section 3.4, Theorem 3.7.2 follows. □

3.8 Examples in low dimensions

3.8.1 Dimension 2

The simplest examples to be considered are abelian surfaces with quaternionic multiplication. Let B be an indefinite division quaternion algebra over $\mathbb{Q}$ of discriminant $D = p_1 \cdot \ldots \cdot p_{2r}$, for p_i prime numbers and $r \geq 1$. Let $\mathcal{O} \subset B$ be a maximal order in B. Since $h(\mathbb{Q}) = 1$, $\mathrm{Pic}_\ell(\mathcal{O})$ is trivial by Proposition 1.2.8. Thus, all left $\mathcal{O}$-ideals are principal. Moreover, the maximal order $\mathcal{O}$ is unique up to conjugation.

By Corollary 3.6.6, any abelian surface A with $\mathrm{End}(A) = \mathcal{O}$ admits a principal polarization. In fact, A admits principal line bundles of each index

0, 1 and 2. In order to compute $\pi_i(A)$ for $i = 0, 1$ and 2 we may use Theorems 3.4.1 and 3.7.2.

Let $d \in \mathbb{Z}\backslash\mathbb{Z}^2$. As is well known, the quadratic orders in $\mathbb{Q}(\sqrt{d})$ containing $\mathbb{Z}[\sqrt{d}]$ are $\mathbb{Z}[\sqrt{d}]$ and $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ if $d \equiv 1(\mod 4)$ or $\mathbb{Z}[\sqrt{d}]$ if $d \not\equiv 1(\mod 4)$. Furthermore, for $S = \mathbb{Z}[\sqrt{d}]$ or $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$,

$$e_S = \begin{cases} 0 & \text{if } d > 0 \text{ and there are units of negative norm in } S, \\ 1 & \text{otherwise.} \end{cases}$$

and $e_S^+ = 0$ in any case. We conclude that

$$\pi_0(A) = \pi_2(A) = \begin{cases} \frac{h(-4D)+h(-D)}{2} & \text{if } D \equiv 3 \quad (\mod 4), \\ \frac{h(-4D)}{2} & \text{otherwise} \end{cases}$$

and

$$\pi_1(A) = \begin{cases} \varepsilon_{4D} h(4D) + \varepsilon_D h(D) & \text{if } D \equiv 1 \quad (\mod 4), \\ \varepsilon_{4D} h(4D) & \text{otherwise,} \end{cases}$$

where ε_D, $\varepsilon_{4D} = 1$ or $\frac{1}{2}$ is computed from the above formula for e_S.

As a simple example, the number of isomorphism classes of principal polarizations on an abelian surface A with QM by a maximal order in a quaternion algebra of discriminant $D = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ is $\pi_0(A) = 1040$. This also implies the existence of 1040 pair-wise nonisomorphic smooth algebraic curves $\{C_1, ..., C_{1040}\}$ of genus 2 such that their respective Jacobian varieties are isomorphic as abelian surfaces: $J(C_1) \simeq \cdots \simeq J(C_{1040})$.

Using a programming package like PARI (cf. [PA89]) and our results, we obtain the following table for the numbers of isomorphism classes of principal invertible sheaves of index 0, 1 and 2 respectively on an abelian surface with QM by a maximal order in a quaternion algebra of discriminant D. Recall that $\pi(A) = \pi_0(A) + \pi_1(A) + \pi_2(A)$ and that $\pi_0(A) = \pi_2(A)$.

Table 3.8.1: Principal invertible sheaves on abelian surfaces with QM								
Disc (B)	$\pi_0(A)$	$\pi_1(A)$	$\pi_2(A)$		Disc (B)	$\pi_0(A)$	$\pi_1(A)$	$\pi_2(A)$
6	1	1	1		111	8	2	8
10	1	1	1		115	4	2	4
14	2	1	2		118	3	1	3
15	2	2	2		119	10	2	10
21	2	2	2		122	5	1	5
22	1	1	1		123	4	2	4
26	3	1	3		129	6	2	6
33	2	2	2		133	2	2	2
34	2	2	2		134	7	1	7
35	4	2	4		141	4	4	4
38	3	1	3		142	2	3	2
39	4	2	4		143	10	2	10
46	2	1	2		145	4	4	4
51	4	2	4		146	8	2	8
55	4	2	4		155	8	2	8
57	2	2	2		158	4	1	4
58	1	1	1		159	10	2	10
62	4	1	4		161	8	2	8
65	4	2	4		166	5	1	5
69	4	2	4		177	2	2	2
74	5	1	5		178	4	2	4
77	4	2	4		183	8	2	8
82	2	2	2		185	8	2	8
85	2	2	2		187	4	2	4
86	5	1	5		194	10	2	10
87	6	2	6		201	6	2	6
91	4	2	4		202	3	1	3
93	2	2	2		203	8	2	8
94	4	1	4		205	4	4	4
95	8	2	8		206	10	1	10
106	3	1	3		209	10	2	10

3.8.2 Dimension 4

Let F be the real quadratic field $\mathbb{Q}(\sqrt{2})$. Its ring of integers is $\mathbb{Z}[\sqrt{2}]$ and all its ideals are principal and can be generated by totally positive elements. That is, the narrow class number of F is $h_+(F) = 1$. The group of units is $\mathbb{Z}[\sqrt{2}]^* = \{\pm(1+\sqrt{2})^n\}_{n\in\mathbb{Z}}$. Representatives for R_F^*/R_F^{*2} are $\{\pm 1, \pm(1+\sqrt{2})\}$.

Let B be the quaternion algebra over F that ramifies exactly at the two prime ideals $(3 \pm \sqrt{2})$ above 7. By the second remark to Theorem 3.4.1, the orders $S_\pm = \mathbb{Z}[\sqrt{2}, \sqrt{\pm 7(1+\sqrt{2})}]$ are the maximal rings of integers in their respective quotient fields. With the help of the programming package PARI we have that the class number is $h = 2$ and $|\mathbb{Z}[\sqrt{2}]^*/N(S_\pm^*)| = 4$ in both cases.

A similar inspection yields that the conductor of the $\mathbb{Z}[\sqrt{2}, \sqrt{7}]$ in its quotient fields is the dyadic ideal $\mathfrak{f} = (\sqrt{2})$. This means that the maximal ring of integers properly contains this order but also that there are no intermediate orders between them.

The ring of integers of $\mathbb{Q}(\sqrt{2}, \sqrt{7})$ is $\widetilde{S} = \mathbb{Z}[\sqrt{2}, \frac{3+\sqrt{7}}{\sqrt{2}}]$ and its class number is $h(\widetilde{S}) = 1$. Generators of the unit group, up to torsion, are $1 + \sqrt{2}$, $\frac{3+\sqrt{7}}{\sqrt{2}}$ and $2\sqrt{2} + \sqrt{7}$. Computing their norm over $\mathbb{Q}(\sqrt{2})$ we obtain that $|\mathbb{Z}[\sqrt{2}]^*/N(\widetilde{S}^*)| = 4$.

We can compute the class number of $S = \mathbb{Z}[\sqrt{2}, \sqrt{7}]$ in terms of that of the field $\mathbb{Q}(\sqrt{2}, \sqrt{7})$ by the formula given in Section 1.1. We have that $h(S) = 1$ and $|\mathbb{Z}[\sqrt{2}]^*/N(S^*)| = 4$.

Finally, the ring of integers of $\mathbb{Q}(\sqrt{2}, \sqrt{-7})$ is $\widetilde{S} = \mathbb{Z}[\sqrt{2}, \frac{1-2\sqrt{2}-\sqrt{-7}}{2}]$ and the conductor of $S = \mathbb{Z}[\sqrt{2}, \sqrt{-7}]$ is $\mathfrak{f} = 2$. Hence there is exactly one order $\hat{S}$ that sits between S and $\widetilde{S}$. We have that $\widetilde{S}^* = \{\pm(1+\sqrt{2})^n\}_{n\in\mathbb{Z}}$, $h(S) = h(\hat{S}) = h(\widetilde{S}) = 2$ and $e_S = e_{\hat{S}} = e_{\widetilde{S}} = 2$.

By applying Theorems 3.4.1, 3.5.3 and 3.7.2, we conclude that for any abelian variety A of dimension 4 such that $\mathrm{End}(A)$ is a maximal order in the quaternion algebra $B/\mathbb{Q}(\sqrt{2})$ of discriminant 7, the number of isomorphism classes of principal line bundles of index 0, 1, 2, 3 and 4 respectively are:

$\pi_0(A)$	$\pi_1(A)$	$\pi_2(A)$	$\pi_3(A)$	$\pi_4(A)$
6	4	4	4	6

3.9 Asymptotic behaviour of $\pi_0(A)$

In this section, we combine Theorem 3.7.2 with analytical tools to estimate the asymptotic behaviour of $\log(\pi_0(A))$. This will yield a stronger version of part 1 of Theorem 3.1.3 in the introduction.

For any number field L, we let D_L and Reg_L stand for the absolute value of the discriminant and the regulator, respectively. For any two sequences of real numbers $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, we write $\{a_n\} \sim \{b_n\}$ if and only if $\lim_{n\to\infty}\dfrac{a_n}{b_n} = 1$.

Theorem 3.9.1. *Let F be a totally real number field of degree n. Let A range over a sequence of principally polarizable abelian varieties of dimension $2n$ with* $\mathrm{End}(A)$ *a maximal order in a quaternion algebra B/F of discriminant $D \in F_+^*$ with $|N_{F/\mathbb{Q}}(D)| \to \infty$. Then*

$$\log \pi_0(A) \sim \log \sqrt{|N_{F/\mathbb{Q}}(D)| \cdot D_F}.$$

The proof of Theorem 3.9.1 adapts an argument of Horie-Horie ([HoHo90]) on estimates of relative class numbers of CM-fields. We first show that there indeed exist families of abelian varieties satisfying the properties quoted in the theorem.

By Čebotarev's Density Theorem, we can find infinitely many pairwise different totally positive principal prime ideals $\{\wp_i\}_{i\geq 1}$ in F. We can also choose them such that $(\wp_i, \vartheta_{F/\mathbb{Q}}) = 1$. We then obtain principal ideals $(D_j) = \wp_1 \cdot \wp_2 \cdot ... \cdot \wp_{2j-1} \cdot \wp_{2j}$ with $D_j \in F_+^*$ and $(D_j, \vartheta_{F/\mathbb{Q}}) = 1$.

According to [Vi80], p. 74, there exists a totally indefinite quaternion algebra B_j over F of discriminant D_j for any $j \geq 1$. Then, Proposition 3.6.2 asserts that there exists an abelian variety A_j of dimension $2n$ such that $\mathrm{End}(A_j)$ is a maximal order in B_j and $\Pi_0(A_j) \neq \emptyset$.

Proof of Theorem 3.9.1. Let A be a principally polarizable abelian variety with quaternion multiplication by a maximal order in a totally indefinite division quaternion algebra B over F of discriminant $D \in F_+^*$.

For any totally positive unit $u_k \in R^*_{F+}$, let $L_k = F(\sqrt{-u_k D})$. For any order $S \supseteq R_F[\sqrt{-u_k D_j}]$ in the CM-field L_k, it holds that $h(S) = c_S h(L_k)$ for some positive constant $c_S \in \mathbb{Z}$ which is uniformly bounded by 2^n.

The class number $h(F)$ turns out to divide $h(L_k)$ and the *relative class number* of L_k is defined to be $h^-(L_k) = h(L_k)/h(F)$ (cf. [Lo00]). Since $h_+(F) = 2^m h(F)$ for $m = n - \dim_{\mathbb{F}_2}(\Sigma(R^*_F))$, Theorem 3.7.2 can be rephrased as $\pi_0(A) = \sum 2^{(e^+_S - 1 - m)} c_S h^-(L_k)$.

In order to apply the Brauer-Siegel Theorem, the key point is to relate the several absolute discriminants D_{L_k} and regulators Reg_{L_k} as u_k vary among totally positive units in F.

Firstly, we have the relations $D_{L_k} = |N_{F/\mathbb{Q}}(D_{L_k/F}) \cdot D_F^2| = 2^{p_k}|N_{F/\mathbb{Q}}(D)|D_F^2$ for some $0 \leq p_k \leq 2n$. Secondly, by [Wa82], p. 41, it holds that $\mathrm{Reg}_{L_k} = 2^c \mathrm{Reg}_F$ with $c = n-1$ or $n-2$.

Let ε be a sufficiently small positive number. By the Brauer-Siegel Theorem, it holds that $D_{L_k}^{(1-\varepsilon)/2} \leq h(L_k)\mathrm{Reg}_{L_k} \leq D_{L_k}^{(1+\varepsilon)/2}$ for $D_{L_k} \gg 1$. Thus

$$\frac{D_F^{(1-\varepsilon)/2}}{h(F)\mathrm{Reg}_F}(D_{L_k}/D_F)^{(1-\varepsilon)/2} \leq h^-(L_k) \leq \frac{D_F^{(1+\varepsilon)/2}}{h(F)\mathrm{Reg}_F}(D_{L_k}/D_F)^{(1+\varepsilon)/2}.$$

Fixing an arbitrary CM-field L appearing in the expression for $\pi_0(A)$, this boils down to

$$C_- \cdot \frac{D_F^{(1-\varepsilon)/2}}{h(F)\mathrm{Reg}_F}(D_L/D_F)^{(1-\varepsilon)/2} \leq \pi_0(A) \leq C_+ \cdot \frac{D_F^{(1+\varepsilon)/2}}{h(F)\mathrm{Reg}_F}(D_L/D_F)^{(1+\varepsilon)/2}$$

for some positive constants C_- and C_+. Taking logarithms, these inequalities yield Theorem 3.9.1. □

Remark 3.9.2. The argument above is not effective since it relies on the classical Brauer-Siegel Theorem on class numbers. However, recent work of Louboutin ([Lo00], [Lo02]) on lower and upper bounds for relative class numbers of CM-fields, based upon estimates of residues at $s = 1$ of Dedekind zeta functions, could be used to obtain explicit lower and upper bounds for $\pi_0(A)$.

Finally, we conclude this section with the proof of the second main result of this chapter quoted in Section 3.1.

Proof of Theorem 3.1.3. Part 1 is an immediate consequence of Theorem

3.9.1. Let us explain how part 2 follows. Assume that A is a simple complex abelian variety of odd and square-free dimension g. Then, by Theorem 1.3.5, $\mathrm{End}(A) \simeq S$ is an order in either a totally real number field F or a CM-field L over a totally real number field F. In any case, $[F : \mathbb{Q}] \leq g$.

In the former case, by Theorem 3.1 of Lange in [La88], $\pi_0(A) = |S^*_+/S^{*2}| \leq 2^{g-1}$. In the latter, let $S_0 \subset F$ be the subring of S fixed by complex conjugation. If $\mathcal{L}$ is a principal polarization on A, the Rosati involution precisely induces complex conjugation on $\mathrm{End}(A) \simeq S$ and we have that $\pi_0(A) = |S^*_{0+}/\mathrm{Norm}_{L/F}(S^*)| \leq |S^*_{0+}/S_0^{*2}| \leq 2^{g-1}$, by applying Theorem 1.5 of [La88]. □

Chapter 4

Shimura varieties and forgetful maps

Introduction

Let F be a totally real number field of degree $[F : \mathbb{Q}] = n$ and let B be a totally indefinite quaternion algebra over F. In this chapter we will be concerned with certain Shimura varieties X_B attached to an arithmetic datum arising from B and several maps that occur naturally between them.

As complex manifolds, these varieties can be described throughout quotients of certain bounded symmetric domains by arithmetic groups acting on them and, by the theory of Baily-Borel (cf. [BaBo66]), they become quasi-projective complex algebraic varieties. Shimura discovered a moduli interpretation of these varieties which allowed him to construct canonical models $X_B/\mathbb{Q}$ over the field $\mathbb{Q}$ of rational numbers. Shimura also explored their arithmetic, showing that the coordinates of so-called *Heegner points* on X_B generate certain class fields and that the Galois action on them can be described by explicit reciprocity laws (cf. [Sh63] and [Sh67]).

The nature of the Shimura varieties X_B differs notably depending on the existence or absence of zero divisors in B. When B is the split algebra $\mathrm{M}_2(F)$, the varieties X_B are classically called *Hilbert modular varieties*. These are not complete and suitable compactifications of them can be constructed, though at the cost of producing new singularities. The literature on them is abundant, especially on the low dimensional cases. In dimension 1, these are called modular curves and have become crucial in many aspects of number theory. In dimension 2, a reference to Hilbert modular surfaces is van der Geer's book [vdGe87].

On the other hand, when B is nonsplit, that is, it is a division algebra, then the emerging Shimura varieties $X_B/\mathbb{Q}$ are already projective. This fact makes the study of their arithmetic highly difficult since, in the Hilbert modular case, much of it is encoded in the added cusps. In remarkable contrast to Hilbert modular varieties, Shimura proved that, when $B \not\simeq \mathrm{M}_2(F)$, X_B do not have real points and therefore do not have rational points over any number field that admits a real embedding. In the last years, there has been increasing interest on Shimura curves arising from rational indefinite quaternion algebras, since they play a crucial role in modularity questions (cf. [HaHaMo99] or [Ri90], for instance).

From the modular point of view, there are natural maps

$$\pi : X_B \longrightarrow \mathcal{A}_{g,(d_1,\ldots,d_g)}$$

from X_B into the moduli space of polarized abelian varieties that involve *forgetting* some additional structures. Further, for any totally real field L containing F and embedded in B, these maps factorize through a morphism $\pi_L : X_B \to \mathcal{H}_L$ into a Hilbert modular variety (cf. Section 4.1).

It is the purpose of this chapter to describe in detail the nature of these morphisms and their image in the several moduli spaces of abelian varieties. As we will see, π and π_L are finite maps onto their image and the mere computation of their degree turns out to be unexpectedly subtle. It is based on arithmetic questions on B that were recently studied by Chinburg and Friedman in a series of papers in connection with arithmetic hyperbolic 3-orbifolds (cf. [ChFr86], [ChFr00]).

In Section 4.1, we review the construction due to Shimura of the varieties we will be dealing with and the above mentioned maps. In Section 4.3, we introduce several Atkin-Lehner groups of automorphisms acting on them and describe their modular interpretation. Particularly, in Section 4.3.2 we introduce what we call the *stable* and *twisting Atkin-Lehner groups*.

We state the main result of the chapter as Theorem 4.4.4 in Section 4.4 and we devote Section 4.5 to present its proof.

In closing this introduction, we point out two different applications of the results presented in this chapter. The first one is pursued in Section 4.6 and concerns the geometry of the quaternionic locus $\mathcal{Q}_{\mathcal{O}}$ of abelian varieties admitting multiplication by a maximal order $\mathcal{O}$ in the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of even dimension g. By means of Theorem 4.4.4 and Eichler's theory on optimal embeddings, the number of irreducible components of $\mathcal{Q}_{\mathcal{O}}$ can be related to certain class numbers and conditions can be given for its irreducibility. We also refer the reader to [Ro5] for an exposition of these results.

Secondly, Theorem 4.4.4 can also be used to explore the arithmetic of abelian varieties with quaternionic multiplication. Indeed, in Section 4.7, we investigate the field of moduli k_B of the quaternion multiplication on a principally polarized abelian variety $(A, \mathcal{L})/\bar{\mathbb{Q}}$. This is a fundamental arithmetic invariant of the $\bar{\mathbb{Q}}$-isomorphism class of $(A, \mathcal{L})$ which is in intimate relationship with the possible fields of definition of $(A, \mathcal{L})$.

The main results developed in this chapter can be found in [Ro3] and [Ro4].

4.1 Shimura varieties

Let F be a totally real number field of degree n and let B be a totally indefinite quaternion algebra over F. Let $\mathcal{D} = \mathrm{disc}(B) = \mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_{2r}$.

Fix a datum $(\mathcal{O}, \mathcal{I}, \varrho)$ consisting of a maximal order $\mathcal{O}$ in B, a left $\mathcal{O}$-ideal $\mathcal{I}$, or rather its class in $\mathrm{Pic}_\ell(\mathcal{O})$, and a positive anti-involution $\varrho = \varrho_\mu : B \to B$, $\beta \mapsto \beta^\varrho = \mu^{-1}\overline{\beta}\mu$, with $\mu \in \mathcal{O}$, $\mu^2 + \delta = 0$, $\mathrm{n}(\mu) = \delta \in F_+^*$. We will refer to $(\mathcal{O}, \mathcal{I}, \varrho)$ as a *quaternionic datum attached to* B.

Attached to the datum $(\mathcal{O}, \mathcal{I}, \varrho)$ there is the following moduli problem over $\mathbb{Q}$: classifying isomorphism classes of triplets $(A, \iota, \mathcal{L})$ given by

- An abelian variety A of dimension $g = 2n$.
- A ring homomorphism $\iota : \mathcal{O} \hookrightarrow \mathrm{End}\,(A)$ such that $H_1(A, \mathbb{Z})$, regarded as a left $\mathcal{O}$-module, is isomorphic to the left ideal $\mathcal{I}$.
- A primitive polarization $\mathcal{L}$ on A such that the Rosati involution $\circ : \mathrm{End}^0(A) \to \mathrm{End}^0(A)$ with respect to $\mathcal{L}$ on $\mathrm{End}^0(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ coincides with ϱ when restricted to $\iota(\mathcal{O})$:

$$\circ|_{\iota(\mathcal{O})} = \varrho \cdot \iota.$$

A triplet $(A, \iota, \mathcal{L})$ will be referred to as a polarized abelian variety with multiplication by $\mathcal{O}$. Two triplets $(A_1, \iota_1, \mathcal{L}_1)$, $(A_2, \iota_2, \mathcal{L}_2)$ are isomorphic if there exists an isomorphism $\alpha \in \mathrm{Hom}(A_1, A_2)$ such that $\alpha\iota_1(\beta) = \iota_2(\beta)\alpha$ for any $\beta \in \mathcal{O}$ and $\alpha^*(\mathcal{L}_2) = \mathcal{L}_1 \in \mathrm{NS}(A_1)$. Remark also that, since a priori there is no canonical structure of R_F-algebra on $\mathrm{End}(A)$, the immersion $\iota : \mathcal{O} \hookrightarrow \mathrm{End}(A)$ is just a homomorphism of rings.

As it was proved by Shimura, the corresponding moduli functor is coarsely represented by an irreducible and reduced quasi-projective scheme $X_B/\mathbb{Q} = X_{(\mathcal{O},\mathcal{I},\varrho)}/\mathbb{Q}$ over $\mathbb{Q}$ and of dimension $n = [F : \mathbb{Q}]$. Moreover, if B is division (that is, if $r > 0$), the Shimura variety X_B is complete (cf. [Sh63], [Sh67]).

Complex analytically, the manifold $X_B(\mathbb{C})$ can be described as the quotient of a symmetric space by the action of a discontinuous group as follows. Since B is totally indefinite, we may fix an embedding $B \hookrightarrow B \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \mathrm{M}_2(\mathbb{R}) \oplus ... \oplus \mathrm{M}_2(\mathbb{R})$ and regard the group

$$\Gamma_B = \mathcal{O}^1 = \{\gamma \in \mathcal{O}^* : \mathrm{n}(\gamma) = 1\}$$

as a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})^n$. An element $\gamma = (\gamma_1, ..., \gamma_n) \in \Gamma_B$ acts on the cartesian product $\mathfrak{H}^n$ of n copies of Poincaré's upper half plane $\mathfrak{H} = \{x + yi : x, y \in \mathbb{R}, y > 0\}$ by Moebius transformations:

$$\gamma \cdot (\tau_1, ..., \tau_n)^t = (\frac{a_1\tau_1 + b_1}{c_1\tau_1 + d_1}, ..., \frac{a_n\tau_n + b_n}{c_n\tau_n + d_n})^t$$

where $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Then

$$\Gamma_B\backslash\mathfrak{H}^n \simeq X_B(\mathbb{C}).$$

4.2 Forgetful maps into $\mathcal{A}_g$ and the Hilbert modular varieties $\mathcal{H}_S$ and $\mathcal{H}_F$

It is the purpose of this section to describe several natural maps that occur between the Shimura varieties X_B, the moduli spaces of polarized abelian varieties and several Hilbert modular varieties $\mathcal{H}_S$ and $\mathcal{H}_F$.

The following notion will be convenient for us in order to introduce these Hilbert modular varieties.

Definition 4.2.1. An Eichler pair (S, φ) for $\mathcal{O}$ is a pair consisting of an order S over R_F in a quadratic extension L of F and an optimal embedding $\varphi : S \hookrightarrow \mathcal{O}$ of orders over R_F.

An Eichler pair is said to be totally real if L is.

We recall from Section 1.2.3 than an embedding $\varphi : S \hookrightarrow \mathcal{O}$ is optimal if $\varphi(S) = \varphi(L) \cap \mathcal{O}$. Note that not all orders S in quadratic extensions L of F can be optimally embedded in $\mathcal{O}$. Namely, there exists an embedding φ of the ring of integers R_L of L into $\mathcal{O}$ if and only if any prime ideal $\mathfrak{p}$ of F that ramifies in B either remains inert or ramifies in L. Here, the fact that B is division and splits at least at one archimedean place makes the condition for the embeddability of R_L in $\mathcal{O}$ particularly neat. Otherwise, it depends heavily on the conjugation class of $\mathcal{O}$.

Given an order S over R_F in a totally real quadratic extension L of F and a sequence of positive integers $(1, d_2, ..., d_g)$, $d_{i-1}|d_i$ for $2 \leq i \leq g = 2n$, we may consider the Hilbert modular variety $\mathcal{H}_S = \mathcal{H}_{S,(1,d_2,...,d_g)}$ that classifies isomorphism classes of triplets $(A, i, \mathcal{L})$ given by

- An abelian variety A of dimension $[L : \mathbb{Q}] = 2n$.
- A ring homomorphism $i : S \hookrightarrow \mathrm{End}(A)$.
- A primitive polarization $\mathcal{L}$ of type $(1, d_2, ..., d_g)$ on A.

The scheme $\mathcal{H}_S$ is $2n$-dimensional, noncomplete and defined over $\mathbb{Q}$. Actually, this is a particular case of the Shimura varieties introduced above: $\mathcal{H}_S$ is the union of several irreducible components, all them isomorphic to Shimura varieties $X_{(\mathrm{M}_2(S),\mathcal{I},\varrho)}$ for several $\mathrm{M}_2(S)$-left ideals $\mathcal{I}$ and anti-involutions ϱ.

In addition to these, we will also be interested in the (reduced) scheme $\mathcal{H}_F/\mathbb{Q}$ that coarsely represents the functor attached to the moduli problem of classifying $(1, d_2, ..., d_g)$-polarized abelian varieties A of dimension $2n$ together with an homomorphism $R_F \hookrightarrow \mathrm{End}(A)$. The variety $\mathcal{H}_F$ has dimension $3n$ and $\mathcal{H}_F(\mathbb{C})$ is the quotient of n copies of the 2-Siegel half space $\mathfrak{H}_2$ by a suitable discontinuous group (cf. [Sh63], [LaBi92]).

However, the Hilbert modular variety $\mathcal{H}_F$ does not correspond to any of the Shimura varieties X_B introduced above for any quaternion algebra B, not even $\mathrm{M}_2(F)$.

Notice that, when $F = \mathbb{Q}$, $\mathcal{H}_F = \mathcal{A}_{2,(1,d)}$ is Igusa's three-fold of level $d \geq 1$, the moduli space of $(1, d)$-polarized abelian surfaces.

Let now $(A, \iota, \mathcal{L})$ be a (primitively) polarized abelian variety with multiplication by $\mathcal{O}$ with respect to the datum $(\mathcal{O}, \mathcal{I}, \varrho)$. Note that the type

$(1, d_2, ..., d_g)$ of $\mathcal{L}$ was not specified when we posed the moduli problem attached to the Shimura variety $X_{(\mathcal{O},\mathcal{I},\varrho)}$. However, since $X_{(\mathcal{O},\mathcal{I},\varrho)}$ is connected, the type of $\mathcal{L}$ only depends on the datum $(\mathcal{O}, \mathcal{I}, \varrho)$ and not on the particular triplet $(A, \iota, \mathcal{L})$. Thus $(1, d_2, ..., d_g)$ will often be referred to as the type of $(\mathcal{O}, \mathcal{I}, \varrho)$.

Let $\mathcal{A}_{g,(1,d_2,...,d_g)}/\mathbb{Q}$ be the moduli space of polarized abelian varieties of type $(1, d_2, ..., d_g)$. It is a reduced scheme over $\mathbb{Q}$ of dimension $\frac{g(g+1)}{2}$. The observation above allows us to define a natural morphism

$$\begin{array}{rccc} \pi: & X_{(\mathcal{O},\mathcal{I},\varrho)} & \longrightarrow & \mathcal{A}_{g,(1,d_2,...,d_g)} \\ & (A, \iota, \mathcal{L}) & \mapsto & (A, \mathcal{L}) \end{array}$$

from the Shimura variety to $\mathcal{A}_{g,(1,d_2,...,d_g)}$ that consists of *forgetting* the quaternion endomorphism structure. This morphism is representable, proper and defined over the field $\mathbb{Q}$ of rational numbers. Moreover, as we now explain, the morphism $\pi : X_B \to \mathcal{A}_{g,(1,d_2,...,d_g)}$ factorizes in a natural way through suitable Hilbert modular varieties $\mathcal{H}_{S,(1,d_2,...,d_g)}$ and the Hilbert modular variety $\mathcal{H}_F$.

Indeed, let (S, φ) be any totally real Eichler pair. Imitating the construction of π we obtain a morphism

$$\begin{array}{rccc} \pi_{(S,\varphi)}: & X_{(\mathcal{O},\mathcal{I},\varrho)} & \longrightarrow & \mathcal{H}_S \\ & (A, \iota, \mathcal{L}) & \mapsto & (A, \iota\cdot\varphi, \mathcal{L}) \end{array}$$

where $\iota \cdot \varphi : S \hookrightarrow \mathcal{O} \hookrightarrow \mathrm{End}(A)$.

Note that, although the construction of the scheme $\mathcal{H}_S$ does not depend on the embeddability of the order S in $\mathcal{O}$ nor on the choice of a possibly existing embedding $\varphi : S \hookrightarrow \mathcal{O}$, the morphism $\pi_{(S,\varphi)} : X_{(\mathcal{O},\mathcal{I},\varrho)} \to \mathcal{H}_S$ does indeed depend on it.

Finally, we also have a morphism

$$\pi_F : \quad X_{(\mathcal{O},\mathcal{I},\varrho)} \longrightarrow \mathcal{H}_F$$

from X_B into the Hilbert modular variety $\mathcal{H}_F$. The map $\pi_F : \quad X_{(\mathcal{O},\mathcal{I},\varrho)} \longrightarrow \mathcal{H}_F$ is constructed as above: by restricting the endomorphism structure of a triplet $(A, \iota, \mathcal{L})$ to $\iota|_{R_F} : R_F \hookrightarrow \mathrm{End}(A)$. The whole picture can be summarized with the following commutative diagram of morphisms:

$$
\begin{array}{cccccccc}
 & & & \mathcal{H}_{S_1} & & & & \\
 & & \overset{\pi_{(S_1,\varphi_1)}}{\nearrow} & \vdots & \searrow & & & \\
\pi : & X_B & \overset{\pi_{(S_2,\varphi_2)}}{\longrightarrow} & \mathcal{H}_{S_2} & \longrightarrow & \mathcal{H}_F & \to & \mathcal{A}_{g,(1,d_2,\dots,d_g)} \\
 & & \underset{\pi_{(S_n,\varphi_n)}}{\searrow} & \vdots & \nearrow & & & \\
 & & & \mathcal{H}_{S_n} & & & & \\
\hline
\text{dimension} & n & & 2n & & 3n & & n(2n+1)
\end{array}
$$

Note that, while there is a canonical forgetful map from X_B to $\mathcal{H}_F$, we obtain distinct maps from X_B to $\mathcal{H}_S$ as $\varphi : S \hookrightarrow \mathcal{O}$ varies among all possible Eichler embeddings.

4.3 The Atkin-Lehner group of a Shimura variety

As before, let F be a totally real number field of degree n. Let B be a totally indefinite quaternion algebra over F of discriminant $\mathcal{D} = \mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_{2r}$ and let $\mathcal{O}$ be a maximal order in B. We recall from Section 1.2 the groups $\mathcal{O}^* \supset \mathcal{O}^*_+ \supseteq \mathcal{O}^1$ of units in $\mathcal{O}$, units in $\mathcal{O}$ of totally positive reduced norm and units in $\mathcal{O}$ of reduced norm 1, respectively. The two latter are related by the exact sequence

$$1 \to \mathcal{O}^1 \to \mathcal{O}^*_+ \xrightarrow{\mathrm{n}} R^*_{F+} \to 1.$$

We also recall that we let B^*_+ the subgroup of elements of B^* of totally positive reduced norm and we let $\mathrm{Norm}_{B^*}(\mathcal{O})$ and $\mathrm{Norm}_{B^*_+}(\mathcal{O})$ denote the normalizer group of $\mathcal{O}$ in B^* and B^*_+, respectively.

