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SHIMURA CURVES EMBEDDED IN IGUSA'S THREEFOLD

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Abstract. Let $\mathcal{O}$ be a maximal order in a totally indefinite quaternion algebra over a totally real number field. In this note we study the locus $\mathcal{Q}_{\mathcal{O}}$ of quaternionic multiplication by $\mathcal{O}$ in the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of even dimension g with particular emphasis in the two-dimensional case. We describe $\mathcal{Q}_{\mathcal{O}}$ as a union of Atkin-Lehner quotients of Shimura varieties and we compute the number of irreducible components of $\mathcal{Q}_{\mathcal{O}}$ in terms of class numbers of CM-fields.

Introduction

Let A be an abelian variety over a fixed algebraic closure $\bar{\mathbb{Q}}$ of the field of rational numbers. By Poincaré's Decomposition Theorem, the algebra of endomorphisms $\mathrm{End}(A)\otimes\mathbb{Q}$ of A decomposes as a direct sum

$$\mathrm{End}(A)\otimes\mathbb{Q}\simeq\oplus_{i=1}^{s}\mathrm{M}_{n_i}(B_i)$$

of matrix algebras over a division algebra B_i.

The ranks $[B_i:\mathbb{Q}]$ are bounded in terms of the dimension of A and, by a classical theorem of Albert, the algebras B_i are isomorphic to either a totally real field, a quaternion algebra over a totally real field or a division algebra over a CM-field (cf. [8], [7]).

We will focus our attention on abelian varieties with totally indefinite quaternionic multiplication. More precisely, let F be a totally real number field of degree $[F:\mathbb{Q}]=n\geq 1$, let R_F be its ring of integers and let ϑ_F denote the different ideal of F over $\mathbb{Q}$. We also let F_+^* denote the group of totally positive elements of F^* and $R_{F+}^*=R_F^*\cap F_+^*$.

Let B be a totally indefinite division quaternion algebra over F, i.e. a division algebra of rank 4 over F such that $B\otimes_{\mathbb{Q}}\mathbb{R}\simeq\oplus_{i=1}^{n}\mathrm{M}_2(\mathbb{R})$, and let $\mathrm{disc}(B)$ denote the reduced discriminant ideal of B. We assume for simplicity that ϑ_F and $\mathrm{disc}(B)$ are coprime ideals of F. Since B is totally indefinite and division, it follows from [16] that $\mathrm{disc}(B)=\wp_1\cdot\ldots\cdot\wp_{2r}$ for pairwise different prime ideals $\wp_i$ of F and $r\geq 1$. We shall denote by $\mathrm{n}=\mathrm{n}_{B/F}$ and $\mathrm{tr}=\mathrm{tr}_{B/F}$ the reduced norm and reduced trace on B, respectively. For any subset O of B, we shall write $O_0=\{\beta\in O:\mathrm{tr}(\beta)=0\}$

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for the set of pure quaternions of O. We shall also use the notation $O_+ = \{\beta \in O : \mathrm{n}(\beta) \in F_+^*\}$ and $O^1 = \{\beta \in O : \mathrm{n}(\beta) = 1\}$.

Finally, let $\mathcal{O}$ be a maximal order in B and let $\vartheta(\mathcal{O}) = \{\mu \in \mathcal{O} : \mathrm{disc}(B) | \mathrm{n}(\mu)\}$ denote the reduced different of $\mathcal{O}$ over R_F. This is a two-sided ideal of $\mathcal{O}$ such that $\mathrm{n}(\vartheta(\mathcal{O})) \cdot R_F = \mathrm{disc}(B)$.

Let $\mathcal{A}_g/\mathbb{Q}$ be the moduli space of principally polarized abelian varieties of dimension $g = 2n$. We let $[(A, \mathcal{L})]$ denote the isomorphism class of a principally polarized abelian variety $(A, \mathcal{L})$ regarded as a closed point in $\mathcal{A}_g$.

It is our aim to investigate the nature of the quaternionic locus

$$Q_{\mathcal{O}} \subset \mathcal{A}_g(\mathbb{C})$$

of isomorphism classes of complex principally polarized abelian varieties $[(A, \mathcal{L})]$ of dimension g such that $\mathrm{End}(A) \supseteq \mathcal{O}$.

A reformulation of Proposition 6.2 in [10] yields that the quaternionic locus $Q_{\mathcal{O}}$ is *nonempty* if and only if $\mathrm{disc}(B)$ *is a totally positive principal ideal of* F. In consequence, for the sake of clarity of the exposition, we will assume throughout these notes that the narrow class number of F is $h_+(F) = 1$. This automatically implies the nonemptyness of $Q_{\mathcal{O}}$.

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1. Abelian varieties with quaternionic multiplication

As a first step in the study of the quaternionic locus $Q_{\mathcal{O}}$ in the moduli space $\mathcal{A}_g$, it is necessary to understand the geometry of the objects that $Q_{\mathcal{O}}$ parametrizes. Let us review some of the results that were accomplished in [10] in this direction.

Definition 1.1. An abelian variety with quaternionic multiplication by $\mathcal{O}$ over $\bar{\mathbb{Q}}$ is an abelian variety $A/\bar{\mathbb{Q}}$ such that $\mathrm{End}(A) \simeq \mathcal{O}$ and $\dim(A) = 2[F : \mathbb{Q}] = 2n$.

Let A be an abelian variety with quaternionic multiplication by $\mathcal{O}$ over $\bar{\mathbb{Q}}$ and let $\mathrm{NS}(A)$ denote the free $\mathbb{Z}$-module of rank $3n = 3g/2$ of line bundles on A up to algebraic equivalence. We say that two line bundles $\mathcal{L}$, $\mathcal{L}' \in \mathrm{NS}(A)$ are isomorphic if there exists an automorphism $\alpha \in \mathrm{Aut}(A) \simeq \mathcal{O}^*$ such that $\mathcal{L}' = \alpha^*(\mathcal{L})$.

Theorem 1.2. *Let $A/\bar{\mathbb{Q}}$ be an abelian variety with quaternion multiplication by $\mathcal{O}$ and let $\iota : \mathcal{O} \simeq \mathrm{End}(A)$ be a fixed isomorphism of rings. Then, there is an isomorphism of groups*

$$\begin{array}{ccc} \mathrm{NS}(A) & \xrightarrow{\sim} & \vartheta(\mathcal{O})_0 \\ \mathcal{L} & \mapsto & c_1(\mathcal{L}) = \mu \end{array}$$

between the Néron-Severi group of A and the group of pure quaternions of the reduced different $\vartheta(\mathcal{O})$ of $\mathcal{O}$.

