p-adic L-functions and the rationality of Darmon cycles
Sur les fonctions L p-adiques et la rationalité des cycles de Darmon

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Abstract

Darmon cycles are an higher weight analogue of Stark-Heegner points. They yield local cohomology classes in the Deligne representation associated to a cuspidal form on $\Gamma_0(N)$ of even weight $k_0 \geq 2$. They are conjectured to be the restriction of global cohomology classes in the Bloch-Kato Selmer group defined over narrow ring class fields attached to a real quadratic field. We show that suitable linear combinations of them obtained by genus characters satisfy these conjectures. We also prove p-adic Gross-Zagier type formulas, relating the derivatives of p-adic L-functions of the weight variable attached to imaginary (resp. real) quadratic fields to Heegner cycles (resp. Darmon cycles). Finally we express the second derivative of the Mazur-Kitagawa p-adic L-function of the weight variable in terms of a global cycle defined over a quadratic extension of $\mathbb{Q}$.

Résumé

Les cycles de Darmon sont un analogue de poids supérieur des points de Stark-Heegner. Ils produisent des classes locales de cohomologie dans la représentation de Deligne associée à une forme cuspidale sur $\Gamma_0(N)$ de poids pair $k_0 \geq 2$. Ils sont supposés être la restriction des classes globales de cohomologie dans le groupe de Bloch-Kato Selmer defini sur un corps de classe d'anneaux au sens restraint attachés à un corps quadratique reél. On montre que des combinaisons linéaires convenables obtenues par les caractères quadratiques du genre répondent à ces suppositions. On prouve aussi des formules p-adiques du type Gross-Zagier, qui relient la derivée des fonctions L avec variable poids attachées à un corps imaginaire (resp. réel) quadratique aux cycles d'Heegner (resp. de Darmon). On exprime la seconde dérivée de la fonction de Mazur-Kitagawa de variable poids comme un cycle global defini sur une extension quadratique de $\mathbb{Q}$.

Contents

1 Introduction

Let $S_{k_0}(\Gamma_0(N))$ be the space of modular forms on $\Gamma_0(N)$ of even weight $k_0 \geq 2$ and suppose that $N = pM$ is a decomposition into prime factors with p a rational prime not dividing M. Let $K/\mathbb{Q}$ be a real quadratic field such that p is inert and the primes dividing M are split in K. When $k_0 = 2$ the paper [Da] offers a p-adic construction of local points in the Mordel-Weil $A_f(K_p)$, that are conjectured to be global points and to be subject to a reciprocity law analogous to the one provided by the theory of complex multiplication. Here f is a new modular form and $A_f/\mathbb{Q}$ is the abelian variety attached to it by the Eichler-Shimura construction. The theory as been extended in [Das], where the construction has been lifted to the p-new quotient of the Jacobian $J_0(N)$.

The present paper rather focuses on the higher weight case $k_0 > 2$. In the paper [Or] it is offered a p-adic integration theory which is a higher weight counterpart of Darmon's one. Section 2 presents a lift of this p-adic integration theory in almost the same way as the theory developed in [Das] offers a lift of the theory developed in [Da]. Indeed, by means of this p-adic integration theory and then following the construction of [RoSe, Section 4.2], we are able to construct a monodromy module $\mathbf{D} \in MF_{\mathbb{Q}_p}(\phi, N)$, the category of filtered Frobenius modules over $\mathbb{Q}_p$, that should be thought of as being a realization in the category of filtered Frobenius modules of the p-new part of the motive of weight k_0 modular forms, as we are going to explain. In [RoSe] a different cohomological approach, allow us to develop a p-integration theory which covers the compact case of a more general Shimura curve: the p-adic integration theory developed in [RoSe], when specialized to a modular curve, is shown to be equivalent to the one presented here, and the monodromy modules to be isomorphic (see [RoSe, Section 6]). The existence of this "modular symbol theoretic" p-adic integration theory is essentially encoded in Proposition 2.8, which borrows from the techniques in [RoSe]; but it turns out that it can not be deduced from the results of [RoSe], and one has a priori two independent theories.

Let us a fix a complete field extension $F_p/\mathbb{Q}_p$. Suppose that there exists a prime $q \parallel N$ different from p. We can consider a factorization $N = pN^+N^-$, where N^- is an odd number of primes. By the Jacquet-Langlands correspondence, the Eichler Shimura relations and the Brauer-Nesbitt principle (see for example [IS, Lemma 5.9]) the Deligne representation $V_{[f]}$ attached to the modular form f can be realized inside the p-adic étale cohomology of the Shimura curve $X = X_{N^+,pN^-}$ attached to the indefinite quaternion algebra $\mathcal{B}$ of discriminant pN^- and an Eichler order of level N^+ in $\mathcal{B}$. Set $n := k_0 - 2$ and $n = n/2$. More generally, in [IS, Lemma 5.9], it is explained how to construct a Chow motive $\mathcal{M}_n$ over $\mathbb{Q}$ whose p-adic realization $V(m+1) := H_p\left(\mathcal{M}_{n,\overline{\mathbb{Q}}}, \mathbb{Q}_p(m+1)\right)$ affords representations for all modular forms on $\Gamma_0(N)$ that are new at the primes dividing pN^-. One has a p-adic étale Abel-Jacobi map:

$$cl_{0,F_p}^{m+1} : CH^{m+1}\left(\mathcal{M}_{n,F_p}\right) \to Ext^1_{G_{F_p}}\left(\mathbb{Q}_p, V(m+1)\right),$$

where CH^{m+1} is the Chow group of codimension $m+1$ cycles, $\mathcal{M}_{n,F_p}$ denotes the base change to F_p and we write again V to denote the restriction of the global representation V to the local group G_{F_p}. Let $\mathbb{D} := \mathbb{D}_{st}(V)$ (resp. $\mathbb{D}_{[f]} := \mathbb{D}_{st}\left(V_{[f]}\right)$) be the associated associated filtered Frobenius module. $\mathbb{D}_{[f]} := \mathbb{D}_{st}\left(V_{[f]}\right)$ is indeed a $K_f \otimes \mathbb{Q}_p$-monodromy module (see [IS, Section 7]). The above ext group is explicitly computed in [IS, (49)]:

$$\begin{array}{rlclcl} IS: & Ext^1_{G_{F_p}}\left(\mathbb{Q}_p, V(m+1)\right) & = & Ext^1_{MF}\left(F, \mathbb{D}(m+1)\right) & = & \\ & \mathbb{D}_{F_p}/F^{m+1} & = & M_k\left(X, F_p\right)^\vee & = & M_k\left(\Gamma', F_p\right)^\vee. \end{array}$$

Here $(-)^\vee$ denotes the F_p-dual space, $M_k(X, F_p)$ is the space of weight k-modular forms on X, while $M_k(\Gamma', F_p)$ denotes the space of weight k modular forms on the Mumford curve $\Gamma' \backslash \mathcal{H}_p$, defined over F_p, and the last equality holds assuming $F_p \supset \mathbb{Q}_{p^2}$. Γ' is the arithmetic group defined in subsection 5.3.2: it is associated to the Eichler order of level N^+ in $\mathcal{B}$ and it is a subgroup of the norm one elements in the definite quaternion algebra ramified at the primes ∞N^-. Indeed the last one of the above identifications comes from the identification $X^{an} = \Gamma' \backslash \mathcal{H}_p$ over $\mathbb{Q}_{p^2}$ provided by the Cerednik-Drinfeld Theorem. In this way the p-adic étale Abel-Jacobi can be interpreted as

$$\log \Phi^{AJ} : CH^{m+1}\left(\mathcal{M}_{n,F_p}\right) \rightarrow M_k\left(\Gamma', F_p\right)^{\vee}.$$

We can consider the projection onto the f-isotypic component, thus getting a p-adic Abel-Jacobi map with values in $e_{[f]} M_k\left(\Gamma', F_p\right)^{\vee}$. Here $e_{[f]}$ is the idempotent in the Hecke algebra corresponding to the modular form f.

Let R be the Eichler $\mathbb{Z}[1/p]$-order consisting of matrices in $\mathbb{M}_2(\mathbb{Z}[1/p])$ that are upper triangular mod M, set $\widetilde{\Gamma} := R^{\times}$ and denote by $\Gamma \subset \widetilde{\Gamma}$ the subgroup of matrices with determinant 1. Write $\mathbf{D}_{F_p}$ to denote the filtered F_p-vector space attached to the base change of $\mathbf{D}$ to F_p. Firstly our integration theory is a morphism

$$\Phi : \left(\Delta^0 \otimes \mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_{\Gamma} \rightarrow \mathbf{D}_{F_p}/F^{m+1}.$$

Here $\Delta^0 := \mathrm{Div}^0\, \mathbb{P}^1(\mathbb{Q})$, $F_p^0 := F_p \cap \mathbb{Q}_p^{ur}$, $\mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right)$ denotes the degree zero divisor supported on $\mathbb{Q}_p^{ur} - \mathbb{Q}_p$ that are fixed by the action of the Galois group $G_{\mathbb{Q}_p^{ur}/F_p^0}$, $\mathbf{P}_n$ is the space of polynomials of degree $\leq n = k_0 - 2$ with coefficients in F_p and F^{m+1} is the $m+1$-step in the filtration of our monodromy module. In order to be able to construct the right analogue of the notion of Stark-Heegner points, following the ideas of [Da], we rather lift the above morphism to

$$\Phi^{AJ} : \left(\Delta^0 \otimes \mathrm{Div}\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_{\Gamma} \rightarrow \mathbf{D}_{F_p}/F^{m+1}.$$

The left hand side should be regarded as being a substitute of the local Chow group. Indeed the Darmon cycles are defined as being suitable elements

$$j_{\Psi} \in \left(\Delta^0 \otimes \mathrm{Div}\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_{\Gamma}$$

attached to the optimal embeddings $\Psi : \mathcal{O} \hookrightarrow R$, where $\mathcal{O}$ is an order of K of conductor prime to ND_K, D_K being the discriminant of $K/\mathbb{Q}$. One of the main differences with the weight 2 setting and with the cohomological approach followed in [RoSe, Section 6] is the lack of uniqueness of the lifting Φ^{AJ}. In any case one can show that the values $\Phi^{AJ}(j_{\Psi})$ are well defined quantities, i.e. they do not depend on the choice of the p-adic Abel-Jacobi map Φ^{AJ}. Furthermore the p-adic Abel-Jacobi images $\Phi^{AJ}(j_{\Psi})$ agree with the p-adic Abel-Jacobi images of the Darmon cycles considered in [RoSe, see Section 6].