Definition 4.3.1. The *positive Atkin-Lehner group* W^1 of $\mathcal{O}$ is

$$W^1 = \mathrm{Norm}_{B^*_+}(\mathcal{O})/(F^* \cdot \mathcal{O}^1).$$

In addition to the above, although less relevant for our purposes, we also introduce the groups

$$W = \mathrm{Norm}_{B^*}(\mathcal{O})/(F^* \cdot \mathcal{O}^*)$$

and

$$W_+ = \mathrm{Norm}_{B^*_+}(\mathcal{O})/(F^* \cdot \mathcal{O}^*_+).$$

Let us recall that, by the Skolem-Noether Theorem 1.2.15, the group of automorphisms of $\mathcal{O}$ is precisely the group $\mathrm{Norm}_{B^*}(\mathcal{O})/F^*$. The group W is identified with the group of principal two-sided ideals of $\mathcal{O}$ by the assignation $\omega \in W \mapsto \mathcal{O} \cdot \omega$ and it is a finite abelian 2-group. More precisely, by results of Eichler (cf. [Ei37], [Ei38] and [Vi80], Theorem 5.7), the reduced norm $\mathrm{n} : B^* \to F^*$ induces an isomorphism

$$W \simeq \mathbb{Z}/2\mathbb{Z} \times \overset{2r}{\cdots} \times \mathbb{Z}/2\mathbb{Z}.$$

Moreover, it also follows from loc. cit. that any element $[\omega] \in W$ can be represented by an element $\omega \in \mathcal{O}$ whose reduced norm $\mathrm{n}(\omega)$ is supported at the prime ideals $\mathfrak{p}|\mathcal{D}$.

The group W_+ may and will be regarded as the subgroup $W_+ = \{[\omega] \in W : \mathrm{n}(\omega) \in F^*_+\}$ of W and both coincide whenever $\mathrm{Pic}_+(F) \simeq \mathrm{Pic}(F)$. From the above, we obtain the exact sequence

$$1 \to R^*_{F+}/R^{*2}_F \xrightarrow{\alpha} W^1 \to W_+ \to 1,$$

where the map $\alpha : R^*_{F+}/R^{*2}_F \to W^1$ maps a totally positive unit $u \in R^*_{F+}$ to any $\alpha_u \in \mathcal{O}^*_+$ with $\mathrm{n}(\alpha_u) = u$, whose existence is guaranteed by Proposition 1.2.5. In this way, we obtain that W^1 is isomorphic to the *direct* product

$$W^1 \simeq R^*_{F+}/R^{*2}_F \times W_+ \simeq (\mathbb{Z}/2\mathbb{Z})^s$$

for $s \leq (n-1) + 2r$. The bound for s follows from Dirichlet's Unit Theorem and the inclusion $W_+ \subseteq W$. The precise value of s can be determined in terms of the behaviour of signatures of elements of R^*_F.

4.3.1 Modular interpretation of the positive Atkin-Lehner group

In his Ph.D thesis, Jordan described the modular interpretation of the action of the positive Atkin-Lehner group W^1 on Shimura curves (cf. [Jo81]). As we now explain, this interpretation can be extended to the higher dimensional cases. We keep the same notations of above.

Definition 4.3.2. Let $A/\mathbb{C}$ be an abelian variety with quaternionic multiplication by $\mathcal{O}$. Let $\mathcal{L} \in \mathrm{NS}(A) \otimes \mathbb{Q}$ be a fractional invertible sheaf and let $\omega \in \mathrm{End}(A) = \mathcal{O}$ be a non zero endomorphism of A.

We define $\mathcal{L}_\omega \in \mathrm{NS}(A) \otimes \mathbb{Q}$ to be the fractional invertible sheaf on A such that

$$c_1(\mathcal{L}_\omega) = \omega^{-1} c_1(\mathcal{L}) \omega.$$

In other words, if we regard the first Chern class $c_1(\mathcal{L})$ of $\mathcal{L}$ as an alternate bilinear form E on $V = \mathrm{Lie}(A(\mathbb{C}))$, then $c_1(\mathcal{L}_\omega) : V \times V \longrightarrow \mathbb{R}$, $(u, v) \mapsto E(\frac{\omega}{\mathrm{n}(\omega)}(u), \omega(v))$. Hence, $c_1(\mathcal{L}_\omega) = \frac{1}{\mathrm{n}(\omega)} c_1(\omega^*(\mathcal{L}))$.

From Theorem 3.3.1, it is checked that this does correspond to the first Chern class of a fractional line bundle on A. Moreover, if $\mathcal{L} \in \mathrm{NS}(A)$ and $\omega \in \mathcal{O} \cap \mathrm{Norm}_{B^*}(\mathcal{O})$, then it also follows from Theorem 3.3.1 that $\mathcal{L}_\omega \in \mathrm{NS}(A)$. Finally, if $\mathcal{L} \in \mathrm{NS}(A)$ is a polarization and $\omega \in \mathcal{O} \cap \mathrm{Norm}_{B_+^*}(\mathcal{O})$, we obtain from Theorem 3.5.3 that $\mathcal{L}_\omega \in \mathrm{NS}(A)$ is also a polarization.

Let $\mathcal{I}$ be a left $\mathcal{O}$-ideal and let ϱ be a positive involution on B. The action of $B_+^* \subset \mathrm{GL}_2^+(\mathbb{R})^n$ on $\mathfrak{H}^n$ by Moebius transformations descends to a free action of W^1 on the set $\mathcal{O}^1 \backslash \mathfrak{H}^n$ of complex points of the Shimura variety $X_B = X_{(\mathcal{O}, \mathcal{I}, \varrho)}$,

The following is due to Jordan for the case of abelian surfaces. The proof is along the same lines the one given in [Jo81] and we consequently omit it.

Proposition 4.3.3. *Let $X_{(\mathcal{O},\mathcal{I},\varrho)}/\mathbb{Q}$ be a Shimura variety and let $P = [(A, \iota, \mathcal{L})]$ denote the isomorphism class of a polarized abelian variety with multiplication by $\mathcal{O}$ viewed as a closed point on it.*

Then, an element $\omega \in W^1$ acts on P by

$$\omega(P) = [(A, \iota_\omega, \mathcal{L}_\omega)]$$

where

$$\begin{array}{rccc} \iota_\omega : & \mathcal{O} & \hookrightarrow & \mathrm{End}(A) \\ & \beta & \mapsto & \omega^{-1}\iota(\beta)\omega. \end{array}$$

Therefore, according to Proposition 4.3.3, an Atkin-Lehner involution $\omega \in W^1$ acts on a triplet $[(A, \iota, \mathcal{L})]$ by leaving stable the isomorphism class of the underlying abelian variety A but conjugating the endomorphism structure $\iota : \mathcal{O} \hookrightarrow \mathrm{End}(A)$ and switching the polarization $\mathcal{L}$. It readily follows the compatibilty of $\mathcal{L}$ with ι that the polarization $\mathcal{L}_\omega$ is compatible with ι_ω.

From this interpretation and by standard moduli considerations, it follows that $W^1 \subseteq \mathrm{Aut}(X_B)$ acts on X_B as a subgroup of algebraic involuting automorphisms over $\mathbb{Q}$.

Remark 4.3.4. If $\alpha \in \mathcal{O}_+^*$, $\mathrm{n}(\alpha) = u \in R_{F+}^*$, then α can be simultaneously viewed as an automorphism $\alpha \in \mathrm{Aut}(A) \simeq \mathcal{O}^*$ of A and as a representative of an Atkin-Lehner involution $[\alpha] \in W^1$. For any polarized abelian variety with quaternionic multiplication $[(A, \iota, \mathcal{L})]$, this automorphism induces an isomorphism of triplets $\alpha[(A, \iota, \mathcal{L})] = [(A, \iota_\alpha, \mathcal{L}_\alpha)] = [(A, \iota, \mathcal{L}_\alpha)]$ where $\mathcal{L}_\alpha$ is a polarization on A such that, although $\mathcal{L} \not\simeq \mathcal{L}_\alpha$, both induce the same Rosati involution on $\mathrm{End}(A) \otimes \mathbb{Q}$.

In the literature (cf. [Sh63], [Mi79]), $\mathcal{L}$ and $\mathcal{L}_\alpha$ are called *weakly isomorphic*. It is checked that $\mathcal{O}_+^*/\mathcal{O}^1$ acts freely and transitively on the set of isomorphism classes of a given weak polarization class on A.

Remark 4.3.5. There is a natural moduli theory for polarized abelian varieties with quaternionic multiplication up to weak isomorphism which is also considered in [Sh63]. Both theories coincide in dimension 2 (because it corresponds to $F = \mathbb{Q}$), but for higher dimensions the latter is coarser and less suitable for our purposes.

Remark 4.3.6. We wonder in what circumstances W^1 is the full group of automorphisms of X_B. The impression is that, *generically*, it does hold that $\mathrm{Aut}(X_B) = W^1$, but of course the term *generic* should be made precise in any case. The split case of an order $\mathcal{O}$ in $B = \mathrm{M}_2(\mathbb{Q})$ was classically studied by Ogg, who found that the modular curve $X_0(37)$ is an interesting exception to the prediction that, whenever the genus of a modular curve is at least 2, $W = W^1 = \mathrm{Aut}(X_\mathcal{O})$. For Shimura curves attached to a maximal order in a rational division quaternion algebra $B/\mathbb{Q}$, this question is investigated in Section 6.1 of Chapter 6. For higher dimensional Shimura varieties, it seems to hold that whenever $\mathcal{O}^*$ does not contain torsion units (besides ± 1), $W^1 = \mathrm{Aut}(X_{\mathcal{O},\mathcal{I},\varrho})$ (cf. Theorem 6.1.2).

4.3.2 The twisting and stable Atkin-Lehner subgroups

Let B be a totally indefinite quaternion algebra over a totally real field F. Let us assume that the reduced discriminant is a totally positive principal ideal of F and let $D \in F_+^*$ be such that $\mathrm{disc}(B) = (D)$. This is an assumption that we will naturally encounter in several situations and which is always satisfied whenever the narrow class number $h_+(F) = 1$ of F is trivial. In particular, this is always satisfied by indefinite rational quaternion algebras $B/\mathbb{Q}$.

The following definition is motivated by Theorems 3.3.1 and 3.5.3.

Definition 4.3.7. A *polarized maximal order* of B is a pair $(\mathcal{O}, \mu)$ where $\mathcal{O} \subset B$ is a maximal order and $\mu \in \mathcal{O}$ is a pure quaternion of totally positive reduced norm $\mathrm{n}(\mu) \in F_+^*$.

If $(\mathrm{n}(\mu)) = (D)$, we say that $(\mathcal{O}, \mu)$ is a *principally polarized order* of B.

Definition 4.3.8. Let $(\mathcal{O}, \mu)$ be a polarized maximal order in B. A *twist* of $(\mathcal{O}, \mu)$ is an element $\chi \in \mathcal{O} \cap \mathrm{Norm}_{B^*}(\mathcal{O})$ such that $\chi^2 + \mathrm{n}(\chi) = 0$ and $\mu\chi = -\chi\mu$.

In particular, if $(\mathcal{O}, \mu)$ is a principally polarized maximal order for some $\mu \in \mathcal{O}$ with $\mu^2 + uD = 0$ and $u \in R_{F+}^*$, a twist of $(\mathcal{O}, \mu)$ is a pure quaternion $\chi \in \mathcal{O} \cap \mathrm{Norm}_{B^*}(\mathcal{O})$ such that

$$B = F + F\mu + F\chi + F\mu\chi = (\frac{-uD, -\mathrm{n}(\chi)}{F}).$$

We say that a principally polarized maximal order $(\mathcal{O}, \mu)$ in B is *twisting* if it admits some twist χ in $\mathcal{O}$. We will say that a maximal order $\mathcal{O}$ is *twisting* if there exists $\mu \in \mathcal{O}$ such that $(\mathcal{O}, \mu)$ is a twisting principally polarized order. Finally, we will say that B is *twisting* if there exists a twisting maximal order $\mathcal{O}$ in B. Note that B is twisting if and only if $B \simeq (\frac{-uD,m}{F})$ for some $u \in R_{F+}^*$ and $m \in F^*$ such that $m|D$.

Definition 4.3.9. Let $(\mathcal{O}, \mu)$ be a polarized maximal order of B. A *twisting involution* $\omega \in W^1$ is an Atkin-Lehner involution such that $[\omega] = [\chi] \in W$ is represented in B^* by a twist χ of $(\mathcal{O}, \mu)$.

We let $V_0(\mathcal{O}, \mu)$ denote the subgroup of W^1 of twisting involutions of $(\mathcal{O}, \mu)$.

For any subring $S \subset \mathcal{O}$, we say that χ is a twist of $(\mathcal{O}, \mu)$ in S if $\chi \in S$. We say that a twisting involution $[\omega]$ is in $S \subseteq \mathcal{O}$ if it can be represented by a twist $\chi \in S$. We will let $V_0(\mathcal{O}, \mu, S)$ denote the subgroup of W^1 generated by the twisting involutions of $(\mathcal{O}, \mu)$ in S, or simply $V_0(S)$ if the polarized order $(\mathcal{O}, \mu)$ is well understood. We will also simply write V_0 for the set of twists of $(\mathcal{O}, \mu)$ in $S = \mathcal{O}$.

Let us remark that, since B is totally indefinite, no $\chi \in B_+^*$ can be a twist of a polarized order $(\mathcal{O}, \mu)$ because a necessary condition for $B \simeq (\frac{-n(\mu), -\mathrm{n}(\chi)}{F})$

is that $\mathrm{n}(\chi)$ be totally negative. In fact, twisting involutions $\omega \in W^1$ are always represented by twists $\chi \in B^*$ of totally negative reduced norm.

For a polarized order $(\mathcal{O}, \mu)$, let $R_\mu = F(\mu) \cap \mathcal{O}$ and let $\Omega = \Omega(R_\mu) = \{\xi \in R_\mu : \xi^f = 1, f \geq 1\}$ denote the finite group of roots of unity in the CM-order R_μ.

Definition 4.3.10. The *stable group* of a polarized order $(\mathcal{O}, \mu)$ is the subgroup

$$W_0 = U_0 \cdot V_0$$

of W^1 generated by

$$U_0 = U_0(\mathcal{O}, \mu) = \mathrm{Norm}_{F(\mu)^*}(\mathcal{O})/(F^* \cdot \Omega(R_\mu)),$$

and the group V_0 of twisting involutions of $(\mathcal{O}, \mu)$.

4.4 Main theorem

As remarked in the introduction, the type of the primitive polarizations of the abelian varieties parametrized by the Shimura variety $X_{(\mathcal{O},\mathcal{I},\varrho)}$ is determined by the datum $(\mathcal{O}, \mathcal{I}, \varrho)$. The following has been proved in Chapter 3 and makes this observation explicit.

Proposition 4.4.1. *The polarizations of the abelian varieties with quaternionic multiplication $(A, \iota, \mathcal{L})$ parametrized by $X_{(\mathcal{O},\mathcal{I},\varrho)}$ are* principal *if and only if:*

(i) $\mathrm{Disc}(B) = (D)$ *is a principal ideal of F generated by a totally positive element $D \in F_+^*$.*

(ii) $\mathrm{n}_{B/F}(\mathcal{I})$ *and $\vartheta_{F/\mathbb{Q}}^{-1}$ lie in the same ideal class in* $\mathrm{Pic}(F)$.

(iii) The positive anti-involution on B is $\varrho = \varrho_\mu : B \to B$, $\beta \mapsto \mu^{-1}\overline{\beta}\mu$ for some $\mu \in \mathcal{O}$ such that $\mu^2 + uD = 0$, $u \in R_{F+}^*$.

For the rest of this chapter, we will focus on moduli spaces of principally polarized abelian varieties and therefore we place ourselves under the conditions on $(\mathcal{O}, \mathcal{I}, \varrho_\mu)$ of Proposition 4.4.1. We thus assume in particular that $\mathrm{disc}(B)$ is a totally positive principal ideal and that we can choose a generator $D \in F_+^*$ of it such that $\mu \in \mathcal{O}$ satisfies $\mu^2 + D = 0$. In this case we say that $(\mathcal{O}, \mathcal{I}, \varrho_\mu)$ is of *principal type*.

Our main Theorem 4.4.4 below describes how the modular maps introduced in Section 4.2 factorize through the quotient of X_B by certain subgroups of Atkin-Lehner involutions that were introduced in Section 4.3.2.

Let

$$\widetilde{X}_B = \widetilde{X}_{(\mathcal{O},\mathcal{I},\varrho)} = \pi(X_{(\mathcal{O},\mathcal{I},\varrho)}) \hookrightarrow \mathcal{A}_g$$

denote the image in the moduli space $\mathcal{A}_g = \mathcal{A}_{g,(1,\ldots,1)}$ of principally polarized abelian varieties by π of the Shimura variety $X_{(\mathcal{O},\mathcal{I},\varrho)}$. Similarly, define $\widetilde{X}_{B/F} \subset \mathcal{H}_F$ and $\widetilde{X}_{B/(S,\varphi)} \subset \mathcal{H}_S$ respectively to be the algebraic subvarieties $\pi_F(X_{(\mathcal{O},\mathcal{I},\varrho)})$ and $\pi_{(S,\varphi)}(X_{(\mathcal{O},\mathcal{I},\varrho)})$ of the Hilbert modular varieties $\mathcal{H}_F$ and $\mathcal{H}_S$.

Remark 4.4.2. Classical objects in the literature are *Humbert varieties*, which are introduced as quotients of Hilbert modular varieties (cf. [vdGe87]). We note that Humbert varieties can be considered as a degenerate case of the above defined varieties $\widetilde{X}_B$, since they correspond to the split algebra $B = \mathrm{M}_2(F)$.

Definition 4.4.3. A closed point $[(A, \iota, \mathcal{L})]$ on X_B and its image on $\widetilde{X}_{B/(S,\varphi)}$, $\widetilde{X}_{B/F}$ or $\widetilde{X}_B$ is called a *Heegner point* if $\mathrm{End}(A) \otimes \mathbb{Q} \simeq \mathrm{M}_2(M)$ for a CM-field M over F. Equivalently, if $\mathrm{End}(A) \supsetneq \mathcal{O}$.

It follows from the work of Shimura in [Sh67] that the set of Heegner points on these varieties is discrete and dense.

Let us say that a morphism $\pi : X \to Y$ of schemes of non necessarily the same dimension is *quasifinite* if the fibres of π are finite (cf. [Har77], p. 91).

With the same notations as above, we have

Theorem 4.4.4. *Let $(\mathcal{O}, \mathcal{I}, \varrho_\mu)$ be a quaternionic datum of principal type and let $X_B = X_{(\mathcal{O},\mathcal{I},\varrho_\mu)}$ be the Shimura variety attached to it. For any totally real Eichler pair (S, φ), let*

$$\begin{array}{ccccccc} & & & \mathcal{H}_S & & & \\ & & {}^{\pi_{S,\varphi}}\nearrow & & \searrow & & \\ \pi: & X_B & & \xrightarrow{\pi_F} & & \mathcal{H}_F \;\to & \mathcal{A}_g. \end{array}$$

be the diagram of forgetful morphisms introduced above. Then

1. *The morphism $\pi_F : X_B \to \mathcal{H}_F$ is a quasifinite morphism that factorizes over $\mathbb{Q}$ into the natural projection $X_B \to X_B/W_0$ from X_B onto its quotient by the stable group $W_0 \subseteq \mathrm{Aut}(X_B)$ and a birational morphism $b_F : X_B/W_0 \dashrightarrow \widetilde{X}_{B/F}$ onto the image of X_B in $\mathcal{H}_F$.*

   The domain of definition of b_F^{-1} is $\widetilde{X}_{B/F} \setminus \mathcal{T}_F$, where $\mathcal{T}_F$ is a finite set of Heegner points.

2. *The morphism $\pi_{(S,\varphi)} : X_B \to \mathcal{H}_S$ is a quasifinite morphism that factorizes over $\mathbb{Q}$ into the projection $X_B \to X/V_0(\varphi(S))$ of X_B onto its quotient by the finite 2-group $V_0(\varphi(S)) \subseteq W_0$ and a birational morphism $b_{(S,\varphi)} : X_B/V_0(\varphi(S)) \dashrightarrow \widetilde{X}_{B/(S,\varphi)}$ into the image of X_B in $\mathcal{H}_S$ by $\pi_{(S,\varphi)}$.*

   As before, $b_{(S,\varphi)}^{-1}$ is defined on the whole $\widetilde{X}_{B/(S,\varphi)}$ but at a finite set $\mathcal{T}_{(S,\varphi)}$ of Heegner points.

We call the sets $\mathcal{T}_F$ and $\mathcal{T}_{(S,\varphi)}$ the *singular Heegner loci* of $\widetilde{X}_{B/F}$ and $\widetilde{X}_{B/(S,\varphi)}$ respectively, as they are indeed the sets of singular points (of quotient type) of these varieties.

We present the proof of the theorem in the next section; let us now make the statement more explicit and precise.

Recall that for a polarized order $(\mathcal{O}, \mu)$, we let $R_\mu = F(\mu) \cap \mathcal{O}$ be the order in the CM-field $F(\mu) \simeq F(\sqrt{-D})$ that optimally embeds in $\mathcal{O}$. Note that, since $\mu \in \mathcal{O}$, $R_\mu \supseteq R_F[\sqrt{-D}]$. We also let $\Omega = \Omega(R_\mu) = \{\xi \in R_\mu : \xi^f = 1, f \geq 1\}$ and $\Omega_{odd} = \{\xi \in R_\mu : \xi^f = 1, f \text{ odd }\}$. Their cardinalities will respectively be denoted by ω and ω_{odd}. Note that U_0 is indeed a subgroup of W^1 because $\Omega = F(\mu) \cap \mathcal{O}^1$.

Theorem 4.4.5. 1. *Let $(\mathcal{O}, \mu)$ be a non twisting principally polarized maximal order. Then,*

   (i) For any totally real Eichler pair (S, φ), the morphism $\pi_{(S,\varphi)} : X_B \to \mathcal{H}_S$ is a birational equivalence and

   (ii) $\deg(\pi_F : X_B \to \mathcal{H}_F) = 2^{\omega_{odd}}$.

2. *Let $(\mathcal{O}, \mu)$ be a twisting principally polarized maximal order. Then,*

   (i) For any totally real Eichler pair (S, φ) such that $\varphi(S)$ contains no twists of $(\mathcal{O}, \mu)$, the morphism $\pi_{(S,\varphi)}$ is a birational equivalence.

(ii) *For any twist* $\chi \in \mathcal{O}$ *of* $(\mathcal{O}, \mu)$ *and any totally real Eichler pair* (S, φ) *such that* $\varphi(S) \supseteq R_F[\chi]$, *the morphism* $\pi_{R_F[\chi]} : X_B \to \mathcal{H}_{R_F[\chi]}$ *is* $2 : 1$.

(a) $\deg(\pi_F : X_B \to \mathcal{H}_F) = 2^{2\omega_{odd}}$.

In particular, note that the forgetful map $\pi_{\mathbb{Q}} : X_B \longrightarrow \mathcal{A}_2$ from a Shimura curve into Igusa's moduli space of principally polarized abelian surfaces is either of degree 2 or of degree 4 and that a necessary condition for the latter is that

$$B \simeq (\frac{-\mathrm{disc}(B), m}{\mathbb{Q}})$$

for some $m > 0$, $m|\mathrm{disc}(B)$.

It turns out, for instance, that for any choice of a quaternionic datum $(\mathcal{O}, \mathcal{I}, \varrho)$ of principal type of discriminant $D = 6$ or 10, the corresponding forgetful map of the Shimura curve $X_{(\mathcal{O},\mathcal{I},\varrho)}$ into $\mathcal{A}_2$ has degree 4.

In order to prove Theorem 4.4.5, we introduce the following lemma.

Lemma 4.4.6. *Let* $(\mathcal{O}, \mu)$ *be a principally polarized maximal order. Then* $U_0 \simeq C_2^{\omega_{odd}}$.

Proof. Let us identify $F(\mu)$ and $F(\sqrt{-D})$ through any fixed isomorphism. As U_0 naturally embeds in $F(\sqrt{-D})^*/(F^* \cdot \Omega)$, we first show that the maximal 2-torsion subgroup H of $F(\sqrt{-D})^*/(F^* \cdot \Omega)$ is isomorphic to $C_2^{\omega_{odd}}$.

If $\omega \in F(\sqrt{-D})^*$ generates a subgroup of $F(\sqrt{-D})^*/(F^* \cdot \Omega)$ of order 2, then $\omega^2 = \lambda\xi$ for some $\lambda \in F^*$ and some root of unity $\xi \in \Omega$. In particular, note that if $\omega \in F(\sqrt{-D})^*$, then $\omega^2 \in F^*$ if and only if $\omega \in F^* \cup F^*\sqrt{-D}$. Let us write $\overline{H} = H/\langle\sqrt{-D}\rangle$.

We then have that, if $\xi \in \Omega$, there exists at most a single subgroup $\langle\omega\rangle \subseteq \overline{H}$ such that $\omega \in F(\sqrt{-D})^*$, $\omega^2 \in F^*\xi$. Indeed, if ω_1, $\omega_2 \in F(\sqrt{-D})$, $\omega_i^2 = \lambda_i\xi$ for some $\lambda_i \in F^*$ then $\omega_1 \cdot {\omega_2^{-1}}^2 \in F^*$ and hence $\omega_1 \cdot \omega_2^{-1} \in F^* \cup F^*\sqrt{-D}$. This shows that $[\omega_1] = [\omega_2] \in \overline{H}$.

Observe further that, if $\xi_f \in \Omega$ is a root of unity of odd order $f \geq 3$, then $\omega = \xi_f^{\frac{f+1}{2}} \in F(\sqrt{-D})^*$ generates a 2-torsion subgroup of $F(\sqrt{-D})^*/(F^* \cdot \Omega)$ such that $\omega^2 = \xi_f$.

It thus suffices to show that $\overline{\overline{H}} = H/\langle\sqrt{-D}, \{\xi_f^{\frac{f+1}{2}}\}_{f\geq 3 \text{ odd}}\rangle$ is trivial. Let $\omega \in F(\sqrt{-D})^*$, $\omega^2 = \lambda\xi$, ξ a root of unity of primitive order $f \geq 1$. If f is 2 or odd we already know that the class $[\omega] \in \overline{\overline{H}}$ is trivial. Further, it cannot exist any $\xi \in F(\sqrt{-D})$ of order $f = 2^N$, $N \geq 2$, because otherwise $\xi^{2^{N-1}}$ would be a square root of -1 and we would have that $F(\sqrt{-D}) = F(\sqrt{-1})$. This is a contradiction since $DR_F = \wp_1 \cdot ... \cdot \wp_{2r}$, $r > 0$.

Finally, it is also impossible that there should exist $\omega \in F(\sqrt{-D})$, $\omega^2 = \lambda\xi$, $\xi^f = 1$, $f = 2^N f_0$ with $N \geq 1$ and $f_0 \geq 3$ odd. Indeed, since in this case $\xi' = \xi^{2^N} \in F(\sqrt{-D})$ is a primitive root of unity of order f_0, $\omega' = \xi^{\frac{f_0+1}{2}}$ satisfies $\omega'^2 = \xi'$. Then we would have $\omega' \cdot \omega^{-1^2} = (\xi^{2^{N-1}})\xi$ and this would mean that $[\frac{\omega'}{\omega}] = [\omega] \in \overline{H}$, which is again a contradiction. This shows that $\overline{\overline{H}}$ is trivial and therefore $H = \langle\sqrt{-D}, \{\xi_f^{\frac{f+1}{2}}\}_{\xi_f \in \Omega_{odd}}\rangle$. In order to conclude the lemma, we only need to observe that both μ and $\xi_f^{\frac{f+1}{2}} \in F(\mu)$ normalize the maximal order $\mathcal{O}$ for any odd f, because their respective reduced norms divide the discriminant D. □

Proof of Theorem 4.4.5: Let us firstly assume that $(\mathcal{O}, \mu)$ is a non twisting polarized order. It is then clear from the definitions that the groups of twisting involutions $V_0(S)$ are trivial for any subring S of $\mathcal{O}$. In addition, by the above lemma, $W_0 = U_0 \simeq C_2^{\omega_{odd}}$. By Theorem 4.4.4, this yields the first part of Theorem 4.4.5. The second follows from the following lemma, which shows that, when $(\mathcal{O}, \mu)$ is twisting, the situation is more subtle and less homogenous.

Lemma 4.4.7. *Let* $(\mathcal{O}, \mu)$ *be a* twisting *principally polarized maximal order in a totally indefinite quaternion algebra* B *over* F *of discriminant* $\mathrm{disc}(B) = (D)$, $D \in F_+^*$. *Then* $U_0 \subset V_0$ *is a subgroup of* V_0 *and* $V_0/U_0 \simeq U_0$. *In particular,* $W_0 = V_0 \simeq C_2^{2\omega_{odd}}$.

Proof. Let $\omega \in U_0$ be represented by an element $\omega \in \mathrm{Norm}_{F(\mu)^*}(\mathcal{O}) \cap \mathcal{O}$ and let $\nu \in V_0$ be a twisting involution. We know that the class of ν in $\mathrm{Norm}_{B^*}(\mathcal{O})/(F^* \cdot \mathcal{O}^*)$ is represented by a twist $\chi \in \mathrm{Norm}_{B^*}(\mathcal{O}) \cap \mathcal{O}$ that satisfies $\chi^2 + \mathrm{n}(\chi) = 0$ and $\mu\chi = -\chi\mu$. Then we claim that $\omega\nu \in V_0$ is again a twisting involution of $(\mathcal{O}, \mu)$. Indeed, first $\omega\chi \in \mathrm{Norm}_{B^*}(\mathcal{O}) \cap \mathcal{O}$, because both ω and χ do. Second, since $\omega \in F(\mu)$, $\mu(\omega\chi) = \mu\omega\chi = \omega\mu\chi = -\omega\chi\mu = -(\omega\chi)\mu$ and finally, we have $\mathrm{tr}(\mu(\omega\chi)) = \mu\omega\chi + \overline{\omega\chi\mu} = \mu\omega\chi - \overline{\omega\chi}\mu = -\mathrm{tr}(\omega\chi)\mu \in F$ and thus $\mathrm{tr}(\omega\chi) = 0$.

This produces a natural action of U_0 on the set of twisting involutions of $(\mathcal{O}, \mu)$ which is free simply because B is a division algebra. In order to show that it is transitive, let χ_1, χ_2 be two twists. Then $\omega = \chi_1\chi_2^{-1} \in F(\mu)$ because $\mu\omega = \mu\chi_1\chi_2^{-1} = -\chi_1\mu\chi_2^{-1} = \chi_1\chi_2^{-1}\mu = \omega\mu$ and $F(\mu)$ is its own commutator subalgebra of B; further $\omega \in \mathrm{Norm}_{B^*}(\mathcal{O})$ because its reduced norm is supported at the ramifying prime ideals $\wp|\mathrm{disc}(B)$. Let us remark that, in the same way, $\chi_1\chi_2 \in \mathrm{Norm}_{F(\mu)^*}(\mathcal{O})$.

We are now in a position to prove the lemma. Let $\nu \in V_0$ be a fixed twisting involution. Then $U_0 \subset V_0$: for any $\omega \in U_0$ we have already shown that $\omega\nu$ is again a twisting involution and hence $(\omega\nu)\nu = \omega \in V_0$ because V_0 is a 2-torsion abelian group. In addition, the above discussion shows that any element of V_0 either belongs to U_0 or is a twisting involution and that there is a noncanonical isomorphism $V_0/U_0 \simeq U_0$. $\square$

This concludes the proof of Theorem 4.4.5. Observe that the above lemma can be rephrased by asserting that, in the twisting case, U_0 acts freely and transitively on the set of twisting involutions of W^1 with respect to $(\mathcal{O}, \mu)$.

In view of Theorem 4.4.5, the behaviour of the forgetful maps introduced above differs considerably depending on whether $(\mathcal{O}, \mu)$ is a twisting polarized order or not. For a maximal order $\mathcal{O}$ in a totally indefinite quaternion algebra B of principal reduced discriminant $D \in F_+^*$, it is then obvious to ask the following questions.

(i) Is there any $\mu \in \mathcal{O}$, $\mu^2 + D = 0$, such that $(\mathcal{O}, \mu)$ is twisting?

(ii) If $(\mathcal{O}, \mu)$ is twisting, which is its twisting group V_0?

Both questions are particular instances of the ones that we consider in Section 2.1.

4.5 Proof of Theorem 4.4.4

Let $(\mathcal{O}, \mathcal{I}, \varrho)$ be a quaternion datum attached to a totally indefinite quaternion algebra B over a totally real number field F of degree $[F : \mathbb{Q}] = n$. We assume that it is of principal type (cf. Proposition 4.4.1 and below). This means in particular that $\mathrm{disc}(B)$ can be generated by an element $D \in F_+^*$ such that $\varrho = \varrho_\mu$ for some $\mu \in \mathcal{O}$ with $\mu^2 + D = 0$.