Moreover, for any two non trivial line bundles $\mathcal{L}$, $\mathcal{L}' \in \mathrm{NS}(A)$, let $\mu = c_1(\mathcal{L})$ and $\mu' = c_1(\mathcal{L}')$. Then we have that

(1) *$\mathcal{L} \simeq \mathcal{L}'$ if and only if there exists $\alpha \in \mathcal{O}^*$ such that $\mu' = \bar{\alpha}\mu\alpha$.*
(2) $\deg(\mathcal{L}) = \deg(\varphi_{\mathcal{L}} : A \to \hat{A})^{1/2} = \mathrm{N}_{F/\mathbb{Q}}(\mathrm{n}(\mu)/D)$.
(3) *$\mathcal{L}$ is a polarization if and only if $\mathrm{n}(\mu) \in F_+^*$ and μ is* ample *(cf.* [10], *§5).*
(4) *The Rosati involution on $\mathcal{O} \overset{\iota}{\simeq} \mathrm{End}(A)$ with respect to $\mathcal{L}$ is*

$$\begin{array}{cccc} \circ : & \mathcal{O} & \to & \mathcal{O} \\ & \beta & \mapsto & \mu^{-1}\bar{\beta}\mu \end{array}$$

Let us write the isomorphism c_1 in more explicit terms. Fix an immersion of $\bar{\mathbb{Q}}$ into the field $\mathbb{C}$ of complex numbers and let $A(\mathbb{C}) = V/\Lambda$ for some complex vector space V and a lattice Λ. Upon the choice of an isomorphism $\iota : \mathcal{O} \simeq \mathrm{End}(A)$, the lattice Λ is naturally a left $\mathcal{O}$-module and, since $h(F) = h_+(F) = 1$, by a theorem of Eichler [2] we know that $\Lambda \simeq \mathcal{O}$.

By the Appell-Humbert Theorem, a line bundle $\mathcal{L} \in \mathrm{NS}(A)$ on A can be regarded as a Riemann form $E : \Lambda \times \Lambda \to \mathbb{Z}$ on V. Let us identify $\Lambda = \mathcal{O}$ through a fixed isomorphism and let $t \in F^*$ be any generator of the principal ideal ϑ_F. Then, the inverse isomorphism $c_1^{-1} : \vartheta(\mathcal{O})_0 \overset{\sim}{\to} \mathrm{NS}(A)$ maps a pure quaternion μ to the Riemann form $E_\mu : \mathcal{O} \times \mathcal{O} \to \mathbb{Z}$, $(\beta_1, \beta_2) \mapsto -\mathrm{tr}_{F/\mathbb{Q}}(\mathrm{tr}(\mu\beta_1\bar{\beta}_2)/\mathrm{n}(\mu))$.

Remark 1.3. The isomorphism c_1 is canonical in the sense that it does not depend on the choice of any polarization on A. However, we warn the reader that it does depend on the choice of the isomorphism $\iota : \mathcal{O} \simeq \mathrm{End}(A)$.

Theorem 1.4. *Let A be an abelian variety over $\bar{\mathbb{Q}}$ with quaternionic multiplication by $\mathcal{O}$. Let $D \in F_+^*$ be a totally positive generator of* $\mathrm{disc}(B)$.

Then, A is principally polarizable and the number of isomorphism classes of principal polarizations on A is

$$\pi_0(A) = \frac{1}{2}\sum_S h(S),$$

where S runs among the set of orders in the CM-field $F(\sqrt{-D})$ that contain $R_F[\sqrt{-D}]$ and $h(S)$ denotes its class number.

2. Shimura varieties

As in the preceding sections, let B be a totally indefinite division quaternion algebra over a totally real field F of trivial narrow class number. Let $D \in F_+^*$ be a totally positive generator of $\mathrm{disc}(B)$.

Definition 2.1. A *principally polarized maximal order* of B is a pair $(\mathcal{O}, \mu)$ where $\mathcal{O} \subset B$ is a maximal order and $\mu \in \mathcal{O}$ is a pure quaternion such that $\mu^2 + uD = 0$ for some $u \in R_{F+}^*$.

Attached to a principally polarized maximal order $(\mathcal{O}, \mu)$ there is the following moduli problem over $\mathbb{Q}$: classifying isomorphism classes of triplets $(A, \iota, \mathcal{L})$ given by

- An abelian variety A of dimension $g = 2n$.
- A ring homomorphism $\iota : \mathcal{O} \hookrightarrow \mathrm{End}\,(A)$.
- A principal polarization $\mathcal{L}$ on A such that

$$\iota(\beta)^{\circ} = \iota(\mu^{-1}\bar{\beta}\mu)$$

for any $\beta \in \mathcal{O}$, where $\circ : \mathrm{End}(A) \to \mathrm{End}(A)$ is the Rosati involution with respect to $\mathcal{L}$.

A triplet $(A, \iota, \mathcal{L})$ will be referred to as a polarized abelian variety with multiplication by $\mathcal{O}$. Two triplets $(A_1, \iota_1, \mathcal{L}_1)$, $(A_2, \iota_2, \mathcal{L}_2)$ are isomorphic if there exists an isomorphism $\alpha \in \mathrm{Hom}(A_1, A_2)$ such that $\alpha\iota_1(\beta) = \iota_2(\beta)\alpha$ for any $\beta \in \mathcal{O}$ and $\alpha^*(\mathcal{L}_2) = \mathcal{L}_1 \in \mathrm{NS}(A_1)$. Note also that, since a priori there is no canonical structure of R_F-algebra on $\mathrm{End}(A)$, the immersion $\iota : \mathcal{O} \hookrightarrow \mathrm{End}(A)$ is just a homomorphism of rings.

As it was proved by Shimura, the corresponding moduli functor is coarsely represented by an irreducible and reduced quasi-projective scheme $\mathfrak{X}_\mu/\mathbb{Q}$ over $\mathbb{Q}$ and of dimension $n = [F : \mathbb{Q}]$. Moreover, since B is division, the Shimura variety $\mathfrak{X}_\mu$ is complete (cf. [14], [15]).

Let $\mathfrak{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ denote the upper half plane. Complex analytically, the manifold $\mathfrak{X}_\mu(\mathbb{C})$ can be described independently of the choice of μ as the quotient

$$\mathcal{O}^1 \backslash \mathfrak{H}^n \simeq \mathfrak{X}_\mu(\mathbb{C})$$

of the symmetric space $\mathfrak{H}^n$ by the action of the group $\mathcal{O}^1$ regarded suitably as a discontinuous subgroup of $\mathrm{SL}_2(\mathbb{R})^n$. See [11] for details.

In addition, we are also interested in the Hilbert modular reduced scheme $\mathcal{H}_F/\mathbb{Q}$ that coarsely represents the functor attached to the moduli problem of classifying principally polarized abelian varieties A of dimension g together with an homomorphism $R_F \hookrightarrow \mathrm{End}(A)$. The Hilbert modular variety $\mathcal{H}_F$ has dimension $3n$ and $\mathcal{H}_F(\mathbb{C})$ is the quotient of n copies of the three-dimensional Siegel half space $\mathfrak{H}_2$ by a suitable discontinuous group (cf. [14], [7]).