Suppose that $f \in S_{k_0}(\Gamma_0(N))$ is a normalized newform and denote by K_f the field generated by the Fourier coefficients of f. Attached to the modular form f there is a $K_f \otimes \mathbb{Q}_p$-monodromy module $\mathbf{D}_{[f]}$, that appears like a quotient of $\mathbf{D}$ in the category $MF_{\mathbb{Q}_p}(\phi, N)$ of filtered Frobenius modules over $\mathbb{Q}_p$; we can consider the p-adic Abel-Jacobi map $\Phi_{[f]}^{AJ}$ obtained by Φ^{AJ} followed by this projection, taking values in $\mathbf{D}_{[f],F_p}/F^{m+1}$. The construction of these monodromy modules, that follows [RoSe, Section 4.2], is reviewed in subsection 2.3: they are built from a space $\mathbf{MS}^{c,w_\infty}$, which is obtained from the cuspidal part of the space of modular symbols with values in the F_p-dual of $\mathbf{P}_n$ and depends on the choice w_∞ of a sign at infinity. In section 3 we show how to realize our p-adic Abel-Jacobi map as taking values in $\mathbf{MS}^{c,w_\infty,\vee}$:

$$\log\Phi^{AJ}:\left(\Delta^0\otimes\mathrm{Div}\left(\mathcal{H}_p^{ur}\right)\otimes\mathbf{P}_n\right)_\Gamma\rightarrow\mathbf{D}_{F_p}/F^{m+1}\overset{\simeq}{\rightarrow}\mathbf{MS}^{c,w_\infty,\vee}.$$

Our p-adic integration theory can be used to produce local cohomology classes in $Ext^1_{G_{F_p}}\left(\mathbb{Q}_p,V_{[f]}\right)$ as follows. Thanks to a combination of the work of Bertolini, Darmon and Iovita with a result of Colmez (see Theorem 4.11) there is an isomorphism $\varphi:\mathbf{D}_{[f]}\simeq\mathbb{D}_{[f]}$. Let $F=H/K$ be the narrow ring class field attached to the order $\mathcal{O}$ and choose a local embedding $H\hookrightarrow F_p$ (assuming $F_p\supset\mathbb{Q}_{p^2}$). Then we find an identification of the tangent spaces

$$\varphi:\mathbf{D}_{[f],F_p}/F^{m+1}\simeq\mathbb{D}_{[f],F_p}/F^{m+1}\overset{\simeq}{\rightarrow}Ext^1_{G_{F_p}}\left(\mathbb{Q}_p,V_{[f]}\right)$$

where the last identification, provided by the Bloch-Kato exponential map, is indeed an isomorphism in our setting. Let $\chi:G_{H/K}\rightarrow\mathbb{C}^\times$ be a character and set

$$j^\chi:=\textstyle\sum_{\sigma\in G_{H^+_{\mathcal{O}}/K}}\chi^{-1}(\sigma)\,j_{\sigma\Psi}\in\left(\Delta^0\otimes\mathrm{Div}\left(\mathcal{H}_p^{ur}\right)\otimes\mathbf{P}_n\right)_\Gamma\otimes\chi.$$

Here $(-)\otimes\chi$ denotes a suitable scalar extension. For every global field F set $MW_f(F):=\mathrm{Im}\left(e_{[f]}\circ cl^{m+1}_{0,F}\right)$. Let H_χ/K be the extension out by the character χ. Denote by $MW(H_\chi)^\chi$ the χ-part of $MW(H_\chi)$. As in [RoSe, Section 5] one can formulate rationality conjectures asserting that:

$$\varphi\left(\Phi^{AJ}\left(j^\chi\right)\right)\in\mathrm{res}_p\left(MW\left(H_\chi\right)^\chi\right).$$

As it follows from the discussion [RoSe, Section 6], our local cohomology classes are the same as the ones defined there in the more general setting of a Shimura curve, when the theory is specialized to a modular curve. One of the main motivations of this paper is indeed to provide instances where the conjectures formulated there, or rather some of their consequences, can be proved.

Fix once and for all an identification $\mathbb{C}\simeq\mathbb{C}_p$. Denote by $K_{[f]}$ the field generated by the Fourier coefficients of f and all its companion cusp forms. Assuming $F_p\supset K_{[f]}$ the tangent space $\mathbf{D}_{[f],F_p}/F^{m+1}=\mathbf{MS}^{c,w_\infty,\vee}$ (resp. $\mathbb{D}_{[f],F_p}/F^{m+1}=e_{[f]}M_k(\Gamma,F_p)^\vee$) splits into $\sigma(f)$-components corresponding to the companion forms $\sigma(f)$ of f. Write Φ^{AJ}_f (resp. $\log\Phi^{AJ}_f$) to denote the f-component of the above p-adic Abel-Jacobi maps, so that $\Phi^{AJ}_{[f]}=\bigoplus_\sigma\Phi^{AJ}_{\sigma(f)}$ (resp. $\log\Phi^{AJ}_{[f]}=\bigoplus_\sigma\log\Phi^{AJ}_{\sigma(f)}$). Attached to the modular form f there is a modular symbol $I_f\in\mathbf{MS}^{c,w_\infty}$ (resp. a rigid analitic modular form f^{rig}) generating the f-component of $\mathbf{MS}^{c,w_\infty}$ (resp. $M_k(\Gamma,F_p)$).

Let $\chi:G_{H/K}\rightarrow\mathbb{C}^\times$ be a genus character, attached to the pair (χ_1,χ_2) of Dirichlet character. Note that the values $\chi_i(-M)$ do not depend on $i=1,2$.

The identification $\mathbb{C} \simeq \mathbb{C}_p$ determines a prime $\mathfrak{p}$ of K_f above p and we can decompose $V_{[f]}$, $\mathbf{D}_{[f]}$, $\mathbb{D}_{[f]}$ and $MW_f(H_\chi)^\chi$ according to the decomposition $K_f \otimes \mathbb{Q}_p = \bigoplus_{\mathfrak{p}'|p} K_{f,\mathfrak{p}'}$, where $K_{f,\mathfrak{p}'}$ denotes the $\mathfrak{p}'$-adic completion of K_f at $\mathfrak{p}'$. We will write $V_{[f],\mathfrak{p}}$, $\mathbf{D}_{[f],\mathfrak{p}}$, $\mathbb{D}_{[f],\mathfrak{p}}$ and $MW_{f,\mathfrak{p}}(H_\chi)^\chi$ to denote the $\mathfrak{p}$-component, so that $MW_{f,\mathfrak{p}}(H_\chi)^\chi$ is naturally a $K_{f,\mathfrak{p}}$-vector space and the f-component of $\mathbf{D}_{[f],F_p}/F^{m+1}$ (resp. $\mathbb{D}_{[f],F_p}/F^{m+1}$) appears in $\mathbf{D}_{[f],\mathfrak{p},F_p}/F^{m+1}$ (resp. $\mathbb{D}_{[f],\mathfrak{p},F_p}/F^{m+1}$).

One of the main results is the following Theorem, that is implied by the conjectures formulated in [RoSe, Section 5]:

Theorem 1.1 *Suppose $N = pM$, that there exists a prime $q \parallel M$ and that*

$$\chi_i(-M) = (-1)^{\frac{k_0}{2}} w_M$$

Then:

1. *there is $y_f^\chi \in MW_f(H_\chi)^\chi$ such that*

$$\varphi\left(\Phi_{[f]}^{AJ}(j^\chi)\right) = \mathrm{res}_p\left(y_f^\chi\right);$$

2. *if $y_{f,\mathfrak{p}}^\chi \neq 0$ we have*

$$MW_{f,\mathfrak{p}}(H_\chi)^\chi = K_{f,\mathfrak{p}} y_{f,\mathfrak{p}}^\chi.$$

The proof of the above Theorem follows the strategy developed in [BD2] and [BD3] in the weight 2 setting. Indeed we also obtain p-adic Gross-Zagier formulas that are of independent interest and an higher weight analogue of the main results of [BD2] and [BD3].

Let $\mathcal{W} := Hom_{cont}\left(\mathbb{Z}_p^\times, \mathbb{G}_m\right)$ be the weight space, viewed as a rigid analytic space over $\mathbb{Q}_p$. The integers $\mathbb{Z}$ are embedded in $\mathcal{W}$ by sending the integer k to the function $t \mapsto t^{k-2}$; let $U \subset \mathcal{W}$ be a small enough open affinoid disk centered at k_0. We will define p-adic L-functions

$$L_p(f/K, \chi, -) : U \to \mathbb{C}_p,$$
$$L_p(f/K', \chi, -) : U \to \mathbb{C}_p.$$

of the weight variable attached to the real quadratic field K or an imaginary quadratic field K' such that we can write $N = pN^+N^-$, where the primes dividing N^+ are split in K and the primes dividing pN^- are inert, squarefree and in even number.

When $K/\mathbb{Q}$ is a real quadratic field satisfying the above assumptions we obtain the following formula, relating the second derivative of the above p-adic L-function to the p-adic Abel-Jacobi image of the Darmon cycles.

Theorem 1.2 *Let $\chi : G_{H^+/K} \to \mathbb{C}^\times$ be a genus character (here H^+ is the narrow Hilbert ring class field). Then:*

$$\frac{d^2}{d\kappa^2}\left[L_p(f/K, \chi, \kappa)\right]_{\kappa=k_0} = \begin{cases} 2D_K^{\frac{k_0-2}{2}} \log \Phi_f^{AJ}(j^\chi)(I_f)^2 & \text{if} \quad \chi_i(-M) = (-1)^{\frac{k_0}{2}} w_M \\ 0 & \text{if} \quad \chi_i(-M) = (-1)^{\frac{k_0-2}{2}} w_M. \end{cases}$$

Let now $K'/\mathbb{Q}$ be an imaginary quadratic field and consider a factorization $N = pN^+N^-$ as above. We now focus on a genus character χ of the imaginary quadratic field K'. Denote by H_χ/K' the extension cut out by the character χ and by $y^\chi \in CH^{m+1}\left(\mathcal{M}_{n,H'_\chi}\right)^\chi$ the corresponding Heegner cycle. There is a decomposition

$$MW_f\left(H_\chi\right)^\chi = MW_f\left(\mathbb{Q}_{\chi_1}\right)^{\chi_1} \oplus MW_f\left(\mathbb{Q}_{\chi_2}\right)^{\chi_2},$$

where $\mathbb{Q}_{\chi_i}/\mathbb{Q}$ denotes the quadratic extension cut out by the Dirichlet character χ_i. Furthermore $cl_{0,f}^{m+1}\left(y^\chi\right)$ belongs precisely to one between $MW_f\left(\mathbb{Q}_{\chi_1}\right)^{\chi_1}$ and $MW_f\left(\mathbb{Q}_{\chi_2}\right)^{\chi_2}$.

We obtain the following formula, this time relating the second derivative of the above p-adic L-function to the p-adic Abel-Jacobi image of an Heegner cycle.

Theorem 1.3 *Let $\chi : G_{H/K} \to \mathbb{C}^\times$ be a genus character (here H is the Hilbert ring class field). If $cl_{0,f}^{m+1}\left(y^\chi\right) \in MW_f\left(\mathbb{Q}_{\chi_i}\right)^{\chi_i}$ we have*

$$\frac{d^2}{d\kappa^2}\left[L_p\left(f/K,\chi,\kappa\right)\right]_{\kappa=k_0} = \begin{cases} 2\log\Phi_f^{AJ}\left(y^\chi\right)\left(f^{rig}\right)^2 & \text{if} \quad \chi_i\left(p\right) = a_p p^{-\frac{k_0-2}{2}} = -w_p \\ 0 & \text{if} \quad \chi_i\left(p\right) = -a_p p^{-\frac{k_0-2}{2}} = w_p. \end{cases}$$

Hence the second derivative of $L_p\left(f/K,\chi,\kappa\right)$ at k_0 encodes information about the restriction at p of $cl_{0,f}^{m+1}\left(y^\chi\right)$: when $\chi_i\left(p\right) = -w_p$ it is zero precisely when (the f-component of) the restriction of $cl_{0,f}^{m+1}\left(y^\chi\right)$ at p is zero. Information on the exact position of $cl_{0,f}^{m+1}\left(y^\chi\right)$, i.e. which one is the character χ_i in the above statement, are given in Lemma 5.28.

We will also consider the restriction of the Mazur-Kitagawa p-adic L-function $L_p\left(f,\omega,\kappa,s\right)$ to the critical line $L_p\left(f,\omega,\kappa,\kappa/2\right)$.