Let $X_B = X_{(\mathcal{O},\mathcal{I},\varrho)}$ be the Shimura variety attached to $(\mathcal{O}, \mathcal{I}, \varrho)$. Let us first show that $\pi_F : X_B \to \mathcal{H}_F$ is a quasifinite map that factorizes into the natural projection

$$X_B \to X_B/W_0$$

from X_B onto its quotient by the stable group $W_0 = U_0 \cdot V_0 \subseteq \mathrm{Aut}_{\mathbb{Q}}(X_B)$ attached to $(\mathcal{O}, \mu)$.

To this end, we firstly make the following

Claim 4.5.1. *There is a free and transitive action of the stable group W_0 on the geometric fibres of the morphism $\pi_F : X_B \to \mathcal{H}_F$ at the set of non Heegner points of $\widetilde{X}_{B/F}$.*

Proof of the claim. Let $(A, j_F, \mathcal{L})$ be a complex principally polarized abelian variety together with a homomorphism of $\mathbb{Z}$-algebras $j_F : R_F \hookrightarrow \mathrm{End}(A)$. Without making any further mention of it, we will regard $\mathrm{End}(A)$ as an R_F-algebra through the given immersion j_F.

The isomorphism class $[(A, j_F, \mathcal{L})]$ may be interpreted as a closed point in $\mathcal{H}_F$. If it is nonempty, the elements in X_B of the fibre of π_F at this point can then be interpreted as those isomorphism classes of triplets $(A, \iota, \mathcal{L})$ where $(A, \mathcal{L})$ is a principally polarized abelian variety of dimension $2n$ and $\iota : \mathcal{O} \hookrightarrow \mathrm{End}(A)$ is a homomorphism of R_F-algebras such that the Rosati involution that $\mathcal{L}$ induces on $\mathcal{O}$, via ι, coincides with ϱ_μ.

Choose a triplet $(A, \iota, \mathcal{L})$ and assume that $\iota : \mathcal{O} \simeq \mathrm{End}(A)$ is actually an isomorphism, that is, $P = [(A, \iota, \mathcal{L})]$ is a non Heegner point of X_B. Through the first Chern class, Theorem 3.3.1 identifies $\mathrm{NS}(A)$ with a lattice in B_0 and it allows us to regard polarizations on A as pure quaternions of B. In particular, since $\circ|_{\iota(\mathcal{O})} = \varrho_\mu \cdot \iota$, we have that $c_1(\mathcal{L}) = \mu$ up to multiplication by elements in F^*.

Recall now that $W_0 = U_0 \cdot V_0$ for certain subgroups U_0 and $V_0 \subseteq W^1$ that were defined in Section 4.3.

Let $\omega \in U_0$. Let $L = F(\mu) \simeq F(\sqrt{-D})$ be the CM-field generated by $\mu \in B$ over F and let $S \supseteq R_F[\mu]$ be the order in L at which μ is optimal. Then ω is represented by an element that we still denote $\omega \in S$ and we wish to show that the closed points $[(A, \iota, \mathcal{L})]$ and $\omega[(A, \iota, \mathcal{L})]$ in X_B lie on the same fibre of π_F.

From Proposition 4.3.3, the isomorphism class of $\omega[(A, \iota, \mathcal{L})]$ is represented by the triplet $(A, \iota_\omega, \mathcal{L}_\omega)$ where $\iota_\omega = \omega^{-1}\iota\omega : \mathcal{O} \hookrightarrow \mathrm{End}(A)$ and $c_1(\mathcal{L}_\omega) = \omega^{-1}c_1(\mathcal{L})\omega$. Since $c_1(\mathcal{L}) = \mu \in B^*/F^*$ and ω belong to the same quadratic algebra L embedded in B, they commute and it holds that

$$c_1(\mathcal{L}_\omega) = \omega^{-1}c_1(\mathcal{L})\omega = c_1(\mathcal{L}).$$

We then conclude that $(A, \mathcal{L})$ and $(A, \mathcal{L}_\omega)$ are isomorphic polarized varieties. Moreover, ι and ι_ω coincide when restricted to the centre R_F of $\mathcal{O}$ and we obtain that $(A, \iota|_{R_F}, \mathcal{L}) \simeq (A, \iota_\omega|_{R_F}, \mathcal{L}_\omega)$. Therefore, we obtain that the group U_0 acts on the geometric fibres of the morphism $\pi_F : X_B \to \mathcal{H}_F$ at non Heegner points.

Let now $\omega \in V_0$ be an element represented by $\omega \in \mathcal{O}_+$ with $\mathrm{n}(\omega) = m \in F_+^*$ and such that

$$B = F + F\mu + F\chi + F\mu\chi$$

for some $\chi \in \mathcal{O}$, $\chi^2 = m$, $\mu\chi = -\chi\mu$. Let $[(A, \iota, \mathcal{L})] \in X_B$ be a closed point over $[(A, j_F, \mathcal{L})] \in \mathcal{H}_F$. We have that $\omega[(A, \iota, \mathcal{L})] = [(A, \iota_\omega, \mathcal{L}_\omega)]$ and we must show that $(A, \iota|_{R_F}, \mathcal{L}) \simeq (A, \iota_\omega|_{R_F}, \mathcal{L}_\omega)$. Again, since $\iota|_{R_F} = \iota_\omega|_{R_F}$, we only need to see that $\mathcal{L}$ and $\mathcal{L}_\omega$ are isomorphic polarizations on A.

To do so, we first note that $\alpha = \omega^{-1} \cdot \chi$ is a unit in $\mathcal{O}$ of reduced norm $\mathrm{n}(\alpha) = -1$. This is due to the fact that the principal ideal mR_F is supported at the prime ideals $\wp|D$, as can be checked locally.

Let us identify α with the automorphism $\iota(\alpha)$ of A. We now show that $\alpha^*(\mathcal{L}_\omega) = \mathcal{L} \in \mathrm{NS}(A)$. Indeed, since $c_1(\mathcal{L})$ and χ anti-commute, we have by Theorem 3.3.1 that $c_1(\alpha^*(\mathcal{L}_\omega)) = \overline{\alpha}\omega^{-1}c_1(\mathcal{L})\omega\alpha = \frac{1}{m}\overline{\chi}c_1(\mathcal{L})\chi = -\chi^{-1}c_1(\mathcal{L})\chi = \chi^{-1}\chi c_1(\mathcal{L}) = c_1(\mathcal{L})$. Since $W_0 = U_0 \cdot V_0$, this proves part of our claim.

Let us now show that the action of W_0 on the fibres of π_F at non Heegner points is free and transitive. By the Skolem-Noether Theorem, if a closed point $[(A, \iota, \mathcal{L})] \in X_B(\mathbb{C})$ lies over $[(A, j_F, \mathcal{L})]$, any element on the fibre of π_F at this point must be represented by a triplet $(A, \omega^{-1}\iota\omega, \mathcal{L}_\omega)$ for some $\omega \in \mathrm{Norm}_{B^*}(\mathcal{O})$. Moreover, since $\mathcal{L}_\omega$ is a polarization on A, it follows from Theorem 3.5.3 that $\omega \in B_+^*$.

Since $(A, \iota, \mathcal{L}) \simeq (A, \iota_\omega, \mathcal{L}_\omega)$ for any $\omega \in F^* \cdot \mathcal{O}^1$, we deduce that there exists a subgroup of $W^1 = \mathrm{Norm}_{B_+^*}(\mathcal{O})/(F^* \cdot \mathcal{O}^1)$ that acts freely and transitively on the fibre of π_F at any non Heegner point. It follows from the discussion above that this subgroup must contain W_0.

Let us now see that it cannot be larger than W_0. Assume that $\omega \in W^1$ is such that $\mathcal{L} \simeq \mathcal{L}_\omega$. Then there exists $\alpha \in \mathcal{O}^* \overset{\iota}{\simeq} \mathrm{Aut}(A)$ such that $\overline{\alpha}\omega^{-1}\mu\omega\alpha = \mu$. Taking reduced norms, this already implies that $\mathrm{n}(\alpha)^2 = 1$.

If $\mathrm{n}(\alpha) = 1$, then $\overline{\alpha} = \alpha^{-1}$ and the above yields that $\mu\omega\alpha = \omega\alpha\mu$. This means that $\omega\alpha \in F(\mu) \cap \mathcal{O} = S$ and therefore $\omega \in U_0$. If $\mathrm{n}(\alpha) = -1$, then $\mu\omega\alpha = -\omega\alpha\mu$. Write $\chi = \omega\alpha$. We then have that $\mathrm{tr}(\mu\chi) = \mu\chi + \overline{\chi\mu} = \mu\chi - \overline{\chi}\mu = -\mathrm{tr}(\chi)\mu \in F$. Thus, since $\mathrm{tr}(\chi) \in F$, we deduce that actually $\mathrm{tr}(\chi) = 0$. Since $\mathrm{n}(\chi) = \mathrm{n}(\omega)\mathrm{n}(\alpha) = -\mathrm{n}(\omega)$, we obtain that $\chi^2 = \mathrm{n}(\omega)$. This says that $\omega \in V_0$. $\square$

Since the action of $W_0 \subseteq \mathrm{Aut}_{\mathbb{Q}}(X_B)$ is algebraic, it extends to an action on the fibres of π_F at the points in the closure $\widetilde{X}_{B/F}$ of the dense subset of non Heegner points of $\widetilde{X}_{B/F}$. Similarly, the action must be free and transitive on the fibres at points in an algebraic subset $\mathcal{U}_F$ of $\widetilde{X}_{B/F}$.

We obtain as a consequence that $\pi_F : X_B \to \mathcal{H}_F$ factorizes through the natural projection of X_B onto the quotient X_B/W_0 and a morphism $b_F : X_B/W_0 \to \mathcal{H}_F$ that is one-to-one outside the Heegner locus of X_B/W_0. Since Heegner points are isolated and b_F is algebraic, b_F must be a birational equivalence between X_B/W_0 and its image in $\mathcal{H}_F$ whose inverse is defined everywhere but at a finite set $\mathcal{T}_F$ of Heegner points.

Moreover, since $W_0 \subseteq \mathrm{Aut}_{\mathbb{Q}}(X_B)$, the projection $X_B \to X_B/W_0$ is defined over $\mathbb{Q}$. Since π_F is also a morphism over $\mathbb{Q}$ which is the composition of the above projection and the birational equivalence b_F, it follows that b_F is also defined over $\mathbb{Q}$. This finishes the proof of the first part of Theorem 4.4.4.

Now let (S, φ) be an Eichler pair for $\mathcal{O}$ and identify S with its image $\varphi(S)$ in $\mathcal{O}$. As we saw, it induces a natural morphism $\pi_{S,\varphi} : X_B \to \mathcal{H}_S$ from X_B into the Hilbert modular variety $\mathcal{H}_S$ in such a way that we have

$$\pi_F : X_B \overset{\pi_{(S,\varphi)}}{\to} \mathcal{H}_S \to \mathcal{H}_F.$$

As the situation is very similar to the one studied above, we will limit ourselves to showing that the subgroup $V_0(S)$ of the stable group W_0 acts freely and transitively on the fibre of $\pi_{(S,\varphi)}$ at any non Heegner point of $\pi_{(S,\varphi)}(X_B) = \widetilde{X}_{B/(L,\varphi)}$ in $\mathcal{H}_S$.

Let $[(A, \iota, \mathcal{L})]$ thus be a closed point on X_B represented by a principally polarized abelian variety with quaternionic multiplication. From the above discussion, it is clear that any other point at the same geometric fibre of $\pi_{(S,\varphi)}$ as $[(A, \iota, \mathcal{L})]$ is represented by $(A, \iota_\omega, \mathcal{L}_\omega)$ for some $\omega \in W_0$.

If $(A, \iota \cdot \varphi, \mathcal{L}) \simeq (A, \iota_\omega \cdot \varphi, \mathcal{L}_\omega)$ is an isomorphism given by $\alpha \in \mathcal{O}^*$, then, as we already saw, $\mathrm{n}(\alpha) = \pm 1$. Write $\chi = \omega\alpha$.

If $\mathrm{n}(\alpha) = 1$, then $\chi \in S \subset F(\mu)$. At the same time, we must have that, for any $\beta \in S \subset \mathcal{O}$, $\chi\beta = \beta\chi$. This means that χ commutes element-wise with L and $F(\mu)$. Since the two quadratic extensions are distinct because the first is totally real while the second purely imaginary, $\chi \in R_F^*$ and thus ω is trivial in W_0.

If $\mathrm{n}(\alpha) = -1$, then $\alpha^*(\mathcal{L}_\omega) = \mathcal{L}$ implies that

$$B = F + F\mu + F\chi + F\mu\chi$$

while $\alpha^{-1}\iota_\omega|_S\alpha = \iota|_S$ says that $\chi \in S$. Thus $\omega \in V_0(S)$. The converse also holds and the theorem follows as before. □

4.6 The quaternionic locus in the moduli space of principally polarized abelian varieties

Let $g = 2n$ for some positive integer n and let $\mathcal{A}_g/\mathbb{Q}$ be the moduli space of principally polarized abelian varieties of dimension g. As in the preceding sections, we let $[(A, \mathcal{L})]$ denote the isomorphism class of a principally polarized abelian variety $(A, \mathcal{L})$ regarded as a closed point in $\mathcal{A}_g$.

Let F be a totally real number field of degree $[F : \mathbb{Q}] = n$. Let $\mathcal{O}$ be a maximal order in a totally indefinite division quaternion algebra B over F. As in Chapter 3, assume that $\vartheta_{F/\mathbb{Q}}$ and $\mathrm{disc}(B)$ are coprime ideals.

It is our aim to use the results of the preceding chapters to investigate the nature of the following object.

Definition 4.6.1. The quaternionic locus

$$Q_{\mathcal{O}} \subset \mathcal{A}_g(\mathbb{C})$$

is the locus of complex principally polarized abelian varieties $[(A, \mathcal{L})]$ with $\mathrm{End}(A) \supseteq \mathcal{O}$.

We naturally wonder about the geometry of $Q_{\mathcal{O}}$. Let us remark that the algebraicity of $Q_{\mathcal{O}}$ is not granted from the definition. A reformulation of Proposition 3.6.2 in Chapter 3 yields

Proposition 4.6.2. *The quaternionic locus $Q_{\mathcal{O}}$ is non empty if and only if* $\mathrm{disc}(B)$ *is a totally positive principal ideal of F.*

Proof. By Propositions 1.2.5 and 1.2.8, there exists a unique $\mathcal{O}$-left ideal $\mathcal{I}_\vartheta$ such that $\mathrm{n}(\mathcal{I}_\vartheta) = \vartheta^{-1}$. From Proposition 3.6.2, there exist principally polarized abelian varieties A with quaternionic multiplication by $\mathcal{O}$ if and only if $H_1(A, \mathbb{Z}) \simeq \mathcal{I}_\vartheta$ as left $\mathcal{O}$-modules and $\mathrm{disc}(B)$ is trivial in $\mathrm{Pic}_+(F)$.

In consequence, and for the sake of simplicity, we assume that $h_+(F) = 1$. This automatically implies that the quaternionic locus $Q_{\mathcal{O}}$ is non empty. We fix a totally positive generator $D \in F_+^*$ of $\mathrm{disc}(B)$. Note that $Q_{\mathcal{O}}$ is the disjoint union of the set of principally polarized abelian varieties $[(A, \mathcal{L})]$

with quaternionic multiplication by $\mathcal{O}$ and the set of Heegner points. The former are distinguished by the fact that $\mathrm{End}(A) \simeq \mathcal{O}$ while the latter satisfy that $\mathrm{End}(A) \supsetneq \mathcal{O}$.

Let $\mathcal{I}_\vartheta$ be the $\mathcal{O}$-left ideal as in the above proof and let $\mu \in \mathcal{O}_0$ be a pure quaternions such that $\mathrm{n}(\mu) = uD$ for some unit $u \in R^*_{F+}$. We also note that the set $\widetilde{X}_\mu(\mathbb{C})$ of complex points of the Shimura variety $\widetilde{X}_{(\mathcal{O},\mathcal{I}_\vartheta,\varrho_\mu)}/\mathbb{Q}$ attached in Section 4.4 to the datum $(\mathcal{O},\mathcal{I}_\vartheta,\varrho_\mu)$ sits inside $Q_\mathcal{O}$.

Proposition 4.6.3. *Let μ, $\mu' \in \mathcal{O}_0$ be two pure quaternions such that $\mathrm{n}(\mu) = uD$ and $\mathrm{n}(\mu') = u'D$ for some units u, $u' \in R^*_{F+}$. If $\widetilde{X}_\mu(\mathbb{C})$ and $\widetilde{X}_{\mu'}(\mathbb{C})$ are different subvarieties of $\mathcal{A}_g(\mathbb{C})$, then $\widetilde{X}_\mu(\mathbb{C}) \cap \widetilde{X}_{\mu'}(\mathbb{C})$ is a finite set of Heegner points.*

Proof. Assume that the isomorphism class $[A,\mathcal{L}]$ of a principally polarized abelian variety falls at the intersection of $\widetilde{X}_\mu$ and $\widetilde{X}_{\mu'}$ in $\mathcal{A}_g$. Write $[A,\mathcal{L}] = \pi([A,\iota,\mathcal{L}]) = \pi([A',\iota',\mathcal{L}'])$ as the image by π of points in X_μ and $X_{\mu'}$, respectively. Since $[(A,\mathcal{L})] = [(A',\mathcal{L}')] \in Q_\mathcal{O}$, we can identify the pair $(A,\mathcal{L}) = (A',\mathcal{L}')$ through a fixed isomorphism of polarized abelian varieties.

Let us assume that $[A,\mathcal{L}] = [A',\mathcal{L}']$ was not a Heegner point. Then $\iota : \mathcal{O} \simeq \mathrm{End}(A)$ would be an isomorphism of rings such that $c_1(\mathcal{L}) = \mu$. We then would have by Theorem 3.3.1 and Corollary 3.3.8 that $c_1(\mathcal{L}) = c_1(\mathcal{L}') = \mu = \mu'$ up to multiplication by elements in F^*. Since $\widetilde{X}_\mu = \widetilde{X}_{u\mu}$ for all units $u \in R^*_F$, this would contradict the statement. Since the set of Heegner points in $\mathcal{A}_g(\mathbb{C})$ is discrete, we conclude that X_μ and $X_{\mu'}$ meet at a finite set of Heegner points. $\square$

Proposition 4.6.4. *1. The locus $Q_\mathcal{O}$ is the set of complex points $\mathcal{Q}_\mathcal{O}(\mathbb{C})$ of a reduced complete subscheme $\mathcal{Q}_\mathcal{O}$ of $\mathcal{A}_g$ defined over $\mathbb{Q}$.*

2. Let $\rho(\mathcal{O})$ be the number of absolutely irreducible components of $\mathcal{Q}_\mathcal{O}$. Then there exist quaternions $\mu_k \in \mathcal{O}_0$ with $\mu_k^2 + u_k D = 0$ for $u_k \in R^*_{F+}$, $1 \leq k \leq \rho(\mathcal{O})$, such that

$$\mathcal{Q}_\mathcal{O} = \bigcup \widetilde{X}_{\mu_k}.$$

is the decomposition of $\mathcal{Q}_\mathcal{O}$ into irreducible components.

Proof. Let $[(A,\mathcal{L})] \in Q_\mathcal{O}$ be the isomorphism class of a complex principally polarized abelian variety such that $\mathrm{End}(A) \simeq \mathcal{O}$. As in the proof of

Proposition 4.6.2, we have that $H_1(A,\mathbb{Z}) \simeq \mathcal{I}_\vartheta$. By Proposition 3.6.2, the Rosati involution with respect to $\mathcal{L}$ on $\mathcal{O}$ must be of the form $\varrho = \varrho_\mu$ for some $\mu \in \mathcal{O}$ with $\mu^2 + uD = 0$, $u \in R^*_{F+}$. Thus $[(A,\mathcal{L})] \in \widetilde{X}_\mu(\mathbb{C})$, namely the set of complex points on a reduced, irreducible, complete and possibly singular scheme over $\mathbb{Q}$ (cf. [Sh63], [Sh67]). Since the set of Heegner points $[(A,\mathcal{L})] \in \widetilde{X}_\mu(\mathbb{C})$ is a set of isolated points lying on the Zariski closure of their complement, we conclude that $Q_{\mathcal{O}}$ is the union of the Shimura varieties $\widetilde{X}_\mu(\mathbb{C})$ as μ varies among pure quaternions satisfying the above properties.

Let us now show that $Q_{\mathcal{O}}$ is actually covered by finitely many Shimura varieties $\widetilde{X}_{\mu_k}(\mathbb{C})$. Let $A/\mathbb{C}$ be an arbitrary abelian variety with quaternionic multiplication by $\mathcal{O}$ such that $H_1(A,\mathbb{Z}) \simeq \mathcal{I}_\vartheta$ and fix an isomorphism $\iota : \mathcal{O} \simeq \mathrm{End}(A)$. Let $(\mathcal{O},\mu)$ be any principally polarized pair. Since $h_+(F) = 1$, there exists a unit $u \in R^*_F$ such that $u\mu$ is an ample quaternion in the sense of [Ro2], §5. Let $\mathcal{L} \in \mathrm{NS}(A)$ be the line bundle on A such that $c_1(\mathcal{L})^{-1} = u\mu$. From Corollary 3.3.8, Theorem 3.5.3 and Corollary 3.6.6, it follows that $\mathcal{L}$ is a principal polarization on A such that the isomorphism class of the triplet $(A,\iota,\mathcal{L})$ corresponds to a closed point in $X_\mu(\mathbb{C})$ and hence $[A,\mathcal{L}] \in \widetilde{X}_\mu$. Since, by Proposition 4.6.3, the intersection points of two different Shimura varieties $\widetilde{X}_\mu(\mathbb{C})$ and $\widetilde{X}_{\mu'}(\mathbb{C})$ in $\mathcal{A}_g(\mathbb{C})$ are Heegner points, this shows that for every irreducible component of $Q_{\mathcal{O}}$ there exists at least one principal polarization $\mathcal{L}$ on A such that $[A,\mathcal{L}]$ lies on it. Consequently, the number $\pi_0(A)$ of isomorphism classes of principal polarizations on A is an upper bound for the number $\rho(\mathcal{O})$ of irreducible components of $Q_{\mathcal{O}}$. Since, by Theorem 3.7.2, the number $\pi_0(A)$ is a finite number, this yields the proof of the proposition. □

In view of Proposition 4.6.4, it is natural to pose the following

Question 4.6.5. What is the number $\rho(\mathcal{O})$ of irreducible components of $\mathcal{Q}_{\mathcal{O}}$? When is $\mathcal{Q}_{\mathcal{O}}$ irreducible?

Let us relate Question 4.6.5 to the following problem. In Chapter 3, Theorem 3.7.2, we computed the number $\pi_0(A)$ of principal polarizations on an abelian variety A with quaternion multiplication by $\mathcal{O}$ as a finite sum of relative class numbers of suitable orders in the CM-fields $F(\sqrt{-uD})$ for $u \in R^*_{F+}/R^{*2}_F$. This has the following modular interpretation:

Let $\mathcal{L}_1$, ..., $\mathcal{L}_{\pi_0(A)}$ be representatives of the $\pi_0(A)$ distinct principal polarizations on A. Then the pairwise nonisomorphic principally polarized abelian varieties $[(A,\mathcal{L}_1)]$, ..., $[(A,\mathcal{L}_{\pi_0(A)})]$ correspond to all closed points in

$\mathcal{Q}_{\mathcal{O}}$ whose underlying abelian variety is isomorphic to A. We then naturally ask the following

Question 4.6.6. Let A be an abelian variety with quaternionic multiplication by $\mathcal{O}$. How are the distinct principal polarizations $[(A, \mathcal{L}_j)]$ distributed among the irreducible components $\widetilde{X}_{\mu_k}$ of $\mathcal{Q}_{\mathcal{O}}$?

It turns out that the two questions above are very related. The linking ingredient is provided by the definition below, which establishes a slightly coarser equivalence relationship on polarizations than the one considered in Chapter 3. Indeed, in Chapter 3 we agreed to saying that two polarizations $\mathcal{L}$, $\mathcal{L}'$ on an abelian variety A with quaternionic multiplication by $\mathcal{O}$ are isomorphic if there is an automorphism $\alpha \in \mathrm{Aut}(A)$ such that $\alpha^*(\mathcal{L}) = \mathcal{L}'$. Later, we remarked in Section 4.3.1 of Chapter 4 that Shimura also considered weak isomorphisms of polarizations. Namely, in our language, $\mathcal{L}$ and $\mathcal{L}'$ are weakly isomorphic if $c_1(\mathcal{L}) \simeq mc_1(\mathcal{L}') \in \mathrm{NS}(A)$ for some $m \in F_+^*$. We shall denote it $\mathcal{L} \simeq_w \mathcal{L}'$.

Definition 4.6.7. Let A be an abelian variety with quaternionic multiplication by $\mathcal{O}$. Two principal polarizations $\mathcal{L}$ and $\mathcal{L}'$ on A are *Atkin-Lehner isogenous*, denoted by $\mathcal{L} \sim \mathcal{L}'$, if there is an isogeny $\omega \in \mathcal{O} \cap \mathrm{Norm}_{B_+^*}(\mathcal{O})$ of A such that

$$\omega^*(\mathcal{L}) \simeq_w \mathcal{L}'.$$

We note that the above definition bears a closed relationship with the modular interpretation of the positive Atkin-Lehner group W^1 given in Proposition 4.3.3.

Definition 4.6.8. Let A be an abelian variety with quaternionic multiplication by $\mathcal{O}$. We let $\hat{\Pi}_0(A)$ be the set of principal polarizations on A up to Atkin-Lehner isogeny and we let $\hat{\pi}_0(A) = \sharp\hat{\Pi}_0(A)$ denote its cardinality.

Theorem 4.6.9 (Distribution of principal polarizations). *Let A be an abelian variety with quaternionic multiplication by $\mathcal{O}$ and let $\mathcal{L}_1$, ..., $\mathcal{L}_{\pi_0(A)}$ be representatives of the $\pi_0(A)$ distinct principal polarizations on A.*

Then, two closed points $[A, \mathcal{L}_i]$ and $[A, \mathcal{L}_j]$ lie on the same irreducible component of $\mathcal{Q}_{\mathcal{O}}$ if and only if the polarizations $\mathcal{L}_i$ and $\mathcal{L}_j$ are Atkin-Lehner isogenous.

Proof. We know from Proposition 4.6.4 that any irreducible component of $\mathcal{Q}_{\mathcal{O}}$ is $\widetilde{X}_\mu = \widetilde{X}_{(\mathcal{O},\mathcal{I}_\vartheta,\varrho_\mu)}$ for some $\mu \in \mathcal{O}$, $\mu^2 + uD = 0$, $u \in R^*_{F+}$. We single out and fix one of them.

Let $\mathcal{L}$ be a principal polarization on A such that $[(A, \mathcal{L})]$ lies on $\widetilde{X}_\mu$ and let $\mathcal{L}'$ be a second principal polarization on A. We claim that $[A, \mathcal{L}'] \in \widetilde{X}_\mu$ if and only if there exists $\omega \in \mathrm{Norm}_{B^*_+}(\mathcal{O})$ such that $\mathcal{L}'$ and $\omega^*(\mathcal{L})$ are weakly isomorphic.

Assume first that $\mathcal{L}' \simeq_w \omega^*(\mathcal{L})$ for some $\omega \in \mathcal{O} \cap \mathrm{Norm}_{B^*_+}(\mathcal{O})$. This amounts to say that $\overline{\omega} c_1(\mathcal{L})\omega = m c_1(\mathcal{L}')$ for some $m \in F^*$. Since both $\omega^*(\mathcal{L})$ and $\mathcal{L}'$ are polarizations, we deduce from Theorem 3.5.3 that $m \in F^*_+$. Moreover, since $\mathcal{L}$ and $\mathcal{L}'$ are principal, we obtain from Proposition 3.3.3 that $m = u\mathrm{n}(\omega)$ for some $u \in R^*_{F+}$.

Note that $(A, \iota_\omega, \mathcal{L}')$ is a principally polarized abelian variety with quaternionic multiplication such that the Rosati involution that $\mathcal{L}'$ induces on $\iota_\omega(\mathcal{O})$ is ϱ_μ. Indeed, this follows because $\iota_\omega(\beta)^{\circ_{\mathcal{L}'}} = \iota((\omega^{-1}\beta\omega))^{\circ_{\mathcal{L}'}} = \iota((\omega^{-1}\mu\omega)^{-1}\overline{\omega^{-1}\beta\omega}(\omega^{-1}\mu\omega)) = \iota_\omega(\mu^{-1}\overline{\beta}\mu)$. This shows that, if $\mathcal{L}'$ and $\omega^*(\mathcal{L})$ are weakly isomorphic for some $\omega \in \mathrm{Norm}_{B^*_+}(\mathcal{O})$, then $[A, \mathcal{L}'] \in \widetilde{X}_\mu$.

Conversely, let us assume that $[A, \mathcal{L}'] \in \widetilde{X}_\mu$. Let $\iota' : \mathcal{O} \hookrightarrow \mathrm{End}(A)$ be such that $[A, \iota', \mathcal{L}'] \in X_{(\mathcal{O},\mathcal{I}_\vartheta,\varrho_\mu)}$. By the Skolem-Noether Theorem, it holds that $\iota' = \omega^{-1}\iota\omega$ for some $\omega \in \mathrm{Norm}_{B^*}(\mathcal{O})$; we can assume that $\omega \in \mathcal{O}$ by suitably scaling it. Since it holds that $\iota_\omega(\beta)^{\circ_{\mathcal{L}'}} = \iota_\omega(\mu^{-1}\overline{\beta}\mu)$ for any $\beta \in \mathcal{O}$, we have that $c_1(\mathcal{L}') = u\omega^{-1}c_1(\mathcal{L})\omega$ for some $u \in R^*_F$ such that $u\mathrm{n}(\omega) \in F^*_+$. Since $\mathrm{n}(\mathcal{O}^*) = R^*_F$, we can find $\alpha \in \mathcal{O}^*$ with reduced norm $\mathrm{n}(\alpha) = u^{-1}$ and thus $\omega\alpha \in B^*_+$. The automorphism $\alpha \in \mathcal{O}^* = \mathrm{Aut}(A)$ induces an isomorphism between the polarizations $\mathcal{L}_{\omega\alpha}$ and $\mathcal{L}'$, since $c_1(\alpha^*(\mathcal{L}')) = \overline{\alpha}(u\omega^{-1}c_1(\mathcal{L})\omega)\alpha = c_1(\mathcal{L}_{\omega\alpha})$. Since, according to our Definition 4.3.2, $\mathcal{L}_{\omega\alpha}$ is weakly equivalent to $\mathcal{L}$, this concludes our claim above and also proves the theorem. □

Corollary 4.6.10. *The number of irreducible components of $\mathcal{Q}_{\mathcal{O}}$ is*

$$\rho(\mathcal{O}) = \hat{\pi}_0(A),$$

independently of the choice of A.

For any irreducible component $\widetilde{X}_{\mu_k}$ of $\mathcal{Q}_{\mathcal{O}}$, let $\Pi_0^{(k)}(A) \subset \Pi_0(A)$ denote the set of isomorphism classes of the isogeny class of principal polarizations lying on $\widetilde{X}_{\mu_k}$.

As another immediate consequence of Theorem 4.6.9, we obtain the following corollary, which establishes an internal structure on the set $\Pi_0(A)$.

Roughly, it asserts that $\Pi_0(A) = \bigcup_{k=1}^{\rho(\mathcal{O})} \Pi_0^{(k)}(A)$ is the union of several isogeny classes that are equipped with a free and transitive action of a 2-torsion finite abelian group.

Corollary 4.6.11. *Let A be an abelian variety with quaternionic multiplication by $\mathcal{O}$. Let $\widetilde{X}_{\mu_k}$ be an irreducible component of $\mathcal{Q}_{\mathcal{O}}$ and let $W_0^{(k)} \subseteq W^1$ be the stable subgroup attached to the polarized order $(\mathcal{O}, \mu_k)$.*

Then there is a free and transitive action of $W^1/W_0^{(k)}$ on $\Pi_0^{(k)}(A)$.

In the case of a non twisting maximal order $\mathcal{O}$, we have that the stable group $W_0(\mathcal{O}, \mu)$ attached to a principally polarized pair $(\mathcal{O}, \mu)$ is $U_0(\mathcal{O}, \mu)$. The following corollary follows from Lemma 4.4.6 in Section 4.7.

Corollary 4.6.12. *Let $\mathcal{O}$ be a non twisting maximal order in B and assume that, for any $u \in R_{F+}^*$, all primitive roots of unity of odd order in the CM-field $F(\sqrt{-uD})$ are contained in the order $R_F[\sqrt{-uD}]$.*

Let A be an abelian variety with quaternionic multiplication by $\mathcal{O}$. Then the distinct isomorphism classes of principally polarized abelian varieties $[(A, \mathcal{L}_1)]$, ..., $[(A, \mathcal{L}_{\pi_0(A)})]$ are equidistributed *among the $\rho(\mathcal{O})$ irreducible components of $\mathcal{Q}_{\mathcal{O}}$.*

In particular, it then holds that

$$\pi_0(A) = \frac{|W^1|}{|W_0|} \cdot \rho(\mathcal{O}).$$

4.6.1 Shimura curves embedded in Igusa's threefold

The whole picture becomes particularly neat when we consider the case of abelian surfaces and Shimura curves. Let B be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $D = p_1 \cdot ... \cdot p_{2r}$ and let $\mathcal{O}$ be a maximal order in B. Since $h(\mathbb{Q}) = 1$, there is a single choice of $\mathcal{O}$ up to conjugation by B^*. Moreover, all left ideals of $\mathcal{O}$ are principal and isomorphic to $\mathcal{O}$ as left $\mathcal{O}$-modules.