Notice that, when $F = \mathbb{Q}$, $\mathcal{H}_F = \mathcal{A}_2$ is Igusa's three-fold, the moduli space of principally polarized abelian surfaces.

There are natural morphisms

$$\begin{array}{cccccc} \pi : & \mathfrak{X}_\mu & \xrightarrow{\pi_F} & \mathcal{H}_F & \longrightarrow & \mathcal{A}_g \\ & (A, \iota, \mathcal{L}) & \mapsto & (A, \iota|_{R_F}, \mathcal{L}) & \mapsto & (A, \mathcal{L}) \end{array}$$

from the Shimura variety $\mathfrak{X}_\mu$ to the Hilbert modular variety $\mathcal{H}_F$ and the moduli space $\mathcal{A}_g$ that consist of gradually *forgetting* the quaternionic endomorphism structure. These morphisms are representable, proper and defined over the field $\mathbb{Q}$ of rational numbers.

As it will be convenient for our purposes in the rest of this paper, we introduce the variety $\widetilde{\mathfrak{X}}_\mu = \pi(\mathfrak{X}_\mu)$ to be the image of the Shimura variety $\mathfrak{X}_\mu$ attached to a pure quaternion $\mu \in \mathcal{O}$ with $\mu^2 + uD = 0$ for some $u \in R^*_{F+}$ in the moduli space $\mathcal{A}_g$ by the forgetful map π. It is important to remark that although the complex analytical structure of $\mathfrak{X}_\mu$ does not depend on the choice of μ, the construction of the forgetful map π and the subvariety $\widetilde{\mathfrak{X}}_\mu$ of $\mathcal{A}_g$ do.

The varieties $\widetilde{\mathfrak{X}}_\mu$ are reduced, irreducible, complete and possibly singular schemes over $\mathbb{Q}$ of dimension n. The set of singularities of $\widetilde{\mathfrak{X}}_\mu$ is a finite set and all the singularities are of *quotient type* (cf. [3] for the terminology).

3. The birational class of the forgetful maps

It is our aim now to describe the forgetful maps $\pi : \mathfrak{X}_\mu \xrightarrow{\pi_F} \mathcal{H}_F \longrightarrow \mathcal{A}_g$ as the projection of the Shimura varieties $\mathfrak{X}_\mu$ onto their quotient by suitable Atkin-Lehner groups up to a birational equivalence. The following groups of Atkin-Lehner involutions were introduced in [12]. We keep the notations and assumptions of the Introduction.

Definition 3.1. The *Atkin-Lehner group* W of the maximal order $\mathcal{O}$ is

$$W = \mathrm{Norm}_{B^*_+}(\mathcal{O})/F^* \cdot \mathcal{O}^1.$$

It was shown in [11] that $W \simeq \mathbb{Z}/2\mathbb{Z} \times \overset{2r}{\cdots} \times \mathbb{Z}/2\mathbb{Z}$, where $2r$ is the number of ramified prime ideals of B.

Definition 3.2. Let $(\mathcal{O}, \mu)$ be a principally polarized maximal order in B. A *twist* of $(\mathcal{O}, \mu)$ is an element $\chi \in \mathcal{O} \cap \mathrm{Norm}_{B^*}(\mathcal{O})$ such that $\chi^2 + \mathrm{n}(\chi) = 0$ and $\mu\chi = -\chi\mu$.

In other words, a twist of $(\mathcal{O}, \mu)$ is a pure quaternion $\chi \in \mathcal{O} \cap \mathrm{Norm}_{B^*}(\mathcal{O})$ such that

$$B = F + F\mu + F\chi + F\mu\chi = (\frac{-uD, -\mathrm{n}(\chi)}{F}).$$

We say that a principally polarized maximal order $(\mathcal{O}, \mu)$ in B is *twisting* if it admits some twist χ in $\mathcal{O}$. We say that a maximal order $\mathcal{O}$ is *twisting* if there exists $\mu \in \mathcal{O}$ such that $(\mathcal{O}, \mu)$ is a twisting principally polarized order. Finally, we say that B is *twisting* if there exists a twisting maximal order $\mathcal{O}$ in B. Note that B is twisting if and only if $B \simeq (\frac{-uD, m}{F})$ for some $u \in R^*_{F+}$ and $m \in F^*$ such that $m|D$.

Definition 3.3. A *twisting involution* $\omega \in W$ of $(\mathcal{O}, \mu)$ is an Atkin-Lehner involution such that $[\omega] = [\chi] \in W$ is represented in B^* by a twist χ of $(\mathcal{O}, \mu)$.

We let $V_0 = V_0(\mathcal{O}, \mu)$ denote the subgroup of W generated by the twisting involutions of $(\mathcal{O}, \mu)$. For a principally polarized maximal order $(\mathcal{O}, \mu)$, let $R_\mu = F(\mu) \cap \mathcal{O}$ and let $\Omega = \Omega(R_\mu) = \{\xi \in R_\mu : \xi^f = 1, f \geq 1\}$ denote the finite group of roots of unity in the CM-quadratic order R_μ over R_F.

Definition 3.4. The *stable group* of $(\mathcal{O}, \mu)$ is the subgroup

$$W_0 = U_0 \cdot V_0$$

of W generated by

$$U_0 = U_0(\mathcal{O}, \mu) = \mathrm{Norm}_{F(\mu)^*}(\mathcal{O})/F^* \cdot \Omega(R_\mu),$$

and the group V_0 of twisting involutions of $(\mathcal{O}, \mu)$.

As it was also shown in [11], for any principally polarized maximal order $(\mathcal{O}, \mu)$, there are natural monomorphisms of groups $V_0 \subseteq W_0 \subseteq W \subseteq \mathrm{Aut}_{\mathbb{Q}}(\mathfrak{X}_\mu) \subseteq \mathrm{Aut}_{\bar{\mathbb{Q}}}(\mathfrak{X}_\mu \otimes \bar{\mathbb{Q}})$. The question whether the two latter immersions are actually isomorphisms was studied by the author for the case of Shimura curves in [9].

The following was proved in [11].

Theorem 3.5. *Let $(\mathcal{O}, \mu)$ be a principally polarized maximal order in B and let $\mathfrak{X}_\mu$ be the Shimura variety attached to it. Then there is a commutative diagram of finite maps*

$$\begin{array}{ccc} \mathfrak{X}_\mu & \xrightarrow{\pi_F} & \mathcal{H}_F \\ & \searrow \qquad \nearrow^{b_F} & \\ & \mathfrak{X}_\mu/W_0, & \end{array}$$

where $\mathfrak{X}_\mu \to \mathfrak{X}_\mu/W_0$ is the natural projection and $b_F : \mathfrak{X}_\mu/W_0 \to \pi_F(\mathfrak{X}_\mu)$ is a birational morphism from $\mathfrak{X}_\mu/W_0$ onto the image of $\mathfrak{X}_B$ in $\mathcal{H}_F$.