Theorem 1.4 *Suppose that there exists $q \parallel M$ and let ω be a quadratic Dirichlet character such that*

$$\omega\left(-N\right) = \left(-1\right)^{\frac{k_0-2}{2}} w_N \text{ and } \omega\left(p\right) = a_p p^{-\frac{k_0-2}{2}} = -w_p.$$

Then:

1. *the p-adic L-function $L_p\left(f/K,\omega,\kappa,\kappa/2\right)$ vanishes to order*

$$\mathrm{ord}_{\kappa=k_0} L_p\left(f,\omega,\kappa,\kappa/2\right) \geq 2;$$

2. *there exists $y^\omega \in CH^{m+1}\left(\mathcal{M}_{n,\mathbb{Q}_\omega}\right)^\omega$ and $t \in K_f^\times$ such that*

$$\frac{d^2}{d\kappa^2}\left[L_p\left(f/K,\omega,\kappa,\kappa/2\right)\right]_{\kappa=k_0} = t\cdot\log\Phi_f^{AJ}\left(y^\omega\right)\left(f^{rig}\right)^2;$$

3. *If $cl_{0,f}^{m+1}\left(y_\mathfrak{p}^\omega\right) \neq 0$ then $MW_{f,\mathfrak{p}}\left(\mathbb{Q}_\omega\right)^\omega = K_{f,\mathfrak{p}} cl_{0,f}^{m+1}\left(y_\mathfrak{p}^\omega\right)$.*

Again $\mathbb{Q}_\omega$ is the extension cut out by the character ω, while $(-)^\omega$ denotes the ω-component. Hence again the second derivative of the Mazur-Kitagawa p-adic L-function $L_p(f,\omega,\kappa,\kappa/2)$ at k_0 encodes information on (the f-component of) the restriction at p of $cl_{0,f}^{m+1}(y^\omega)$, whose $\mathfrak{p}$-component generates $MW_{f,\mathfrak{p}}(\mathbb{Q}_\omega)^\omega$ when non-zero. In particular

$$\frac{d^2}{d\kappa^2}\left[L_p(f/K,\omega,\kappa,\kappa/2)\right]_{\kappa=k_0}\neq 0 \;\Rightarrow\; MW_{f,\mathfrak{p}}(\mathbb{Q}_\omega)^\omega = K_{f,\mathfrak{p}}cl_{0,f}^{m+1}\left(y_{\mathfrak{p}}^\omega\right).$$

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2 p-adic integration theory, $\mathcal{L}$-invariants and the monodromy module of weight k_0 modular forms

Let S_k be the space of modular forms of even weight $k>2$, endowed with the $\mathbb{GL}_2^+(\mathbb{Q})$-action

$$f\mid\gamma := \det\gamma^{k-1}(cz+d)^{-k}f(\gamma z).$$

For every integer N denote as usual by $S_k(\Gamma_0(N)) := S_k^{\Gamma_0(N)}$ the $\Gamma_0(N)$-invariants, i.e. the space of weight k modular forms on $\Gamma_0(N)$. Let $\mathbb{T}_N$ be the Hecke $\mathbb{Q}$-algebra generated by the operators T_l for $l\nmid N$ and U_l for $l\mid N$ acting on $S_k(\Gamma_0(N))$. Then $\dim_{\mathbb{Q}}\mathbb{T}_N=\dim_{\mathbb{C}}S_k(\Gamma_0(N))$ (see [Sh, Theorem 3.51]). The number field generated by the Fourier coefficients of a normalized modular form f is denoted by K_f. The spaces $S_k(\Gamma_0(N))$ are endowed with the Petersson inner product $\langle -,-\rangle_k$.

Let $\mathbf{P}_{k-2}$ be the space of polynomials of degree $\leq k-2$, endowed with the following right $\mathbb{GL}_2$-action:

$$P(X)\mathbf{M} := (cX+d)^{k-2}P\left(\frac{aX+b}{cX+d}\right) \text{ for } P\in\mathbf{P}_{k-2}(K_p), \tag{1}$$

where $\mathbf{M}=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathbb{GL}_2$.

Usually we do not specify any field in the notation and we write $\mathbf{P}_{k-2}=\mathbf{P}_{k-2}(F)$ when such a choice has been made. The dual space $\mathbf{V}_{k-2}(F) := Hom_F(\mathbf{P}_{k-2}(F),F)$ is then endowed with a natural $\mathbb{GL}_2$-left action by the rule

$$(\mathbf{M}\Lambda)(P) := \Lambda(P\mathbf{M}).$$

The same notation $\mathbf{V}_{k-2}$ will be in force to mean that the choice of a field has been made. Indeed, whenever $\mathbf{V}$ and $\mathbf{W}$ are vector spaces over some field F, we set $\mathbf{V}^\vee := Hom_F(\mathbf{V},F)$ and $\mathbf{V}\otimes\mathbf{W}$ without reference to the field.

We recall that $\mathbf{P}_{k-2}$ and $\mathbf{V}_{k-2}$ carry a non-degenerate $\mathbb{GL}_2$-invariant bilinear form (see for example [BDIS, Sec. 1.2] or [IS, (33)]):

$$\begin{aligned} \langle -,-\rangle_{\mathbf{P}_{k-2}} &: \mathbf{P}_{k-2}\otimes\mathbf{P}_{k-2}\rightarrow F, \\ \langle -,-\rangle_{\mathbf{V}_{k-2}} &: \mathbf{V}_{k-2}\otimes\mathbf{V}_{k-2}\rightarrow F. \end{aligned}$$

Let $\Delta := \operatorname{Div}\mathbb{P}^1(\mathbb{Q})$ and $\Delta^0 := \operatorname{Div}^0\mathbb{P}^1(\mathbb{Q})$ be respectively the space of divisors and degree zero divisors supported at the cusps with coefficients in some field F, endowed with their natural action by fractional transformation by $\mathbb{GL}_2(\mathbb{Q})$. For any space $\mathbf{V}$ endowed with an action by $G\subset\mathbb{GL}_2(\mathbb{Q})$ (a congruence group of $\mathbb{SL}_2(\mathbb{Z})$ in the applications) set $\mathcal{BS}(\mathbf{V}) := Hom(\Delta,\mathbf{V})$ and $\mathcal{MS}(\mathbf{V}) := Hom\left(\Delta^0,\mathbf{V}\right)$, equipped with the natural induced actions. There is a canonical exact sequence

$$0\rightarrow\mathbf{V}\rightarrow\mathcal{BS}(\mathbf{V})\rightarrow\mathcal{MS}(\mathbf{V})\rightarrow 0. \tag{2}$$

We also write $\mathcal{BS}_G(\mathbf{V}) := \mathcal{BS}(\mathbf{V})^G$ and $\mathcal{MS}_G(\mathbf{V}) := \mathcal{MS}(\mathbf{V})^G$ to denote the G-invariants. Finally when $\mathbf{V}=\mathbf{V}_{k-2}=\mathbf{V}_{k-2}(F)$ we will occasionally write $\mathcal{MS}^k=\mathcal{MS}^k(F)$ (and $\mathcal{MS}_G^k=\mathcal{MS}_G^k(F)$ for the invariants).

Recall the Bruhat-Tits tree $\mathcal{T}$ at p, whose vertices $\mathcal{V}=\mathcal{V}(\mathcal{T})$ are the homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$. Let $L_* := \mathbb{Z}_p^2$ be the standard $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$ and set $L_\infty := \mathbb{Z}_p\oplus p\mathbb{Z}_p$. Write $\mathcal{E}=\mathcal{E}(\mathcal{T})$ to denote the set of ordered edges and choose the following orientation $\mathcal{E}=\mathcal{E}^+\sqcup\mathcal{E}^-$: write $\mathcal{V}^+$ (resp. $\mathcal{V}^-$) to denote the set of those vertices v such that $d(v,v_*)$ is even (resp. odd), where $v_* := [L_*]$; define $\mathcal{E}^+$ (resp. $\mathcal{E}^-$) to be the set of those edges e such that $s(e)\in\mathcal{V}^+$ (resp. $s(e)\in\mathcal{V}^-$).

We denote by $\mathcal{C}_0(\mathcal{E},\mathbf{V})$ (resp. $\mathcal{C}(\mathcal{V},\mathbf{V})$) the space of maps $c:\mathcal{E}\rightarrow\mathbf{V}$ such that $c(\overline{e})=-c(e)$ (resp. $\mathcal{C}(\mathcal{V},\mathbf{V})$ the set of all maps $c:\mathcal{V}\rightarrow\mathbf{V}$). The set of harmonic cocycles $\mathcal{C}_{har}(\mathcal{E},\mathbf{V})$ is defined by the following exact sequence (see [Gr, Lemma 24] for the right exactness):

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathcal{C}_{har}(\mathcal{E},\mathbf{V}) & \rightarrow & \mathcal{C}_0(\mathcal{E},\mathbf{V}) & \overset{\varphi_s}{\rightarrow} & \mathcal{C}(\mathcal{V},\mathbf{V}) & \rightarrow & 0 \\ & & & & \varphi_s(c)(v) & := & \sum_{s(e)=v} c(e). & & \end{array} \tag{3}$$

It will be also useful to consider the following exact sequence:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathbf{V} & \rightarrow & \mathcal{C}(\mathcal{V},\mathbf{V}) & \overset{\partial^*}{\rightarrow} & \mathcal{C}_0(\mathcal{E},\mathbf{V}) & \rightarrow & 0 \\ & & & & (\partial^* c)(e) & := & c(s(e))-c(t(e)). & & \end{array} \tag{4}$$

Let $F_p/\mathbb{Q}_p$ be any complete local field. $\mathcal{A}_{k-2}\left(\mathbb{P}^1(\mathbb{Q}_p)\right) := \mathcal{A}_{k-2}\left(\mathbb{P}^1(\mathbb{Q}_p),F_p\right)$ denotes the space of F_p-valued locally analytic functions on $\mathbb{Q}_p$ with a pole of order at most $k-2$ at ∞. The same formula (1) endows it with a $\mathbb{GL}_2(\mathbb{Q}_p)$-module structure. This space sits in the following exact sequence

$$0\rightarrow\mathbf{P}_{k-2}\rightarrow\mathcal{A}_{k-2}\left(\mathbb{P}^1(\mathbb{Q}_p)\right)\rightarrow\mathcal{A}_{k-2}\left(\mathbb{P}^1(\mathbb{Q}_p)\right)/\mathbf{P}_{k-2}\rightarrow 0.$$

Define the spaces $\mathcal{D}^0_{k-2}\left(\mathbb{P}^1\left(\mathbb{Q}_p\right)\right)$, $\mathcal{D}_{k-2}\left(\mathbb{P}^1\left(\mathbb{Q}_p\right)\right)$ and $\mathbf{V}_{k-2}$ by taking the (continuous) F_p-dual exact sequence:

$$0 \to \mathcal{D}^0_{k-2}\left(\mathbb{P}^1\left(\mathbb{Q}_p\right)\right) \to \mathcal{D}_{k-2}\left(\mathbb{P}^1\left(\mathbb{Q}_p\right)\right) \to \mathbf{V}_{k-2} \to 0.$$

It will also be convenient to consider the subspace

$$\mathcal{D}^{0,b}_{k-2}\left(\mathbb{P}^1\left(\mathbb{Q}_p\right)\right) \subset \mathcal{D}_{k-2}\left(\mathbb{P}^1\left(\mathbb{Q}_p\right)\right)$$

of bounded distributions, as defined for example in [RoSe].

We recall that there is a standard basis for the topology on $\mathbb{P}^1\left(\mathbb{Q}_p\right)$ obtained from the open compact subsets $U_e \subset \mathbb{P}^1\left(\mathbb{Q}_p\right)$ corresponding to the ends of $\mathcal{T}$ originating from e.

Note that with the only possible exception of S_k, the above spaces are endowed with an action by the full group $\mathbb{GL}_2\left(\mathbb{Q}\right)$. Hence the matrix $W_\infty = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ acts on these spaces; furthermore, since it normalizes the groups of the form $\Gamma_0\left(N\right)$, the cohomology groups $H^i\left(\Gamma_0\left(N\right), -\right)$ are endowed with a natural W_∞-action. Suppose that $\mathbf{V}$ is a characteristic $\neq 2$ vector space endowed with a W_∞-action (the characteristic will be 0 in our applications): we denote by $\mathbf{V}^{w_\infty}$ the direct summand of $\mathbf{V} = \mathbf{V}^+ \oplus \mathbf{V}^-$ on which $W_\infty = w_\infty \in \{\pm 1\}$.