Let A be a complex abelian surface with quaternionic multiplication by $\mathcal{O}$. By Theorem 3.1.2, A is principally polarizable and the number of isomorphism classes of principal polarizations on A is

$$\pi_0(A) = \frac{\widetilde{h}(-D)}{2},$$

where, for any nonzero squarefree integer d, we write

$$\widetilde{h}(d) = \begin{cases} h(4d) + h(d) & \text{if } d \equiv 1 \mod 4, \\ h(4d) & \text{otherwise.} \end{cases}$$

For any integral element $\mu \in \mathcal{O}$ such that $\mu^2 + D = 0$, let now $X_\mu = X_{(\mathcal{O},\mathcal{O},\varrho_\mu)}$ be the Shimura curve that coarsely represents the functor which classifies principally polarized abelian surfaces $(A, \iota, \mathcal{L})$ with quaternionic multiplication by $\mathcal{O}$ such that the Rosati involution with respect to $\mathcal{L}$ on $\mathcal{O}$ is ϱ_μ. This is an algebraic curve over $\mathbb{Q}$ whose isomorphism class does not actually depend on the quaternion μ, but only on the discriminant D (cf. [Sh67]). Hence, it is usual to simply denote this isomorphism class as X_D.

Let $W^1 = W = \{\omega_m : m|D\} \simeq (\mathbb{Z}/2\mathbb{Z})^{2r}$ be the Atkin-Lehner group attached to $\mathcal{O}$ in Section 4.3. From Section 4.3.1, we know that $W \subseteq \mathrm{Aut}_{\mathbb{Q}}(X_D)$ is a subgroup of the group of automorphisms of the Shimura curve X_D.

Let now $\mathcal{A}_2$ be the moduli space of principally polarized abelian surfaces. By the work of Igusa (cf. [Ig60]), it is an affine scheme over $\mathbb{Q}$ which contains the moduli space $\mathcal{M}_2$ of curves of genus 2 as a Zariski open and dense subset, immersed in $\mathcal{A}_2$ via the Torelli embedding.

Sitting in $\mathcal{A}_2$ there is the quaternionic locus $\mathcal{Q}_{\mathcal{O}}$ of isomorphism classes of principally polarized abelian surfaces $[(A, \mathcal{L})]$ such that $\mathrm{End}(A) \supseteq \mathcal{O}$. Since all maximal orders $\mathcal{O}$ in B are pairwise conjugate, the quaternionic locus $\mathcal{Q}_{\mathcal{O}}$ does not actually depend on the choice of $\mathcal{O}$ and we may simply denote it by $\mathcal{Q}_{\mathcal{O}} = \mathcal{Q}_D$.

There are forgetful finite morphisms $\pi : X_\mu \to \mathcal{Q}_D \subset \mathcal{A}_2$ which map the Shimura curve X_μ onto an irreducible component $\widetilde{X}_\mu$ of $\mathcal{Q}_D$. Although the isomorphism class of $X_\mu/\mathbb{Q}$ does not depend on the choice of μ, the image $\widetilde{X}_\mu \subset \mathcal{Q}_D$ *does depend* on this choice.

Let us now compare the non twisting and twisting case, respectively. We firstly assume that

$$B \not\simeq (\frac{-D, m}{\mathbb{Q}})$$

for all positive divisors $m|D$ of D. Then all principally polarized pairs $(\mathcal{O}, \mu)$ in B are *non twisting* and the stable subgroup attached to $(\mathcal{O}, \mu)$ is

$$W_0 = U_0 = \langle \omega_D \rangle \subset W,$$

independently of the choice of μ. By Theorem 4.4.4, we have that any irreducible component $\widetilde{X}_\mu$ of $\mathcal{Q}_D$ is birationally equivalent to the Atkin-Lehner quotient $X_D/\langle\omega_D\rangle$ and thus the quaternionic locus $\mathcal{Q}_D$ in $\mathcal{A}_2$ is the union of pairwise birationally equivalent Shimura curves $\widetilde{X}_{\mu_1}$, ...,$\widetilde{X}_{\mu_{\rho(B)}}$, meeting at a finite set of Heegner points.

Moreover, for any abelian surface A with quaternionic multiplication by $\mathcal{O}$, it follows from Theorem 4.6.9 that the closed points $\{[(A,\mathcal{L}_j)]\}_{j=1}^{\pi_0(A)}$ are equidistributed among the $\rho(\mathcal{O})$ irreducible components of $\mathcal{Q}_D$. In addition, Corollary 4.6.12 ensures that $|W/W_0| = 2^{2r-1}|\pi_0(A)$, as genus theory for binary quadratic forms already ensures. This two facts yield that the number of irreducible components of the quaternionic locus in the non twisting case is

$$\rho(\mathcal{O}) = \frac{\widetilde{h}(-D)}{2^{2r}}.$$

On the other hand, let us assume that

$$B \simeq (\frac{-D,m}{\mathbb{Q}})$$

for some $m|D$. Note that this may hold for several positive divisors of D. In this case, there can be several different birational classes of irreducible components on $\mathcal{Q}_D$. Indeed, the assumption means that there exist pure quaternions $\mu \in \mathcal{O}$, $\mu^2 + D = 0$, such that $(\mathcal{O},\mu)$ is a *twisting* polarized order. Then

$$W_0(\mathcal{O},\mu) = \langle\omega_m,\omega_D\rangle$$

and $\widetilde{X}_\mu$ is birationally equivalent to $X_D/\langle\omega_m,\omega_D\rangle$. We may refer to $\widetilde{X}_\mu$ as a twisting irreducible component of $\mathcal{Q}_D$.

In addition to these, there may exist non twisting polarized orders $(\mathcal{O},\mu)$ such that the corresponding irreducible components $\widetilde{X}_\mu$ of $\mathcal{Q}_D$ are birationally equivalent to $X_D/\langle\omega_D\rangle$. We may refer to these as the non twisting irreducible components of $\mathcal{Q}_D$.

We then have the following lower and upper bounds for the number of irreducible components of $\mathcal{Q}_D$:

$$\frac{\widetilde{h}(-D)}{2^{2r}} < \quad \rho(\mathcal{O}) \quad \leq \frac{\widetilde{h}(-D)}{2^{2r-1}}.$$

Summing up, we obtain the following

Theorem 4.6.13. *Let B be an indefinite division quaternion algebra over $\mathbb{Q}$ of discriminant $D = p_1 \cdot \ldots \cdot p_{2r}$. Then, the quaternionic locus $\mathcal{Q}_D$ in $\mathcal{A}_2$ is irreducible if and only if*

$$\widetilde{h}(-D) = \begin{cases} 2^{2r-1} & \text{if } B \simeq (\frac{-D,m}{\mathbb{Q}}) \text{ for some } m|D, \\ 2^{2r} & \text{otherwise.} \end{cases}$$

Proof. If B is not a twisting quaternion algebra, we already know from the above that the number of irreducible components of $\mathcal{Q}_D$ is $\dfrac{\widetilde{h}(-D)}{2^{2r}}$. Hence, in this case, the quaternionic locus of discriminant D in $\mathcal{A}_2$ is irreducible if and only if $\widetilde{h}(-D) = 2^{2r}$. If on the other hand B is twisting, it follows from the above inequalities that $\mathcal{Q}_D$ is irreducible if and only if $\widetilde{h}(-D) = 2^{2r-1}$. $\square$

In view of Theorem 4.6.13, there arises a closed relationship between the irreductibility of the quaternionic locus in Igusa's threefold and the genus theory of integral binary quadratic forms and the classical *numeri idonei* studied by Euler, Schinzel and others. We refer the reader to [Ar95] and [Sch59] for the latter.

4.6.2 Hashimoto-Murabayashi's families

As the simplest examples to be considered, let B_6 and B_{10} be the rational quaternion algebras of discriminant $D = 2 \cdot 3 = 6$ and $2 \cdot 5 = 10$, respectively. Hashimoto and Murabayashi [HaMu95] exhibited two families of principally polarized abelian surfaces with quaternionic multiplication by a maximal order in these quaternion algebras. Namely, let

$$C^{(6)}_{(s,t)} : Y^2 = X(X^4 + PX^3 + QX^2 + RX + 1)$$

be the family of curves with

$$P = 2s + 2t, \quad Q = \frac{(1+2t^2)(11-28t^2+8t^4)}{3(1-t^2)(1-4t^2)}, \quad R = 2s - 2t$$

over the base curve

$$g^{(6)}(t,s) = 4s^2t^2 - s^2 + t^2 + 2 = 0.$$

And let

$$C^{(10)}_{(s,t)} : Y^2 = X(P^2X^4 + P^2(1+R)X^3 + PQX^2 + P(1-R)X + 1)$$

be the family of curves with

$$P = \frac{4(2t+1)(t^2-t-1)}{(t-1)^2}, \quad Q = \frac{(t^2+1)(t^4+8t^3-10t^2-8t+1)}{t(t-1)^2(t+1)^2}$$

and

$$R = \frac{(t-1)s}{t(t+1)(2t+1)}$$

over the base curve

$$g^{(10)}(t,s) = s^2 - t(t-2)(2t+1) = 0.$$

Let $J^{(6)}_{(s,t)} = \mathrm{Jac}(C^{(6)}_{(s,t)})$ and $J^{(10)}_{(s,t)} = \mathrm{Jac}(C^{(10)}_{(s,t)})$ be the Jacobian surfaces of the fibres of the families of curves above respectively. It was proved in [HaMu95] that their ring of endomorphisms contain a maximal order in B_6 and B_{10}, respectively.

Both $B_6 = (\frac{-6,2}{\mathbb{Q}})$ and $B_{10} = (\frac{-10,2}{\mathbb{Q}})$ are twisting quaternion algebras. Moreover, it turns out from our formula for $\pi_0(A)$ above that any abelian surface A with quaternionic multiplication by a maximal order in either B_6 or B_{10} admits a *single* isomorphism class of principal polarizations. This implies that $\rho(B_6) = \hat{\pi}_0(A) = \pi_0(A) = 1$ and $\rho(B_{10}) = \hat{\pi}_0(A) = \pi_0(A) = 1$, respectively.

Moreover, the Shimura curves $X_6/\mathbb{Q}$ and $X_{10}/\mathbb{Q}$ have genus 0, although they are not isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$ because there are no rational points on them. However, it is easily seen that $X_6/W_0 = X_6/W \simeq \mathbb{P}^1_{\mathbb{Q}}$ and $X_{10}/W_0 = X_{10}/W \simeq \mathbb{P}^1_{\mathbb{Q}}$, respectively.

As it is observed in [HaMu95], the base curves $g^{(6)}$ and $g^{(10)}$ are curves of genus 1 and not of genus 0 as it should be expected. This is explained by the fact that there are obvious isomorphisms between the fibres of the families $C^{(6)}$ and $C^{(10)}$, respectively.

Ibukiyama, Katsura and Oort [IbKaOo86] proved that the supersingular locus in $\mathcal{A}_2/\overline{\mathbb{F}}_p$ is irreducible if and only if $p \leq 11$. As a corollary to their work, Hashimoto and Murabayashi obtained that the reduction mod 3 and 5 of the family of Jacobian surfaces with quaternionic multiplication by B_6 and B_{10} respectively yield the single irreducible component of the supersingular locus in $\mathcal{A}_2/\overline{\mathbb{F}}_3$ and $\mathcal{A}_2/\overline{\mathbb{F}}_5$ respectively. The following statement may be considered as a lift to characteristic 0 of these results.

Theorem 4.6.14. *1. The quaternionic locus $\mathcal{Q}_6$ in $\mathcal{A}_2/\mathbb{Q}$ is absolutely irreducible and birationally equivalent to $\mathbb{P}^1_{\mathbb{Q}}$ over $\mathbb{Q}$. The generic element of $\mathcal{Q}_6$ over $\bar{\mathbb{Q}}$ is given by Hashimoto-Murabayashi's family $C^{(6)}$.*

2. The quaternionic locus $\mathcal{Q}_{10}$ in $\mathcal{A}_2/\mathbb{Q}$ is absolutely irreducible and birationally equivalent to $\mathbb{P}^1_{\mathbb{Q}}$ over $\mathbb{Q}$. The generic element of $\mathcal{Q}_{10}$ over $\bar{\mathbb{Q}}$ is given by Hashimoto-Murabayashi's family $C^{(10)}$.

Proof. This follows from Theorem 4.6.13 and the discussion above. □

In particular, we obtain from Theorem 4.6.14 that every principally polarized abelian surface $(A, \mathcal{L})$ over $\bar{\mathbb{Q}}$ with quaternionic multiplication by a maximal order of discriminant 6 or 10 is isomorphic over $\bar{\mathbb{Q}}$ to the Jacobian variety of one of the curves $C^{(6)}_{(s,t)}$ or $C^{(10)}_{(s,t)}$, except for finitely many degenerate cases.

4.7 The field of moduli of the quaternionic multiplication on an abelian variety

The aim of this section is to provide an application of the main results in this chapter to the arithmetic of abelian varieties with quaternionic multiplication, as it was done in [Ro4].

Let $\bar{\mathbb{Q}}$ be a fixed algebraic closure of the field $\mathbb{Q}$ of rational numbers and let $(A, \mathcal{L})/\bar{\mathbb{Q}}$ be a polarized abelian variety. The field of moduli of $(A, \mathcal{L})/\bar{\mathbb{Q}}$ is the minimal number field $k_{A,\mathcal{L}} \subset \bar{\mathbb{Q}}$ such that $(A, \mathcal{L})$ is isomorphic (over $\bar{\mathbb{Q}}$) to all its Galois conjugates $(A^\sigma, \mathcal{L}^\sigma)$, $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/k_{A,\mathcal{L}})$.

The field of moduli $k_{A,\mathcal{L}}$ is an essential arithmetic invariant of the $\bar{\mathbb{Q}}$-isomorphism class of $(A, \mathcal{L})$. It is contained in all possible fields of definition of $(A, \mathcal{L})$ and, unless $(A, \mathcal{L})$ admits a rational model over $k_{A,\mathcal{L}}$ itself, there is not a unique minimal field of definition for $(A, \mathcal{L})$. In this regard, we have the following theorem of Shimura [Sh72].

Theorem 4.7.1. *A generic polarized abelian variety of odd dimension admits a model over its field of moduli. For a generic polarized abelian variety of even dimension, the field of moduli is not a field of definition.*

We will place ourselves in the even dimensional-case since we shall be interested in abelian varieties with quaternionic multiplication.

Definition 4.7.2. Let $(A, \mathcal{L})/\bar{\mathbb{Q}}$ be a polarized abelian variety and let $S \subseteq \mathrm{End}_{\bar{\mathbb{Q}}}(A)$ be a subring of $\mathrm{End}_{\bar{\mathbb{Q}}}(A)$. The field of moduli of S is the minimal number field $k_S \supseteq k_{A,\mathcal{L}}$ such that, for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/k_S)$, there is an isomorphism $\varphi_\sigma/\bar{\mathbb{Q}} : A \rightarrow A^\sigma$, $\varphi_\sigma^*(\mathcal{L}^\sigma) = \mathcal{L}$, of polarized abelian varieties that induces commutative diagrams

$$\begin{array}{ccc} A & \longrightarrow & A^\sigma \\ \beta \downarrow & & \downarrow \beta^\sigma \\ A & \longrightarrow & A^\sigma \end{array}$$

for any $\beta \in S$.

We remark that, as a consequence of the very basic definitions, the field of moduli of the multiplication-by-n endomorphisms on A is exactly $k_{\mathbb{Z}} = k_{A,\mathcal{L}}$. But in the case that $\mathrm{End}(A) \supsetneq \mathbb{Z}$, little is known on the chain of Galois extensions $k_{\mathrm{End}(A)} \supseteq k_S \supseteq k_{A,\mathcal{L}}$.

Our main Theorem 4.7.4 in this direction concerns the case of abelian varieties with quaternionic multiplication. It is a consequence of the results obtained in Section 4.4 on modular embeddings of Shimura varieties in Hilbert modular varieties and the moduli spaces of principally polarized abelian varieties.

Let $(A, \mathcal{L})/\bar{\mathbb{Q}}$ be a principally polarized abelian variety with quaternionic multiplication by a maximal order in a totally indefinite division quaternion algebra B over a totally real number field F of degree $[F : \mathbb{Q}] = n$. In Chapter 3, it was shown that the first Chern class c_1 induces a monomorphism of the Néron-Severi group $\mathrm{NS}(A)$ of A into the additive group B_0 of quaternions of B of null reduced trace

$$c_1 : \mathrm{NS}(A) \hookrightarrow B_0.$$

Moreover, it was proved that there exists a left $\mathcal{O}$-ideal $\mathcal{I}_\vartheta$ in B such that $H_1(A, \mathbb{Z}) \simeq \mathcal{I}_\vartheta$ and $\mathrm{n}(\mathcal{I}_\vartheta) = \vartheta_{F/\mathbb{Q}}^{-1}$ and that the discriminant ideal $\mathrm{disc}(B)$ of B is necessarily principal and totally positive. Moreover, it can be generated by an element $D \in F_+^*$ such that, as it was shown in Corollary 3.3.8, the quaternion $\mu = c_1(\mathcal{L})^{-1} \in B_0$ satisfies that $\mu^2 + D = 0$.

Therefore, naturally attached to $(A, \mathcal{L})$ there is the principally polarized order $(\mathcal{O}, \mu)$. In turn, as it was shown in Section 4.1, attached to $(\mathcal{O}, \mu)$ there is a Shimura variety $X_{(\mathcal{O}, \mathcal{I}_\vartheta, \mu)}/\mathbb{Q}$ which solves the coarse moduli problem of classifying triplets principally polarized abelian varieties $(A, \iota, \mathcal{L})$ with a monomorphism of rings $\iota : \mathcal{O} \hookrightarrow \mathrm{End}(A)$ satisfying a certain compatibility between $\mathcal{L}$ and μ (cf. Section 4.1 for details).

The motivation for recalling this Shimura variety at this point is the following standard modular interpretation of the field of moduli $k_{\mathcal{O}}$.

Proposition 4.7.3. *The field of moduli $k_{\mathcal{O}}$ of the quaternionic multiplication on the principally polarized abelian variety $(A, \mathcal{L})$ is*

$$k_{\mathcal{O}} = \mathbb{Q}(P),$$

the extension over $\mathbb{Q}$ generated by the coordinates of the point $P = [A, \iota, \mathcal{L}]$ on Shimura's canonical model $X_{(\mathcal{O}, \mathcal{I}_\vartheta, \mu)}/\mathbb{Q}$ that represents the $\bar{\mathbb{Q}}$-isomorphism class of the triplet.

A similar construction holds for the totally real subalgebras of B. Indeed, let L/F be a totally real quadratic extension of F and $\varphi : L \hookrightarrow B$ an immersion of L in B. Then $S = \varphi^{-1}(\varphi(L) \cap \mathcal{O})$ is an order of L over R_F which is optimally embedded in $\mathcal{O}$. By identifying S with a subring of the ring of endomorphisms of A, we again have that

$$k_S = \mathbb{Q}(P_{|S})$$

is the extension over $\mathbb{Q}$ generated by the coordinates of the point $P_{|S} = [A, \iota_{|S}, \mathcal{L}]$ on the Hilbert variety $\mathcal{H}_S/\mathbb{Q}$ that solves the coarse moduli problem of classifying abelian varieties of dimension $2n$ with multiplication by S.

Along the same lines, the field of moduli k_{R_F} of the central endomorphisms of A is the extension $\mathbb{Q}(P_{|R_F})$ of $\mathbb{Q}$ generated by the coordinates of the point $P_{|R_F} = [A, \iota_{|R_F}, \mathcal{L}]$ on the Hilbert variety $\mathcal{H}_F/\mathbb{Q}$ which solves the coarse moduli problem of classifying abelian varieties of dimension $2n$ with multiplication by R_F.

Note that there are infinitely many choices of totally real quadratic orders S in $\mathcal{O}$ over R_F.

The tool for studying the Galois extensions $k_{\mathcal{O}}/k_S/k_{R_F}$ is provided by the forgetful modular maps

$$\begin{array}{rcccccc} \pi_F: & X_{(\mathcal{O},\mathcal{I}_\vartheta,\mu)} & \overset{\pi_{S,\varphi}}{\longrightarrow} & \mathcal{H}_S & \longrightarrow & \mathcal{H}_F \\ & P & \mapsto & P_{|S} & \mapsto & P_{|R_F}. \end{array}$$

It was shown in Theorem 4.4.4 that π_F and $\pi_{S,\varphi}$ are quasifinite maps over $\mathbb{Q}$. Let us identify S with its image $\varphi(S)$ in $\mathcal{O}$. Let $V_0(S)$ denote the Atkin-Lehner subgroup of twisting involutions in S and let W_0 denote the Atkin-Lehner stable subgroup introduced in Section 4.3.2. Both are 2-torsion finite abelian subgroups of the positive Atkin-Lehner group W^1. Furthermore, it was proved that, up to a birational equivalence over $\mathbb{Q}$ which is biregular away of a finite set of Heegner points, these maps are the natural projections

$$X_{(\mathcal{O},\mathcal{I}_\vartheta,\mu)} \rightarrow X_{(\mathcal{O},\mathcal{I}_\vartheta,\mu)}/V_0(S) \rightarrow X_{(\mathcal{O},\mathcal{I}_\vartheta,\mu)}/W_0.$$

In consequence, the Galois group $G = \mathrm{Gal}(k_{\mathcal{O}}/k_{R_F})$ of the extension of fields of moduli $k_{\mathcal{O}}/k_{R_F}$ is naturally embedded in W_0: any $\sigma \in G$ acts on a principally polarized abelian variety with quaternionic multiplication $(A, \iota, \mathcal{L})$ by leaving the $\bar{\mathbb{Q}}$-isomorphism class of $\pi_F(A, \iota, \mathcal{L}) = (A, \iota_{|R_F} : R_F \hookrightarrow \mathrm{End}_{\bar{\mathbb{Q}}}(A), \mathcal{L})$ invariant. Similarly, $\mathrm{Gal}(k_{\mathcal{O}}/k_S)$ embeds in $V_0(S)$ for any totally real order S embedded in $\mathcal{O}$.

As a corollary of Theorem 4.4.5, the above discussion yields the main result of this section.

Theorem 4.7.4. *Let $(A, \mathcal{L})/\bar{\mathbb{Q}}$ be a principally polarized abelian variety of dimension $2n$ such that $\mathrm{End}(A) \simeq \mathcal{O}$ is a maximal order in a totally indefinite division quaternion algebra B over a totally real number field F of degree $[F : \mathbb{Q}] = n$.*

(i) If $(\mathcal{O}, \mu)$ is non twisting, then $k_{\mathcal{O}} = k_S$ for any totally real quadratic order $S \subset \mathcal{O}$ over R_F and

$$\mathrm{Gal}(k_{\mathcal{O}}/k_{R_F}) \subseteq C_2^{\omega_{odd}}.$$

(ii) If $(\mathcal{O}, \mu)$ is twisting, let $\{\chi_1, ..., \chi_{s_0}\}$ be representatives of the finite set of twists of $(\mathcal{O}, \mu)$ up to multiplicative elements in F^*. Then, for any real quadratic order $S \subset \mathcal{O}$, $S \not\subset F(\chi_i)$, $1 \le i \le s_0$,

$$k_{\mathcal{O}} = k_S.$$

On the other hand, $k_{\mathcal{O}}/k_{S_i}$ is (at most) a quadratic extension for any totally real order $S \subset F[\chi_i]$.

Moreover, $k_{\mathcal{O}} = k_{S_1} \cdot ... \cdot k_{S_{s_0}}$ and

$$\mathrm{Gal}(k_{\mathcal{O}}/k_{R_F}) \subseteq C_2^{2\omega_{odd}}.$$

Proof. The non twisting claim (i) follows from Theorem 4.4.5 and the discussion previous to it. Let us assume that $(\mathcal{O}, \mu)$ is a twisting pair. If $S \not\subset F(\chi_i)$ for any twist χ_i, $i = 1, ..., s_0$, of $(\mathcal{O}, \mu)$, $V_0(S)$ is trivial and hence, by Theorem 4.4.5, $\mathrm{Gal}(k_{\mathcal{O}}/k_S)$ is also trivial. If, on the other hand, $S = R_F[\chi_i]$ (or any other order in $F(\chi_i) \cap \mathcal{O}$), then $V_0(S) \simeq C_2$ is generated by the twisting involution associated to χ_i. Again, we deduce that in this case $k_{\mathcal{O}}/k_S$ is at most a quadratic extension.

With regard to the last statement, note that $U_0 \supseteq \langle [\mu] \rangle$ is at least of order 2. It then follows from Lemma 4.4.6 that there exist two noncommuting twists χ, $\chi' \in \mathcal{O}$. Then $R_F[\chi, \chi']$ is a suborder of $\mathcal{O}$ and, since they both generate B over $\mathbb{Q}$, the fields of moduli $k_{\mathcal{O}}$ and $k_{R_F[\chi,\chi']}$ coincide. This shows that $k_{\mathcal{O}} \subseteq k_{S_1} \cdot ... \cdot k_{S_{s_0}}$. The converse inclusion is obvious.

Finally, we deduce from Theorem 4.4.5 and Lemma 4.4.7 that $k_{\mathcal{O}}/k_{R_F}$ is a $(2, ..., 2)$-extension of degree at most $2^{2\omega_{odd}}$. □

Theorem 4.7.4 combine with the following theorem of Shimura [Sh75] to determine in many cases the Galois groups $\mathrm{Gal}(k_{\mathcal{O}}/k_{R_F})$ and $\mathrm{Gal}(k_{\mathcal{O}}/k_S)$ completely.

Theorem 4.7.5. *The field of moduli $k_{\mathcal{O}}$ of totally indefinite quaternionic multiplication on an abelian variety is totally imaginary.*

Corollary 4.7.6. *Let $(\mathcal{O}, \mu)$ be a non twisting principally polarized maximal order and assume that $F(\sqrt{-D})$ is a CM-field with no purely imaginary roots of unity. Then, for any real quadratic order S over R_F, $k_{\mathcal{O}}/k_{R_F} = k_S/k_{R_F}$ is at most a quadratic extension.*

If, in addition, k_{R_F} admits a real embedding, then $k_{\mathcal{O}}$ is indeed a quadratic extension of k_{R_F}.

Remark 4.7.7. In the twisting case, the field of moduli of quaternion multiplication is already generated by the field of moduli of any maximal real commutative multiplication but for finitely many exceptional cases. This homogeneity does not occur in the non twisting case.

In view of Theorem 4.7.4, the shape of the fields of moduli of the endomorphisms of the polarized abelian variety $(A, \mathcal{L})$ differs considerably depending on whether it gives rise to a twisting polarized order $(\mathcal{O}, \mu)$ or not. For more details on this question, we refer the reader to the discussion at the end of Section 4.4 and Section 2.1

Chapter 5

Arithmetic of abelian surfaces with quaternionic multiplication

Introduction

In this chapter we provide an insight to the arithmetic of abelian surfaces with quaternionic multiplication over number fields. Whereas a lot of research has been carried out in the last century on abelian varieties with complex or totally real multiplication, it is remarkable that there is very little knowledge on the arithmetic of abelian varieties with QM. Consequently, we will be concerned with some of the fundamental questions that firstly arise in this regard.

In Section 5.1, we review some general results due to Silverberg and Ribet and we investigate the Galois theoretical properties of the minimal field of definition K_B of the quaternionic multiplication on an abelian surface A/K over a number field K. In Theorem 5.1.3 of Section 5.1.1, we firstly prove a very general result that holds for arbitrary abelian varieties whose ring of endomorphisms is an order in an arbitrary quaternion algebra over a field F. Subsequently, in Theorems 5.1.8 and 5.1.10 of Section 5.1.2, we prove a more precise statement for abelian surfaces. In order to accomplish that, we study Galois representations on the ring of endomorphisms $\mathrm{End}_{\bar{\mathbb{Q}}}(A)$ and on the Néron-Severi group $\mathrm{NS}(A_{\bar{\mathbb{Q}}})$ of these abelian varieties.

In Section 5.2, we combine our results with those of Mestre and Jordan to make the relationship between the field of moduli and field of definition of the quaternionic multiplication explicit for the case of Jacobian surfaces.

Finally, we consider in Section 5.3 several concrete examples of curves C/K of genus two over a number field such that their Jacobian varieties $J(C)/K$ have quaternionic multiplication over $\bar{K}$ due to Hashimoto, Murabayashi and Tsunogai and we explore their diophantine properties.

The results presented in this chapter are partly contained in [Ro4], [DiRo1] and [DiRo2].

5.1 The field of definition of the quaternionic multiplication on an abelian surface

Fix $\bar{\mathbb{Q}} \subset \mathbb{C}$ an algebraic closure of the field $\mathbb{Q}$ of rational numbers and let $K \subset \bar{\mathbb{Q}}$ be a number field. The following result is due to Ribet [Ri75].

Proposition 5.1.1. *Let A/K be an abelian variety over K and let $S \subseteq \mathrm{End}_{\bar{\mathbb{Q}}}(A)$ be a subring of endomorphisms of A. Then there is a* unique minimal extension *K_S/K such that $S \subseteq \mathrm{End}_{K_S}(A)$.*

The extension K_S/K is normal and unramified at the prime ideals of K of semistable reduction of A.

We remark that the minimal field of definition of a subring S of endomorphisms of A only depends on the algebra $L = S \otimes \mathbb{Q} \subseteq \mathrm{End}_{\bar{\mathbb{Q}}}(A) \otimes \mathbb{Q}$ and not on the particular choice of the integral order sitting in $L \cap \mathrm{End}_{\bar{\mathbb{Q}}}(A)$. Hence, we will often denote it simply by $K_L = K_S$.

Let now F be a totally real number field and let R_F denote its ring of integers. Let $\mathcal{O}$ be a maximal order over R_F in a totally indefinite quaternion algebra B over F. Let A/K be an abelian variety of dimension $2[F : \mathbb{Q}]$ and assume that there is an isomorphism $\iota : \mathcal{O} \xrightarrow{\sim} \mathrm{End}_{\bar{\mathbb{Q}}}(A)$.

According to Definition 3.2.1, we say that A/K is an abelian variety with quaternionic multiplication by $\mathcal{O}$, although attention must be paid to the fact that the quaternion multiplication need not be defined over the field K itself.

In regard to Proposition 5.1.1, we remark that A/K has potential good reduction and therefore no places of K of bad reduction of A are semistable. This is a consequence of Grothendieck's Potential Good Reduction Theorem.

For any algebra $L \subseteq B$ over $\mathbb{Q}$ and in accordance to the above, we let K_L/K denote the minimal field of definition of the ring of endomorphisms $S = L \cap \mathrm{End}_{\bar{\mathbb{Q}}}(A)$ on A over K. Note that we have $K_{\mathbb{Q}} = K$.

With particular interest on abelian surfaces, it is the aim of this section to study

1. The field extension K_B/K given by the minimal field of definition K_B of the quaternionic multiplication on A over K.

2. The filtration of intermediate endomorphism algebras $F \subseteq \mathrm{End}_{\tilde{K}}(A) \otimes \mathbb{Q} \subseteq B$ for any subfield $K_F \subseteq \tilde{K} \subseteq K_B$.

The first question has been studied in a more general context by several authors. In [Si92], Silverberg also gave an explicit upper bound for the degree $[K_B : K]$ in terms of certain combinatorial numbers. In the particular case of abelian surfaces with quaternionic multiplication, [Si92], Proposition 4.3, predicts that $[K_B : K] \leq 48$. As our results will show, these bounds are not sharp: see Theorem 5.1.3 for arbitrary orders $\mathcal{O}$ and Theorem 5.1.8 for maximal orders.

The study of the Galois extension K_B/K can be divided into that of the extensions K_F/K and K_B/K_F. The former have been studied by several authors in the totally real case. See [Ri80] and [Ri94]) for details. Here there are some known general facts.

Proposition 5.1.2. *Let A/K be an abelian variety over a number field K and assume that the algebra of endomorphisms $\mathrm{End}_{\bar{\mathbb{Q}}}(A) \otimes \mathbb{Q}$ of A over $\bar{\mathbb{Q}}$ contains a totally real number field F of arbitrary degree $[F : \mathbb{Q}] = n$. Then there is a monomorphism of groups $\mathrm{Gal}(K_F/K) \hookrightarrow \mathrm{Aut}(F/\mathbb{Q})$ and, in particular, $[K_F : K] \leq n$.*

Therefore, we will focus ourselves on the extension K_B/K_F given by the field of definition of the quaternionic multiplication on an abelian variety A over K_F.

We devote next two subsections to carefully state and prove our results. This is accomplished by studying the absolute Galois action on the ring of endomorphisms and on the Néron-Severi group of an abelian variety, respectively.