The domain of definition of b_F^{-1} is $\pi_F(\mathfrak{X}_\mu) \setminus \mathcal{T}_F$, where $\mathcal{T}_F$ is a finite set (of Heegner points).

4. The quaternionic locus

As above, we let $\mathcal{O}$ be a maximal order in a totally indefinite division quaternion algebra B over a totally real field F of trivial narrow class number and we fix a generator $D \in F_+^*$ of $\mathrm{disc}(B)$.

It is the aim of this section to use the preceding results to study the quaternionic locus $Q_{\mathcal{O}}$ in $\mathcal{A}_g(\mathbb{C})$. As we mentioned in the Introduction, since $h_+(F) = 1$, the set $Q_{\mathcal{O}}$ is not empty.

Definition 4.1. A Heegner point in $Q_{\mathcal{O}}$ is an isomorphism class $[(A, \mathcal{L})]$ of a principally polarized abelian variety such that $\mathrm{End}(A) \supsetneq \mathcal{O}$.

According to the Definitions 1.1 and 4.1, we note that $Q_{\mathcal{O}}$ is the disjoint union of the set of principally polarized abelian varieties $[(A,\mathcal{L})]$ with quaternionic multiplication by $\mathcal{O}$ and the set of Heegner points. The former is a subset of $Q_{\mathcal{O}}$ whose closure with respect to the analytical topology is $Q_{\mathcal{O}}$ itself. The latter is also a dense but discrete subset of $Q_{\mathcal{O}}$ (cf. [15]).

In order to understand the nature of the locus $Q_{\mathcal{O}}$, we observe that for any principally polarized pair $(\mathcal{O},\mu)$, the set $\widetilde{\mathfrak{X}}_\mu(\mathbb{C})$ of complex points of the Shimura variety $\widetilde{\mathfrak{X}}_\mu/\mathbb{Q}$ attached to $(\mathcal{O},\mu)$ sits inside $Q_{\mathcal{O}}$.

Proposition 4.2. *Let μ, $\mu'\in\mathcal{O}_0$ be two pure quaternions such that $\mathrm{n}(\mu)=uD$ and $\mathrm{n}(\mu')=u'D$ for some units u, $u'\in R^*_{F+}$. If $\widetilde{\mathfrak{X}}_\mu(\mathbb{C})$ and $\widetilde{\mathfrak{X}}_{\mu'}(\mathbb{C})$ are different subvarieties of $\mathcal{A}_g(\mathbb{C})$, then $\widetilde{\mathfrak{X}}_\mu(\mathbb{C})\cap\widetilde{\mathfrak{X}}_{\mu'}(\mathbb{C})$ is a finite set of Heegner points.*

Proof. Assume that the isomorphism class $[A,\mathcal{L}]$ of a principally polarized abelian variety falls at the intersection of $\widetilde{\mathfrak{X}}_\mu$ and $\widetilde{\mathfrak{X}}_{\mu'}$ in $\mathcal{A}_g$. Write $[A,\mathcal{L}]=\pi([A,\iota,\mathcal{L}])=\pi([A',\iota',\mathcal{L}'])$ as the image by π of points in $\mathfrak{X}_\mu$ and $\mathfrak{X}_{\mu'}$, respectively. Since $[(A,\mathcal{L})]=[(A',\mathcal{L}')]\in\mathcal{Q}_{\mathcal{O}}$, we can identify the pair $(A,\mathcal{L})=(A',\mathcal{L}')$ through a fixed isomorphism of polarized abelian varieties.

Let us assume that $[A,\mathcal{L}]=[A',\mathcal{L}']$ was not a Heegner point. Then $\iota:\mathcal{O}\simeq\mathrm{End}(A)$ would be an isomorphism of rings such that $c_1(\mathcal{L})=\mu$. We then would have by Theorem 1.2, §4, that $c_1(\mathcal{L})=c_1(\mathcal{L}')=\mu=\mu'$ up to multiplication by elements in F^*. Since $\widetilde{\mathfrak{X}}_\mu=\widetilde{\mathfrak{X}}_{u\mu}$ for all units $u\in R^*_F$, this would contradict the statement. Since the set of Heegner points in $\mathcal{A}_g(\mathbb{C})$ is discrete, we conclude that any two irreducible components of $\mathcal{Q}_{\mathcal{O}}$ meet at a finite set of Heegner points. $\square$

Proposition 4.3. (1) *The locus $Q_{\mathcal{O}}$ is the set of complex points $\mathcal{Q}_{\mathcal{O}}(\mathbb{C})$ of a reduced complete subscheme $\mathcal{Q}_{\mathcal{O}}$ of $\mathcal{A}_g$ defined over $\mathbb{Q}$.*

(2) *Let $\rho(\mathcal{O})$ be the number of absolutely irreducible components of $\mathcal{Q}_{\mathcal{O}}$. Then there exist quaternions $\mu_k\in\mathcal{O}_0$ with $\mu_k^2+u_kD=0$ for $u_k\in R^*_{F+}$, $1\le k\le\rho(\mathcal{O})$, such that*

$$\mathcal{Q}_{\mathcal{O}}=\bigcup\widetilde{\mathfrak{X}}_{\mu_k}.$$

is the decomposition of $\mathcal{Q}_{\mathcal{O}}$ into irreducible components.

Proof. Let $[(A,\mathcal{L})]\in Q_{\mathcal{O}}$ be the isomorphism class of a complex principally polarized abelian variety such that $\mathrm{End}(A)\simeq\mathcal{O}$. Fix an isomorphism $\iota:\mathcal{O}\simeq\mathrm{End}(A)$. By Theorem 1.2, §4, the Rosati involution with respect to $\mathcal{L}$ on $\mathcal{O}$ must be of the form $\varrho_\mu:\mathcal{O}\to\mathcal{O}$, $\beta\mapsto\mu^{-1}\bar\beta\mu$ for some $\mu\in\mathcal{O}$ with $\mu^2+uD=0$, $u\in R^*_{F+}$. Thus $[(A,\mathcal{L})]=\pi([A,\iota,\mathcal{L}])\in\widetilde{\mathfrak{X}}_\mu(\mathbb{C})$, namely the set of complex points on a reduced, irreducible, complete and possibly singular scheme over $\mathbb{Q}$ (cf. [14], [15]). Since the set of Heegner points $[(A,\mathcal{L})]\in\widetilde{\mathfrak{X}}_\mu(\mathbb{C})$ is a discrete set which lies on the Zariski closure of its complement, we conclude that $Q_{\mathcal{O}}$ is the union of the

Shimura varieties $\widetilde{\mathfrak{X}}_\mu(\mathbb{C})$ as μ varies among pure quaternions satisfying the above properties.