We recall that there is a $\mathbb{GL}_2^+\left(\mathbb{Q}\right)$-equivariant map

$$\begin{aligned} &\widetilde{I}_- : S_k \otimes_{\mathbb{R}} \mathbb{C} \to \mathcal{MS}^k\left(\mathbb{C}\right) \\ &\widetilde{I}_f\left\{x - y\right\}\left(P\right) := 2\pi i \int_x^y f\left(z\right) P\left(z, 1\right) dz \in \mathbb{C}. \end{aligned} \tag{5}$$

The composition of this morphism with the boundary map δ arising from the exact sequence (2) by taking the $\Gamma_0\left(N\right)$-invariants identifies $S_k\left(\Gamma_0\left(N\right)\right) \otimes_{\mathbb{R}} \mathbb{C}$ with the image of $H^1_c\left(\Gamma_0\left(N\right), \mathbf{V}_{k-2}\left(\mathbb{C}\right)\right)$ in $H^1\left(\Gamma_0\left(N\right), \mathbf{V}_{k-2}\left(\mathbb{C}\right)\right)$, usually called the parabolic cohomology subgroup $H^1_{par}\left(\Gamma_0\left(N\right), \mathbf{V}_{k-2}\left(\mathbb{C}\right)\right)$. The identification

$$\delta \circ \widetilde{I}_- : S_k\left(\Gamma_0\left(N\right)\right) \otimes_{\mathbb{R}} \mathbb{C} \overset{\simeq}{\to} H^1_{par}\left(\Gamma_0\left(N\right), \mathbf{V}_{k-2}\left(\mathbb{C}\right)\right) \tag{6}$$

is called the Eichler-Shimura isomorphism. Since $\mathbf{V}_{k-2}$ is an irreducible $\Gamma_0\left(N\right)$-module (in light of the assumption $k > 2$, see for example [Hi, 6.1 Lemma 2]), the following sequence is exact by definition of the parabolic cohomology, and Hecke equivariant:

$$0 \to \mathcal{BS}^k_{\Gamma_0(N)}\left(\mathbb{C}\right) \to \mathcal{MS}^k_{\Gamma_0(N)}\left(\mathbb{C}\right) \to H^1_{par}\left(\Gamma_0\left(N\right), \mathbf{V}_{k-2}\left(\mathbb{C}\right)\right) \to 0. \tag{7}$$

More generally we define $H^1_{par}\left(G, \mathbf{V}\right) := \delta\left(\mathcal{MS}_G\left(\mathbf{V}\right)\right)$.

We recall the following Theorem of Shimura.

Proposition 2.1 *There exist complex periods $\Omega_f^{\pm} \in \mathbb{C}$ such that*

$$I_f^{\pm} := \left(\Omega_f^{\pm}\right)^{-1} \widetilde{I}_f^{\pm} \in \mathcal{MS}_{\Gamma_0(M)}^{k,\pm}\left(K_f\right)$$

The periods $\Omega_f^{\pm}$ can be chosen such that

$$\Omega_f^{+}\Omega_f^{-} = \langle f, f\rangle_k \,.$$

Once we make the choice of a sign $w_\infty \in \{\pm 1\}$ we set

$$\Omega_f := \Omega_f^{w_\infty} \text{ and } I_f := I_f^{w_\infty}.$$

As in the introduction we let $k_0 > 2$ be a fixed even weight and set for shortness $n := k_0 - 2$, $m := n/2 = (k_0 - 2)/2$.

2.1 Decomposition into Eisenstein and cuspidal parts

Whenever M is a $\mathbb{T}_N$-module we say that it admits an Eisenstein/cuspidal decomposition if there exists a Hecke operator T_l for some $l \nmid N$ such that:

(a) we can write $M = M^e \oplus M^c$;

(b) the operator $t_l := T_l - l^{k-1} - 1$ is zero on M^e and is invertible on M^c.

The following Lemmas are easily established.

Lemma 2.2 *Whenever $M = M^e \oplus M^c$ admits an Eisenstein/cuspidal decomposition, $M^* \subset M$ with $* = e, c$ is a $\mathbb{T}_N$-submodule and furthermore the decomposition is unique.*

Let M_1 (resp. M_2) be a $\mathbb{T}_N$-module (resp. $\mathbb{T}_M$-module). If $f : M_1 \to M_2$ is a Hecke equivariant morphism (i.e. a morphism such that $T_l f = f T_l$ for every $l \nmid MN$) and there exists T_l with $l \nmid MN$ such that the properties (a) and (b) are satisfied by M_1 and M_2,

$$\begin{aligned} &f = f^e \oplus f^c : M_1^e \oplus M_1^c \to M_2^e \oplus M_2^c \\ &\text{with } f^* : M_1^* \to M_2^* \text{ for } * = e, c. \end{aligned}$$

In particular

$$\ker(f) = \ker(f^e) \oplus \ker(f^c) \ \text{ and } \ \operatorname{coker}(f) = \operatorname{coker}(f^e) \oplus \operatorname{coker}(f^c)$$

admit an Eisenstein/cuspidal decomposition.

Lemma 2.3 *Suppose that we are given an exact sequence*

$$0 \to E \to M \to C \to 0$$

of Hecke modules such that $t_l = 0$ on E and is invertible on C. Then there exists a unique Hecke equivariant section $C \hookrightarrow M$, $M = M^e \oplus M^c$ admits an Eisenstein/cuspidal decomposition, $M^e = E$ and $M^c = C$.

We are now going to describe the Eisenstein/cuspidal decompositions of some spaces that will be of interest to us. Recall the groups $\Gamma_0(M)$, $\Gamma_0(pM)$ and Γ from the Introduction.

Eisenstein/cuspidal decomposition of $\mathcal{MS}_{\Gamma_0(M)}(\mathbf{V}_n)$, $\mathcal{MS}_{\Gamma_0(pM)}(\mathbf{V}_n)$, and $\mathcal{MS}_{\Gamma}(\mathcal{C}_{har}(\mathcal{E},\mathbf{V}_n))$

The exact sequence (7) endows $\mathcal{MS}_{\Gamma_0(M)}(\mathbf{V}_n)$ and $\mathcal{MS}_{\Gamma_0(pM)}(\mathbf{V}_n)$ with Eisenstein/cuspidal decompositions in light of Lemma 2.3: indeed in [Or, Section 7.2] a careful study of the action of the Hecke operators on $\mathcal{BS}_{\Gamma_0(M)}(\mathbf{V}_n)$ shows the existence of l such that $t_l = 0$ on $\mathcal{BS}_{\Gamma_0(M)}(\mathbf{V}_n)$ and $\mathcal{BS}_{\Gamma_0(pM)}(\mathbf{V}_n)$; on the other hand by the Ramanujan-Petersson conjecture proved by Deligne this Hecke operator is invertible on $H^1_{par}(\Gamma_0(N),\mathbf{V}_n)$.

Taking the Γ-invariants from the exact sequence (3) with $\mathbf{V} = \mathcal{MS}^n$ (and using Shapiro's Lemma) gives the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \to & \mathcal{C}_{har}(\mathcal{E},\mathcal{MS}(\mathbf{V}_n))^{\Gamma} & \to & \mathcal{C}_0(\mathcal{E},\mathcal{MS}(\mathbf{V}_n))^{\Gamma} & \to & \mathcal{C}(\mathcal{V},\mathcal{MS}(\mathbf{V}_n))^{\Gamma} \\ & & \| & & \| & & \| \\ 0 & \to & \mathcal{MS}^{har}_{\Gamma_0(pM)}(\mathbf{V}_n) & \to & \mathcal{MS}_{\Gamma_0(pM)}(\mathbf{V}_n) & \to & \mathcal{MS}_{\Gamma_0(M)}(\mathbf{V}_n)^2, \end{array} \tag{8}$$

where $\mathcal{MS}^{har}_{\Gamma_0(pM)}(\mathbf{V}_n)$ is by definition the image of $\mathcal{C}_{har}(\mathcal{E},\mathcal{MS}(\mathbf{V}_n))^{\Gamma}$ under Shapiro's isomorphism. The lower right arrow can be described explicitly in terms of corestriction as in [Gr, Section 3.2]. Thanks to Lemma 2.2 we can endow $\mathcal{MS}_{\Gamma}(\mathcal{C}_{har}(\mathcal{E},\mathbf{V}_n)) = \mathcal{C}_{har}(\mathcal{E},\mathcal{MS}(\mathbf{V}_n))^{\Gamma}$ with a natural Eisenstein/cuspidal decomposition.

Remark 2.4 *Let $\mathbb{T} := \mathbb{T}^{p-new}_{pM}$ be the p-new quotient of the Hecke algebra $\mathbb{T}_{pM}$. It follows from Lemma 2.2 and the Eichler-Shimura isomorphism (6) that $\mathcal{MS}_{\Gamma}(\mathcal{C}_{har}(\mathcal{E},\mathbf{V}_n(F)))^c$ is a free rank two module over $\mathbb{T}_F := \mathbb{T}\otimes_{\mathbb{Q}} F$.*

Eisenstein/cuspidal decomposition of $\mathcal{MS}_{\Gamma}(\mathbf{V}_n)$, $H^1(\Gamma,\mathcal{MS}(\mathbf{V}_n))$ and $H_1\left(\Gamma,\Delta^0\otimes\mathbf{P}_n\right) = H^1(\Gamma,\mathcal{MS}(\mathbf{V}_n))^{\vee}$

Sequence (4) (together with Shapiro's Lemma) produces the long exact sequence

$$\begin{aligned} 0 \quad \to \quad & \mathcal{MS}_{\Gamma}(\mathbf{V}_n) \to \mathcal{MS}_{\Gamma_0(M)}(\mathbf{V}_n)^2 \to \mathcal{MS}_{\Gamma_0(pM)}(\mathbf{V}_n) \\ & \xrightarrow{\delta} H^1(\Gamma,\mathcal{MS}(\mathbf{V}_n)) \to 0, \end{aligned} \tag{9}$$

where the zero on the right is a consequence of $H^1(\Gamma_0(M),\mathcal{MS}(\mathbf{V}_n)) = 0$ (see [Or, Section 7.1]). Thanks to Lemma 2.2 we can endow $\mathcal{MS}_{\Gamma}(\mathbf{V}_n)$ and $H^1(\Gamma,\mathcal{MS}(\mathbf{V}_n))$ with an Eisensten/cuspidal decomposition. It follows that $H_1\left(\Gamma,\Delta^0\otimes\mathbf{P}_n\right) = H^1(\Gamma,\mathcal{MS}(\mathbf{V}_n))^{\vee}$ is naturally endowed with a cuspidal decomposition too.

Eisenstein/cuspidal decomposition of $H^1(\Gamma, Hom(\mathbf{P}_n,\mathbf{V}))$

The groups $H^1(G,\mathbf{V}_n)$ with $G=\Gamma_0(N),\Gamma_0(pN)$ have an Eisenstein/cuspidal decomposition. The long exact sequence obtained from (4) and Shapiro's Lemma gives $H^1(\Gamma,\mathbf{V}_n)$ an Eisenstein/cuspidal decomposition too. Let $\mathbf{V}$ be a finite dimensional vector space endowed with the trivial Γ-action. By the universal coefficient Theorem

$$H^1(\Gamma, Hom(\mathbf{P}_n,\mathbf{V})) = H^1(\Gamma,\mathbf{V}_n)\otimes\mathbf{V},$$

and the Eisenstein/cuspidal decomposition on $H^1(\Gamma,\mathbf{V}_n)$ induces an Eisenstein/cuspidal decomposition on $H^1(\Gamma, Hom(\mathbf{P}_n,\mathbf{V}))$.

Lemma 2.5 *We have* $H^1(\Gamma, Hom(\mathbf{P}_n,\mathbf{V}))^c = H_{par}(\Gamma, Hom(\mathbf{P}_n,\mathbf{V}))=0$.