5.1.1 The action of $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ on the endomorphism ring.

In this section we use Chinburg-Friedman's recent classification of the finite subgroups of maximal arithmetic Kleinian groups in [ChFr00F] to describe the field of definition of the quaternionic multiplication on an abelian variety. We will be able to present a result that holds true for abelian varieties whose ring of endomorphisms is an arbitrary order $\mathcal{O}$ in an arbitrary quaternion algebra B over an arbitrary field F.

Let A/K be an abelian variety over a number field K and assume that $\mathrm{End}_{\bar{\mathbb{Q}}}(A) \simeq \mathcal{O}$ is an order in a quaternion algebra B over a field F. By Theorem 1.3.5, we know that F is either a totally real field or a CM-field. Moreover, if F is totally real, B is either totally definite or totally indefinite over F and $2[F:\mathbb{Q}] \,|\, \dim(A)$.

Let K_F be the minimal field of definition of the central endomorphisms on A. The absolute Galois group $G_{K_F} = \mathrm{Gal}(\bar{\mathbb{Q}}/K_F)$ acts in a natural way on the full ring of endomorphisms $\mathrm{End}_{\bar{\mathbb{Q}}}(A) = \mathcal{O}$ and induces a Galois representation

$$\gamma : G_{K_F} \longrightarrow \mathrm{Aut}(\mathcal{O}).$$

By Skolem-Noether's Theorem 1.2.15, for any $\tau \in G_{K_F}$ there exists an automorphism $[\gamma_\tau] : B \to B$ of B such that $\beta^\tau = \gamma_\tau^{-1}\beta\gamma_\tau$ for any $\beta \in \mathrm{End}_{\bar{\mathbb{Q}}}(A) = \mathcal{O}$. We obtain an exact sequence of groups

$$1 \to G_{K_B} \to G_{K_F} \to \mathrm{Norm}_{B^*}(\mathcal{O})/F^*$$

and thus a monomorphism $\mathrm{Gal}(K_B/K_F) \hookrightarrow \mathrm{Norm}_{B^*}(\mathcal{O})/F^*$.

The next theorem easily follows from the work of Chinburg anf Friedman [ChFr00F].

Theorem 5.1.3. *Let A/K be an abelian variety such that $\mathrm{End}_{\bar{\mathbb{Q}}}(A) \simeq \mathcal{O}$ is a non necessarily maximal order in an arbitrary quaternion algebra B over a field F. Let K_B (respectively K_F) be the minimal extension of K over which all (respectively central) endomorphisms of A are defined.*

Then K_B/K_F is a Galois extension with $\mathrm{Gal}(K_B/K_F)$ isomorphic to either a cyclic group, a dihedral group, the alternating groups A_4, A_5 or the symmetric group S_4.

1. *If* $\mathrm{Gal}(K_B/K_F) \simeq C_n$ *or* D_n *is either cyclic of order* n *or a dihedral group of order* $2n$*, then there exists a* n*-th primitive root of unity* $\zeta_n \in B^*$*.*

2. *If* $\mathrm{Gal}(K_B/K_F) \simeq A_4$ *or* S_4*, then* $B \simeq (\frac{-1,-1}{F})$ *is the totally definite Hamilton's quaternion algebra.*

3. *If* $\mathrm{Gal}(K_B/K_F) \simeq A_5$*, then* $B \simeq (\frac{-1,-1}{F})$ *and* $\sqrt{5} \in F$*.*

Proof. As it was already observed, $\mathrm{Gal}(K_B/K_F)$ is a finite subgroup of $\mathrm{Norm}_{B^*}(\mathcal{O})/F^* \subset B^*/F^*$. Chinburg and Friedman [ChFr00F] proved that the only possible finite subgroups of B^*/F^* are the cyclic groups C_n, the dihedral groups D_n and S_4, A_4 and A_5. By [ChFr00F], Lemma 2.8, a necessary condition for B^*/F^* to contain either S_4 or A_4 is that $B = (\frac{-1,-1}{F})$. Also, if B^*/F^* contains a finite subgroup isomorphic to the alternating group A_5, then $B = (\frac{-1,-1}{F})$ and $F \supseteq \mathbb{Q}(\sqrt{5})$.

Moreover, Lemma 2.1 in [ChFr00F] shows that B^*/F^* contains a cyclic group of order $n > 2$ if and only if there exists $\zeta_n \in B^*$ satisfying $\zeta_n^n = 1$, $\zeta_n^{n/d} \neq 1$ for any proper divisor d of n. In this case, any subgroup $C_n \subseteq B^*/F^*$ is conjugate to $\langle [1+\zeta_n] \rangle$. Moreover, Chinburg and Friedman prove that B^*/F^* contains a dihedral subgroup D_n, $n \geq 2$, if and only if it contains a cyclic group C_n (cf. [ChFr00F], Lemma 2.3). If $n = 2$, any subgroup of $B^*/\mathbb{Q}^*$ isomorphic to $D_2 = C_2 \times C_2$ is of the form $\langle [x], [y] \rangle \subset B^*/F^*$ with $x, y \in B^*$, $x^2 = d$, $y^2 = m$, $xy = -yx$ for some $d, m \in F^*$. This yields the proof of the theorem. □

Corollary 5.1.4. *Let* A/K *be an abelian surface with quaternionic multiplication by a non necessarily maximal order* $\mathcal{O}$ *in a quaternion algebra* B*. Then* $\mathrm{Gal}(K_B/K) \simeq C_n$ *or* D_n *with* $n = 1, 2, 3, 4$ *or* 6*.*

Proof. Since B is an indefinite quaternion algebra over $\mathbb{Q}$, we have that $F = \mathbb{Q}$ and $K_{\mathbb{Q}} = K$. By the theorem above, $\mathrm{Gal}(K_B/K) \simeq C_n$ or D_n for some $n \geq 1$ such that there exists a n-th primitive root of unity $\zeta_n \in B^*$. However, any $\zeta_n \in B^*$ generates either a trivial or a quadratic field extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ and this is only possible for $n = 1, 2, 3, 4$ and 6. □

5.1.2 The action of $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ on the Néron-Severi group of abelian surfaces

In this subsection we prove a more accurate result on the field of definition of the quaternionic multiplication on an abelian surface. Moreover, we show that there are very restrictive conditions for a quadratic field $\mathbb{Q}(\sqrt{d})$ to be realized as the algebra of endomorphims $\mathrm{End}^0_K(A)$ of an abelian surface A/K over a number field with quaternionic multiplication over $\bar{\mathbb{Q}}$.

Let $\mathcal{O}$ be a maximal order in an indefinite division quaternion algebra B over $\mathbb{Q}$. Assume that A is an abelian surface defined over a number field K together with an isomorphism of rings $\iota : \mathcal{O} \xrightarrow{\sim} \mathrm{End}_{\bar{\mathbb{Q}}}(A)$. In Chapter 3, the absolute Néron-Severi group $\mathrm{NS}(A_{\mathbb{C}}) \simeq \mathrm{NS}(A_{\bar{\mathbb{Q}}})$ was largely studied: it was seen that the first Chern class allows us to regard $\mathrm{NS}(A_{\mathbb{C}})$ as a sublattice of the 3-dimensional vector space B_0 of pure quaternions of B in a way that fundamental properties of invertible sheaves $\mathcal{L}$ on A can be interpreted in terms of the arithmetic of B. This is the case of the degree $\deg(\mathcal{L})$, the behaviour under pull-backs by endomorphisms, the index $i(\mathcal{L})$ and the ampleness of an invertible sheaf. We summarize these results in the following statement; see Theorem 3.3.1, Proposition 3.3.3 and Theorem 3.5.3 for more details.

Theorem 5.1.5. *Let $A/\bar{\mathbb{Q}}$ be an abelian surface with $\mathrm{End}_{\bar{\mathbb{Q}}}(A) \overset{\iota}{\simeq} \mathcal{O}$ a maximal order in a quaternion algebra of discriminant D. Then A is principally polarizable and there is an isomorphism of additive groups*

$$\begin{array}{rccc} c_1 : & \mathrm{NS}(A_{\bar{\mathbb{Q}}}) & \to & \mathcal{O}_0^\sharp \\ & \mathcal{L} & \mapsto & c_1(\mathcal{L}) \end{array}$$

such that

1. $\deg(\mathcal{L}) = D \cdot \mathrm{n}(c_1(\mathcal{L}))$.
2. *For any endomorphism $\alpha \in \mathcal{O} = \mathrm{End}_{\bar{\mathbb{Q}}}(A)$, $c_1(\alpha^*(\mathcal{L})) = \overline{\alpha} c_1(\mathcal{L})\alpha$.*
3. *An invertible sheaf $\mathcal{L} \in \mathrm{NS}(A_{\bar{\mathbb{Q}}})$ is a polarization if and only if $\mathrm{n}(c_1(\mathcal{L})) > 0$ and $\det(\nu_{\mathcal{L}}) > 0$ where $\nu_{\mathcal{L}} \in \mathrm{GL}_2(\mathbb{R})$ is any matrix such that $\nu_{\mathcal{L}}^{-1} \cdot c_1(\mathcal{L}) \cdot \nu_{\mathcal{L}} \in \mathbb{Q}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.*

We recall from Section 1.2 that $\mathcal{O}^\sharp = \{\beta \in B : \mathrm{tr}(\mathcal{O}\beta) \subseteq \mathbb{Z}\}$ denotes the codifferent ideal of $\mathcal{O}$ in B. By $\mathcal{O}_0^\sharp$ we mean the subgroup $\mathcal{O}^\sharp \cap B_0$ of pure quaternions of $\mathcal{O}^\sharp$. For our purposes in this section, we only need to know that it is a lattice in B_0 and in particular the Picard number $\rho(A_{\bar{\mathbb{Q}}}) = 3$.

Since A_K is always attached with a nonnecessarily principal polarization, we have that $1 \leq \rho(A_K) \leq 3$. Both three cases are possible and each possibility has a direct translation in terms of the algebra of endomorphisms:

- $\rho(A_K) = 1$ if and only if $\mathrm{End}^0_K(A)$ is $\mathbb{Q}$ or an imaginary quadratic field,
- $\rho(A_K) = 2$ if and only if $\mathrm{End}^0_K(A)$ is a real quadratic field and
- $\rho(A_K) = 3$ if and only if $\mathrm{End}^0_K(A) = \mathrm{End}_L(A) = B$.

The proof of these facts follows from [Mu70] and [LaBi92].

We now consider the action of the Galois group $G_K = \mathrm{Gal}(\bar{\mathbb{Q}}/K)$ on $\mathrm{NS}(A_{\bar{\mathbb{Q}}})$ given by $\mathcal{L}(\Theta)^\tau = \mathcal{L}(\Theta^\tau)$ for any invertible sheaf $\mathcal{L}$ on A represented by a Weil divisor Θ and $\tau \in G_K$. From Theorem 5.1.5, any automorphism of $\mathrm{NS}(A_{\bar{\mathbb{Q}}})$ can be regarded as a linear automorphism of B_0. Moreover, since the Galois action conserves the degree of invertible sheaves and the first Chern class is a monomorphism of quadratic modules $c_1 : (\mathrm{NS}(A_{\bar{\mathbb{Q}}}), \deg) \hookrightarrow (B_0, D \cdot \mathrm{n})$, we obtain a Galois representation

$$\begin{array}{rccccc} \eta: & G_K & \longrightarrow & \mathrm{Aut}(\mathrm{NS}(A_{\bar{\mathbb{Q}}}), \deg) & \subset & \mathrm{Aut}(B_0, D \cdot \mathrm{n}) \\ & \tau & \mapsto & & & \eta_\tau \end{array}$$

We have that

1. For any $\alpha \in \mathcal{O} = \mathrm{End}_L(A)$, $(\alpha^*(\mathcal{L})^\tau) = (\alpha^\tau)^*(\mathcal{L}^\tau)$.
2. The index $i(\mathcal{L})$ only depends on the G_K-orbit of $\mathcal{L}$, that is, $i(\mathcal{L}^\tau) = i(\mathcal{L})$, for any $\tau \in G_K$. In particular $\mathcal{L}^\tau$ is a polarization if and only if $\mathcal{L}$ is.

The following relates the Galois actions on $\mathrm{End}_{\bar{\mathbb{Q}}}(A)$ and on the Néron-Severi group $\mathrm{NS}(A_{\bar{\mathbb{Q}}})$ by means of a reciprocity law.

Theorem 5.1.6. *Let A/K be an abelian surface with quaternionic multiplication by a maximal order $\mathcal{O}$ of discriminant D in a quaternion algebra B.*

Let

$$\begin{array}{rccc} \gamma: & G_K & \to & \mathrm{Aut}(\mathrm{End}_{\bar{\mathbb{Q}}}(A)) \\ & \tau & \mapsto & \\ & & & [\gamma_\tau]: \begin{array}{ccc} \mathcal{O} & \to & \mathcal{O} \\ \beta & \mapsto & {\gamma_\tau}^{-1}\beta\gamma_\tau \end{array} \end{array}$$

be the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ *on the ring of endomorphisms of* A. *Define* $\varepsilon_\tau = \mathrm{sign}(\mathrm{n}(\gamma_\tau)) \in \{\pm 1\}$. *Then*

$$c_1(\mathcal{L}^\tau) = \varepsilon_\tau \cdot {\gamma_\tau}^{-1} c_1(\mathcal{L})\gamma_\tau$$

for any invertible sheaf $\mathcal{L} \in \mathrm{NS}(A_{\bar{\mathbb{Q}}})$ *and any* $\tau \in G_K$.

Proof. Fix $\tau \in G_K$. We firstly claim that $\eta_\tau : B_0 \to B_0$ is given by $\mu \mapsto \tilde{\varepsilon}\tilde{\gamma}^{-1}\mu\tilde{\gamma}$ for some $\tilde{\varepsilon} = \pm 1$, $\tilde{\gamma} \in B^*$. Indeed, any linear endomorphism of B_0 extends uniquely to an endomorphism of B and $\mathrm{End}(B) \simeq B \otimes B$ with $\tilde{\gamma}_1 \otimes \tilde{\gamma}_2 : B \to B$, $\beta \mapsto \tilde{\gamma}_1\beta\tilde{\gamma}_2$. We must have in addition that $\mathrm{tr}(\tilde{\gamma}_1\mu\tilde{\gamma}_2) = \mathrm{tr}(\tilde{\gamma}_2\tilde{\gamma}_1\mu) = 0$ for any $\mu \in B_0$. This automatically implies that $\tilde{\gamma}_2\tilde{\gamma}_1 \in \mathbb{Q}$.

Since the action of G_K on $\mathrm{NS}(A_{\bar{\mathbb{Q}}})$ conserves the degree of invertible sheaves , we deduce from Theorem 5.1.5 that

$$\mathrm{n}(\eta_\tau(\mu)) = \mathrm{n}(\tilde{\gamma}_1\mu\tilde{\gamma}_2) = \mathrm{n}(\tilde{\gamma}_1)\mathrm{n}(\mu)\mathrm{n}(\tilde{\gamma}_2) = \mathrm{n}(\mu)$$

for all $\mu \in B_0$. Hence $\mathrm{n}(\tilde{\gamma}_2) = \mathrm{n}(\tilde{\gamma}_1)^{-1}$ and thus $\tilde{\gamma} := \tilde{\gamma}_2 = \tilde{\varepsilon}\tilde{\gamma}_1^{-1}$ for some $\tilde{\varepsilon} = \pm 1$. This proves the claim.

We now show that $\tilde{\gamma} = \gamma_\tau \in B^*/\mathbb{Q}^*$ and $\tilde{\varepsilon} = \varepsilon_\tau$. We know that $(\alpha^*(\mathcal{L})^\tau) = (\alpha^\tau)^*(\mathcal{L}^\tau)$ for any $\alpha \in \mathcal{O}$. Taking Theorem 5.1.5 into account, this implies that $\eta_\tau(\overline{\alpha}\mu\alpha) = \overline{[\gamma_\tau](\alpha)}\eta_\tau(\mu)[\gamma_\tau](\alpha)$ and thus $\tilde{\varepsilon}\tilde{\gamma}^{-1}(\overline{\alpha}\mu\alpha)\tilde{\gamma} = \tilde{\varepsilon}\overline{({\gamma_\tau}^{-1}\alpha\gamma_\tau)}\tilde{\gamma}^{-1}\mu\tilde{\gamma}({\gamma_\tau}^{-1}\alpha\gamma_\tau)$ for any $\alpha \in B$, $\mu \in B_0$. Choosing $\alpha = \mu$ and bearing in mind that ${\gamma_\tau}^{-1} = \overline{\gamma_\tau}\mathrm{n}(\gamma_\tau)^{-1}$, this says that $\tilde{\gamma}^{-1}\mu\tilde{\gamma} = {\gamma_\tau}^{-1}$ $\mu^{-1}\gamma_\tau\tilde{\gamma}^{-1}\mu\tilde{\gamma}\ {\gamma_\tau}^{-1}\mu\gamma_\tau$ and thus $\mu(\omega\mu\omega^{-1}) = (\omega\mu\omega^{-1})\mu$, where we write $\omega = \gamma_\tau\tilde{\gamma}^{-1}$. The centralizer of $\mathbb{Q}(\mu)$ in B is $\mathbb{Q}(\mu)$ itself and therefore $(\omega\mu\omega^{-1}) \in \mathbb{Q}(\mu)$. But $\mathrm{tr}(\mu) = \mathrm{tr}(\omega\mu\omega^{-1}) = 0, \mathrm{n}(\mu) = \mathrm{n}(\omega\mu\omega^{-1})$ and this implies that $\mu = \pm\omega\mu\omega^{-1}$. Since this must hold for any $\mu \in B_0$, it follows that $\omega \in \mathbb{Q}^*$ and thus $\tilde{\gamma} = \gamma_\tau \in B^*/\mathbb{Q}^*$ as we wished.

We then already have that $\eta_\tau : B_0 \to B_0$ is given by $\mu \mapsto \tilde{\varepsilon}{\gamma_\tau}^{-1}\mu\gamma_\tau$ for some $\tilde{\varepsilon} = \pm 1$. If $\mu = c_1(\mathcal{L})$ for a polarization $\mathcal{L}$ on A, this means that $c_1(\mathcal{L}^\tau) = \tilde{\varepsilon}{\gamma_\tau}^{-1}\mu\gamma_\tau$. Since $\mathcal{L}^\tau$ is still an ample invertible sheaf we have, according to Theorem 5.1.5, §3, that $\tilde{\varepsilon} = \mathrm{sign}(\mathrm{n}(\gamma_\tau))$. □

In Section 4.3.2, we introduced the *twists* and *principal twists* of a polarized order in a quaternion algebra. In the case of a rational indefinite quaternion algebra, let us recall and refine our definitions as follows.

Definition 5.1.7. Let $\mathcal{O}$ be a maximal order in an indefinite quaternion algebra B over $\mathbb{Q}$ and let $D = \operatorname{disc}(B)$. We say that $\mathcal{O}$ admits a *twist of degree* $\delta \geq 1$ and *norm* $m \in \mathbb{Z}$, $m|D$, if

$$B = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij = (\frac{-D\delta, m}{\mathbb{Q}})$$

with i, $j \in \mathcal{O}$, $i^2 = -D\delta$, $j^2 = m$ and $ij = -ji$.

For any positive integer $\delta \geq 1$, let $\mathcal{N}_\delta = \{m_1, ..., m_t\}$, $0 < m_i|D$, denote the (possibly empty) set of norms of the twists of degree δ on $\mathcal{O}$. It is easy to show that the set $\mathcal{N}_1$ of norms of principal twists is either $\mathcal{N}_1 = \emptyset$ or $\mathcal{N}_1 = \{m, D/m\}$ for some $m|D$. In other cases, $\mathcal{N}_\delta$ can be larger. Indeed, if $\delta = D$ for instance, then $\mathcal{N}_D$ is either empty or equal to the set of sums of two squares $m = m_1^2 + m_2^2$ that divide D. We finally note that a quaternion order $\mathcal{O}$ can very well admit twists of several different degrees.

In practice, the computation of a finite number of Hilbert symbols suffices to decide whether a given indefinite order is twisting of certain degree δ. Let us just quote that a necessary and sufficient condition for B to contain a maximal order $\mathcal{O}$ admitting a twist of degree δ and norm m is that $m > 0$, $m|D = \operatorname{disc}(\mathcal{O}) = \operatorname{disc}(B)$ and that for any odd prime $p|D$: $m \not\in \mathbb{F}_p^{*^2}$ if $p \nmid m$ ($D/m \not\in \mathbb{F}_p^{*^2}$ if $p|m$ respectively).

Examples of quaternion algebras with principally twisting maximal orders are $B = (\frac{-6,2}{\mathbb{Q}})$ and $B = (\frac{-10,2}{\mathbb{Q}})$ of discriminant $D = 6$ and $D = 10$, respectively.

We can now prove the following result.

Theorem 5.1.8. *Let A/K be an abelian surface defined over a number field K such that $\operatorname{End}_{\bar{\mathbb{Q}}} \simeq \mathcal{O}$ is a maximal order $\mathcal{O}$ in a quaternion algebra B of discriminant D and let K_B/K be the minimal extension of K such that $\operatorname{End}_{K_B}(A) \simeq \mathcal{O}$. Fix a polarization $\mathcal{L}_0$ on A_K and let $\delta = \deg(\mathcal{L}_0)$ be its degree.*

A *(a) If δ is not equal to D neither to $3D$ up to squares, then $\operatorname{Gal}(K_B/K) \simeq \{1\}$, C_2 or $D_2 = C_2 \times C_2$.*

(b) *If* $\delta = Dk^2$ *for some* $k \in \mathbb{Z}$, *then* $\mathrm{Gal}(K_B/K) \simeq C_n$ *or* D_n *with* $n = 1, 2$ *or* 4.

(c) *If* $\delta = \frac{Dk^2}{3}$ *for some* $k \in \mathbb{Z}$, *then* $\mathrm{Gal}(K_B/K) \simeq C_n$ *or* D_n *for* $n = 1, 2, 3$ *or* 6.

B *In any of the cases above, if* $\mathcal{O}$ *does not admit any twist of degree* δ, *then* $\mathrm{Gal}(K_B/K)$ *is necessarily cyclic.*

C (a) *If* $\mathrm{Gal}(K_B/K) \simeq C_2$, *then* $\mathrm{End}_K^0(A) \simeq \mathbb{Q}(\sqrt{-D\delta})$ *or* $\mathbb{Q}(\sqrt{m_i})$ *for* $m_i \in \mathcal{N}_\delta$ *a norm of a twist of degree* δ *on* $\mathcal{O}$.

(b) *If* $\mathrm{Gal}(K_B/K) = C_3$ *or* C_6, *then* $\mathrm{End}_K^0(A) \simeq \mathbb{Q}(\sqrt{-3})$.

(c) *If* $\mathrm{Gal}(K_B/K) = C_4$, *then* $\mathrm{End}_K^0(A) \simeq \mathbb{Q}(\sqrt{-1})$.

(d) *If* $\mathrm{Gal}(K_B/K) = D_n$, *then* $\mathrm{End}_K(A) \simeq \mathbb{Q}$.

Proof. Recall that, according to Theorem 5.1.3, $\mathrm{Gal}(K_B/K) \simeq C_n$ or D_n with $n = 2, 3, 4$ or 6. Let $\mathcal{L}_0$ be a polarization on A_K and let $\mu = c_1(\mathcal{L}_0) \in \mathcal{O}_0$. It satisfies that $\mu^2 + D\delta = 0$ by Theorem 5.1.5, §1. Since $\mathcal{L}_0 \in \mathrm{NS}(A_{\bar{\mathbb{Q}}})^{G_K}$, it is invariant under the action of $\mathrm{Gal}(K_B/K)$. Fix $\tau \in \mathrm{Gal}(K_B/K)$ and let $\gamma_\tau \in B^*$ the quaternion associated to τ in Section 5.1.1. Suitably scaling it, we can (and we do) choose a representative in $\mathrm{Norm}_{B^*}(\mathcal{O})/\mathbb{Q}^*$ such that $\gamma_\tau \in \mathcal{O}$ and $\mathrm{n}(\gamma_\tau)$ is a square-free integer. Then, since γ_τ must normalize $\mathcal{O}$, we know that $\mathrm{n}(\gamma_\tau)|D$.

Since $\mathcal{L}_0$ in $\mathrm{NS}(A_{\bar{\mathbb{Q}}})$ is G_K-invariant, it follows from Theorem 5.1.6 that $\mu = c_1(\mathcal{L}_0) = c_1(\mathcal{L}_0^\tau) = \varepsilon_\tau {\gamma_\tau}^{-1}\mu\gamma_\tau$.

If $\varepsilon_\tau = -1$, then the above expression implies that $\mu\gamma_\tau = -\gamma_\tau\mu$. Since $\mathrm{tr}(\mu\gamma_\tau) = \mu\gamma_\tau - \overline{\gamma_\tau}\mu = -\mathrm{tr}(\gamma_\tau)\mu \in \mathbb{Q}^*$, we deduce that $\mathrm{tr}(\gamma_\tau) = 0$. This means that ${\gamma_\tau}^2 = m$ for some $m|D$ and $B = \mathbb{Q}+\mathbb{Q}\mu+\mathbb{Q}\gamma_\tau+\mathbb{Q}\mu\gamma_\tau = (\frac{-D\delta, m}{\mathbb{Q}})$. The indefiniteness of B forces m to be positive. We obtain that in this case $\langle[\gamma_\tau]\rangle \simeq C_2$.

On the other hand, if $\varepsilon_\tau = 1$, then $\gamma_\tau \in \mathbb{Q}(\mu) \simeq \mathbb{Q}(\sqrt{-D\delta}) = \mathbb{Q}(\sqrt{-\underline{D\delta}})$, where we let $\underline{D\delta}$ denote the square-free part of $D\delta$. In this case, and bearing in mind that γ_τ must generate a finite subgroup of $B^*/\mathbb{Q}^*$, we deduce that either

- $\gamma_\tau = \pm\sqrt{\underline{D\delta}/D\delta} \cdot \mu$ and hence $\underline{D\delta}|D$ and $\langle[\gamma_\tau]\rangle \simeq C_2$,
- $\gamma_\tau = 1 + \zeta_n$ for some n^{th}-primitive root of unity $\zeta_n \in B^*$, $n = 3$ or 6, and hence $\underline{D\delta} = 3$ and $\langle[\gamma_\tau]\rangle \simeq C_n$ or

- $\gamma_\tau = 1 + \zeta_4$ for some 4^{th}-primitive root of unity $\zeta_4 \in B^*$ and hence $\underline{D\delta} = 1$ and $\langle[\gamma_\tau]\rangle \simeq C_4$.

We conclude that a necessary condition for $\mathrm{Gal}(K_B/K)$ to contain a cyclic subgroup of order $n \geq 3$ is $\underline{D\delta} = 1$ or 3 which amounts to saying that $\deg(\mathcal{L}_0) = \delta$ is D or $3D$ up to squares respectively. Also, if $\deg(\mathcal{L}_0) = Dk^2$, then necessarily $\mathrm{Gal}(K_B/K) \simeq C_n$ or D_n with $n = 1, 2$ or 4 and an analogous statement holds if $\deg(\mathcal{L}_0) = 3D$ up to squares. Further, if $B \not\simeq (\frac{-D\delta, m}{\mathbb{Q}})$ for any $0 < m|D$, then it follows from the discussion above that $\epsilon_\tau = 1$ for any $\tau \in \mathrm{Gal}(K_B/K)$ and, as a consequence, $\mathrm{Gal}(K_B/K) \subset \mathbb{Q}(\mu)^*/\mathbb{Q}^*$. Since the only finite subgroups of $\mathbb{Q}(\mu)^*/\mathbb{Q}^*$ are cyclic, the proof of parts **A** and **B** is completed.

As for part **C**, assume first that $\mathrm{Gal}(K_B/K) = \langle[\gamma_\tau]\rangle \simeq C_2$. Then $\gamma_\tau \in B^*$ satisfies ${\gamma_\tau}^2 = -\mathrm{n}(\gamma_\tau) \in \mathbb{Q}^*$ and we already saw that the only possibilities are, up to squares, $\mathrm{n}(\gamma_\tau) = D\delta$ or $m \in \mathcal{N}_\delta$. In any of these cases, $\mathrm{End}^0_K(A) = \{\beta \in \mathrm{End}_{K_B}(A) : \beta^\tau = \beta\} = \{\beta \in \mathrm{End}_{K_B}(A) : \beta\gamma_\tau = \gamma_\tau\beta\} = \mathbb{Q}(\gamma_\tau)$ and this implies our first assertion of part **C**. Similarly, if $\mathrm{Gal}(K_B/K) = \langle 1+\zeta_n\rangle \simeq C_n$ with $n = 3$, 4 or 6 then $\mathrm{End}^0_K(A) = \mathbb{Q}(1 + \zeta_n) \simeq \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$ depending on the cases. Finally, if $\mathrm{Gal}(K_B/K) = \langle\gamma_\tau, \gamma_{\tau'}\rangle \simeq D_n$ with $\langle\gamma_\tau\rangle \simeq C_n$ and $\langle\gamma_{\tau'}\rangle \simeq C_2$, then $\mathrm{End}^0_K(A) = \{\beta \in \mathbb{Q}(\gamma_\tau) : \beta^{\tau'} = \beta\} = \mathbb{Q}$. Here, the last equality holds because it is not possible that γ_τ and $\gamma_{\tau'}$ commute. □

The following lemma may be useful in many situations in order to apply Theorem 5.1.8. It easily follows from Theorem 5.1.5.

Lemma 5.1.9. *Let $A/\bar{\mathbb{Q}}$ be an abelian surface with* $\mathrm{End}(A)$ *a maximal order in a quaternion algebra of discriminant D. If there exist prime numbers $p, q|D$ such that p splits in $\mathbb{Q}(\sqrt{-1})$ and q splits in $\mathbb{Q}(\sqrt{-3})$, then no polarizations on A have degree Dk^2 or $3Dk^2$ for any $k \in \mathbb{Z}$.*

When particularized to the Jacobian variety of a curve, Theorem 5.1.8 asserts the following.

Theorem 5.1.10. *Let C/K be a curve of genus 2 defined over a number field K and let $J(C)$ be its Jacobian variety. Assume that* $\mathrm{End}_{\bar{\mathbb{Q}}}(J(C)) = \mathcal{O}$ *is a maximal order in a division quaternion algebra B of discriminant D.*

Let K_B/K be the minimal extension of K over which all endomorphisms of $J(C)$ are defined. Then

1. *K_B/K is an abelian extension with $G =$ $\mathrm{Gal}(K_B/K) \simeq (1)$, C_2 or $D_2 = C_2 \times C_2$, where C_2 denotes the cyclic group of order two.*

2. *If* $B \not\simeq (\frac{-D,m}{\mathbb{Q}})$ *for any* $m|D$*, then* K_B/K *is at most a quadratic extension of* K*. In this case,* $\mathrm{End}_K(A) \simeq \mathbb{Q}(\sqrt{-D})$*.*

3. *If* $B = (\frac{-D,m}{\mathbb{Q}})$ *for some* $m|D$*, then* $\mathrm{End}_K(A)$ *is isomorphic to either* $\mathcal{O}$*, an order in* $\mathbb{Q}(\sqrt{-D})$*,* $\mathbb{Q}(\sqrt{m})$ *or* $\mathbb{Q}(\sqrt{D/m})$*, or* $\mathbb{Z}$*. In each case, we respectively have* $\mathrm{Gal}(K_B/K) \simeq (1)$*,* C_2 *and* D_2*.*

5.2 Fields of moduli and fields of definition of Jacobian varieties of genus two curves

Let $C/\bar{\mathbb{Q}}$ be a smooth irreducible curve of genus 2 and let $(J(C), \Theta_C)$ denote its principally polarized Jacobian variety. Assume that $\mathrm{End}_{\bar{\mathbb{Q}}}(J(C)) = \mathcal{O}$ is a maximal order in an indefinite quaternion algebra B over $\mathbb{Q}$ of reduced discriminant $D = p_1 \cdot \ldots \cdot p_{2r}$. Recall that $\mathcal{O}$ is unique up to conjugation or, equivalently by the Skolem-Noether Theorem, up to isomorphism.

Attached to $(J(C), \Theta_C)$ is the polarized order $(\mathcal{O}, \mu)$, where $\mu = c_1(\Theta_C) \in \mathcal{O}$ is a pure quaternion of reduced norm D. As we have seen, a necessary condition for $(\mathcal{O}, \mu)$ to be twisting is that $B \simeq (\frac{-D,m}{\mathbb{Q}})$ for some $m|D$. The isomorphism occurs if and only if for any $p|D$, the integer m (respectively D/m) is not a square mod p if $p \nmid m$ (if $p|m$, respectively).

In the rational case, the Atkin-Lehner and the positive Atkin-Lehner groups coincide and $W = W^1 = \{\omega_d : d|D\} \simeq C_2^{2r}$ is generated by elements $\omega_d \in \mathcal{O}$, $\mathrm{n}(\omega_d) = d|D$. Moreover, $U_0 = \langle \omega_D \rangle \simeq C_2$.

If $(\mathcal{O}, \mu)$ is a non twisting polarized order, then the field of moduli of quaternionic multiplication $k_{\mathcal{O}}$ is at most a quadratic extension over the field of moduli k_C of the curve C by Theorem 4.7.4.