Let us now show that $Q_{\mathcal{O}}$ is actually covered by finitely many pairwise different Shimura varieties. Let $A/\mathbb{C}$ be an arbitrary abelian variety with quaternionic multiplication by $\mathcal{O}$ and fix an isomorphism $\iota : \mathcal{O} \simeq \mathrm{End}(A)$.

Let $(\mathcal{O}, \mu)$ be any principally polarized pair. Since $h_+(F) = 1$, there exists a unit $u \in R_F^*$ such that $u\mu$ is an ample quaternion in the sense of [10], §5. Let $\mathcal{L} \in \mathrm{NS}(A)$ be the line bundle on A such that $c_1(\mathcal{L})^{-1} = u\mu$. From Theorem 1.2, it follows that $\mathcal{L}$ is a principal polarization on A such that the isomorphism class of the triplet $(A, \iota, \mathcal{L})$ corresponds to a closed point in $\mathfrak{X}_\mu(\mathbb{C})$ and hence $[A, \mathcal{L}] \in \widetilde{\mathfrak{X}}_\mu$.

Since, by Proposition 4.2, the intersection points of two different Shimura varieties $\widetilde{\mathfrak{X}}_\mu(\mathbb{C})$ and $\widetilde{\mathfrak{X}}_{\mu'}(\mathbb{C})$ in $\mathcal{A}_g(\mathbb{C})$ are Heegner points, this shows that for every irreducible component of $Q_{\mathcal{O}}$ there exists at least one principal polarization $\mathcal{L}$ on A such that $[A, \mathcal{L}]$ lies on it. Consequently, the number $\pi_0(A)$ of isomorphism classes of principal polarizations on A is an upper bound for the number $\rho(\mathcal{O})$ of irreducible components of $Q_{\mathcal{O}}$. Since, by Theorem 1.4, the number $\pi_0(A)$ is a finite number, this yields the proof of the proposition. □

In view of Proposition 4.3, it is natural to pose the following

Question 4.4. What is the number $\rho(\mathcal{O})$ of irreducible components of $\mathcal{Q}_{\mathcal{O}}$? When is $\mathcal{Q}_{\mathcal{O}}$ irreducible?

5. The distribution of principal polarizations on an abelian variety in $\mathcal{Q}_{\mathcal{O}}$

Let us relate Question 4.4 to the following problem. In Theorem 1.4, we computed the number $\pi_0(A)$ of principal polarizations on an abelian variety A with quaternion multiplication by $\mathcal{O}$ as the finite sum of relative class numbers of suitable orders in the CM-fields $F(\sqrt{-uD})$ for $u \in R_{F+}^*/R_F^{*2}$. This has the following modular interpretation:

Let $\mathcal{L}_1$, ..., $\mathcal{L}_{\pi_0(A)}$ be representatives of the $\pi_0(A)$ distinct principal polarizations on A. Then the pairwise nonisomorphic principally polarized abelian varieties $[(A, \mathcal{L}_1)]$, ..., $[(A, \mathcal{L}_{\pi_0(A)})]$ correspond to all closed points in $\mathcal{Q}_{\mathcal{O}}$ whose underlying abelian variety is isomorphic to A. We then naturally ask the following

Question 5.1. Let A be an abelian variety with quaternionic multiplication by $\mathcal{O}$. How are the distinct principal polarizations $[(A, \mathcal{L}_j)]$ distributed among the irreducible components $\widetilde{\mathfrak{X}}_{\mu_k}$ of $\mathcal{Q}_{\mathcal{O}}$?

It turns out that the two questions above are related. The linking ingredient is provided by the definition below, which establishes a slightly coarser equivalence relationship on polarizations than the one considered in Theorem 1.2, §1.

Definition 5.2. Let A be an abelian variety with quaternionic multiplication by $\mathcal{O}$ over $\bar{\mathbb{Q}}$.

(1) Two polarizations $\mathcal{L}$ and $\mathcal{L}'$ on A are weakly isomorphic if $c_1(\mathcal{L}) \simeq mc_1(\mathcal{L}') \in \mathrm{NS}(A)$ for some $m \in F_+^*$. We shall denote it $\mathcal{L} \simeq_w \mathcal{L}'$.
(2) Two principal polarizations $\mathcal{L}$ and $\mathcal{L}'$ on A are *Atkin-Lehner isogenous*, denoted $\mathcal{L} \sim \mathcal{L}'$, if there is an isogeny $\omega \in \mathcal{O} \cap \mathrm{Norm}_{B_+^*}(\mathcal{O})$ of A such that

$$\omega^*(\mathcal{L}) \simeq_w \mathcal{L}'.$$

We note that there is a closed relationship between the above definition and the modular interpretation of the Atkin-Lehner group W given in [11].

Definition 5.3. Let A be an abelian variety with quaternionic multiplication by $\mathcal{O}$ over $\bar{\mathbb{Q}}$. We let $\hat{\Pi}_0(A)$ be the set of principal polarizations on A up to Atkin-Lehner isogeny and we let $\hat{\pi}_0(A) = \sharp\hat{\Pi}_0(A)$ denote its cardinality.

Theorem 5.4 (Distribution of principal polarizations)**.** *Let A be an abelian variety with quaternionic multiplication by $\mathcal{O}$ over $\bar{\mathbb{Q}}$ and let $\mathcal{L}_1$, ..., $\mathcal{L}_{\pi_0(A)}$ be representatives of the $\pi_0(A)$ distinct principal polarizations on A.*

Then, two closed points $[A, \mathcal{L}_i]$ and $[A, \mathcal{L}_j]$ lie on the same irreducible component of $\mathcal{Q}_{\mathcal{O}}$ if and only if the polarizations $\mathcal{L}_i$ and $\mathcal{L}_j$ are Atkin-Lehner isogenous.

Proof. We know from Proposition 4.3 that any irreducible component of $\mathcal{Q}_{\mathcal{O}}$ is $\widetilde{\mathfrak{X}}_\mu$ for some principally polarized pair $(\mathcal{O}, \mu)$. We single out and fix one of these.

Let $\mathcal{L}$ be a principal polarization on A such that $[(A, \mathcal{L})]$ lies on $\widetilde{\mathfrak{X}}_\mu$ and let $\mathcal{L}'$ be a second principal polarization on A. We claim that $[A, \mathcal{L}'] \in \widetilde{\mathfrak{X}}_\mu$ if and only if there exists $\omega \in \mathcal{O} \cap \mathrm{Norm}_{B_+^*}(\mathcal{O})$ such that $\mathcal{L}'$ and $\omega^*(\mathcal{L})$ are weakly isomorphic.