Proof. The claim is reduced $\mathbf{V}=K$, and we may apply [RoSe, Lemma 3.10]. ■

Taking the cuspidal parts from the exact sequence (9) and the applying Lemma 2.2 we get the exact sequence:

$$\begin{aligned} 0 \quad \rightarrow \quad & \mathcal{MS}_{\Gamma}(\mathbf{V}_n)^c \rightarrow \mathcal{MS}_{\Gamma_0(M)}(\mathbf{V}_n)^{2,c} \rightarrow \mathcal{MS}_{\Gamma_0(pM)}(\mathbf{V}_n)^c \\ & \stackrel{\delta^c}{\rightarrow} H^1(\Gamma,\mathcal{MS}(\mathbf{V}_n))^c \rightarrow 0. \end{aligned} \tag{10}$$

Lemma 2.6 *The boundary map δ^c restricts to give an isomorphism:*

$$\delta^c : \mathcal{MS}^{p-new}_{\Gamma_0(pM)}(\mathbf{V}_n)^c \stackrel{\simeq}{\rightarrow} H^1(\Gamma,\mathcal{MS}(\mathbf{V}_n))^c.$$

Proof. The proof is analogous to [RoSe, Lemma 2.9] ■

2.2 p-adic integration theory

Until the end of this section we fix a complete field extension $F_p/\mathbb{Q}_p$ and we will work over this field. Consider the natural map

$$\begin{aligned} R &: \mathcal{D}^0_n\left(\mathbb{P}^1(\mathbb{Q}_p)\right) \rightarrow \mathcal{C}_{har}(\mathcal{E},\mathbf{V}_n) \\ R(\mu)_e(P) &:= \mu\left(P\chi_{U_e}\right) \end{aligned}$$

It induces a map

$$R:\mathcal{MS}\left(\mathcal{D}^0_n\left(\mathbb{P}^1(\mathbb{Q}_p)\right)\right) \rightarrow \mathcal{MS}(\mathcal{C}_{har}(\mathcal{E},\mathbf{V}_n)) = \mathcal{C}_{har}(\mathcal{E},\mathcal{MS}(\mathbf{V}_n)).$$

Write $\Gamma_{e_*}\backslash\Gamma = \bigsqcup_{e\in\mathcal{E}}\Gamma_{e_*}\gamma_e$ where $\gamma_e e = e_*$ and $\Gamma_{e_*}=\Gamma_0(pM)$ is the stabilizer in Γ of the edge e_*. Whenever $\mathbf{V}$ is a Γ_{e_*}-module endowed with a (possibly infinite) norm $|-|$, define the following norm on $\mathcal{C}_0(\mathcal{E},\mathbf{V})$:

$$\|c\| := \sup_{e\in\mathcal{E}^+}|\gamma_e c(e)| \in \mathbb{R}\cup\infty.$$

The above definition does not depend on the choice of the coset representatives.

Lemma 2.7 *Taking the invariants, the Shapiro isomorphism*

$$\mathcal{C}_0(\mathcal{E},\mathbf{V})^{\Gamma} \simeq \mathbf{V}^{\Gamma_{e_*}}$$
$$c \mapsto c(e_*)$$

respects the norms $\|-\|$ *and* $|-|$.

Proof. The Γ-module identification $\mathcal{C}_0(\mathcal{E},\mathbf{V}) = \mathcal{C}(\mathcal{E}^+,\mathbf{V})$ respects the norms defined on the right hand side by the same formula. $\mathcal{C}(\mathcal{E}^+,\mathbf{V})$ is identified with the induced module $\mathrm{Ind}_{\Gamma_{e_*}}^{\Gamma}\mathbf{V}$. Mapping v to $c^v(e) := \gamma_e^{-1}e$ gives an explicit inverse to the Shapiro isomorphism. The claim easily follows from the definition of the norms. ■

Proposition 2.8 *Taking* Γ*-invariants yields an isomorphism*

$$\mathcal{MS}_{\Gamma}\left(\mathcal{D}_n^{0,b}\left(\mathbb{P}^1(\mathbb{Q}_p)\right)\right) \overset{\simeq}{\to} \mathcal{MS}_{\Gamma}\left(\mathcal{C}_{har}(\mathcal{E},\mathbf{V}_n)\right) = \mathcal{C}_{har}\left(\mathcal{E},\mathcal{MS}(\mathbf{V}_n)\right)^{\Gamma}.$$

Proof. Let $|-|$ be a (finite) Γ_{e_*}-invariant norm on $\mathbf{V}_n$, that must exists since $\mathbf{V}_n$ is finite dimensional and $\Gamma_{e_*} \subset \mathbb{GL}_2(L_*)$ is contained in a compact subgroup of $\mathbb{GL}_2(\mathbb{Q}_p)$. Endow $\mathcal{C}_0(\mathcal{E},\mathbf{V}_n)$ with the same norm $\|-\|$ considered in the Lemma 2.7. Let $\mathcal{C}_0^b(\mathcal{E},\mathbf{V}_n)$ (resp. $\mathcal{C}_{har}^b(\mathcal{E},\mathbf{V}_n)$) be the subspace of those elements of $\mathcal{C}_0(\mathcal{E},\mathbf{V}_n)$ (resp. $\mathcal{C}_{har}(\mathcal{E},\mathbf{V}_n)$) having finite norm.

Consider the Γ-modules

$$Hom\left(\Delta^0,\mathcal{C}_*(\mathcal{E},\mathbf{V}_n)\right) = \mathcal{C}_*\left(\mathcal{E},Hom\left(\Delta^0,\mathbf{V}_n\right)\right) \text{ with } * = 0, har,$$

Define on $Hom\left(\Delta^0,\mathbf{V}_n\right)$ a norm by the formula:

$$|m|' := \sup_{x,y\in\mathbb{P}^1(\mathbb{Q})} |m(x-y)|.$$

Note that the above formula defines a Γ_{e_*}-invariant norm on $Hom(D,\mathbf{V}_n)$, since the norm on $\mathbf{V}_n$ was Γ_{e_*}-invariant. Furthermore, taking the Γ_{e_*}-invariants we see that the above norm is finite on $Hom_{\Gamma_{e_*}}\left(\Delta^0,\mathbf{V}_n\right)$. Indeed for every $\gamma \in \Gamma_{e_*}$, thanks to the Γ_{e_*}-invariance of the norm on $\mathbf{V}_n$, we have

$$\left|m\left(\gamma^{-1}x-\gamma^{-1}y\right)\right| = \left|\gamma m\left(\gamma^{-1}x-\gamma^{-1}y\right)\right| = \left|(\gamma m)(x-y)\right|;$$

hence, whenever $m \in Hom_{\Gamma_{e_*}}\left(\Delta^0,\mathbf{V}_n\right)$, the sup can be taken over all a set of representatives for the set of Γ_{e_*}-equivalence classes of $\mathbb{P}^1(\mathbb{Q})$. Thanks to Lemma 2.7 we also know that, setting

$$\|m\|' := \sup_{e\in\mathcal{E}^+} |\gamma_e m(e)|',$$

defines a finite norm on $\mathcal{C}_0\left(\mathcal{E},Hom\left(\Delta^0,\mathbf{V}_n\right)\right)^{\Gamma}$, and hence also on the subset

$$Hom_{\Gamma}\left(\Delta^0,\mathcal{C}_{har}(\mathcal{E},\mathbf{V}_n)\right) \subset Hom_{\Gamma}\left(\Delta^0,\mathcal{C}_0(\mathcal{E},\mathbf{V}_n)\right).$$

Making explicit the definition of the above norms we see that

$$\begin{aligned} \|m\|' \quad : \quad &= \sup_{e\in\mathcal{E}^+} |\gamma_e m(e)|' = \sup_{\substack{e\in\mathcal{E}^+ \\ x,y\in\mathbb{P}^1(\mathbb{Q}_p)}} |(\gamma_e m(e))(x-y)| = \\ &= \sup_{\substack{e\in\mathcal{E}^+ \\ x,y\in\mathbb{P}^1(\mathbb{Q}_p)}} \left|\gamma_e m(e)\left(\gamma_e^{-1}x - \gamma_e^{-1}y\right)\right| = \\ &= \sup_{\substack{e\in\mathcal{E}^+ \\ x,y\in\mathbb{P}^1(\mathbb{Q}_p)}} |\gamma_e m(e)(x-y)| \end{aligned}$$

must be finite on $\mathcal{C}_{har}(\mathcal{E},\mathcal{MS}(\mathbf{V}_n))^{\Gamma}$. In particular, for every $x,y\in\mathbb{P}^1(\mathbb{Q})$ (and every element of $Hom_{\Gamma}\left(\Delta^0,\mathcal{C}_{har}(\mathcal{E},\mathbf{V}_n)\right)$) we find

$$\|m(x-y)\| = \sup_{e\in\mathcal{E}^+} |\gamma_e m(e)(x-y)| \le \|m\|' < \infty.$$

In other words for every $x,y\in\mathbb{P}^1(\mathbb{Q}_p)$ we have $m(x-y)\in\mathcal{C}_{har}^b(\mathcal{E},\mathbf{V}_n)$, so that the natural inclusion of $Hom_{\Gamma}\left(\Delta^0,\mathcal{C}_{har}^b(\mathcal{E},\mathbf{V}_n)\right)$ in $Hom_{\Gamma}\left(\Delta^0,\mathcal{C}_{har}(\mathcal{E},\mathbf{V}_n)\right)$ is really an identity

$$Hom_{\Gamma}\left(\Delta^0,\mathcal{C}_{har}^b(\mathcal{E},\mathbf{V}_n)\right) = Hom_{\Gamma}\left(\Delta^0,\mathcal{C}_{har}(\mathcal{E},\mathbf{V}_n)\right). \tag{11}$$

By the Theorem of Amice-Velu-Teitelbaum (see [DT] for the apropriate formulation) the morphism R restricts to give an isomorphism $\mathcal{D}_n^{0,b}\left(\mathbb{P}^1(\mathbb{Q}_p)\right)\simeq\mathcal{C}_{har}^b(\mathcal{E},\mathbf{V})$ and the claim follows from (11). ■

Recall our fixed working field F_p and let $F_p^0 := F_p\cap\mathbb{Q}_p^{ur}$ be the maximal absolutely unramified subfield of F_p. Write $\mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right)$ (resp. $\mathrm{Div}\left(\mathcal{H}_p^{ur}\right)$) to denote the degree zero divisors (resp. the divisors) supported on $\mathbb{Q}_p^{ur}-\mathbb{Q}_p$ that are fixed by the action of the Galois group $G_{\mathbb{Q}_p^{ur}/F_p^0}$.

Definition 2.9 *Define pairings*

$$\begin{array}{ccc} \left(\Delta^0\otimes\mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right)\otimes\mathbf{P}_n\right)\otimes\mathcal{MS}\left(\mathcal{D}_n^0\left(\mathbb{P}^1(\mathbb{Q}_p)\right)\right) & \rightarrow & F_p \\ (r-s)\otimes(\tau_2-\tau_1)\otimes P\otimes\mu & \mapsto & \int_{\tau_1}^{\tau_2} P\omega_\mu^{\log}\{r\rightarrow s\} \\ \left(\Delta^0\otimes Div^0(\mathcal{H}_p)\otimes\mathbf{P}_n\right)\otimes\mathcal{MS}\left(\mathcal{D}_n^0\left(\mathbb{P}^1(\mathbb{Q}_p)\right)\right) & \rightarrow & F_p \\ (r-s)\otimes(\tau_2-\tau_1)\otimes P\otimes\mu & \mapsto & \int_{\tau_1}^{\tau_2} P\omega_\mu^{\mathrm{ord}}\{r\rightarrow s\} \end{array}$$

where

$$\int_r^s\int_{\tau_1}^{\tau_2} P\omega_\mu^{\log} := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_p\left(\frac{t-\tau_2}{t-\tau_1}\right) P(t)\,d\mu\{r\rightarrow s\}(t)$$

and

$$\int_r^s\int_{\tau_1}^{\tau_2} P\omega_\mu^{\mathrm{ord}} := \sum_{e:red(\tau_1)\rightarrow red(\tau_2)} \int_{U(e)} P(t)\,d\mu\{r\rightarrow s\}(t).$$

Since the pairings are Γ-invariant they give pairings

$$\Psi^{\log}, \Psi^{\text{ord}} : \left(\Delta^0 \otimes \text{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma \otimes \mathcal{MS}_\Gamma\left(\mathcal{D}_n^0\left(\mathbb{P}^1\left(\mathbb{Q}_p\right)\right)\right) \to F_p.$$

Hence there are two morphisms

$$\Psi^{\log}, \Psi^{\text{ord}} : \left(\Delta^0 \otimes \text{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma \to \mathcal{MS}_\Gamma\left(\mathcal{D}_n^0\left(\mathbb{P}^1\left(\mathbb{Q}_p\right)\right)\right)^\vee.$$

From now on we will identify, via Proposition 2.8,

$$\mathbf{MS} := \mathcal{MS}_\Gamma\left(\mathcal{D}_n^{0,b}\left(\mathbb{P}^1\left(\mathbb{Q}_p\right)\right)\right) = \mathcal{MS}_\Gamma\left(\mathcal{C}_{har}\left(\mathcal{E}, \mathbf{V}_n\right)\right).$$

Consider the exact sequence

$$0 \to \Delta^0 \otimes \text{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n \to \Delta^0 \otimes \text{Div}\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n \to \Delta^0 \otimes \mathbf{P}_n \to 0,$$

yielding the boundary map

$$H_1\left(\Gamma, \Delta^0 \otimes \mathbf{P}_n\right) \stackrel{\partial}{\to} \left(\Delta^0 \otimes \text{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma$$

Recall that we have introduced Eisenstein/cuspidal decompositions on both $H_1\left(\Gamma, \Delta^0 \otimes \mathbf{P}_n\right)$ and $\mathbf{MS}$. Let p^c be the projection onto the cuspidal part of $\mathbf{MS}^\vee$.