On the other hand, if $(\mathcal{O}, \mu)$ is twisting and $B = (\frac{-D,m}{\mathbb{Q}})$ for $m|D$, then $V_0 = \{1, \omega_m, \omega_{D/m}, \omega_D\} \simeq C_2^2$, where we can choose representatives ω_m, $\omega_{D/m}$ in $\mathcal{O}$ such that $\mu\omega_m = -\omega_m\mu$ and $\mu\omega_{D/m} = -\omega_{D/m}\mu$. Note that, up to multiplication by non zero rational numbers, ω_m and $\omega_{D/m}$ are the only twists of $(\mathcal{O}, \mu)$. We obtain by Theorem 4.7.4 that

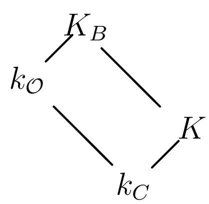
- $k_{\mathcal{O}}/k_C$ is at most a quartic abelian extension.
- $k_{\mathcal{O}} = k_S$ for any real quadratic order $S \not\subset \mathbb{Q}(\omega_m) \simeq \mathbb{Q}(\sqrt{m})$ or $\mathbb{Q}(\omega_{D/m}) \simeq \mathbb{Q}(\sqrt{D/m})$.

- $k_{\mathbb{Z}[\omega_m]}$ and $k_{\mathbb{Z}[\omega_{D/m}]}$ are at most quadratic extensions of k_C and these are such that $k_{\mathcal{O}} = k_{\mathbb{Z}[\omega_m]} \cdot k_{\mathbb{Z}[\omega_{D/m}]}$.

Mestre [Me90] studied the relation between the field of moduli $k_C = k_{(J(C),\Theta_C)}$ of a curve of genus 2 and its possible fields of definition, under the hypothesis that the hyperelliptic involution is the only automorphism on the curve. Mestre constructed an obstruction in $\mathrm{Br}_2(k_C)$ for C to be defined over its field of moduli. On identifying this obstruction with a quaternion algebra H_C over k_C, he showed that C admits a model over a number field K such that $k_C \subseteq K$ if and only if $H_C \otimes K \simeq \mathrm{M}_2(K)$.

If $\mathrm{Aut}(C) \supsetneq C_2$, Cardona and Quer [CaQu02] have recently proved that C always admits a model over its field of moduli k_C.

Assume now, as before, that $\mathrm{End}_{\bar{\mathbb{Q}}}(J(C)) \simeq \mathcal{O}$ is a maximal order in an indefinite division quaternion algebra B over $\mathbb{Q}$. Let K be a field of definition of C; note that, since $\mathrm{End}_{\bar{\mathbb{Q}}}(J(C)) \otimes \mathbb{Q} = B$ is division, $\mathrm{Aut}(C) \simeq C_2$ and therefore k_C may not be a field of definition of the curve. Having made the choice of a model C/K, there is a minimal (Galois) field extension L/K of K such that $\mathrm{End}_L(J(C)) \simeq \mathcal{O}$. This gives rise to a diagram of Galois extensions



The nature of the Galois extensions K_B/K was studied in Section 5.1, while the relation between the field of moduli $k_{\mathcal{O}}$ and the possible fields of definition K_B of the quaternionic multiplication was investigated by Jordan in [Jo86]. The combination of all these facts yields the following statement.

Proposition 5.2.1. *Let C/K be a smooth curve of genus 2 over a number field K and assume that $\mathrm{End}_{\bar{\mathbb{Q}}}(J(C))$ is a maximal quaternion order $\mathcal{O}$. Let K_B/K the minimal extension of K over which all endomorphisms of $J(C)$ are defined.*

Then $\mathrm{Gal}(K_B/K) \simeq \mathrm{Gal}(k_{\mathcal{O}}/k_C) \simeq \{1\}$, C_2 *or* $D_2 = C_2 \times C_2$.

Proof. Assume first that $k_C = K$ is a field of definition of the curve. Then $k_{\mathcal{O}} = K_B$ is a field of definition of all endomorphisms of $J(C)$: if it was not, there would be infinitely many pairwise different extensions L_α/K

such that $\mathrm{End}_{L_\alpha}(J(C)) = \mathcal{O}$. This would contradict Silverberg's result that states that such extensions are unique (cf. [Si92]).

If, on the contrary, Mestre's obstruction H_C is non trivial in $\mathrm{Br}_2(k_C)$, then C admits a model over any quadratic extension K/k_C that splits H_C but not over k_C itself. We then have that any field of definition K_B of all endomorphisms of $J(C)$ must be a quadratic extension of k_C that strictly contains it. Indeed, by [Jo81], we know that $[K_B : k_C] \leq 2$. Suppose that $K_B = k_C$. Then, K_B would contain all fields of definition H of C and this is not possible.

In either case, the possibilities for $\mathrm{Gal}(K_B/K)$ are $\{1\}$, C_2 and D_2, by Theorem 4.7.4. □

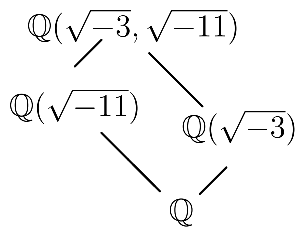
Remark 5.2.2. The above argument actually yields more than this: it either holds that $k_C \subsetneq K \subsetneq k_{\mathcal{O}} \subsetneq K_B$ is a chain of quadratic extensions or $K_B = k_{\mathcal{O}} \cdot K$. Moreover, the first of these possibilities only arises when $[K_B : K] = 4$ and for the finitely many subextensions K/k_C of $k_{\mathcal{O}}$ such that $H_C \otimes_{k_C} K$ is trivial.

Example 5.2.3. Let C be the smooth projective curve of hyperelliptic model

$$Y^2 = \frac{1}{48} X(9075X^4 + 3025(3+2\sqrt{-3})X^3 - 6875X^2 + 220(-3+2\sqrt{-3})X + 48).$$

Let $A = J(C)/K$ be the Jacobian variety of C over $K = \mathbb{Q}(\sqrt{-3})$. By [HaMu95], A is an abelian surface with quaternionic multiplication by a maximal order in the quaternion algebra of discriminant 6. See also Section 4.6.2. As it is explicitly shown in [HaMu95], there is an isomorphism between C and the conjugate curve C^τ over $\mathbb{Q}$. Hence, the field of moduli $k_C = \mathbb{Q}$ is the field of rational numbers.

In addition, it was shown in [DiRo1] that $K_B = \mathbb{Q}(\sqrt{-3}, \sqrt{-11})$ is the minimal field of definition of the quaternionic multiplication on A. By our last proposition and remark, we must have that $K_B = K \cdot k_{\mathcal{O}}$ with $[k_{\mathcal{O}} : k_C] = 2$. In addition, by Shimura's Theorem 4.7.5, $k_{\mathcal{O}}$ must be an imaginary quadratic extension of $\mathbb{Q}$. This shows that $k_{\mathcal{O}} = \mathbb{Q}(\sqrt{-11})$ and the picture of fields of moduli and definition of A is completed:



5.3 Explicit examples of Jacobian varieties of genus two curves

Let $K \subset \bar{\mathbb{Q}}$ be a number field in an algebraic closure $\bar{\mathbb{Q}}$ of the field $\mathbb{Q}$ of rational numbers and let C/K be an irreducible smooth algebraic curve of genus 2 over K. Assume that $\mathrm{End}_{\bar{\mathbb{Q}}}(J(C)) = \mathcal{O}$ is a maximal order in an indefinite division quaternion algebra B of discriminant D. As it was shown in Theorem 5.1.10, the minimal field of definition K_B of all endomorphisms of $J(C)$ is either a trivial, a quadratic or a biquadratic extension of K. Moreover, we have that

- (Trivial case) If $K_B = K$, then $\mathrm{End}_K(J(C)) = \mathcal{O}$,
- (Quadratic case) If $[K_B : K] = 2$, then $\mathrm{End}_K(J(C)) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-D})$, $\mathbb{Q}(\sqrt{m})$ or $\mathbb{Q}(\sqrt{D/m})$, for some positive proper divisor $m|D$,
- (Biquadratic case) If $[K_B : K] = 4$, then $\mathrm{End}_K(J(C)) = \mathbb{Z}$.

In this section, we show that Theorem 5.1.10 is sharp by exhibiting several explicit examples which attain all the possibilities allowed by the theorem. We obtain them by considering several particular fibres of Hashimoto-Murabayashi's and Hashimoto-Tsunogai's families of curves of genus two whose Jacobian varieties are abelian surfaces with quaternionic multiplication (cf. Section 4.6.1, [HaMu95] and [HaTs99]).

As a by-product, we also show that all computations performed in [HaTs99] supporting an analogue of the Sato-Tate conjecture for these abelian surfaces are unconditionally correct (cf. [HaTs99] for details).

Being these examples part of a joint work with L. Dieulefait, we omit the details of the proofs. Instead, we refer the reader to [DiRo2].

Example 5.3.1. [Biquadratic case]

I. Let $C_1/\mathbb{Q}(\sqrt{2})$ be a smooth projective model of the curve

$$Y^2 = (X^2+5)((-1/6+\sqrt{2})X^4+20X^3-490/6X^2+100X+25(-1/6-\sqrt{2})).$$

Then, the Jacobian variety $J_1 = J(C_1)/\mathbb{Q}(\sqrt{2})$ of C_1 has multiplication by a maximal order $\mathcal{O}$ in the quaternion algebra B_6 of discriminant 6 over the quartic extension $K_B = \mathbb{Q}(\sqrt{-2}, \sqrt{-1}, \sqrt{-5})$ of $\mathbb{Q}(\sqrt{2})$. Moreover,

- $\mathrm{End}_{\mathbb{Q}(\sqrt{2},\sqrt{-5})}(J_1) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{2})$,
- $\mathrm{End}_{\mathbb{Q}(\sqrt{2},\sqrt{5})}(J_1) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{3})$,
- $\mathrm{End}_{\mathbb{Q}(\sqrt{2},\sqrt{-1})}(J_1) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-6})$ and
- $\mathrm{End}_{\mathbb{Q}(\sqrt{2})}(J_1) = \mathbb{Z}$.

II. Let $C_2/\mathbb{Q}$ be a smooth projective model of the curve

$$Y^2 = (X^2+7/2)(83/30X^4+14X^3-1519/30X^2+49X-1813/120)$$

and let $J_2 = J(C_2)/\mathbb{Q}$ be its Jacobian variety. Then,

- $\mathrm{End}_L(J_2) = \mathcal{O}$ is a maximal order in B_6 for $L = \mathbb{Q}(\sqrt{-6}, \sqrt{-14})$,
- $\mathrm{End}_{\mathbb{Q}(\sqrt{-14})}(J_2) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{2})$,
- $\mathrm{End}_{\mathbb{Q}(\sqrt{21})}(J_2) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{3})$,
- $\mathrm{End}_{\mathbb{Q}(\sqrt{-6})}(J_2) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-6})$ and
- $\mathrm{End}_{\mathbb{Q}}(J_2) = \mathbb{Z}$.

Note that $B_6 = (\frac{-6,2}{\mathbb{Q}}) = (\frac{-6,3}{\mathbb{Q}})$ is a twisting quaternion algebra. Hence, the above two examples are in perfect concordance with Theorem 5.1.10.

Example 5.3.2. [Quadratic case]

Let C_3 be the smooth algebraic curve of genus 2 of hyperelliptic model

$$Y^2 = \frac{1}{48}X(9075X^4+3025(3+2\sqrt{-3})X^3-6875X^2+220(-3+2\sqrt{-3})X+48).$$

Let J_3/K be the Jacobian variety of C_3 over $K = \mathbb{Q}(\sqrt{-3})$. By [HaMu95], the ring of endomorphisms of J_3 over $\bar{\mathbb{Q}}$ is a maximal order in the quaternion algebra of discriminant $D = 10$ over $\mathbb{Q}$. As is shown in [DiRo2], $L = \mathbb{Q}(\sqrt{-3}, \sqrt{-11})$ is the minimal field of definition of the quaternion endomorphisms of J_3 and

$$\mathrm{End}_K(J_3) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}(\sqrt{5}).$$

Finally, let us conclude with a pair of examples of abelian surfaces A/K with quaternionic multiplication over a quadratic imaginary field K such that all quaternion endomorphisms of A are defined over K itself. Obvious examples can be obtained by suitably extending the base field of the Jacobian varieties of the curves in the above examples. Indeed, $J_3/\mathbb{Q}(\sqrt{-3}, \sqrt{-11})$ is an example of this phenomenon.

Below, we present two non trivial examples of abelian surfaces A/K with $\mathrm{End}_K(A)$ a maximal quaternion order that cannot be obtained by base extension from a subfield of K.

Example 5.3.3. [Trivial case]

Let C_4/K_4, C_5/K_5 be the fibres of the Hashimoto-Tsunogai's family $S_6(t, s)$ at the values $t_4 = 2$ and $t_5 = 2/3$ over $K_4 = \mathbb{Q}(\sqrt{-379})$ and $K_5 = \mathbb{Q}(\sqrt{-19})$, respectively. See Section 4.6.2 for details. Then, $\mathrm{End}_K(J(C_i)) \simeq \mathcal{O}$ is a maximal order in B_6.

The computation of the absolute Igusa invariants of these curves show that there does not exist any curve $C/\mathbb{Q}$ such that $C \simeq C_3$ nor $C \simeq C_4$ over $\overline{\mathbb{Q}}$. See [CaQu02] for details.

Remark 5.3.4. The above two examples are defined over completely imaginary fields $L = K$. This is not a coincidence since, by a result of Shimura (cf. [Sh75]), there are not curves C/K of genus 2 over a number field K admitting a real archimedean place such that $\mathrm{End}_K(J(C))$ is a quaternion order.

To close this section devoted to present some examples, we wish to illustrate several modularity questions. We recall that an abelian variety $A/\mathbb{Q}$ of GL_2-type over the field $\mathbb{Q}$ of rational numbers is an abelian variety whose algebra of endomorphisms $\mathrm{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q} = F$ is a number field F of degree $[F : \mathbb{Q}] = \dim(A)$. The generalized Shimura-Taniyama-Weil conjecture predicts that any abelian variety of GL_2-type over $\mathbb{Q}$ is modular: it is isogenous over $\mathbb{Q}$ to a factor of the Jacobian variety $J_1(N)$ of the modular curve $X_1(N)$ for some positive integer N. As is well known, as a consequence of the work of Wiles, Taylor, Diamond, Conrad, Breuil and others, any elliptic curve $E/\mathbb{Q}$ is modular (cf. [Wi95]). In dimension two, Ellenberg [Ell] has shown that any abelian surface $A/\mathbb{Q}$ of GL_2-type over $\mathbb{Q}$ with sufficiently good reduction at 3 and 5 is modular.

We note that any elliptic curve E/K gives rise to an abelian surface with multiplication by the split quaternion order $\mathrm{M}_2(\mathbb{Z})$ by considering the power $A = E^2$. In the literature, abelian surfaces with quaternionic multiplication are often called *fake elliptic curves*. The motivation for this terminology is the strong analogy between the arithmetic of elliptic curves and fake elliptic curves. We refer the reader to [Bu96], [Jo81], [Jo86], [JoLi85], [DiRo1], [DiRo2], [Oh74], [Se68] for these analogies, but also for some interesting differences. In this regard and in view of Wiles et al.'s Theorem, it is natural to wonder:

Question 5.3.5. Is any abelian surface $A/\mathbb{Q}$ with quaternionic multiplication modular?

A necessary condition for $A/\mathbb{Q}$ to be modular is that $A/\mathbb{Q}$ be of GL_2-type. Moreover, according to Shimura-Taniyama-Weil's conjecture, this is also a sufficient condition. Then, the above question translates to ask whether any abelian surface $A/\mathbb{Q}$ with quaternionic multiplication is of GL_2-type over $\mathbb{Q}$. It has recently been suggested that this could be the case. However, our previous examples show that there exist fake elliptic curve over $\mathbb{Q}$ that fail to be modular. Indeed, we obtain the following

Corollary 5.3.6. *The Jacobian variety $J_2/\mathbb{Q}$ is a nonmodular abelian surface with quaternionic multiplication.*

Chapter 6

Diophantine properties of Shimura curves

Introduction

In this chapter, we explore the diophantine properties of Shimura curves.

Let B be an indefinite rational quaternion algebra and choose a maximal order $\mathcal{O}$ of B. Let $D = p_1 \cdot \ldots \cdot p_{2r}$, p_i prime numbers, be the discriminant of B. We can view $\Gamma = \{\gamma \in \mathcal{O} : n(\gamma) = 1\}$ as an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{R})$ through an identification $\Psi : B \otimes \mathbb{R} \simeq M_2(\mathbb{R})$ and consider the Riemann surface $\Gamma \backslash \mathcal{H}$, where $\mathcal{H}$ denotes the upper-half plane of Poincaré. As it was pointed out in Chapter 4, Shimura showed in [Sh67] that this is the set of complex points of an algebraic curve $X_B = X_D/\mathbb{Q}$ over $\mathbb{Q}$ which parametrizes abelian surfaces with quaternionic multiplication by $\mathcal{O}$.

The classical modular case arises when we consider the split quaternion algebra $B = M_2(\mathbb{Q})$ of discriminant $D = 1$. In this case, $X_1 = \mathbf{A}^1_{\mathbb{Q}}$ is the j-line that classifies elliptic curves or, by squaring, abelian surfaces with multiplication by $M_2(\mathbb{Z})$. Throughout, we limit ourselves to *non split* quaternion algebras, that is, $D \neq 1$. In this case, Γ has no parabolic elements and $\Gamma \backslash \mathcal{H}$ is already compact so there are no cusps and the automorphic forms on X_D do not admit Fourier expansions. In this regard, see [Mor95] and [Ba02].

As it was introduced in Chapter 4, the elements of the Atkin-Lehner group $W^1 = W = \{\omega_m : m|D\} \simeq C_2^{2r}$, where C_2 is the cyclic group of order two, act as rational involutions on the Shimura curve X_D and there is a natural inclusion $W \subseteq \mathrm{Aut}_{\mathbb{Q}}(X_D)$. It is the aim of Section 6.1 to examine the full group of automorphisms $\mathrm{Aut}(X_D \otimes \mathbb{C})$ of these curves. We firstly study the fields of definition of the automorphisms $\omega \in \mathrm{Aut}(X_D \otimes \mathbb{C})$ and describe the possible group structures of $\mathrm{Aut}(X_D \otimes \mathbb{C})$. In many cases, we prove that $\mathrm{Aut}(X_D \otimes \mathbb{C}) = W$.

Let us recall that Ogg [Ogg77] studied the group of automorphisms of the modular curves $X_0(N)$ for square-free level N. There, the action of $\mathrm{Aut}(X_0(N))$ on the set of cusps played a fundamental role. When $D > 1$, the difficulty lies precisely in the absence of cusps on X_D.

Next, in Section 6.2, the family of Shimura curves X_D that admit bielliptic involutions is determined. The hyperelliptic problem was already settled by Michon and Ogg independently in [Mic81], [Ogg74] and [Ogg83]. Also, the family of bielliptic modular curves $X_0(N)$ was given in [Bar99]. Since $\mathrm{Aut}(X_0(N))$ is largely understood (cf. [Ogg77] and [KeMo88]), the main point in [Bar99] was to count the number of fixed points of the non Atkin-Lehner involutions that appear when $4|N$ or $9|N$. In our case this difficulty does not arise, but on the other hand the automorphism groups of the Shimura curves X_D are less known.

In Section 6.3, we derive some arithmetical consequences from the above results concerning the set of rational points on X_D over quadratic fields. Recall that by a fundamental theorem of Shimura, there are no real points on Shimura curves and therefore quadratic imaginary fields are the simplest fields over which these curves may have rational points. Our main theorem in this section completely solves a question posed and studied by Kamienny in [Ka90]:

Question 6.0.7. Which Shimura curves X_D of genus $g \geq 2$ admit infinitely many quadratic points?

This question is motivated by Faltings' Theorem on Mordell's conjecture. In Theorem 6.3.3 and Table 6.3.3, we give a complete and explicit answer to question 6.0.7 by listing the finitely many discriminants D such that

$$\sharp\{P \in X_D(\bar{\mathbb{Q}}) : [\mathbb{Q}(P) : \mathbb{Q}] \leq 2\} = +\infty.$$

Our method is based upon ideas of Abramovich, Harris and Silverman (see [AbHa91] and [HaSi91]). We also note that our Theorem 6.3.3 can be used to derive interesting consequences on several questions concerning modular abelian surfaces with extra-twist.

Finally, in Section 6.4, we use the theory of Čerednik-Drinfeld to compute equations of elliptic Atkin-Lehner quotients of Shimura curves. Table 6.4 in Section 6.4 gives a Weierstrass equation of *all* elliptic curves of the form $X_D/\langle w\rangle$ where $w \in \mathrm{Aut}(X_D)$ is any $\mathbb{Q}$-bielliptic involution on the curve. Some examples were already given in [Rob89].

The main tools used in this chapter come from the reduction of Shimura curves at both good and bad places. Drinfeld constructed a projective model M_D over $\mathbb{Z}$ of the Shimura curve X_D which extends the moduli interpretation given by Shimura to abelian schemes over arbitrary bases (cf. [Dr76], [Bu96] and [BoCa91]). Morita [Mo81] showed that M_D has good reduction at all primes $p \nmid D$ and Shimura determined in [Sh67] the zeta function of the special fibre of M_D at p. Moreover, the Čerednik-Drinfeld theory (cf. [BoCa91], [Ce76], [Dr76] and [JoLi85]) provides a good account of the behaviour of the reduction of $M_D \pmod p$ when $p|D$.

Our results in this chapter were presented in [Ro02].

6.1 The group of automorphisms of Shimura curves

Throughout, X_D will denote the canonical model over $\mathbb{Q}$ of the Shimura curve of discriminant $D = p_1 \cdot \ldots \cdot p_{2r} \neq 1$. It is a proper smooth scheme over $\mathbb{Q}$ of dimension 1. Let $\mathrm{Aut}_{\mathbb{Q}}(X_D)$ be the group of $\mathbb{Q}$-automorphisms of X_D that sits inside the full group of geometric automorphisms $\mathrm{Aut}_{\mathbb{C}}(X_D \otimes \mathbb{C})$ of the complex algebraic curve $X_D \otimes \mathbb{C}$.

Proposition 6.1.1. *If* $g(X_D) \geq 2$*, then*

1. *All automorphisms of* $X_D \otimes \mathbb{C}$ *are defined over* $\mathbb{Q}$*. That is:* $\mathrm{Aut}_{\mathbb{C}}(X_D \otimes \mathbb{C}) = \mathrm{Aut}_{\mathbb{Q}}(X_D)$.
2. $\mathrm{Aut}_{\mathbb{Q}}(X_D) \simeq C_2^s$*,* $s \geq 2r$*.*

Proof. Ribet [Ri75] proved that all the endomorphisms of an abelian variety A/K with semistable reduction over a number field K are defined over an unramified extension of K. The Jacobian variety $J_D/\mathbb{Q}$ of X_D has good reduction at primes $p \nmid D$ and, from [JoLi86], we know that the identity's connected component of the reduction mod p, $p|D$, of the Néron model of

J_D is a torus. Hence, J_D has semistable reduction over $\mathbb{Q}$ and all its endomorphisms are rational because $\mathbb{Q}$ has no nontrivial unramified extensions.

Since, by Hurwitz Theorem, $\mathrm{Aut}_{\mathbb{C}}(X_D \otimes \mathbb{C})$ is a finite group, all automorphisms of $X_D \otimes \mathbb{C}$ are defined over $\bar{\mathbb{Q}}$. Moreover, the natural map $\mathrm{Aut}_{\bar{\mathbb{Q}}}(X_D \otimes \bar{\mathbb{Q}}) \to \mathrm{Aut}_{\bar{\mathbb{Q}}}(J_D \otimes \bar{\mathbb{Q}})$ is injective and $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariant and therefore we conclude from above that all automorphisms of $X_D \otimes \mathbb{C}$ are rational.

For the second part, let $X_0(D)/\mathbb{Q}$ be the modular curve of level D and consider the new part $J_0(D)^{new}/\mathbb{Q}$ of its Jacobian variety $J_0(D)$. It is well known (cf. [Ri75]) that $\mathrm{End}^0_{\mathbb{Q}}(J_0(D)^{new}) \simeq \mathbb{T}\otimes\mathbb{Q} \simeq \prod_{i=1}^t K_i$, where $\mathbb{T}$ denotes the Hecke algebra of level D and K_i are totally real number fields. Ribet's Isogeny Theorem (cf. [Ri80]) states the existence of an isogeny

$$\varphi : J_D \longrightarrow J_0(D)^{new}$$

between J_D and $J_0(D)^{new}$. This isogeny is Hecke invariant (but signinterchanging for the Atkin-Lehner action) and defined over $\mathbb{Q}$. Hence, the ring of endomorphisms $\mathrm{End}_{\mathbb{Q}}(J_D)$ is an order in $\prod_{i=1}^t K_i$. An automorphism of the curve X_D induces an automorphism of finite order on J_D. Moreover, the group of integral units in $\prod_{i=1}^t K_i$ is isomorphic to C_2^t. Thus $\mathrm{Aut}_{\mathbb{Q}}(X_D) \simeq C_2^s$ with $2r \leq s \leq t$, the first inequality holding just because $W \subseteq \mathrm{Aut}_{\mathbb{Q}}(X_D)$. □

We conclude that any automorphism of X_D acts as a rational involution on it. In view of the above proposition, we will simply denote the group $\mathrm{Aut}_{\mathbb{C}}(X_D \otimes \mathbb{C}) = \mathrm{Aut}_{\mathbb{Q}}(X_D)$ by $\mathrm{Aut}(X_D)$. Naturally we ask whether the Atkin-Lehner group is the full group of automorphisms of the curve, provided that $g(X_D) \geq 2$. This is the case for modular curves $X_0(N)$ of square free level N, $N \neq 37$ (cf. [Ogg77] and [KeMo88]).

Recall that an elliptic point on the curve X_D is a branched point of the natural projection

$$\mathcal{H} \to \Gamma\backslash\mathcal{H} \simeq X_D(\mathbb{C}).$$

The stabilizers of those elliptic points in $\Gamma/\{\pm 1\}$ are of order 2 or 3. 2-elliptic points (respectively 3-elliptic points) correspond to Γ-conjugacy classes of embeddings of the quadratic order $\mathbb{Z}[i]$, $i^2 = -1$ (respectively $\mathbb{Z}[\rho]$, $\rho^3 = 1$) in the quaternion order $\mathcal{O}$. Their cardinality is given by

$$e_2 = \prod_{\ell | D} (1 - \left(\frac{-4}{\ell}\right)),$$

$$e_3 = \prod_{\ell | D} (1 - \left(\frac{-3}{\ell}\right)),$$

where $(\frac{\cdot}{\cdot})$ denotes the Kronecker symbol.

Theorem 6.1.2. *Let X_D be the Shimura curve of discriminant D. If it has no elliptic points, then* $\mathrm{Aut}(X_D) = W$.

Proof. If there are no elliptic points on $X_D(\mathbb{C})$, then the natural projection $\mathcal{H} \to \Gamma \backslash \mathcal{H} \simeq X_D(\mathbb{C})$ is the universal cover of the Riemann surface $X_D(\mathbb{C})$ so $\mathrm{Aut}(X_D) \simeq \mathrm{Norm}_{\mathrm{PGL}_2^+(\mathbb{R})}(\Gamma)/\Gamma$. Here the superindex + denotes matrices with positive determinant. It is known that $W \simeq \mathrm{Norm}_{B^\times}(\Gamma)/\mathbb{Q}^\times \Gamma \simeq C_2^{2r}$ (cf. [Mic81] and [Ogg83]). We observe now that the $\mathbb{Q}$-vector space spanned by Γ is $\langle \Gamma \rangle_{\mathbb{Q}} = B$. Indeed, since the reduced norm n is indefinite on the space of pure quaternions B_0, we can find linearly independent elements ω_1, ω_2, $\omega_3 \in B_0$ such that $\mathbb{Z}[\omega_i] \subset B$ is a real quadratic order in B. Then, by solving the corresponding Pell equations, we find units $\gamma_i \in \mathbb{Z}[\omega_i] \cap \Gamma$, $\gamma_i \neq \pm 1$ such that $\{1, \gamma_1, \gamma_2, \gamma_3\}$ is a $\mathbb{Q}$-basis of B.

Hence, any $\alpha \in \mathrm{Norm}_{\mathrm{GL}_2^+(\mathbb{R})}(\Gamma)$ will actually normalize B^*. By the Skolem-Noether Theorem, α induces an inner automorphism of B so that $\alpha \in \mathbb{R}^* \mathrm{Norm}_{B^*}(\Gamma)$. This shows that $\mathrm{Aut}(X_D) = W$. □

Remark 6.1.3. In proving the above theorem, we have also shown that the Atkin-Lehner group W of an arbitrary Shimura curve X_D is exactly the subgroup of automorphisms that lift to a Möbius transformation on $\mathcal{H}$ through the natural uniformization $\mathcal{H} \to \Gamma \backslash \mathcal{H} \simeq X_D(\mathbb{C})$.

The proof remains valid for Eichler orders of square-free level N and therefore it generalizes an analogous result of Lehner and Newman for discriminant $D = 1$ (cf. [LeNe64]).

The next theorem is similar in spirit to Theorem 6.1.2 and requires a previous lemma due to Ogg [Ogg77].

Lemma 6.1.4. *Let K be a field and $\mu(K)$ its group of roots of unity. Let $p = \max(1, \mathrm{char} K)$ the characteristic exponent of K. Let C be an irreducible curve defined over K and $P \in C(K)$ a regular point on it. Let G be a finite* *group of K-automorphisms acting on C and fixing the point P. Then there is a homomorphism $f : G \to \mu(K)$ whose kernel is a p-group.*

Theorem 6.1.5. *Let $D = 2p$, $3p$; p a prime number. If $g(X_D) \geq 2$, then* $\mathrm{Aut}(X_D) = W \simeq C_2 \times C_2$.

Proof. Suppose first that $D = 2p$ with $p \equiv 3 \pmod 4$. In this case, the fixed points on X_D of the Atkin-Lehner involution ω_2 are *Heegner points* (see e.g. [Al99] for a general account). Their coordinates on Shimura's canonical model X_D generate certain class fields. More precisely, if the genus $g(X_D)$ is even, then ω_2 exactly fixes two points P, P' with complex multiplication by the quadratic order $\mathbb{Z}[i]$ and hence (cf. [ShTa61]) P, $P' \in X_D(\mathbb{Q}(i))$. If $g(X_D)$ is odd, then, besides P and P', ω_2 fixes two more points Q, $Q' \in X_D(\mathbb{Q}(\sqrt{-2}))$ which have complex multiplication by $\mathbb{Z}[\sqrt{-2}]$. As we have seen, $\mathrm{Aut}(X_D)$ is an abelian group so it acts on the set of fixed points of ω_2 on X_D. Since all automorphisms are rational, they must keep the field of rationality of these points so that $\mathrm{Aut}(X_D)$ actually acts on $\{P, P'\}$. It follows from the previous lemma that the order of the stabilizer of P or P' in $\mathrm{Aut}(X_D)$ is at most 2. Hence, $\#\mathrm{Aut}(X_D) \leq 4$ and $\mathrm{Aut}(X_D) = W$.

Suppose now that $D = 2p$, $p \equiv 1 \pmod 4$ or $D = 3p$, $p \equiv 1 \pmod 3$. By the theory of Čerednik-Drinfeld, the special fibre $M_D \otimes \mathbb{F}_p$ of the reduction mod p of the integral model M_D of our Shimura curve consists of two rational irreducible components Z, Z' defined over $\mathbb{F}_{p^2}$. The complete local rings of the intersection points of Z and Z' over the maximal unramified extension $\mathbb{Z}_p^{unr}$ of $\mathbb{Z}_p$ are isomorphic to $\mathbb{Z}_p^{unr}[x, y]/(xy - p^\ell)$ for some length $\ell \geq 1$. The reduction mod p of the Atkin-Lehner involution ω_p switches Z and Z', fixing the double points of intersection. Among these double points, there is exactly one, say $\widetilde{Q}$, which has *length* 2, as it follows from [Ku79]. Thus $\mathrm{Aut}(X_D)$ acting on $M_D \otimes \mathbb{F}_p$ must fix $\widetilde{Q}$. Recalling now that $\mathrm{Aut}(X_D) \simeq C_2^s$, we again apply Ogg's Lemma 6.1.4 to the curve $Z/\mathbb{F}_{p^2}$ $(p \neq 2)$ and the point $\widetilde{Q}$ to obtain that $\#\mathrm{Aut}(X_D) \leq 4$. Therefore $\mathrm{Aut}(X_D) = W$.

In the remaining case, namely when $D = 3p$, $p \equiv -1 \pmod 3$, we observe the curious phenomenon that $\ell = 109$ is a prime of good reduction for the Shimura curve X_D that yields

$$\# M_D \otimes \mathbb{F}_{109}(\mathbb{F}_{109}) \not\equiv 0 \pmod 4$$

except for the two exceptional cases $D = 3 \cdot 89$ and $D = 3 \cdot 137$. In any case, we check that $\# M_{3\cdot 89} \otimes \mathbb{F}_{67}(\mathbb{F}_{67}) = 94$ and $\# M_{3\cdot 137} \otimes \mathbb{F}_{103}(\mathbb{F}_{103}) = 98$.