Assume first that $\mathcal{L}' \simeq_w \omega^*(\mathcal{L})$ for some $\omega \in \mathcal{O} \cap \mathrm{Norm}_{B_+^*}(\mathcal{O})$. This amounts to saying that $\bar{\omega} c_1(\mathcal{L})\omega = mc_1(\mathcal{L}')$ for some $m \in F^*$. Since both $\omega^*(\mathcal{L})$ and $\mathcal{L}'$ are polarizations, we deduce from Theorem 1.2, §3, that $m \in F_+^*$. Moreover, since $\mathcal{L}$ and $\mathcal{L}'$ are principal, we obtain from Theorem 1.2, §2, that $m = u\mathrm{n}(\omega)$ for some $u \in R_{F+}^*$.

Note that $(A, \iota_\omega, \mathcal{L}')$ is a principally polarized abelian variety with quaternionic multiplication such that the Rosati involution that $\mathcal{L}'$ induces on $\iota_\omega(\mathcal{O})$ is ϱ_μ. Indeed, this follows because $\iota_\omega(\beta)^{\circ_{\mathcal{L}'}} = \iota((\omega^{-1}\beta\omega))^{\circ_{\mathcal{L}'}} = \iota((\omega^{-1}\mu\omega)^{-1}\overline{\omega^{-1}\beta\omega}(\omega^{-1}\mu\omega))$- $= \iota_\omega(\mu^{-1}\bar{\beta}\mu)$. This shows that, if $\mathcal{L}'$ and $\omega^*(\mathcal{L})$ are weakly isomorphic for some $\omega \in \mathcal{O} \cap \mathrm{Norm}_{B_+^*}(\mathcal{O})$, then $[A, \mathcal{L}'] \in \widetilde{\mathfrak{X}}_\mu$.

Conversely, let us assume that $[A, \mathcal{L}'] \in \widetilde{\mathfrak{X}}_\mu$. Let $\iota' : \mathcal{O} \hookrightarrow \mathrm{End}(A)$ be such that $[A, \iota', \mathcal{L}'] \in \mathfrak{X}_{(\mathcal{O}, \mathcal{I}_\vartheta, \varrho_\mu)}$. By the Skolem-Noether Theorem, it holds that $\iota' = \omega^{-1}\iota\omega$ for some $\omega \in \mathrm{Norm}_{B^*}(\mathcal{O})$; we can assume that $\omega \in \mathcal{O}$ by suitably scaling it. Since it holds that $\iota_\omega(\beta)^{\circ_{\mathcal{L}'}} = \iota_\omega(\mu^{-1}\bar{\beta}\mu)$ for any $\beta \in \mathcal{O}$, we have that $c_1(\mathcal{L}') = u\omega^{-1}c_1(\mathcal{L})\omega$ for some $u \in R_F^*$ such that $u\mathrm{n}(\omega) \in F_+^*$. Since $\mathrm{n}(\mathcal{O}^*) = R_F^*$, we can find $\alpha \in \mathcal{O}^*$ with reduced norm $\mathrm{n}(\alpha) = u^{-1}$ and thus $\omega\alpha \in B_+^*$.

Let $\mathcal{L}_{\omega\alpha}$ be the polarization on A such that $c_1(\mathcal{L}_{\omega\alpha}) = \frac{u}{\mathrm{n}(\omega)} c_1((\omega\alpha)^*(\mathcal{L}))$. The automorphism $\alpha \in \mathcal{O}^* = \mathrm{Aut}(A)$ induces an isomorphism between the polarizations $\mathcal{L}_{\omega\alpha}$ and $\mathcal{L}'$, since $c_1(\alpha^*(\mathcal{L}')) = \bar{\alpha}(u\omega^{-1}c_1(\mathcal{L})\omega)\alpha = c_1(\mathcal{L}_{\omega\alpha})$. Hence $\mathcal{L}'$ is weakly isomorphic to $(\omega\alpha)^*(\mathcal{L})$. This concludes our claim above and also proves the theorem. □

Corollary 5.5. *The number of irreducible components of $\mathcal{Q}_{\mathcal{O}}$ is*

$$\rho(\mathcal{O}) = \hat{\pi}_0(A),$$

independently of the choice of A.

For any irreducible component $\widetilde{\mathfrak{X}}_{\mu_k}$ of $\mathcal{Q}_{\mathcal{O}}$, let $\Pi_0^{(k)}(A) \subset \Pi_0(A)$ denote the set of isomorphism classes of the isogeny class of principal polarizations lying on $\widetilde{\mathfrak{X}}_{\mu_k}$.

As another immediate consequence of Theorem 5.4, the following corollary establishes an internal structure on the set $\Pi_0(A)$. Roughly, it asserts that $\Pi_0(A) = \bigcup_{k=1}^{\rho(\mathcal{O})} \Pi_0^{(k)}(A)$ is the disjoint union of the sets $\Pi_0^{(k)}(A)$, which are equipped with a free and transitive action of a 2-torsion finite abelian group.

Corollary 5.6. *Let A be an abelian variety with quaternionic multiplication by $\mathcal{O}$. Let $\widetilde{\mathfrak{X}}_{\mu_k}$ be an irreducible component of $\mathcal{Q}_{\mathcal{O}}$ and let $W_0^{(k)} \subseteq W$ be the stable subgroup attached to the polarized order $(\mathcal{O}, \mu_k)$.*

Then there is a free and transitive action of $W/W_0^{(k)}$ on the Atkin-Lehner isogeny class $\Pi_0^{(k)}(A)$ of principal polarizations lying on $\widetilde{\mathfrak{X}}_{\mu_k}$.

In the case of a non twisting maximal order $\mathcal{O}$, we have that the stable group $W_0(\mathcal{O}, \mu)$ attached to a principally polarized pair $(\mathcal{O}, \mu)$ is $U_0(\mathcal{O}, \mu)$. The following corollary follows from the proof of Lemma 2.10 in [12].

Corollary 5.7. *Let $\mathcal{O}$ be a non twisting maximal order in B and assume that, for any $u \in R_{F+}^*$, any primitive root of unity of odd order in the CM-field $F(\sqrt{-uD})$ is contained in the order $R_F[\sqrt{-uD}]$.*

Let A be an abelian variety with quaternionic multiplication by $\mathcal{O}$. Then the distinct isomorphism classes of principally polarized abelian varieties $[(A, \mathcal{L}_1)]$, ..., $[(A, \mathcal{L}_{\pi_0(A)})]$ are equidistributed *among the $\rho(\mathcal{O})$ irreducible components of $\mathcal{Q}_{\mathcal{O}}$.*

In particular, it then holds that

$$\pi_0(A) = \frac{|W|}{|W_0|} \cdot \rho(\mathcal{O}).$$

6. Shimura curves embedded in Igusa's threefold

The whole picture becomes particularly neat when we consider the simplest case of quaternion algebras over $\mathbb{Q}$. Let then B be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $D = p_1 \cdot \ldots \cdot p_{2r}$ and let $\mathcal{O}$ be a maximal order in B. Since

$h(\mathbb{Q}) = 1$, there is a single choice of $\mathcal{O}$ up to conjugation by B^*. Moreover, all left ideals of $\mathcal{O}$ are principal and hence isomorphic to $\mathcal{O}$ as left $\mathcal{O}$-modules.