Theorem 2.10 *The morphism*

$$p^c \circ \Psi_\partial^{\text{ord}} := p^c \circ \Psi^{\text{ord}} \circ \partial : H_1\left(\Gamma, \Delta^0 \otimes \mathbf{P}_n\right) \to \mathbf{MS}^{c,\vee}$$

is surjective and induces an isomorphism

$$H_1\left(\Gamma, \Delta^0 \otimes \mathbf{P}_n\right)^c \stackrel{\simeq}{\to} \mathbf{MS}^{c,\vee}.$$

Proof. The proof is just a copy of [RoSe, Theorem 3.11] with the obvious modifications and Lemma 2.6 in place of [RoSe, Lemma 2.9]. ■

Definition 2.11 *The morphisms*

$$\Phi^{\log}, \Phi^{\text{ord}} : \left(\Delta^0 \otimes \text{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma \to \mathbf{MS}^{c,\vee}$$

are by definition $\Phi^* := p_c \circ \Psi^*$ *with* $* = \log, \text{ord}$.

The above Theorem allows us to define the Orton $\mathcal{L}$-invariant.

Corollary 2.12 *There exists a unique* $\mathcal{L} \in End_{\mathbb{T}_{\mathbb{Q}_p}}\left(\mathbf{MS}^{c,\vee}\right)$ *such that*

$$\Phi^{\log} \circ \partial = \mathcal{L} \circ \Phi^{\text{ord}} \circ \partial : H_1\left(\Gamma, \Delta^0 \otimes \mathbf{P}_n\right) \to \mathbf{MS}^{c,\vee}.$$

Proof. The Corollary can be deduced from Theorem 2.10 exactly as [RoSe, Corollary 3.13] is deduced from [RoSe, Theorem 3.11]. ■

2.3 The monodromy module of weight k_0 modular forms

Choose a sign w_∞ and set

$$\mathbf{D} = \mathbf{D}^{w_\infty} := \mathbf{MS}^{c,\vee,w_\infty} \oplus \mathbf{MS}^{c,\vee,w_\infty}.$$

Note that $\mathbf{D}$ is a free rank two module over $\mathbb{T}_{F_p}$ by Remark 2.4.

Define

$$\Phi := -\Phi^{\log} \oplus \Phi^{\mathrm{ord}} : \left(\Delta^0 \otimes \mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma \to \mathbf{D}.$$

According to Corollary 2.12 and Theorem 2.10

$$F^{m+1} := \left\{(-\mathcal{L}x, x) : x \in \mathbf{MS}^{c,\vee,w_\infty}\right\} = \mathrm{Im}\left(\Phi \circ \partial\right) \tag{12}$$

is a free rank one $\mathbb{T}_{F_p}$-submodule.

Let σ be the absolute Frobenius automorphism of F_p^0. Write $\mathbf{D}\left(F_p^0\right) := \mathbf{MS}^{c,\vee,w_\infty}\left(F_p^0\right)^2$.Then we can consider the σ-linear automorphism $\sigma_{\mathbf{D}} := 1 \otimes \sigma$ on $\mathbf{D}\left(F_p^0\right) = \mathbf{D}\left(\mathbb{Q}_p\right) \otimes_{\mathbb{Q}_p} F_p^0$.

We define a structure of filtered Frobenius module over F_p on $\mathbf{D}$ as follows.

(a) The filtration is

$$\begin{aligned} &\mathbf{D} = F^0 \supsetneq F^1 = ... = F^{k-1} \supsetneq F^k = 0, \\ &F^{m+1} = \left\{(-\mathcal{L}x, x) : x \in \mathbf{MS}^{c,\vee,w_\infty}\right\}, \text{ for } m := (k-2)/2. \end{aligned}$$

(b) The Frobenius operator φ is defined on $\mathbf{D}\left(F_p^0\right)$ by the equation $\varphi = U_p \otimes \sigma_{\mathbf{D}} \oplus pU_p \otimes \sigma_{\mathbf{D}}$, i.e.

$$\varphi\left(x, y\right) := \left(U_p\sigma_{\mathbf{D}}\left(x\right), pU_p\sigma_{\mathbf{D}}\left(y\right)\right).$$

(c) The monodromy operator N is defined on $\mathbf{D}\left(F_p^0\right)$ by the rule

$$N\left(x, y\right) = \left(y, 0\right).$$

It is easily checked that the above conditions define indeed a filtered Frobenius module structure on $\mathbf{D}$, defined over $\mathbb{Q}_p$ if we have taken $F_p = \mathbb{Q}_p$. The filtered Frobenius module $\mathbf{D}$ over F_p is indeed obtained from the one over $\mathbb{Q}_p$ by base change from $MF_{\mathbb{Q}_p}\left(\phi, N\right)$ to $MF_{F_p}\left(\phi, N\right)$. Since the Hecke algebra $\mathbb{T}$ is commutative, every element of this ring commutes with φ and N. Furthermore $F^{m+1} \subset \mathbf{D}$ is a (rank one) $\mathbb{T}_{F_p}$-submodule. Indeed $\mathbf{D}$ is a rank two $\mathbb{T}_{\mathbb{Q}_p}$-monodromy module over F_p.

Let $f \in S_k\left(\Gamma_0\left(pM\right)\right)$ be a normalized p-new weight k eigenform. Denote by $I_f^{w_\infty} \in \mathcal{MS}^k_{\Gamma_0(M)}\left(K_f\right)$ the modular symbol attached to the choice of the sign w_∞ that was chosen to define $\mathbf{D}$, appropriately normalized by means of Proposition 2.1. Let $K_{[f]}$ be the composition of the fields K_{f^σ}, where f^σ is the modular form obtained from f by applying the automorphism $\sigma \in G_{\mathbb{Q}}$ to the Fourier coefficients of f. Up to extending F_p we can fix an embedding $K_{[f]} \subset F_p$.

Let

$$\mathbf{MS}_f^{c,w_\infty} := F_p I_f^{w_\infty} \hookrightarrow \mathbf{MS}^{c,w_\infty}$$

be the f-eigencomponent of $\mathbf{MS}^{c,w_\infty}$, on which the Hecke algebra acts through

$$\mathbb{T} \to K_f \subset F_p. \tag{13}$$

Write

$$\mathbf{MS}_{[f]}^{c,w_\infty} := \bigoplus_\sigma \mathbf{MS}_{f^\sigma}^{c,w_\infty}$$

Note that the above sum can be indexed by the $[K_f : \mathbb{Q}]$ embeddings of K_f in $\overline{\mathbb{Q}}$. The inclusion $\mathbf{MS}_{[f]}^{c,w_\infty} \subset \mathbf{MS}^{c,w_\infty}$ gives rise to a morphism

$$e_{[f]} : \mathbf{D} \twoheadrightarrow \mathbf{D}_{[f]},$$

where we define:

$$\mathbf{D}_{[f]} := \mathbf{MS}_{[f]}^{c,\vee,w_\infty} \oplus \mathbf{MS}_{[f]}^{c,\vee,w_\infty}.$$

We also note that $\mathbf{D}_{[f]} = \bigoplus_\sigma \mathbf{D}_{f^\sigma}$, where $\mathbf{D}_f$ is similarly defined.

Hence we can consider

$$\Phi_{[f]} : \left(\Delta^0 \otimes \mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma \xrightarrow{\Phi} \mathbf{D} \overset{e_{[f]}}{\twoheadrightarrow} \mathbf{D}_{[f]}.$$

Since $\mathbf{MS}_{[f]}^{c,w_\infty} \subset \mathbf{MS}^{c,w_\infty}$ is an Hecke submodule, setting $F_{[f]}^{m+1} := e_{[f]}\left(F^{m+1}\right)$ it is easily checked that $\mathbf{D}_{[f]}$ gets a structure of filtered F_p-vector space with multiplication by $K_f \otimes \mathbb{Q}_p$. The same remark applies to $\mathbf{D}_f$, the Hecke algebra acting though (13). In this way

$$\mathbf{D}_{[f]} = \bigoplus_\sigma \mathbf{D}_{f^\sigma} \tag{14}$$

is a decomposition of filtered F_p-vector spaces endowed with multiplication by the Hecke algebra.

We write $\mathcal{L}_{[f]} \in End_{\mathbb{T}_{\mathbb{Q}_p}}\left(\mathbf{MS}_{[f]}^{c,w_\infty}\right)$ (resp. $\mathcal{L}_f \in End_{\mathbb{T}_{\mathbb{Q}_p}}\left(\mathbf{MS}_f^{c,w_\infty}\right)$) to denote the $\mathcal{L}$-invariant corresponding to the modular form f (of course depending a priori on the choice of w_∞), i.e. the image of $\mathcal{L}$ acting on $\mathbf{MS}_{[f]}^{c,w_\infty}$ (resp. $\mathbf{MS}_f^{c,w_\infty}$). It is also characterized by exploiting a property similar to the one of Corollary 2.12 (see [RoSe, Section 4.3] for details). We have $\mathcal{L}_{[f]} \in K_f \otimes \mathbb{Q}_p$ and $\mathcal{L}_f \in F_p$ is the image of it under (13). Then we have

$$\begin{aligned} F_{[f]}^{m+1} &= \left\{\left(-\mathcal{L}_{[f]}x, x\right) : x \in \mathbf{MS}_{[f]}^{c,\vee,w_\infty}\right\} \subset \mathbf{D}_{[f]}, \\ F_f^{m+1} &= \left\{\left(-\mathcal{L}_f x, x\right) : x \in \mathbf{MS}_f^{c,\vee,w_\infty}\right\} \subset \mathbf{D}_f. \end{aligned}$$

Remark 2.13 $\mathbf{D}_{[f]}$ *has indeed a natural $\mathbb{Q}_p$-structure compatible that can be used to define on $\mathbf{D}_{[f]}$ the structure of a $K_f \otimes \mathbb{Q}_p$-monodromy module over $\mathbb{Q}_p$,* *i.e. we could have taken $F_p = \mathbb{Q}_p$. In this way $e_{[f]}$ becomes an epimorphism in $MF_{F_p}(\phi, N)$*

On the other hand $\mathbf{D}_f$ is only defined when $K_f \subset F_p$. Assuming $K_{[f]} \subset F_p$, the decomposition (14) of filtered F_p-vector spaces endowed with multiplication by the Hecke algebra produces a decomposition

$$\mathbf{D}_{[f]}/F_{[f]}^{m+1} = \bigoplus_\sigma \mathbf{D}_{f^\sigma}/F_{f^\sigma}^{m+1} \tag{15}$$

of the tangent space of $\mathbf{D}_{[f]} \in MF_{F_p}(\phi, N)$. The Hecke algebra acts through (13) on the f-component.

We can also consider

$$\Phi_{[f]} : \left(\Delta^0 \otimes \mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma \xrightarrow{\Phi} \mathbf{D}/F^{m+1} \overset{e_{[f]}}{\twoheadrightarrow} \mathbf{D}_{[f]}/F^{m+1}.$$

The same construction holds for the inclusion $\mathbf{MS}_f^{c,w_\infty} \subset \mathbf{MS}^{c,w_\infty}$ and produces the analogous morphisms Φ_f. We will write e_f to denote the projection onto the f-component.