This is carried out by using the explicit formula for the number of rational points over finite fields of the reduction of Shimura curves at good places given by Jordan and Livné in [JoLi85]. From this we proceed as above: since all automorphisms of X_D are defined over $\mathbb{Q}$, their reduction mod ℓ must preserve the $\mathbb{F}_\ell$-rational points on $M_D \otimes \mathbb{F}_\ell$ and we apply Ogg's Lemma 6.1.4 to the regular curve $M_D \otimes \mathbb{F}_\ell$ to conclude that $\mathrm{Aut}(X_D) = W$. $\square$

Remark 6.1.6. The first argument can be adapted for more general discriminants in an obvious way. For instance, if $D = p\delta$ where p is a prime integer, $p \equiv 3 \pmod 8$, and $(\frac{-p}{\ell}) = -1$ for any $\ell|\delta$, then we again obtain that $\mathrm{Aut}(X_D) = W$ because the Hilbert class field of $\mathbb{Q}(\sqrt{-p})$ is strictly contained in the ring class field of conductor 2 and, by genus theory, $h(-p)$ is odd.

Example 6.1.7. Shimura curve quotient $X_{291}^+ = X_{291}/W$ has genus 2 and therefore it is hyperelliptic. However, the hyperelliptic involution on X_{291}^+ is exceptional: it does not lift to a Möbius transformation on $\mathcal{H}$ through $\pi : \mathcal{H} \to X_{291}^+(\mathbb{C}) = \Gamma \cdot W\backslash\mathcal{H}$, while all automorphisms of X_{291} are of Atkin-Lehner type by Theorem 6.1.5. This is caused by the fact that π is not the universal cover of X_{291}^+.

6.2 Bielliptic Shimura curves

Recall that an algebraic curve C of genus $g \geq 2$ is bielliptic if it admits a degree 2 map $\varphi : C \to E$ onto a curve E of genus 1. We will ignore fields of rationality until the next section. Alternatively, C is bielliptic if and only if there is an involutive automorphism acting on it with $2g-2$ fixed points. We present now some facts about bielliptic curves C such that, like Shimura curves, $\mathrm{Aut}(C) \simeq C_2^s$.

Lemma 6.2.1. *Let C/K, char $K \neq 2$, be a bielliptic curve of genus g with $\mathrm{Aut}(C) \simeq C_2^s$ for some $s \geq 1$. For any $w \in \mathrm{Aut}(C)$, let $n(w)$ denote the number of fixed points of w on C. Let v be a bielliptic involution on C and for any $w \in \mathrm{Aut}(C)$, $w \neq 1$ or v, denote $w' = v \cdot w$.*

1. *If g is even, then $n(w) = 2$ and $n(w') = 6$, or viceversa. If g is odd, then $\{n(w), n(w')\} = \{0,0\}, \{0,8\}$ or $\{4,4\}$ as non ordered pairs.*
2. *If g is even, then $s \leq 3$. If g is odd, then $s \leq 4$.*

3. If $g \geq 6$, then the bielliptic involution v is unique.

Proof. It follows from Hurwitz's Theorem applied to the projection of the curve C onto its quotient by suitable groups generated by involutions acting on it. □

Remark 6.2.2. Observe that if D is odd, then $g(X_D)$ is always odd, as we check from Eichler's formula for the genus (see e.g. [Ogg83]).

Obviously, the main source for possible bielliptic involutions on the curves X_D is the Atkin-Lehner group. From Eichler's formula for $n(w)$, $w \in W$ (see [Ogg83]), it is a routine exercise to check whether X_D has bielliptic involutions of Atkin-Lehner type. An alternative way to compute $n(w)$ is to read *backwards* the last column of Table 5 in [Ant75]. This is because Ribet's isogeny $\varphi : J_D \to J_0(D)^{new}$ switches the sign of the Atkin-Lehner action. But, first, we should focus on possible extra involutions and also bound the bielliptic discriminants D. Following Ogg's method in [Ogg74], we give such an upper bound in the next

Proposition 6.2.3. *If $D > 547$, X_D is not bielliptic.*

Proof. Suppose that the curve X_D is bielliptic: there is a degree 2 map $\varphi : X_D \to E$ onto a curve E of genus 1. By Proposition 6.1.1, both φ and E are defined over $\mathbb{Q}$ although E may not be an elliptic curve over $\mathbb{Q}$ since it may fail to have rational points. See Section 6.3 for examples. Choose a prime of good reduction $\ell \nmid D$ of X_D, let K_ℓ be the quadratic unramified extension of $\mathbb{Q}_\ell$ and let R_ℓ denote its ring of integers. As follows from [JoLi85], $X_D(K_\ell) \neq \emptyset$ and hence E is an elliptic curve over K_ℓ. Moreover, due to Ribet's Isogeny Theorem, E also has good reduction over ℓ. By the universal property of the Néron model of E over R_ℓ, φ extends to the minimal smooth model $M_D \otimes R_\ell$ of X_D and we can reduce the bielliptic structure mod ℓ to obtain a $2:1$ map $\tilde{\varphi} : M_D \otimes \mathbb{F}_{\ell^2} \to \widetilde{E}$. From Weil's estimate, $N_{\ell^2} = \# M_D \otimes \mathbb{F}_{\ell^2}(\mathbb{F}_{\ell^2}) \leq 2 \cdot \# \widetilde{E}(\mathbb{F}_{\ell^2}) \leq 2(\ell+1)^2$. Besides, we obtain from [JoLi85] that $\frac{(\ell-1)}{12} \prod_{p|D} (p-1) \leq N_{\ell^2}$ so N_{ℓ^2} grows as D tends to infinity. Applying these inequalities for $\ell = 2, 3, 5, 7, 11$, we conclude that, if $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \nmid D$, then $D \leq 546$. But, if $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 | D$, then $s \geq 5$ and therefore, by Lemma 6.2.1, X_D cannot be bielliptic. □

We are now able to prove

Theorem 6.2.4. *There are exactly thirty-two values of D for which X_D is bielliptic. In each case, the bielliptic involutions are of Atkin-Lehner type. These values, together with the genus $g = g(X_D)$ and the bielliptic involutions are given in Table 6.2.4 below.*

Table 6.2.4: Bielliptic Shimura curves

D	g	ω_m		D	g	ω_m		D	g	ω_m
26	2	ω_2,ω_{13}		82	3	ω_{82}		210	5	ω_{30},ω_{42},
35	3	ω_7		85	5	ω_{17}				ω_{70},ω_{105},
38	2	ω_2,ω_{19}		94	3	ω_2				ω_{210}
39	3	ω_{13}		106	4	ω_{53},ω_{106}		215	15	ω_{215}
51	3	ω_3		115	6	ω_{23}		314	14	ω_{314}
55	3	ω_5		118	4	ω_{59},ω_{118}		330	5	ω_3,ω_{22}
57	3	ω_{57}		122	6	ω_{122}				ω_{33},ω_{165},
58	2	ω_2,ω_{58}		129	7	ω_{129}				ω_{330}
62	3	ω_2		143	12	ω_{143}		390	9	ω_{390}
65	5	ω_{65}		166	6	ω_{166}		462	9	ω_{154}
69	3	ω_3		178	7	ω_{89}		510	9	ω_{510}
77	5	ω_{11},ω_{77}		202	8	ω_{101}		546	13	ω_{546}

Proof. Since we need only consider discriminants $D \leq 546$, we can first use any programming package to build up the list of Atkin-Lehner bielliptic involutions on Shimura curves X_D. These computations yield Table 6.2.4 above. In order to ensure that no extra bielliptic involutions arise, we observe that the above results, and particularly Theorem 6.1.5, imply that any bielliptic involution on X_D, for most of the discriminants $D \leq 546$, must be of Atkin-Lehner type. There are exactly three cases, namely $D = 55$, $D = 85$ and $D = 145$, for which none of the previous results and their obvious generalizations seem to apply.

Ad hoc arguments can be worked out for them. Firstly, the Jacobian varieties of the curves X_{55} and X_{85} have just one $\mathbb{Q}$-isogeny class of sub-abelian varieties of dimension 1, so there can be at most one bielliptic involution on these curves. But ω_5 (respectively ω_{17}) is already a bielliptic involution on X_{55} (respectively X_{85}).

More interesting is the curve X_{145} of genus 9. It is not bielliptic by any Atkin-Lehner involution although $J_{145} \sim_{\mathbb{Q}} E \times S \times A_3 \times A_3'$, where each factor has dimension 1, 2, 3 and 3 respectively. We check that $n(\omega_5) = n(\omega_{29}) = 0$ and $n(\omega_{145}) = 8$, so if there was a bielliptic involution v on X_{145} then $n(w'_{145}) = 0$, by Lemma 5.1. It follows from Lefschetz's fixed point formula (see [LaBi92]) that the rational traces of these three involutions on the Jacobian J_{145} would be $\mathrm{tr}(\omega_5) = \mathrm{tr}(\omega_{29}) = \mathrm{tr}(w'_{145}) = 2$. Moreover, involutions on J_{145} must be of the form $\{\pm 1_E\} \times \{\pm 1_S\} \times \{\pm 1_{A_3}\} \times \{\pm 1_{A_3'}\}$, up to conjugation by φ, thus tr $= 2$ can only be attained by two different involutions. Therefore v cannot exist and X_{145} is not bielliptic.

It can be showed that actually $\mathrm{Aut}(X_{145}) = W$: from the decomposition of J_{145} we know that $W \simeq C_2^2 \subseteq \mathrm{Aut}(X_{145}) \subseteq C_2^4$. Since X_{145} is neither hyperelliptic (cf. [Ogg83]) nor bielliptic (as we have just seen), it follows that the involutions $\{-1_E\} \times \{-1_S\} \times \{-1_{A_3}\} \times \{-1_{A_3'}\}$ and $\{+1_E\} \times \{-1_S\} \times \{-1_{A_3}\} \times \{-1_{A_3'}\}$ cannot be induced from $\mathrm{Aut}(X_{145})$. Thus it is a subgroup of index at least 4 in C_2^4 and $\mathrm{Aut}(X_{145}) = W$. $\square$

6.3 Infinitely many quadratic points on Shimura curves

Shimura [Sh75] proved that $X_D(\mathbb{R}) = \emptyset$ and in particular there are no $\mathbb{Q}$-rational points on Shimura curves X_D. Jordan and Livné [JoLi85] gave explicit criteria for deciding whether the curves X_D do have rational points over the p-adic fields $\mathbb{Q}_p$ for any finite prime p.

Less is known about rational points over global fields. Jordan [Jo86] proved that for a fixed quadratic imaginary field K, with class number $h_K \neq 1$, there are only finitely many discriminants D for which K splits the quaternion algebra B of discriminant D and $X_D(K) \neq \emptyset$. In this section we solve a question that is to an extent reciprocal: which Shimura curves X_D, $g(X_D) \geq 2$, have infinitely many quadratic points over $\mathbb{Q}$?

That is,

$$\#X_D(\mathbb{Q}, 2) = \#\{P \in X_D(\bar{\mathbb{Q}}) : [\mathbb{Q}(P) : \mathbb{Q}] \leq 2\} = +\infty.$$

We will say that an algebraic curve C/K of genus $g \geq 2$ is *hyperelliptic* over K (respectively *bielliptic* over K) if there is an involution $v \in \mathrm{Aut}_K(C)$ such that the quotient curve $C/\langle v\rangle$ is K-isomorphic to $\mathbb{P}^1_K$ (respectively an elliptic curve E/K). Notice that in both cases $C/\langle v\rangle(K) \neq \emptyset$ while it perfectly well happen that $C(K) = \emptyset$.

The following theorem of Abramovich and Harris [AbHa91] shows that the question above is closely related to the diophantine problem of determining the family of hyperelliptic and bielliptic Shimura curves over $\mathbb{Q}$.

Theorem 6.3.1. *Let C be an algebraic curve of genus greater than or equal to 2, defined over a number field K. Then $C(K, 2) = \#\infty$ if and only if C is* *either hyperelliptic over K or bielliptic over K mapping to an elliptic curve E of* $\mathrm{rank}_K(E) \geq 1$.

Ogg [Ogg83], [Ogg84] gave the list of hyperelliptic Shimura curves over $\mathbb{Q}$. In what follows, we will determine which bielliptic Shimura curves from Table 6.2.4 are bielliptic over $\mathbb{Q}$.

We first observe that the map $X_D \to X_D/\langle w \rangle$ is always defined over $\mathbb{Q}$ since we showed in Proposition 6.1.1 that $\mathrm{Aut}_{\mathbb{C}}(X_D \otimes \mathbb{C}) = \mathrm{Aut}_{\mathbb{Q}}(X_D)$. In order to check whether $X_D/\langle w \rangle(\mathbb{Q}) \neq \emptyset$ for each pair (D,w) in Table 6.2.4, we can disregard those in which $X_D/\langle w \rangle$ fails to have rational points over some completion $\mathbb{Q}_v$ of $\mathbb{Q}$. This is done by using the precise results in that direction given by Jordan and Livné in [JoLi85] and Ogg in [Ogg83] and [Ogg84]; the conclusions are compiled in the following table. Let us say that a field L is *deficient* for an algebraic curve C defined over a subfield $K \subset L$ if $C(L) = \emptyset$.

Table 6.3: Deficient completions L of $\mathbb{Q}$ for $X_D/\langle\omega_m\rangle$

D	ω_m	L		D	ω_m	L		D	ω_m	L
35	ω_7	$\mathbb{Q}_5$		115	ω_{23}	$\mathbb{Q}_5$		330	ω_3	$\mathbb{R}$,$\mathbb{Q}_2$
39	ω_{13}	$\mathbb{R}$,$\mathbb{Q}_3$		178	ω_{89}	$\mathbb{R}$,$\mathbb{Q}_2$				$\mathbb{Q}_5$,$\mathbb{Q}_{11}$
51	ω_3	$\mathbb{Q}_{17}$		210	ω_{30}	$\mathbb{R}$,$\mathbb{Q}_3$		330	ω_{22}	$\mathbb{R}$,$\mathbb{Q}_2$,$\mathbb{Q}_3$
55	ω_5	$\mathbb{R}$,$\mathbb{Q}_{11}$		210	ω_{42}	$\mathbb{R}$,$\mathbb{Q}_2$,$\mathbb{Q}_3$				$\mathbb{Q}_5$,$\mathbb{Q}_{11}$
62	ω_2	$\mathbb{R}$,$\mathbb{Q}_{31}$				$\mathbb{Q}_5$,$\mathbb{Q}_7$		330	ω_{33}	$\mathbb{R}$,$\mathbb{Q}_2$
69	ω_3	$\mathbb{R}$,$\mathbb{Q}_{23}$		210	ω_{70}	$\mathbb{R}$,$\mathbb{Q}_2$				$\mathbb{Q}_3$,$\mathbb{Q}_5$
77	ω_{11}	$\mathbb{R}$,$\mathbb{Q}_7$				$\mathbb{Q}_3$,$\mathbb{Q}_5$		330	ω_{165}	$\mathbb{Q}_2$,$\mathbb{Q}_3$
85	ω_{17}	$\mathbb{Q}_5$		210	ω_{105}	$\mathbb{R}$,$\mathbb{Q}_2$				$\mathbb{Q}_5$,$\mathbb{Q}_{11}$
94	ω_2	$\mathbb{R}$,$\mathbb{Q}_{47}$				$\mathbb{Q}_7$		462	ω_{154}	$\mathbb{R}$,$\mathbb{Q}_{11}$

On the genus 1 Atkin-Lehner quotients $X_D/\langle\omega_m\rangle$ that do have rational points over all completions of $\mathbb{Q}$, we can try to construct a $\mathbb{Q}$-rational point by means of the theory of complex multiplication. That is, a Heegner point $P \in X_D(K)$ with CM by a quadratic imaginary order R, $R \otimes \mathbb{Q} = K$, $h(R) = 1$, will project onto a $\mathbb{Q}$-rational point on $X_D/\langle\omega_m\rangle$ if and only if $\omega_m(P) = \overline{P}$, where $\overline{P}$ is the complex conjugate of P on $X_D(K)$. From [Jo81], 3.1.4, we deduce that $\omega_m(P) = \overline{P}$ if m is the product of the primes $p|D$ that are inert in K.

Performing the necessary computations, it follows that among those pairs (D, w) that $X_D/\langle w\rangle(\mathbb{Q}_v) \neq \emptyset$ for every completion $\mathbb{Q}_v$ of $\mathbb{Q}$, it is always possible to produce a $\mathbb{Q}$-rational point on $X_D/\langle w\rangle$ by the above means, except for two interesting cases: (X_{26}, ω_2) and (X_{58}, ω_2).

Since $g(X_{26}) = g(X_{58}) = 2$, we may apply a result of Kuhn (cf. [Kuhn88]) to deduce that there are also rational points on the quotients $X_{26}/\langle\omega_2\rangle$ and $X_{58}/\langle\omega_2\rangle$. Therefore, the Hasse-Minkowsky principle is never violated for the Atkin-Lehner quotients from Table 6.2.4 and those pairs $X_D/\langle\omega_m\rangle$ that do not appear in Table 6.3 are bielliptic curves over $\mathbb{Q}$. There are only eighteen values of D for which X_D admits a bielliptic involution over $\mathbb{Q}$.

It still remains to compute the Mordell-Weil rank of the elliptic curves $X_D/\langle w\rangle$ over $\mathbb{Q}$. Using Cremona's tables [Cre92], switching the sign of the Atkin-Lehner action as explained above, we can determine the $\mathbb{Q}$-isogeny class of these elliptic curves.

This is enough to compute their Mordell-Weil rank but we can use a beautiful idea of Roberts (cf. [Rob89]) to compute the $\mathbb{Q}$-isomorphism class and hence a Weierstrass equation for them as follows: Cremona's tables give the Kodaira symbols of the reduction of elliptic curves E at the primes $p|\mathrm{cond}(E)$. This is done by using Tate's algorithm which makes use of a Weierstrass equation of the curve. This is not available in our case, but we can instead use Čerednik-Drinfeld theory to compute the Kodaira symbols for the reduction mod p, $p|D$, of $X_D/\langle w\rangle$ and contrast them with Cremona's tables. This procedure uniquely determines the $\mathbb{Q}$-isomorphism class of the curves.

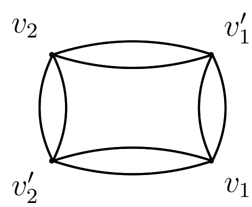
Example 6.3.2. Curve X_{210} has genus 5 and is bielliptic by the Atkin-Lehner involution ω_{210}. From Eichler's Theorem 1.2.18, the quadratic order $\mathbb{Z}[\sqrt{-43}]$ embeds in the quaternion algebra B of discriminant 210. Such an embedding produces a point $P \in X_{210}(\mathbb{Q}(\sqrt{-43}))$. From the above, it follows that $\omega_{210}(P) = \overline{P}$. Therefore, $X_{210}/\langle\omega_{210}\rangle(\mathbb{Q}) \neq \emptyset$ and we obtain that (X_{210}, ω_{210}) is a bielliptic pair over $\mathbb{Q}$. A glance at Cremona's Table 3, p. 249-250, shows that the elliptic curve $X_{210}/\langle\omega_{210}\rangle$ falls in the $\mathbb{Q}$-isogeny class $210D$ because it is the only one that corresponds to a newform $f \in H^0(\Omega^1, J_{210})$ such that $\omega_{210}^*(f) = f$ (recall that the sign for the Atkin-Lehner action is opposite to the classical modular case!). Therefore, from Cremona's Table 4, $\mathrm{rank}_{\mathbb{Q}}(X_{210}/\langle\omega_{210}\rangle) = 1$.

In order to determine a Weierstrass equation for $X_{210}/\langle\omega_{210}\rangle$ we may compute the Kodaira symbols of its reduction mod p, $p|210$. It suffices to study the reduction at $p = 3$. The Čerednik-Drinfeld theory asserts that $M_{210} \otimes \mathbb{F}_3$ is reduced and its irreducible components are all rational and defined over $\mathbb{F}_9$. Moreover, $M_{210} \otimes \mathbb{Z}_3$ is a (minimal) regular model over $\mathbb{Z}_3$. This is because over the quadratic unramified integral extension R_3 of $\mathbb{Z}_3$, $M_{210} \otimes R_3$ is a Mumford curve uniformized by a (torsion-free) Schottky group, as one checks from Čerednik-Drinfeld's explicit description of this group and the congruences $5 \equiv -1 \pmod 3$ and $7 \equiv -1 \pmod 4$.

In a way, Čerednik-Drinfeld's description of the reduction of Shimura curves at $p|D$ is not so different from Deligne-Rappoport's for the modular curves $X_0(N)$ at $p \parallel N$ because $M_{p\delta} \otimes \mathbb{F}_p$ is again the union of two copies of the Shimura curve -also called Gross curve- $M_\delta \otimes \mathbb{F}_p$, defined in terms of a *definite* quaternion algebra.

Let $h(\delta, \nu)$ denote the class number of an (arbitrary) Eichler order of level ν in the quaternion algebra of discriminant δ. The dual graph $\mathcal{G}$ of $M_{210} \otimes \mathbb{F}_3$ has as vertices the irreducible components of $M_{210} \otimes \mathbb{F}_3$. There are $2h(\frac{210}{3}, 1) = 2h(70, 1) = 4$ of them. Two vertices v, $\tilde{v}$ in $\mathcal{G}$ are joined by as many edges as there are intersection points between the corresponding components Z, $\widetilde{Z}$ in $M_{210} \otimes \mathbb{F}_3$. In our case, there are $h(\frac{210}{3}, 3) = 8$ edges in $\mathcal{G}$, that is, 8 double points in $M_{210} \otimes \mathbb{F}_3$.

We may label the 4 vertices v_1, v_1', v_2, v_2' so that $\omega_3(v_i) = v_i'$, where ω_3 still denotes the involution ω_3 now acting on $\mathcal{G}$. There are no edges joining v_1 and v_2, and the same holds for v_1' and v_2'. The total number of edges joining v_1 with v_1' and v_2 with v_2' is 4, as Kurihara (cf. [Ku79]) deduced from trace formulae of Brandt matrices. Since there must also be $p+1 = 4$ edges at the star of any vertex, it turns out that the dual graph $\mathcal{G}$ must be



Dual graph of $M_{210} \otimes \mathbb{F}_3$

Since $3|210$, $\omega_{210}(\{v_1, v_2\}) = \{v_1', v_2'\}$ and therefore $\mathcal{G}/\langle\omega_{210}\rangle$ is a graph with two vertices joined by two edges, which corresponds to the Kodaira symbol I_2. The only elliptic curve in the $\mathbb{Q}$-isogeny class $210D$ whose reduction type at $p = 3$ is I_2 is $210D2$. Hence, a Weierstrass equation for $X_{210}/\langle\omega_{210}\rangle$ is $y^2 + xy = x^3 + x^2 - 23x + 33$.

Performing similar computations, we obtain the list of bielliptic Shimura curves (X_D, w) over $\mathbb{Q}$ such that the genus 1 Atkin-Lehner quotient $X_D/\langle w\rangle$ is an elliptic curve with infinitely many rational points. With this procedure, we also give a Weierstrass equation for the elliptic curves $X_D/\langle w\rangle$. Together with the hyperelliptic Shimura curves over $\mathbb{Q}$ given by Ogg, we obtain the family of Shimura curves of genus $g(X_D) \geq 2$ with infinitely many quadratic points. Summing up, we obtain the following

Theorem 6.3.3. *There are only finitely many D for which X_D has infinitely many quadratic points over $\mathbb{Q}$. These curves, together with their rational or elliptic quotients, are listed below.*

Table 6.3.3: Shimura curves X_D, $g(X_D) \geq 2$, with $\#X_D(\mathbb{Q}, 2) = +\infty$

D	ω_m	$g(X_D/\langle\omega_m\rangle)$		D	ω_m	$g(X_D/\langle\omega_m\rangle)$		D	ω_m	$g(X_D/\langle\omega_m\rangle)$
26	ω_{26}	0		77	ω_{77}	1		143	ω_{143}	1
35	ω_{35}	0		82	ω_{82}	1		146	ω_{146}	0
38	ω_{38}	0		86	ω_{86}	0		159	ω_{159}	0
39	ω_{39}	0		87	ω_{87}	0		166	ω_{166}	1
51	ω_{51}	0		94	ω_{94}	0		194	ω_{194}	0
55	ω_{55}	0		95	ω_{95}	0		206	ω_{206}	0
57	ω_{57}	1		106	ω_{106}	1		210	ω_{210}	1
58	ω_{29}	0		111	ω_{111}	0		215	ω_{215}	1
	ω_{58}	1		118	ω_{118}	1		314	ω_{314}	1
62	ω_{62}	0		119	ω_{119}	0		330	ω_{330}	1
65	ω_{65}	1		122	ω_{122}	1		390	ω_{390}	1
69	ω_{69}	0		129	ω_{129}	1		510	ω_{510}	1
74	ω_{74}	0		134	ω_{134}	0		546	ω_{546}	1

6.4 Equations of elliptic quotients of Shimura curves

By using the method explained in the preceding section, we obtain the following complete list of Weierstrass equations of quotients $X_D/\langle\omega\rangle$ of degree 2 of a Shimura curve X_D which are elliptic curves over $\mathbb{Q}$. The instances corresponding to the discriminants $D \leq 60$ were already given in [Rob89]. In the second column, we quote the isomorphism class of the elliptic curve in Cremona's notation. Note that in the table below there are missing some of the Atkin-Lehner quotients $X_D/\langle\omega_m\rangle/\mathbb{Q}$ of genus 1 compiled in Table 6.2.4. These are exactly those genus 1 quotients which fail to have rational points (see Table 6.3). Let us also remark that there are several examples of elliptic quotients $X_D/\langle\omega_m\rangle$ of Mordell-Weil rank 0 over $\mathbb{Q}$. This explains why they do not appear in Table 6.3.3.

Table 6.4: Equations of all elliptic quotients of Shimura curves of degree 2		
$X_D/\langle\omega_m\rangle$	Cremona symbol	Weierstrass equation
$X_{26}/\langle\omega_2\rangle$	$26B_2$	$y^2+xy+y=x^3-x^2-213x-1257$
$X_{26}/\langle\omega_{13}\rangle$	$26A_1$	$y^2+xy+y=x^3-5x-8$
$X_{38}/\langle\omega_2\rangle$	$38B_2$	$y^2+xy+y=x^3+x^2-70x-279$
$X_{38}/\langle\omega_{19}\rangle$	$38A_3$	$y^2+xy+y=x^3-16x+22$
$X_{57}/\langle\omega_{57}\rangle$	$57A_1$	$y^2+y=x^3-x^2-2x+2$
$X_{58}/\langle\omega_2\rangle$	$58B_2$	$y^2+xy+y=x^3+x^2-455x-3951$
$X_{58}/\langle\omega_{58}\rangle$	$58A_1$	$y^2+xy=x^3-x^2-x+1$
$X_{65}/\langle\omega_{65}\rangle$	$65A_1$	$y^2+xy=x^3-x$
$X_{77}/\langle\omega_{77}\rangle$	$77A_1$	$y^2+y=x^3+2x$
$X_{82}/\langle\omega_{82}\rangle$	$82A_1$	$y^2+xy+y=x^3-2x$
$X_{106}/\langle\omega_{53}\rangle$	$106D_1$	$y^2+xy=x^3+x^2-27x-67$
$X_{106}/\langle\omega_{106}\rangle$	$106B_1$	$y^2+xy=x^3+x^2-7x+5$
$X_{118}/\langle\omega_{59}\rangle$	$118D_1$	$y^2+xy=x^3+x^2+56x-192$
$X_{118}/\langle\omega_{118}\rangle$	$118A_1$	$y^2+xy=x^3+x^2+x+1$
$X_{122}/\langle\omega_{122}\rangle$	$122A_1$	$y^2+xy+y=x^3+2x$
$X_{129}/\langle\omega_{129}\rangle$	$129A_1$	$y^2+y=x^3-x^2-19x+39$
$X_{143}/\langle\omega_{143}\rangle$	$143A_1$	$y^2+y=x^3-x^2-x-2$
$X_{166}/\langle\omega_{166}\rangle$	$166A_1$	$y^2+xy=x^3+x^2-6x+4$
$X_{202}/\langle\omega_{101}\rangle$	$202A_1$	$y^2+xy=x^3-x^2+4x-176$
$X_{210}/\langle\omega_{210}\rangle$	$210D_2$	$y^2+xy=x^3+x^2-23x+33$
$X_{215}/\langle\omega_{215}\rangle$	$215A_1$	$y^2+y=x^3-8x-12$
$X_{314}/\langle\omega_{314}\rangle$	$314A_1$	$y^2+xy=x^3-x^2+13x-11$
$X_{330}/\langle\omega_{330}\rangle$	$330E_2$	$y^2+xy=x^3+x^2-102x+324$
$X_{390}/\langle\omega_{390}\rangle$	$390A_2$	$y^2+xy=x^3+x^2-33x-63$
$X_{510}/\langle\omega_{510}\rangle$	$510D_2$	$y^2+xy+y=x^3+x^2-421x-3157$
$X_{546}/\langle\omega_{546}\rangle$	$546C_2$	$y^2+xy+y=x^3-137x+380$

Chapter 7

Resum en català

Introducció

En aquest treball estudiem diferents qüestions sobre la geometria i l'aritmètica de les àlgebres de quaternions, les varietats abelianes i les varietats de Shimura, amb l'objectiu d'investigar les estretes relacions existents entre elles.

Més concretament, l'estudi se centra en varietats abelianes A tals que el seu anell d'endomorfismes $\mathrm{End}(A)$ és un ordre maximal en una àlgebra de quaternions B totalment indefinida sobre un cos de nombres F totalment real i en les varietats de Shimura $X_B/\mathbb{Q}$ que sorgeixen de manera natural com als seus espais de moduli. Tal i com pretenem mostrar, moltes de les propietats aritmètiques i geomètriques d'aquestes varietats abelianes estan codificades o bé en l'àlgebra de quaternions B o bé en les varietats de Shimura X_B. Alhora, no és possible portar a terme un estudi d'aquestes varietats de Shimura sense un bon coneixement dels objectes que parametritzen.

Del treball de Shimura [Sh63] i la classificació d'àlgebres de divisió involutives deguda a Albert (cf. [Mu70]), se'n segueix que hi ha un ventall limitat d'anells que es realitzen com l'anell d'endomorfismes d'una varietat abeliana. En efecte, si A és una varietat abeliana simple sobre un cos algebraicament tancat, aleshores $\mathrm{End}(A)$ és un ordre en un cos totalment real, una àlgebra de quaternions sobre un cos totalment real o una àlgebra de divisió sobre un cos de multiplicació complexa. Molts dels aspectes de la geometria i l'aritmètica de les varietats abelianes es poden interpretar en els seus anells d'endomorfismes. De fet, és remarcable que, en molts sentits, varietats abelianes amb anells d'endomorfismes diferents tenen comportaments diferents.

Hi ha un nombre considerable de treballs sobre varietats abelianes amb multiplicació complexa. Sense cap ànim de donar-ne un llistat exhaustiu, algunes de les aportacions més rellevants es deuen a Shimura i Taniyama [ShTa61], Lang [La88] o Mumford [Mu70]. L'impacte d'aquests treballs en branques cabdals de la teoria de nombres, com ara la teoria de cossos de classes o la conjectura de Birch i Swinnerton-Dyer, ha estat enorme.

També hi ha literatura abundant sobre varietats abelianes amb multiplicació totalment real. Destaquem els treballs de Humbert [Hu93], Ribet [Ri80], [Ri94], Lange [La88] i Wilson [Wi02]. En aquest cas, el desenvolupament d'aquesta teoria ha estat tradicionalment motivat per la seva relació evident amb les conjectures generalitzades de Shimura-Taniyama-Weil. En efecte, les varietats abelianes modulars $A_f/\mathbb{Q}$ associades a una forma modular cuspidal $f \in S_2(\Gamma_0(N))$ de pes 2 i caràcter trivial tenen multiplicació real sobre $\mathbb{Q}$. El lector pot consultar [HaHaMo99] i [Ri90] per a més detalls.

En canvi, l'estudi de les varietats abelianes amb multiplicació quaterniònica ha estat portat a terme per menys autors. En aquest cas, l'aritmètica de les àlgebres d'endomorfismes d'aquestes varietats abelianes és més complexa que en el cas commutatiu i aquest fet té com a conseqüència que el grup de Néron-Severi de les varietats abelianes amb multiplicació quaterniònica és més inaccessible. Sigui com sigui, adrecem el lector a [No01], [JoMo94], [HaMu95], [HaHaMo99], [Oh74], [DiRo1] i [DiRo2] per a algunes contribucions recents.