Let A be a complex abelian surface with quaternionic multiplication by $\mathcal{O}$. By Theorem 1.4, A is principally polarizable and the number of isomorphism classes of principal polarizations on A is

$$\pi_0(A) = \frac{\tilde{h}(-D)}{2},$$

where, for any nonzero squarefree integer d, we write

$$\tilde{h}(d) = \begin{cases} h(4d) + h(d) & \text{if } d \equiv 1 \mod 4, \\ h(4d) & \text{otherwise.} \end{cases}$$

For any integral element $\mu \in \mathcal{O}$ such that $\mu^2 + D = 0$, let now $\mathfrak{X}_\mu$ be the Shimura curve that coarsely represents the functor which classifies principally polarized abelian surfaces $(A, \iota, \mathcal{L})$ with quaternionic multiplication by $\mathcal{O}$ such that the Rosati involution with respect to $\mathcal{L}$ on $\mathcal{O}$ is ϱ_μ. This is an algebraic curve over $\mathbb{Q}$ whose isomorphism class does not actually depend on the choice of the quaternion μ, but only on the discriminant D (cf. [15]). Hence, it is usual to simply denote this isomorphism class as $\mathfrak{X}_D$.

Let $W = \{\omega_m : m|D\} \simeq (\mathbb{Z}/2\mathbb{Z})^{2r}$ be the Atkin-Lehner group attached to $\mathcal{O}$ in Section 3. We know that $W \subseteq \mathrm{Aut}_{\mathbb{Q}}(\mathfrak{X}_D)$ is a subgroup of the group of automorphisms of the Shimura curve $\mathfrak{X}_D$.

Let now $\mathcal{A}_2$ be Igusa's three-fold, the moduli space of principally polarized abelian surfaces. By the work of Igusa (cf. [6]), it is a scheme over $\mathbb{Q}$ that contains, as a Zariski open and dense subset, the moduli space $\mathcal{M}_2$ of curves of genus 2, immersed in $\mathcal{A}_2$ via the Torelli embedding.

Sitting in $\mathcal{A}_2$ there is the quaternionic locus $\mathcal{Q}_{\mathcal{O}}$ of isomorphism classes of principally polarized abelian surfaces $[(A, \mathcal{L})]$ such that $\mathrm{End}(A) \supseteq \mathcal{O}$. Since all maximal orders $\mathcal{O}$ in B are pairwise conjugate, the quaternionic locus $\mathcal{Q}_{\mathcal{O}}$ does not actually depend on the choice of $\mathcal{O}$ and we may simply denote it by $\mathcal{Q}_{\mathcal{O}} = \mathcal{Q}_D$.

As explained in Section 2 and 3, there are forgetful finite morphisms $\pi : \mathfrak{X}_\mu \to \mathcal{Q}_D \subset \mathcal{A}_2$ which map the Shimura curve $\mathfrak{X}_\mu$ onto an irreducible component $\widetilde{\mathfrak{X}}_\mu$ of $\mathcal{Q}_D$. We insist on the fact that the image $\widetilde{\mathfrak{X}}_\mu \subset \mathcal{Q}_D$ *does depend* on the choice of the quaternion μ.

Let us now compare the non twisting and twisting case, respectively. We first assume that

$$B \not\simeq (\frac{-D, m}{\mathbb{Q}})$$

for all positive divisors $m|D$ of D. Then all principally polarized pairs $(\mathcal{O},\mu)$ in B are *non twisting* and the stable subgroup attached to $(\mathcal{O},\mu)$ is

$$W_0 = U_0 = \langle \omega_D \rangle \subset W,$$

independently of the choice of μ. By Theorem 3.5, we deduce that any irreducible component $\widetilde{\mathfrak{X}}_\mu$ of $\mathcal{Q}_D$ is birationally equivalent to the Atkin-Lehner quotient $\mathfrak{X}_D/\langle\omega_D\rangle$ and thus the quaternionic locus $\mathcal{Q}_D$ in $\mathcal{A}_2$ is the union of pairwise birationally equivalent Shimura curves $\widetilde{\mathfrak{X}}_{\mu_1}$, ...,$\widetilde{\mathfrak{X}}_{\mu_{\rho(\mathcal{O})}}$, meeting at a finite set of Heegner points.

Moreover, for any abelian surface A with quaternionic multiplication by $\mathcal{O}$, it follows from Theorem 5.4 that the closed points $\{[(A,\mathcal{L}_j)]\}_{j=1}^{\pi_0(A)}$ are equidistributed among the $\rho(\mathcal{O})$ irreducible components of $\mathcal{Q}_D$. In addition, Corollary 5.7 ensures that

$$|W/W_0| = 2^{2r-1}|\pi_0(A),$$

as genus theory for binary quadratic forms already predicts.

Finally, we obtain that the number of irreducible components of $\mathcal{Q}_D$ in the non twisting case is

$$\rho(\mathcal{O}) = \frac{\tilde{h}(-D)}{2^{2r}}.$$

On the other hand, let us assume that

$$B \simeq (\frac{-D,m}{\mathbb{Q}})$$

for some $m|D$. In this case, there can be two different birational classes of irreducible components on $\mathcal{Q}_D$. Indeed, the assumption means that there exist pure quaternions $\mu\in\mathcal{O}$, $\mu^2+D=0$, such that $(\mathcal{O},\mu)$ is a *twisting* principally polarized maximal order. Then

$$W_0(\mathcal{O},\mu) = \langle \omega_m, \omega_D \rangle$$

and $\widetilde{\mathfrak{X}}_\mu$ is birationally equivalent to $\mathfrak{X}_D/\langle\omega_m,\omega_D\rangle$. We may refer to $\widetilde{\mathfrak{X}}_\mu$ as a twisting irreducible component of $\mathcal{Q}_D$.

In addition to these, there may exist non twisting polarized orders $(\mathcal{O},\mu)$ such that the corresponding irreducible components $\widetilde{\mathfrak{X}}_\mu$ of $\mathcal{Q}_D$ are birationally equivalent to $\mathfrak{X}_D/\langle\omega_D\rangle$. We may refer to these as the non twisting irreducible components of $\mathcal{Q}_D$.