2.4 The p-adic Abel-Jacobi maps in the Darmon setting

Consider the exact sequence

$$\begin{aligned} \ldots \quad &\to \quad H_i\left(\Delta^0 \otimes \mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right) \to H_i\left(\Delta^0 \otimes \mathrm{Div}\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right) \\ &\to \quad H_i\left(\Delta^0 \otimes \mathbf{P}_n\right) \to \ldots. \end{aligned}$$

obtained from the short exact sequence

$$0 \to \Delta^0 \otimes \mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n \to \Delta^0 \otimes \mathrm{Div}\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n \to \Delta^0 \otimes \mathbf{P}_n \to 0.$$

Let $\mathbf{V}$ be any F_p-vector space, regarded like a trivial Γ-module. The application of $Hom(-, \mathbf{V})$ produces the following exact sequence:

$$\begin{aligned} &Hom\left(\left(\Delta^0 \otimes \mathbf{P}_n\right)_\Gamma, \mathbf{V}\right) \to Hom\left(\left(\Delta^0 \otimes \mathrm{Div}\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma, \mathbf{V}\right) \to \\ &Hom\left(\left(\Delta^0 \otimes \mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma, \mathbf{V}\right) \to Hom\left(H_1\left(\Gamma, \Delta^0 \otimes \mathbf{P}_n\right), \mathbf{V}\right) \end{aligned} \tag{16}$$

It will be convenient to give the following:

Definition 2.14 *A $\mathbf{V}$-valued definite integration theory is an element*

$$\Phi \in Hom\left(\left(\Delta^0 \otimes \mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma, \mathbf{V}\right).$$

A $\mathbf{V}$-valued semidefinite integration theory lifting Φ is an element

$$\Phi^{AJ} \in Hom\left(\left(\Delta^0 \otimes \mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma, \mathbf{V}\right)$$

such that the image via the middle arrow of (16) is Φ. One can also define the notion of $\mathbf{V}$-valued positive oriented definite integration theory and the notion of $\mathbf{V}$-valued positive oriented semidefinite integration theory by means of the exact sequence

$$0 \to \Delta^0 \otimes \mathrm{Div}^0 \left(\mathcal{H}_p^{ur+}\right) \otimes \mathbf{P}_n \to \Delta^0 \otimes \mathrm{Div} \left(\mathcal{H}_p^{ur+}\right) \otimes \mathbf{P}_n \to \Delta^0 \otimes \mathbf{P}_n \to 0, \quad (17)$$

where $\mathcal{H}_p^{ur+}$ denotes the subset of those $\tau \in \mathcal{H}_p^{ur}$ such that $red(\tau) \in \mathcal{V}^+$.

In particular we can consider the $\mathbf{D}/F^{m+1}$-valued integration theory obtained by Φ followed by the projection onto the quotient $\mathbf{D}/F^{m+1}$, that we will denote again by the same symbol by abuse of notation:

$$\Phi : \left(\Delta^0 \otimes \mathrm{Div}^0 \left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma \to \mathbf{D}/F^{m+1}.$$

Definition 2.15 *A p-adic Abel-Jacobi map (in the Darmon setting) is any $\mathbf{D}/F^{m+1}$-valued semidefinite integration theory lifting the above integration theory Φ (eventually positive oriented).*

Proposition 2.16 *There exists a $\mathbf{D}/F^{m+1}$-valued semidefinite integration theory Φ^{AJ} lifting the $\mathbf{D}/F^{m+1}$-valued integration theory Φ. In particular the restriction of Φ^{AJ} to $\left(\Delta^0 \otimes \mathrm{Div}\left(\mathcal{H}_p^{ur+}\right) \otimes \mathbf{P}_n\right)_\Gamma$ provides a $\mathbf{D}/F^{m+1}$-valued positive oriented semidefinite integration theory lifting the restriction of Φ to the group $\left(\Delta^0 \otimes \mathrm{Div}^0\left(\mathcal{H}_p^{ur+}\right) \otimes \mathbf{P}_n\right)_\Gamma$.*

Proof. The claim follows from (16) specialized to $\mathbf{V} = \mathbf{D}/F^{m+1}$, in light of (12). ■

Remark 2.17 *One of the main differences with the weight 2 setting, as well as with the cohomological approach followed in [RoSe], is in the lack of the uniqueness of a semidefinite integration theory. In fact note that two different liftings differs by an element of*

$$Hom\left(\left(\Delta^0 \otimes \mathbf{P}_n\right)_\Gamma, \mathbf{V}\right),$$

as it follows from the exactness of (16). In any case we will be able to define the p-adic Abel-Jacobi image of the Darmon cycles $j_\Psi \in \left(\Delta^0 \otimes \mathrm{Div}\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma$ by showing that $\Phi^{AJ}(j_\Psi)$ does not depend on the choice of the p-adic Abel-Jacobi map Φ^{AJ} (see the subsequent Proposition 2.22).

2.5 Darmon cycles

Let $K/\mathbb{Q}$ be a real quadratic field of discriminant D_K and recall our factorization $N = pM$. We make the following assumption:

Axiom 2.18 *(Darmon hypothesis) The prime p is inert in K while the primes dividing M are split.*

Choose embeddings:

$$\sigma : K \to \mathbb{R} \text{ and } \sigma_p : K \to K_p$$

that we will use to regard K like a subfield of both $\mathbb{R}$ and the quadratic unramified extension $\mathbb{Q}_{p^2}/\mathbb{Q}_p$. In particular the inequalities $<$ make sense between elements of K and we can consider $\mathcal{H}_p(K) := \mathcal{H}_p \cap K$. We may also view $\sqrt{D_K} \in K_p$ via σ_p.

We denote by $\mathcal{E}mb = \mathcal{E}mb(K, \mathbb{M}_2(\mathbb{Q}))$ the set of all the $\mathbb{Q}$-algebra embeddings of K into $\mathbb{M}_2(\mathbb{Q})$. Whenever $\mathcal{O}$ is a $\mathbb{Z}[1/p]$-order of conductor c prime to $D_K N$ we also denote by $\mathcal{E}mb(\mathcal{O}, \mathcal{R})$ the set of $\mathbb{Z}[1/p]$-embeddings of $\mathcal{O}$ into our fixed Eichler $\mathbb{Z}[1/p]$-order $\mathcal{R}$. Define the $\mathbb{Z}[1/p]$-order associated to $\Psi \in \mathcal{E}mb$ as being $\mathcal{O}_\Psi := \Psi^{-1}(\mathcal{R})$, so that for every fixed $\mathbb{Z}[1/p]$-order $\mathcal{O}$ as above $\mathcal{E}mb(\mathcal{O}, \mathcal{R}) \subset \mathcal{E}mb$ is the subset of those $\Psi \in \mathcal{E}mb$ such that $\mathcal{O}_\Psi = \mathcal{O}$. Attached to the embedding $\Psi \in \mathcal{E}mb$ there are the following data:

- the two fixed points $\tau_\Psi, \overline{\tau}_\Psi \in \mathcal{H}_p$ for the action of $\Psi(K^\times)$ on $\mathcal{H}_p(K)$, ordered in such a way that the action of $K^\times$ on the tangent space at τ_Ψ is through the character $z \mapsto z/\overline{z}$;

- the unique fixed vertex $v_\Psi \in \mathcal{V}$ for the action of $\Psi(K^\times)$ on $\mathcal{V}$, which is nothing but the reduction $red(\tau_\Psi) = red(\overline{\tau}_\Psi)$;

- the unique polynomial up to sign P_Ψ in $\mathbf{P}_2$ which is fixed by the action of $\Psi(K^\times)$ on $\mathbf{P}_2 \otimes \det^{-1}$ and satisfies $\langle P_\Psi, P_\Psi \rangle_{\mathbf{P}_2} = -D_K/4$ (the pairing being defined like in [BDIS]), which we fix by the choice

$$P_\Psi := \mathrm{Tr}\left(\Psi\left(\sqrt{D_K}/2\right) \cdot \begin{pmatrix} X & -X^2 \\ 1 & -X \end{pmatrix}\right) \in \mathbf{P}_2.$$

  The other one is obtained replacing $\sqrt{D_K}/2$ with $-\sqrt{D_K}/2$;

- the stabilizer Γ_Ψ of Ψ in Γ, which is nothing but

$$\Gamma_\Psi = \Psi\left(K^\times\right) \cap \Gamma = \Psi\left(\mathcal{O}_1^\times\right),$$

  where $\mathcal{O}_1^\times$ stands for the subgroup of $\mathcal{O}^\times$ of norm 1 and $\mathcal{O} = \mathcal{O}_\Psi$ is the associated order;

- the generator $\gamma_\Psi \in \Gamma_\Psi/\{\pm 1\} \simeq \mathbb{Z}$ which is the image $\gamma_\Psi := \Psi(u)$ of the unique generator of $u \in \mathcal{O}_1^\times$ such that $\sigma(u) > 1$.

For each $\tau \in \mathcal{H}_p(K) := \mathcal{H}_p \cap K$ (use σ_p to view K as a subfield of K_p), we say that τ has positive orientation at p if $red(\tau) \in \mathcal{V}^+$. We write $\mathcal{H}_p^+(K)$ to denote the set of positive oriented elements in $\mathcal{H}_p(K)$. We say that $\Psi \in \mathcal{E}mb^+ \subset \mathcal{E}mb$ has positive orientation whenever $v_\Psi \in \mathcal{V}^+$, i.e. $\tau_\Psi, \overline{\tau}_\Psi \in \mathcal{H}_p^+(K)$. It is possible to introduce the notion of negative oriented embeddings and then we have $\mathcal{E}mb = \mathcal{E}mb^+ \sqcup \mathcal{E}mb^-$. We also denote by $\mathcal{E}mb^+(\mathcal{O}, \mathcal{R})$ the subset of positive oriented embeddings of conductor c. The group Γ naturally acts on $\mathcal{E}mb$ by conjugation, preserving all the subsets we introduced.

We note that the association

$$\Psi \mapsto (\tau_\Psi, P_\Psi, \gamma_\Psi)$$

satisfies the following property under the conjugation action by $\gamma \in \Gamma$:

$$\left(\tau_{\gamma\Psi\gamma^{-1}}, P_{\gamma\Psi\gamma^{-1}}, \gamma_{\gamma\Psi\gamma^{-1}}\right) = \left(\gamma\tau_\Psi, \gamma P_\Psi := P_\Psi\gamma^{-1}, \gamma\gamma_\Psi\gamma^{-1}\right). \tag{18}$$

Once we fix $x \in \mathbb{P}^1(\mathbb{Q})$ we can consider

$$\begin{array}{rcl} j \ : \ \mathcal{E}mb(\mathcal{O},\mathcal{R}) & \to & \Delta^0 \otimes \mathrm{Div}\left(\mathcal{H}_p\right) \otimes \mathbf{P}_n \\ j_\Psi & := & \gamma_\Psi x - x \otimes \tau_\Psi \otimes D_K^{-\frac{k_0-2}{4}} P_\Psi^m. \end{array}$$

Lemma 2.19 *The image $[j_\Psi]$ of j_Ψ in $\left(\Delta^0 \otimes \mathrm{Div}\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma$ does not depend on the choice of $y \in \Gamma x$ that was made to define it. Furthermore it does not depend on the choice of a representative of the class $[\Psi]$ of Ψ in $\mathcal{E}mb$, so that the above association gives a well defined map*

$$j : \Gamma\backslash\mathcal{E}mb(\mathcal{O},\mathcal{R}) \to \left(\Delta^0 \otimes \mathrm{Div}\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma.$$

Proof. The proof is easy. ■

For later purposes it is useful to remark the following property of the data attached to $\Psi \in \mathcal{E}mb$.

Remark 2.20 *We have*

$$\left(\tau_{\overline{\Psi}}, P_{\overline{\Psi}}, \gamma_{\overline{\Psi}}\right) = \left(\overline{\tau}_\Psi, -P_\Psi, \gamma_\Psi^{-1}\right).$$

Indeed the equality $\left(\tau_{\overline{\Psi}}, P_{\overline{\Psi}}\right) = \left(\overline{\tau}_\Psi, -P_\Psi\right)$ is clear. To see that $\gamma_{\overline{\Psi}} = \gamma_\Psi^{-1}$ simply note that, since the norm of u is one, $u^{-1} = \overline{u}$. Thus

$$\gamma_{\overline{\Psi}} := \overline{\Psi}(u) = \Psi(\overline{u}) = \Psi\left(u^{-1}\right) =: \gamma_\Psi^{-1}.$$

Definition 2.21 *The Darmon cycle attached to the embedding Ψ is the element $[j_\Psi] \in \left(\Delta^0 \otimes \mathrm{Div}\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma$, also denoted by j_Ψ by abuse of notation.*

The following proposition allows us to define the p-adic Abel-Jacobi image of the Darmon cycles.