Shimura [Sh63], [Sh67] va considerar els espais grollers de mòduli de varietats abelianes amb multiplicació quaterniònica i va provar que admeten un model canònic $X_B/\mathbb{Q}$ sobre el cos $\mathbb{Q}$ dels nombres racionals. Com a varietats analítiques, les varietats $X_B(\mathbb{C})$ es poden descriure per mitjà de quocients compactes de certs dominis simètrics i afitats per grups aritmètics que actuen en ells. Shimura va explorar les propietats diofantines de les varietats X_B i va demostrar que les coordenades dels anomenats *punts de Heegner* sobre X_B generen cossos de classes tals que l'acció galoisiana sobre ells pot ser descrita mitjançant lleis de reciprocitat explícites.

En els darrers anys, hi ha hagut un interès creixent en l'estudi de les varietats de Shimura que ha estat cruicial en molts aspectes de la teoria dels nombres.

En efecte, pel que fa a les conjectures modulars en el seu sentit mes ampli, les corbes de Shimura juguen un paper fonamental en la demostració de Ribet de la conjectura Epsilon que, a la vegada, implica que el darrer teorema de Fermat és una conseqüència de la conjectura de Shimura-Taniyama-Weil (cf. [Ri89], [Ri90] i [Pr95]).

En relació a les conjectures de Birch i Swinnerton-Dyer, Vatsal [Va02] i Cornut [Cor02] han provat recentment i de manera independent unes conjectures de Mazur sobre el comportament dels punts de Heegner sobre cossos anticiclotòmics en corbes el·líptiques, tot fent ús de corbes modulars, corbes de Shimura, corbes de Gross i les teories ergòdiques de Ratner. D'altra banda, Bertolini i Darmon [BeDa96], [BeDa98], [BeDa99] han explotat la teoria de Čerednik-Drinfeld sobre les fibres singulars dels models enters de Morita de les corbes de Shimura per provar versions anti-ciclotòmiques de les conjectures de Mazur, Tate i Teitelbaum sobre variants p-àdiques de la conjectura de Birch i Swinnerton-Dyer.

Pel que fa a les conjectures de finitud i de quadratura del grup de Shafarevich-Tate d'una varietat abeliana sobre un cos de nombres, Poonen i Stoll [PoSt99] recentment han fet un estudi de l'aparellament de Cassels-Tate en aquest context i han donat criteris explícits per a la quadratura de la part de torsió del grup de Shafarevich-Tate de les varietats Jacobianes de les corbes algebraiques. Al seu torn, i basant-se en aquests resultats, Jordan i Livné [JoLi99] han mostrat quocients Atkin-Lehner de corbes de Shimura tals que el nombre d'elements de la part finita del grup de Shafarevich-Tate de les seves varietats Jacobianes no és un quadrat perfecte sinó el doble d'un quadrat perfecte.

En un treball recent, Stein [St02] ha proporcionat exemples explícits de varietats abelianes $A/\mathbb{Q}$ tals que $\sharp\mathrm{Sha}(A/\mathbb{Q}) = p \cdot n^2$, $n \in \mathbb{Z}$, per tot nombre primer senar $p < 10000$, $p \neq 37$.

A banda de les seves moltes aplicacions, molts autors han investigat les propietats geomètriques i diofantines de les corbes de Shimura, que són interessants per elles mateixes. D'aquesta manera, els models enters d'aquestes corbes i les seves fibres especials han estat considerats per Morita [Mo81], Boutot i Carayol [BoCa91], Buzzard [Bu96], Čerednik [Ce76], Drinfeld [Dr76] i Zink [Zi81], entre molts d'altres. De gran interès també és la sèrie d'articles de Kudla i Kudla-Rapoport sobre els aparellaments d'altura en les corbes de Shimurs, nombres d'intersecció de 0-cicles especials i els valors de les funcions derivades de certes sèries d'Einsestein al centre del seu puntt de simetria, en la línea dels treballs clàssics de Gross-Zagier i Hirzebruch-Zagier. Vegeu [Kud97] i [KudRa02], per exemple.

Resultats de caire efectiu i computacional sobre les corbes de Shimura han estat portats a terme per Kurihara [Ku79], Elkies [El98], Alsina [Al99] i Bayer [Ba02], entre d'altres. Aquests treballs tenen una vàlua particular, degut a l'absència de punts cuspidals en aquestes corbes, que fa el tractament d'aquestes qüestions més difícil que en el cas modular clàssic.

Un altre resultat remarcable i molt recent és la demostració d'Edixhoven i Yafaev [EdYa02] de la conjectura d'André-Oort sobre la distribució de punts especials en les varietats de Shimura.

En una altra direcció, Ihara [Ih], Jordan-Livné [JoLi85], [Jo86], [JoLi86], Ogg [Ogg83], [Ogg84], Milne [Mi79] i Kamienny [Ka90] han estudiat, sota diferents punts de vista, els conjunts de punts racionals de les corbes de Shimura, els seus quocients Atkin-Lehner i les seves varietats Jacobianes sobre cossos globals, locals i finits. Finalment, adrecem el lector a [Gr02] per a una introducció específica a les superfícies de Shimura.

7.1 Resultats

Introducció

L'objectiu d'aquest apartat és exposar els resultats fonamentals d'aquesta tesi sobre varietats abelianes amb multiplicació quaterniònica de dimensió arbitrària i les varietats de Shimura que sorgeixen com als seus espais de mòduli.

7.1.1 Varietats abelianes amb multiplicació quaterniònica

En aquesta secció exposem els resultats continguts en el capítol 3 d'aquesta memòria.

És ben conegut que les corbes el·líptiques sobre un cos algebraicament tancat qualsevol admeten exactament una polarització principal llevat de translacions. En general, les varietats abelianes de dimensió superior no comparteixen aquesta propietat. Si A és una varietat abeliana, no és una qüestió trivial decidir si A és principalment polarizable i, en aquest cas, és un problema interessant descriure el conjunt $\Pi_0(A)$ de classes d'isomorfisme de polaritzacions principals de A. El teorema de Narasimhan-Nori [NaNo81] assegura que $\Pi_0(A)$ és un conjunt finit i nosaltres denotem per $\pi_0(A)$ el seu cardinal.

Una varietat abeliana principalment polaritzable genèrica admet una única classe de polaritzacions principals. Humbert [Hu93] va exhibir superfícies abelianes complexes i simples amb dues polaritzacions principals no isomorfes. Més tard, Hayashida i Nishi (cf. [HaNi65] i [Ha68]) van calcular el nombre $\pi_0(E_1 \times E_2)$ per parelles de corbes el·líptiques isògenes $E_1/\mathbb{C}$ i $E_2/\mathbb{C}$ amb multiplicació complexa. En característica positiva, Ibukiyama, Katsura i Oort [IbKaOo86] van relacionar el nombre de polaritzacions principals en la potència E^n d'una corba el·líptica supersingular amb el nombre de classes de certes formes hermitianes. Lange [La88] va traduir aquest problema al llenguatge de la teoria de nombres en termes de l'aritmètica de l'anell $\mathrm{End}(A)$ i va produir exemples de varietats abelianes simples de dimensió superior amb diverses polaritzacions principals. Al mateix temps, però, va mostrar que per a les varietats abelianes amb àlgebra d'endomorfismes $\mathrm{End}(A) \otimes \mathbb{Q} = F$ commutativa i totalment real, el nombre $\pi_0(A)$ és uniformement afitat en termes de la dimensió de A: $\pi_0(A) \leq 2^{\dim(A)-1}$. En altres paraules: les varietats abelianes amb multiplicació real poden admetre diverses polaritzacions principals però no un nombre *arbitràriament gran*.

Es podria considerar natural que la fita de Lange, o alguna altra fita per a $\pi_0(A)$, fos vàlida per a *qualsevol* varietat abeliana simple. És doncs natural plantejat la següent

Qüestió. *Sigui $g \geq 1$ un enter positiu. Existeixen varietats abelianes simples de dimensió g amb un nombre arbitràriament gran de polaritzacions principals no isomorfes?*

Tal i com ja hem observat, en dimensió 1 la resposta a aquesta qüestió és negativa. Quan $g = 2$, tans sols es coneixien superfícies abelianes simples amb $\pi_0(A) \leq 2$, degut al treball de Humbert. Un dels nostres resultats principals, enunciat en un cas particular, és el següent.

Teorema. *Sigui F un cos de nombres totalment real de grau $[F : \mathbb{Q}] = n$, sigui R_F el seu anell d'enters i sigui $\vartheta_{F/\mathbb{Q}}$ la diferent de F sobre $\mathbb{Q}$. Sigui A una varietat abeliana complexa de dimensió $2n$ i tal que el seu anell d'endomorfismes $\mathrm{End}(A) \simeq \mathcal{O}$ és un ordre maximal en una àlgebra de divisió de quaternions totalment indefinida B sobre F.*

Suposem que el nombre restringit de classes $h_+(F)$ de F és 1 i que $\vartheta_{F/\mathbb{Q}}$ i $\mathrm{disc}(B)$ són ideals coprimers de F. Aleshores,

1. *A és principalment polaritzable.*

2. *El nombre de classes d'isomorfisme de polaritzacions principals de A és*

$$\pi_0(A) = \frac{1}{2}\sum_S h(S),$$

on S recorre el conjunt finit d'ordres en el cos de multiplicació complexa $F(\sqrt{-D})$ que contenen $R_F[\sqrt{-D}]$, $D \in F_+^*$ és un generador totalment positiu del discriminant reduït $\mathcal{D}$ de B, i $h(S)$ denota el nombre de classes de S.

En particular, si A és una superfície abeliana,

$$\pi_0(A) = \begin{cases} \dfrac{h(-4D)+h(-D)}{2} & \text{if } D \equiv 3 \mod 4, \\ \dfrac{h(-4D)}{2} & \text{altrament.} \end{cases}$$

Per tal d'acometre la demostració del teorema anterior, presentem una aproximació al problema que té els seus orígens en el treball clàssic de Shimura [Sh63] sobre famílies analítiques de varietats abelianes amb anell d'endomorfismes prescrit.

La nostra proposta és essencialment diferent de la presa per Lange en [La88] o Ibukiyama-Katsura-Oort's en [IbKaOo86]. En efecte, mentre que a [La88] i [IbKaOo86] s'explota la interpretació dels fibrats de línea com a endomorfismes simètrics, nosaltres traduïm les qüestions que ens ocupen al llenguatge d'immersions optimals d'ordres quadràtics d'Eichler. Això ens porta a resoldre un problema que té les seves arrels en el treball de O'Connor, Pall i Pollack (cf. [Po60]).

En relació a la qüestió anterior, el nostre segon resultat principal del capítol 3 és el següent.

Teorema. *Sigui g un enter positiu. Aleshores,*

1. *Si g és parell, existeixen varietats abelianes simples A de dimensió g tals que $\pi_0(A)$ és arbitràriament gran.*

2. *Si g és senar i lliure de quadrats, es té que $\pi_0(A) \leq 2^{g-1}$, per tota varietat abeliana simple A de dimensió g sobre $\mathbb{C}$.*

El creixement indefinit de $\pi_0(A)$ quan g és parell se segueix del teorema enunciat anteriorment, en la seva versió precisa enunciada com a Teorema 3.7.2 al capítol 3 combinat amb resultats analítics sobre el comportament assimptòtic dels nombres de classes relatius de cossos CM deguts a Horie-Horie [HoHo90] i Louboutin [Lo00], [Lo02]. La segona part del teorema se segueix de les idees de Lange a [La88].

Del teorema anterior i del fet que tota superfície abeliana simple i principalment polaritzada és la varietat Jacobiana d'una corba llisa de gènere 2 que, pel teorema de Torelli, és única llevat d'isomorfisme.

Corol·lari. *Existeixen conjunts* $\{C_1, ..., C_N\}$ *amb un nombre arbitràriament gran de corbes de gènere* 2 *no isomorfes que posseeixen varietats Jacobianes simples i isomorfes* $J(C_1) \simeq J(C_2) \simeq ... \simeq J(C_N)$.

Notem que l'enunciat del darrer teorema no té en consideració les varietats abelianes de dimensió senar i no lliure de quadrats. La conjectura següent està motivada pel fet que, quan g no és lliure de quadrats, existeixen varietats abelianes tals que el seu anell d'endomorfismes és un ordre en una àlgebra de divisió no commutativa sobre un cos de multiplicació complexa i hi ha una estreta similitud entre l'aritmètica dels grups de Néron-Severi d'aquestes varietats abelianes i de les varietats abelianes amb multiplicació quaterniònica.

Conjectura. *Sigui g un enter positiu no lliure de quadrats. Aleshores, existeixen varietats abelianes simples de dimensió g tals que $\pi_0(A)$ és arbitràriament gran.*

7.1.2 Varietats de Shimura i morfismes d'oblit

Sigui F un cos de nombres totalment real de grau $[F : \mathbb{Q}] = n$ i sigui B una àlgebra de divisió de quaternions totalment indefinida sobre F. En el capítol 4 d'aquesta memòria estudiem certes varietats de Shimura X_B associades a l'àlgebra B i certs morfismes d'oblit que ocorren entre elles.

Per simplicitat en l'exposició, suposarem al llarg de la secció que $h_+(F) = 1$. Sigui $D \in F_+^*$ un generador totalment positiu de $\mathrm{disc}(B)$.

Definició. Un *ordre maximal principalment polaritzat* de B és un parell

$(\mathcal{O}, \mu)$ tal que $\mathcal{O} \subset B$ és un ordre maximal i $\mu \in \mathcal{O}$ és un quaternió pur tal que $\mu^2 + uD = 0$ per alguna unitat $u \in R^*_{F+}$.

Associat a un ordre maximal principalment polaritzat $(\mathcal{O}, \mu)$, podem considerar el problema següent de mòduli sobre $\mathbb{Q}$: classificar les classes d'isomorfisme de tripletes $(A, \iota, \mathcal{L})$ donades per

- Una varietat abeliana A de dimensió $g = 2n$.
- Un homomorfisme d'anells $\iota : \mathcal{O} \hookrightarrow \mathrm{End}(A)$.
- Una polarització principal $\mathcal{L}$ en A tal que

$$\iota(\beta)^\circ = \iota(\mu^{-1}\bar{\beta}\mu)$$

  per tot $\beta \in \mathcal{O}$, on $\circ : \mathrm{End}(A) \to \mathrm{End}(A)$ denota la involució de Rosati respecte $\mathcal{L}$.

Shimura [Sh63], [Sh67] va demostrar que el corresponent functor de moduli es pot representar grollerament per un esquema $X_\mu/\mathbb{Q}$ reduït, irreductible i quasi-projectiu sobre $\mathbb{Q}$ de dimensió $n = [F : \mathbb{Q}]$. A més a més, degut al fet que B és una àlgebra de divisió, la varietat de Shimura X_μ és completa.

Sigui $\mathfrak{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ l'hiperplà superior de Poincaré. Com a varietat analítica, $X_\mu(\mathbb{C})$ es pot descriure, independentment de la tria de μ, com el quocient

$$\mathcal{O}^1 \backslash \mathfrak{H}^n \simeq X_\mu(\mathbb{C})$$

de l'espai simètric $\mathfrak{H}^n$ per l'acció del grup $\mathcal{O}^1$, entès com a subgrup discontinu de $\mathrm{SL}_2(\mathbb{R})^n$.

A més de les varietats de Shimura que tot just hem introduït, sigui $\mathcal{H}_F/\mathbb{Q}$ l'esquema modular de Hilbert que representa de forma grollera el functor associat al problema de mòduli de classificar varietats abelianes principalment polaritzades $(A, \mathcal{L})$ de dimensió g proveïdes d'un morfisme d'anells $R_F \hookrightarrow \mathrm{End}(A)$. La varietat modular de Hilbert $\mathcal{H}_F$ té dimensió $3n$ i el conjunt de punts complexos $\mathcal{H}_F(\mathbb{C})$ és el quocient de $\mathfrak{H}_2^n$, on $\mathfrak{H}_2$ denota l'espai de Siegel de dimensió 3, per un cert grup discontinu (cf. [Sh63], [LaBi92]).

Observem que quan $F = \mathbb{Q}$, aleshores $\mathcal{H}_F = \mathcal{A}_2$ és la 3-varietat d'Igusa, l'espai de mòduli de les superfícies abelianes principament polaritzades.

Existeixen uns morfismes naturals

$$\begin{array}{rccccc} \pi: & X_\mu & \xrightarrow{\pi_F} & \mathcal{H}_F & \longrightarrow & \mathcal{A}_g \\ & (A,\iota,\mathcal{L}) & \mapsto & (A,\iota|_{R_F},\mathcal{L}) & \mapsto & (A,\mathcal{L}) \end{array}$$

de la varietat de Shimura X_μ en la varietat modular de Hilbert $\mathcal{H}_F$ i l'espai de mòduli $\mathcal{A}_g$ que consisteixen en oblidar de manera gradual l'estructura d'endomorfismes quaterniònics. Aquests morfismes són representables, propis i estan definits sobre el cos $\mathbb{Q}$ dels nombres racionals.

Definició. El *grup d'Atkin-Lehner* W d'un ordre maximal $\mathcal{O}$ de B és

$$W = \mathrm{Norm}_{B_+^*}(\mathcal{O})/(F^* \cdot \mathcal{O}^1).$$

Es satisfà que

$$W \simeq \mathbb{Z}/2\mathbb{Z} \times \overset{2r}{\cdots} \times \mathbb{Z}/2\mathbb{Z},$$

on $2r \geq 2$ és el nombre de ideals primers de F que ramifiquen en B.

Definició. Sigui $(\mathcal{O},\mu)$ un ordre maximal principalment polaritzat en B. Un *torçament* de $(\mathcal{O},\mu)$ és un element $\chi \in \mathcal{O} \cap \mathrm{Norm}_{B^*}(\mathcal{O})$ tal que $\chi^2 + \mathrm{n}(\chi) = 0$ i $\mu\chi = -\chi\mu$.

En altres paraules, un torçament de $(\mathcal{O},\mu)$ és un quaternió pur $\chi \in \mathcal{O} \cap \mathrm{Norm}_{B^*}(\mathcal{O})$ tal que

$$B = F + F\mu + F\chi + F\mu\chi = (\frac{-uD, -\mathrm{n}(\chi)}{F}).$$

Diem que un ordre maximal principalment polaritzat $(\mathcal{O},\mu)$ en B és *torçat* si admet un torçament χ en $\mathcal{O}$. Diem que un ordre maximal $\mathcal{O}$ és *torçat* si existeix $\mu \in \mathcal{O}$ tal que $(\mathcal{O},\mu)$ és torçat. Finalment, diem que B és *torçada* si existeix un ordre maximal torçat en B. Observem que B és torçada si, i només si, $B \simeq (\frac{-uD,m}{F})$ per alguna unitat $u \in R_{F+}^*$ i $m \in F^*$ tal que $m|D$.

Definició. Una *involució torçada* $\omega \in W$ de $(\mathcal{O},\mu)$ és una involució d'Atkin-Lehner tal que $[\omega] = [\chi] \in W$ es pot representar en B^* per un torçament χ de $(\mathcal{O},\mu)$.

Per tot ordre maximal principalment polaritzat $(\mathcal{O}, \mu)$, sigui $R_\mu = F(\mu) \cap \mathcal{O}$ i sigui $\Omega = \Omega(R_\mu) = \{\xi \in R_\mu : \xi^f = 1, f \geq 1\}$ el grup finit d'arrels de la unitat en l'ordre de multiplicació complexa R_μ sobre R_F.

Definició. El *grup estable* de $(\mathcal{O}, \mu)$ és el subgrup

$$W_0 = U_0 \cdot V_0$$

de W generat per

$$U_0 = U_0(\mathcal{O}, \mu) = \mathrm{Norm}_{F(\mu)^*}(\mathcal{O})/(F^* \cdot \Omega(R_\mu)),$$

i el grup V_0 generat per les involucions torçades de $(\mathcal{O}, \mu)$.

Disposem dels següents monomorfismes naturals de grups $V_0 \subseteq W_0 \subseteq W \subseteq \mathrm{Aut}_{\mathbb{Q}}(X_\mu) \subseteq \mathrm{Aut}_{\bar{\mathbb{Q}}}(X_\mu \otimes \bar{\mathbb{Q}})$.

Teorema. *Sigui $(\mathcal{O}, \mu)$ un ordre maximal principalment polaritzat en B i sigui X_μ la varietat de Shimura associada a $(\mathcal{O}, \mu)$. Aleshores existeix un diagrama commutatiu de morfismes de fibres finites*

$$\begin{array}{ccc} X_\mu & \xrightarrow{\pi_F} & \mathcal{H}_F \\ & \searrow \qquad \nearrow^{b_F} & \\ & X_\mu/W_0, & \end{array}$$

on $X_\mu \to X_\mu/W_0$ és la projecció natural i $b_F : X_\mu/W_0 \to \pi_F(X_\mu)$ és una equivalència birracional entre X_μ/W_0 i la imatge de X_μ en $\mathcal{H}_F$.

El domini de definició de b_F^{-1} és $\pi_F(X_\mu) \setminus \mathcal{T}_F$, on $\mathcal{T}_F$ és un conjunt finit de punts de Heegner.

Concloem aquesta exposició dels resultats de la tesi en dimensió arbitrària tot indicant dues aplicacions diferents dels resultats que hem presentat.

La primera aplicació ha estat desenvolupada a la secció 4.6 i es refereix a la geometria del *lloc quaterniònic* $\mathcal{Q}_\mathcal{O}$ de les varietats abelianes que admeten multiplicació per un ordre maximal $\mathcal{O}$ en l'espai de mòduli $\mathcal{A}_g$ de varietats abelianes principalment polaritzades de dimensió parell g.

Mitjançant el teorema anterior i la teoria d'Eichler en immersions optimals, el nombre de components irreductibles de $\mathcal{Q}_\mathcal{O}$ es poden relacionar amb certs nombres de classes i es pot estudiar la seva irreductibilitat.

En segon lloc, el teorema anterior també es pot utilitzar per a explorar l'aritmètica de les varietats abelianes amb multiplicació quaterniònica. Efectivament, a la secció 4.7 combinem el teorema anterior amb la seva interpretació modular per tal d'obtenir resultats sobre el cos de mòduli dels endomorfismes en aquestes varietats abelianes.

7.2 Conclusions

Introducció

En aquest apartat exposem els resultats, de caire més acurat, que hem obtingut en els capítols 5 i 6 en l'estudi de les varietats abelianes amb multiplicació quaternionica de dimensió $g = 2$ i les varietats de Shimura de dimensió $n = 1$. Aquests resultats són fruit de la combinació de la teoria desenvolupada en els capítols anteriors i d'eines noves.

7.2.1 Superfícies abelianes amb multiplicació quaterniònica

Fixem una clausura algebraica $\bar{\mathbb{Q}} \subset \mathbb{C}$ del cos $\mathbb{Q}$ dels nombres racionals i sigui $K \subset \bar{\mathbb{Q}}$ un cos de nombres.

Proposició. (Silverberg) *Sigui A/K una varietat abeliana sobre K i sigui $S \subseteq \mathrm{End}_{\bar{\mathbb{Q}}}(A)$ un subanell d'endomorfismes de A. Aleshores hi ha una única extensió minimal K_S/K tal que $S \subseteq \mathrm{End}_{K_S}(A)$.*

L'extensió K_S/K és normal i no ramificada en els ideals primers de K de bona reducció de A.

Teorema. Sigui A/K una superfície abeliana amb multiplicació quaterniònica per un ordre $\mathcal{O}$ en una àlgebra de quaternions B. Aleshores, $\mathrm{Gal}(K_B/K)$ és isomorf a un grup cíclic C_n o un grup dihedral D_n amb $n = 1, 2, 3, 4$ o 6.

Teorema. *Sigui C/K una corba de gènere 2 i definida sobre K. Sigui $J(C)$ la varietat Jacobiana de C i suposem que $\mathrm{End}_{\bar{\mathbb{Q}}}(J(C)) = \mathcal{O}$ és un ordre maximal en una àlgebra de divisió de quaternions B sobre $\mathbb{Q}$ de discriminant D.*

Sigui K_B/K l'extensió minimal de K sobre la qual tots els endomorfismes de $J(C)$ estan definits. Aleshores

1. *K_B/K és una extensió abeliana amb $G = \mathrm{Gal}(K_B/K) \simeq (1)$, C_2 o $D_2 = C_2 \times C_2$.*
2. *Si $B \not\simeq (\frac{-D,m}{\mathbb{Q}})$ per a qualsevol $m|D$, aleshores K_B/K és una extensió trivial o quadràtica de K. En aquest darrer cas, $\mathrm{End}_K(A) \simeq \mathbb{Q}(\sqrt{-D})$.*
3. *Si $B = (\frac{-D,m}{\mathbb{Q}})$ per algun enter positiu $m|D$, aleshores $\mathrm{End}_K(A)$ és isomorf o bé a $\mathcal{O}$, a un ordre de $\mathbb{Q}(\sqrt{-D})$, $\mathbb{Q}(\sqrt{m})$ o $\mathbb{Q}(\sqrt{D/m})$, o $\mathbb{Z}$. En cada cas, tenim respectivement que $\mathrm{Gal}(K_B/K) \simeq (1)$, C_2 i D_2.*

A continuació presentem alguns exemples que il·lustren els nostres resultats.

Exemple I. Sigui $C_I/\mathbb{Q}$ el model projectiu i llis de la corba

$$Y^2 = (X^2 + 7/2)(83/30X^4 + 14X^3 - 1519/30X^2 + 49X - 1813/120)$$

i sigui $J_I = J(C_I)/\mathbb{Q}$ la seva varietat Jacobiana. Aleshores,

- $\mathrm{End}_L(J_I) = \mathcal{O}$ és un ordre maximal de l'àlgebra de quaternions B_6 de discriminant 6 i $L = \mathbb{Q}(\sqrt{-6}, \sqrt{-14})$,
- $\mathrm{End}_{\mathbb{Q}(\sqrt{-14})}(J_I) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{2})$,
- $\mathrm{End}_{\mathbb{Q}(\sqrt{21})}(J_I) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{3})$,
- $\mathrm{End}_{\mathbb{Q}(\sqrt{-6})}(J_I) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-6})$ i
- $\mathrm{End}_{\mathbb{Q}}(J_I) = \mathbb{Z}$.

Si tenim en compte que $B_6 = (\frac{-6,2}{\mathbb{Q}}) = (\frac{-6,3}{\mathbb{Q}})$ és una àlgebra de quaternions torçada, l'exemple anterior està d'acord amb el teorema anterior.

Exemple II. Sigui C_{II} el model projectiu i llis de la corba

$$Y^2 = \frac{1}{48}X(9075X^4+3025(3+2\sqrt{-3})X^3-6875X^2+220(-3+2\sqrt{-3})X+48).$$

Sigui J_{II}/K la varietat Jacobiana de C_{II} sobre $K=\mathbb{Q}(\sqrt{-3})$.

- $\mathrm{End}_L(J_{II})=\mathcal{O}$ és un ordre maximal de l'àlgebra de quaternions B_{10} de discriminant $D=10$ i $L=\mathbb{Q}(\sqrt{-3},\sqrt{-11})$ és el cos minimal de definició dels endomorfismes quaterniònics de J_{II} i
- $\mathrm{End}_K(J_3)\otimes_{\mathbb{Z}}\mathbb{Q}=\mathbb{Q}(\sqrt{5})$.

Exemple III. Sigui C_{III}/K la fibra de la família $S_6(t,s)$ de Hashimoto-Tsunogai [HaTs99] en el paràmetre $t=2$ sobre $K=\mathbb{Q}(\sqrt{-379})$. Aleshores $\mathrm{End}(J_{III})=\mathcal{O}$ és un ordre maximal de l'àlgebra de quaternions B_6 de discriminant $D=6$ i tots els endomorfismes quaterniònics de J_{III} estan definits sobre el mateix cos K.

7.2.2 Propietats diofantines de les corbes de Shimura

Sigui B una àlgebra de divisió de quaternions indefinida sobre $\mathbb{Q}$ i sigui $\mathcal{O}$ un ordre maximal de B. Sigui $D=p_1\cdot\ldots\cdot p_{2r}$ el discriminant de B. Podem veure el grup $\Gamma=\{\gamma\in\mathcal{O}:n(\gamma)=1\}$ com un subgrup aritmètic de $\mathrm{SL}_2(\mathbb{R})$ mitjançant una identificació $\Psi: B\otimes\mathbb{R}\simeq M_2(\mathbb{R})$ i considerar la superfície de Riemann $\Gamma\backslash\mathcal{H}$. Shimura [Sh67] va demostrar que aquest és el conjunt de punts complexos d'una corba algebraica $X_D=X_D/\mathbb{Q}$ sobre $\mathbb{Q}$ que parametritza superfícies abelianes amb multiplicació quaterniònica per $\mathcal{O}$.

En el capítol 6 d'aquesta memòria hem estudiat l'estructura del grup d'automorfismes de les corbes de Shimura. L'eina principal per a obtenir els nostres resultats ha estat la teoria de Čerednik-Drinfeld.

Teorema. *Sigui X_D la corba de Shimura de discriminant D. Si $g(X_D)\geq 2$, aleshores*

(i) *Tots els automorfismes de $X_D\otimes\mathbb{C}$ estan definits sobre $\mathbb{Q}$.*

(ii) $\mathrm{Aut}_{\mathbb{Q}}(X_D) \simeq C_2^s$ *per algun enter* $s \geq 2r$.

(iii) *Si* X_D *no té punts el·líptics,* $\mathrm{Aut}(X_D) \simeq C_2^{2r}$.

(iv) *Si* $D = 2p$ *o* $3p$ *amb* p *un nombre primer,* $\mathrm{Aut}(X_D) \simeq C_2 \times C_2$.

Com a conseqüència d'aquest teorema, en el capítol 6 determinem quines corbes de Shimura són biel·líptiques, és a dir, admeten un morfisme de grau 2 en una corba de gènere 1.

Teorema. *Existeixen exactament trenta-dos valors de* D *tals que la corba* X_D *és biel·líptica. En cada cas, les involucions biel·líptiques són de tipus Atkin-Lehner. Aquests valors, juntament amb el gènere* $g = g(X_D)$ *i el llistat de les involucions biel·líptiques de* X_D*, es proporcionen en la taula següent.*

Corbes de Shimura biel·líptiques										
D	g	ω_m		D	g	ω_m		D	g	ω_m
26	2	ω_2,ω_{13}		82	3	ω_{82}		210	5	ω_{30},ω_{42},
35	3	ω_7		85	5	ω_{17}				ω_{70},ω_{105},
38	2	ω_2,ω_{19}		94	3	ω_2				ω_{210}
39	3	ω_{13}		106	4	ω_{53},ω_{106}		215	15	ω_{215}
51	3	ω_3		115	6	ω_{23}		314	14	ω_{314}
55	3	ω_5		118	4	ω_{59},ω_{118}		330	5	ω_3,ω_{22}
57	3	ω_{57}		122	6	ω_{122}				ω_{33},ω_{165},
58	2	ω_2,ω_{58}		129	7	ω_{129}				ω_{330}
62	3	ω_2		143	12	ω_{143}		390	9	ω_{390}
65	5	ω_{65}		166	6	ω_{166}		462	9	ω_{154}
69	3	ω_3		178	7	ω_{89}		510	9	ω_{510}
77	5	ω_{11},ω_{77}		202	8	ω_{101}		546	13	ω_{546}

Com a conseqüència dels resultats anteriors, ens trobem en disposició de respondre una qüestió de Kamienny amb tota completitud:

Qüestió. (Kamienny) Quines corbes de Shimura admeten infinits punts quadràtics sobre $\mathbb{Q}$?

Aquesta és la pregunta diofantina més natural i senzilla que hom es pot plantejar a la llum del teorema de Shimura [Sh75], que assegura que el conjunt $X_D(\mathbb{R})$ de punts reals de les corbes de Shimura és buit.

Tan sols existeixen un nombre finit de valors de D tals que la corba de Shimura X_D admet infinits punts quadràtics sobre $\mathbb{Q}$. El llistat d'aquestes corbes, juntament amb els seus quocients racionals o el·líptics es proporcionat en la taula següent.

Corbes de Shimura X_D de gènere ≥ 2 amb infinits punts quadràtics										
D	ω_m	$g(X_D/\langle\omega_m\rangle)$		D	ω_m	$g(X_D/\langle\omega_m\rangle)$		D	ω_m	$g(X_D/\langle\omega_m\rangle)$
26	ω_{29}	0		77	ω_{77}	1		143	ω_{143}	1
35	ω_{35}	0		82	ω_{82}	1		146	ω_{146}	0
38	ω_{38}	0		86	ω_{86}	0		159	ω_{159}	0
39	ω_{39}	0		87	ω_{87}	0		166	ω_{166}	1
51	ω_{51}	0		94	ω_{94}	0		194	ω_{194}	0
55	ω_{55}	0		95	ω_{95}	0		206	ω_{206}	0
57	ω_{57}	1		106	ω_{106}	1		210	ω_{210}	1
58	ω_{29}	0		111	ω_{111}	0		215	ω_{215}	1
	ω_{58}	1		118	ω_{118}	1		314	ω_{314}	1
62	ω_{62}	0		119	ω_{119}	0		330	ω_{330}	1
65	ω_{65}	1		122	ω_{122}	1		390	ω_{390}	1
69	ω_{69}	0		129	ω_{129}	1		510	ω_{510}	1
74	ω_{74}	0		134	ω_{134}	0		546	ω_{546}	1

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