We then have the following lower and upper bounds for the number of irreducible components of $\mathcal{Q}_D$:

$$\frac{\tilde{h}(-D)}{2^{2r}} < \quad \rho(\mathcal{O}) \quad \leq \frac{\tilde{h}(-D)}{2^{2r-1}}.$$

Summing up, we obtain the following

Theorem 6.1. *Let B be an indefinite division quaternion algebra over $\mathbb{Q}$ of discriminant $D = p_1 \cdot \ldots \cdot p_{2r}$. Then, the quaternionic locus $\mathcal{Q}_D$ in $\mathcal{A}_2$ is irreducible if and only if*

$$\widetilde{h}(-D) = \begin{cases} 2^{2r-1} & \textit{if } B \simeq (\frac{-D,m}{\mathbb{Q}}) \textit{ for some } m|D, \\ 2^{2r} & \textit{otherwise.} \end{cases}$$

Proof. If B is not a twisting quaternion algebra, we already know from the above that the number of irreducible components of $\mathcal{Q}_D$ is $\dfrac{\widetilde{h}(-D)}{2^{2r}}$. Hence, in this case, the quaternionic locus of discriminant D in $\mathcal{A}_2$ is irreducible if and only if $\widetilde{h}(-D) = 2^{2r}$. If on the other hand B is twisting, it follows from the above inequalities that $\mathcal{Q}_D$ is irreducible if and only if $\widetilde{h}(-D) = 2^{2r-1}$. □

In view of Theorem 6.1, there arises a closed relationship between the irreductibility of the quaternionic locus in Igusa's threefold and the genus theory of integral binary quadratic forms and the classical *numeri idonei* studied by Euler, Schinzel and others. We refer the reader to [1] and [13] for the latter.

7. Hashimoto-Murabayashi's families

As the simplest examples to be considered, let B_6 and B_{10} be the rational quaternion algebras of discriminant $D = 2 \cdot 3 = 6$ and $2 \cdot 5 = 10$, respectively. Hashimoto and Murabayashi [4] exhibited two families of principally polarized abelian surfaces with quaternionic multiplication by a maximal order in these quaternion algebras. Namely, let

$$C^{(6)}_{(s,t)} : Y^2 = X(X^4 + PX^3 + QX^2 + RX + 1)$$

be the family of curves with

$$P = 2s + 2t, \quad Q = \frac{(1+2t^2)(11-28t^2+8t^4)}{3(1-t^2)(1-4t^2)}, \quad R = 2s - 2t$$

over the base curve

$$g^{(6)}(t, s) = 4s^2t^2 - s^2 + t^2 + 2 = 0.$$

And let

$$C^{(10)}_{(s,t)} : Y^2 = X(P^2X^4 + P^2(1+R)X^3 + PQX^2 + P(1-R)X + 1)$$

be the family of curves with

$$P = \frac{4(2t+1)(t^2-t-1)}{(t-1)^2}, \quad Q = \frac{(t^2+1)(t^4+8t^3-10t^2-8t+1)}{t(t-1)^2(t+1)^2}$$

and

$$R = \frac{(t-1)s}{t(t+1)(2t+1)}$$

over the base curve

$$g^{(10)}(t,s) = s^2 - t(t-2)(2t+1) = 0.$$

Let $J^{(6)}_{(s,t)} = \mathrm{Jac}(C^{(6)}_{(s,t)})$ and $J^{(10)}_{(s,t)} = \mathrm{Jac}(C^{(10)}_{(s,t)})$ be the Jacobian surfaces of the fibres of the families of curves above respectively. It was proved in [4] that their ring of endomorphisms contain a maximal order in B_6 and B_{10}, respectively.

Both $B_6 = (\frac{-6,2}{\mathbb{Q}})$ and $B_{10} = (\frac{-10,2}{\mathbb{Q}})$ are twisting quaternion algebras. Moreover, it turns out from our formula for $\pi_0(A)$ in the above section that any abelian surface A with quaternionic multiplication by a maximal order in either B_6 or B_{10} admits a *single* isomorphism class of principal polarizations. This implies that $\rho(B_6) = \hat{\pi}_0(A) = \pi_0(A) = 1$ and $\rho(B_{10}) = \hat{\pi}_0(A) = \pi_0(A) = 1$, respectively.

Moreover, the Shimura curves $\mathfrak{X}_6/\mathbb{Q}$ and $\mathfrak{X}_{10}/\mathbb{Q}$ have genus 0, although they are not isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$ because there are no rational points on them. However, it is easily seen that $\mathfrak{X}_6/W_0 = \mathfrak{X}_6/W \simeq \mathbb{P}^1_{\mathbb{Q}}$ and $\mathfrak{X}_{10}/W_0 = \mathfrak{X}_{10}/W \simeq \mathbb{P}^1_{\mathbb{Q}}$, respectively.

As it is observed in [4], the base curves $g^{(6)}$ and $g^{(10)}$ are curves of genus 1 and not of genus 0 as it should be expected. This is explained by the fact that there are obvious isomorphisms between the fibres of the families $C^{(6)}$ and $C^{(10)}$, respectively.

Ibukiyama, Katsura and Oort [5] proved that the supersingular locus in $\mathcal{A}_2/\bar{\mathbb{F}}_p$ is irreducible if and only if $p \leq 11$. As a corollary to their work, Hashimoto and Murabayashi obtained that the reduction mod 3 and 5 of the family of Jacobian surfaces with quaternionic multiplication by B_6 and B_{10} respectively yield the single irreducible component of the supersingular locus in $\mathcal{A}_2/\bar{\mathbb{F}}_3$ and $\mathcal{A}_2/\bar{\mathbb{F}}_5$ respectively. The following statement may be considered as a lift to characteristic 0 of these results.

Theorem 7.1.
(1) *The quaternionic locus $\mathcal{Q}_6$ in $\mathcal{A}_2/\mathbb{Q}$ is absolutely irreducible and birationally equivalent to $\mathbb{P}^1_{\mathbb{Q}}$ over $\mathbb{Q}$. A universal family over $\bar{\mathbb{Q}}$ is given by Hashimoto-Murabayashi's family $C^{(6)}$.*
(2) *The quaternionic locus $\mathcal{Q}_{10}$ in $\mathcal{A}_2/\mathbb{Q}$ is absolutely irreducible and birationally equivalent to $\mathbb{P}^1_{\mathbb{Q}}$ over $\mathbb{Q}$. A universal family over $\bar{\mathbb{Q}}$ is given by Hashimoto-Murabayashi's family $C^{(10)}$.*

Proof. This follows from Theorem 6.1 and the discussion above. □

In particular, we obtain from Theorem 7.1 that every principally polarized abelian surface $(A, \mathcal{L})$ over $\bar{\mathbb{Q}}$ with quaternionic multiplication by a maximal order of discriminant 6 or 10 is isomorphic over $\bar{\mathbb{Q}}$ to the Jacobian variety of one of the curves $C^{(6)}_{(s,t)}$ or $C^{(10)}_{(s,t)}$, except for finitely many degenerate cases.

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