Proposition 2.22 *For every $\Phi \in Hom\left(\left(\Delta^0 \otimes \mathrm{Div}\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma, \mathbf{V}\right)$ let Φ_i^{AJ} with $i = 1, 2$ be two $\mathbf{V}$-valued indefinite integration theories lifting the integration theory Φ. Then we have*

$$\Phi_1^{AJ}\left([\gamma_\Psi x - x \otimes \tau \otimes P]\right) = \Phi_2^{AJ}\left([\gamma_\Psi x - x \otimes \tau \otimes P]\right),$$
$$\text{for any } \tau \in \mathcal{H}_p \text{ and any } P \in K_p P_\Psi^m.$$

In particular

$$\Phi_1^{AJ}\left([j_\Psi]\right) = \Phi_2^{AJ}\left([j_\Psi]\right).$$

The same result holds for positive oriented $\mathbf{V}$-valued integration theories.

Proof. By Remark 2.17 $\Phi_1^{AJ} - \Phi_2^{AJ}$ belongs to $Hom\left(\left(\Delta^0 \otimes \mathbf{P}_n\right)_\Gamma, \mathbf{V}\right)$. More explicitly this simply means that we may write $\left(\Phi_1^{AJ} - \Phi_2^{AJ}\right) = \Delta \circ \pi$ for some $\Delta \in Hom\left(\left(\Delta^0 \otimes \mathbf{P}_n\right)_\Gamma, \mathbf{V}\right)$, where π is the quotient map with source $\left(\Delta^0 \otimes \mathrm{Div}\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma$ and target $\left(\Delta^0 \otimes \mathbf{P}_n\right)_\Gamma$. In other words, for every $x, y \in \mathbb{P}^1(\mathbb{Q})$, $\tau \in \mathcal{H}_p$ and every $P \in Hom(\mathbf{P}_n, \mathbf{V})$:

$$\Phi_1^{AJ}(x - y \otimes \tau \otimes P) - \Phi_2^{AJ}(x - y \otimes \tau \otimes P) = \Delta(x - y \otimes P).$$

We will show that every element $\Delta \in Hom\left(\left(\Delta^0 \otimes \mathbf{P}_n\right)_\Gamma, \mathbf{V}\right)$ satisfies

$$\Delta(\gamma_\Psi x - x \otimes P) = 0 \text{ for } P \in K_p P_\Psi^m,$$

from which the claim will follow, in light of the above equality.

Consider the function

$$c_x : \gamma \to \Delta(\gamma x - x \otimes -) \in Hom(\mathbf{P}_n, \mathbf{V}).$$

It is a crossed homomorphism from Γ to $Hom(\mathbf{P}_n, \mathbf{V})$ because $\mathbf{V}$ is endowed with the trivial Γ-action. Let $\mathbf{c}_x$ be the class of c_x in $H^1(\Gamma, Hom(\mathbf{P}_n, \mathbf{V}))$.

Consider the exact sequence (2)

$$0 \to Hom(\mathbf{P}_n, \mathbf{V}) \to \mathcal{BS}(Hom(\mathbf{P}_n, \mathbf{V})) \to \mathcal{MS}(Hom(\mathbf{P}_n, \mathbf{V})) \to 0.$$

We claim that $\mathbf{c}_x = -\delta\boldsymbol{\Delta}$, where we regard Δ as an element of

$$Hom\left(\left(\Delta^0 \otimes \mathbf{P}_n\right)_\Gamma, \mathbf{V}\right) = Hom_\Gamma\left(\left(\Delta^0 \otimes \mathbf{P}_n\right), \mathbf{V}\right) = \mathcal{MS}_\Gamma(Hom(\mathbf{P}_n, \mathbf{V})),$$

and δ is the boundary map arising from the above exact sequence. Once we will have established this fact the claim will follow from Lemma 2.5, since then we will know that $\mathbf{c}_x = -\delta\boldsymbol{\Delta} = 0$. But this means that there exists $\Lambda \in Hom(\mathbf{P}_n, \mathbf{V})$ such that $\partial\Lambda = c$, i.e. for every $\gamma \in \Gamma$ and every $P \in \mathbf{P}_n$

$$\Delta(\gamma x - x \otimes P) = c_x(\gamma)(P) = \Lambda\left(\gamma^{-1} P\right) - \Lambda(P).$$

But (18) implies that $K_p P_\Psi^m \subset \mathbf{P}_n^{\Gamma_\Psi}$; evaluating at $\gamma_\Psi x - x \otimes P$ with $P \in K_p P_\Psi^m$ gives $\Delta(\gamma_\Psi x - x \otimes P) = 0$.

Hence it remains to prove the equality $\mathbf{c}_x = -\delta\boldsymbol{\Delta}$. By definition $\delta\Delta$ is obtained choosing $\widetilde{\Delta} \in \mathcal{BS}(Hom(\mathbf{P}_n, \mathbf{V}))$ such that $\widetilde{\Delta}(x - y) = \Delta(x - y)$ for every degree zero divisor $x - y$ and then noticing that

$$\begin{aligned} \gamma \quad \mapsto \quad & \gamma\widetilde{\Delta} - \widetilde{\Delta} = \left(\gamma\widetilde{\Delta}\right)(y) - \widetilde{\Delta}(y) = \\ = \quad & \widetilde{\Delta}\left(\gamma^{-1} y\right)\left(\gamma^{-1} -\right) - \widetilde{\Delta}(y)(-) \in Hom(\mathbf{P}_n, \mathbf{V}) \end{aligned}$$

is a constant function, independent of the choice of the divisor y at which to evaluate it. Taking $y = \gamma x$ for any given γ we find that the above cocycle is

$$\gamma \mapsto \widetilde{\Delta}(x)\left(\gamma^{-1} -\right) - \widetilde{\Delta}(\gamma x)(-) \in Hom(\mathbf{P}_n, \mathbf{V})$$

On the other hand, up to the identification

$$Hom\left(\Delta^0 \otimes \mathbf{P}_n, \mathbf{V}\right) = \mathcal{MS}\left(Hom\left(\mathbf{P}_n, \mathbf{V}\right)\right),$$
$$c_x\left(\gamma\right)\left(P\right) = \Delta\left(\gamma x - x\right)\left(P\right) = \widetilde{\Delta}\left(\gamma x\right)\left(P\right) - \widetilde{\Delta}\left(x\right)\left(P\right).$$

Hence the sum $c_x + \delta\Delta$ to be considered is

$$\widetilde{\Delta}\left(\gamma x\right)\left(P\right) - \widetilde{\Delta}\left(x\right)\left(P\right) + \widetilde{\Delta}\left(x\right)\left(\gamma^{-1}P\right) - \widetilde{\Delta}\left(\gamma x\right)\left(P\right) = \widetilde{\Delta}\left(x\right)\left(\gamma^{-1}P\right) - \widetilde{\Delta}\left(x\right)\left(P\right)$$

and we have to show that this is a coboundary.

But now a coboundary in $H^1\left(\Gamma, Hom\left(\mathbf{P}_n, \mathbf{V}\right)\right)$ is of the form $\gamma \mapsto \gamma\Lambda - \Lambda$ with $\Lambda \in Hom\left(\mathbf{P}_n, \mathbf{V}\right)$, i.e. $\left(\partial\Lambda\right)\left(\gamma\right)\left(P\right) = \Lambda\left(\gamma^{-1}P\right) - \Lambda\left(P\right)$. We can now take $\Lambda = \widetilde{\Delta}\left(x\right)\left(-\right) \in Hom\left(\mathbf{P}_n, \mathbf{V}\right)$, so that

$$\left(\partial\left(\widetilde{\Delta}\left(x\right)\left(-\right)\right)\right)\left(\gamma\right)\left(P\right) = \widetilde{\Delta}\left(x\right)\left(\gamma^{-1}P\right) - \widetilde{\Delta}\left(x\right)\left(P\right).$$

The same proof applies for positive oriented $\mathbf{V}$-valued integration theories, exploiting the long exact sequence obtained from (17) in place of (16). ■

Now we are in the position to define the p-adic Abel-Jacobi image of the Darmon cycles.

Definition 2.23 *The p-adic Abel-Jacobi image of the Darmon cycle attached to the embedding* Ψ *is the element*

$$\Phi^{AJ}\left(j_\Psi\right) = \Phi^{AJ}\left(\left[j_\Psi\right]\right) \in \mathbf{D}/F^{m+1},$$

where Φ^{AJ} *is any p-adic Abel-Jacobi map.*

As in [Da] the set $\Gamma\backslash\mathcal{E}mb^+\left(\mathcal{O}, \mathcal{R}\right)$ is naturally endowed with an action by the (narrow) Picard group $Pic^+\left(\mathcal{O}\right)$ attached to the order $\mathcal{O}$. The class field theory identifies canonically $Pic^+\left(\mathcal{O}\right)$ with the Galois group over K of the narrow ring class field $H^+_{\mathcal{O}}$.

$$rec : Pic^+\left(\mathcal{O}\right) \overset{\simeq}{\to} G_{H^+_{\mathcal{O}}/K}.$$

In this way $G_{H^+_{\mathcal{O}}/K}$ acts on $\Gamma\backslash\mathcal{E}mb^+\left(\mathcal{O}, \mathcal{R}\right)$.

Remark 2.24 *As in [Da, after Lemma 5.7] it is possible to introduce the notion of oriented embeddings* $\mathcal{E}mb^{+\mathfrak{d}}(\mathcal{O}, \mathcal{R})$ *by fixing a homomorphism*

$$\mathfrak{d} : \mathcal{O} \to \mathbb{Z}/M\mathbb{Z}.$$

Then Γ *preserves* $\mathcal{E}mb^{+\mathfrak{d}}(\mathcal{O}, \mathcal{R})$*, so that it makes sense to consider the quotient* $\Gamma\backslash\mathcal{E}mb^{+\mathfrak{d}}(\mathcal{O}, \mathcal{R})$ *and this set becomes a torsor under the action of* $Pic^+\left(\mathcal{O}\right)$*. Furthermore, the Atkin-Lehner involution* W_{l^e} *at the primes dividing* $l^e \parallel M$ *transitively permutes the possible orientations, while the Atkin-Lehner involution* W_p *reverses the orientation at p.*

Let $\chi : G_{H^+_{\mathcal{O}}/K} \to \mathbb{C}^\times$ be a character. It will be convenient to introduce the following linear combination

$$j^\chi := \sum_{\sigma \in G_{H^+_{\mathcal{O}}/K}} \chi^{-1}\left(\sigma\right) j_{\sigma\Psi} \in \left(\Delta^0 \otimes \mathrm{Div}\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)^\chi_\Gamma. \qquad (19)$$

3 Review of the p-adic Abel-Jacobi map in the Darmon setting

Consider the following commutative diagram:

$$\begin{array}{ccc} \left(\Delta^0 \otimes \mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma & \overset{\Phi}{\to} & \mathbf{D}/F^{m+1} \\ \| & & \downarrow f \\ \left(\Delta^0 \otimes \mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma & \overset{\log\Phi}{\to} & \mathbf{MS}^{c,\vee,w_\infty} \end{array} \tag{20}$$

where:

- $f(x,y) = -x - \mathcal{L}y$, which is easily checked to be well defined, i.e. $f\left(F^{m+1}\right) = 0$, and is an isomorphism;
- $\log\Phi := \Phi^{\log} - \mathcal{L}\Phi^{\mathrm{ord}}$.

Note also that, by Corollary 2.12,

$$\log\Phi \circ \partial = \left(\Phi^{\log} - \mathcal{L}\Phi^{\mathrm{ord}}\right) \circ \partial = 0,$$

a fact that can also be deduced by the commutativity of the diagram and the equality $\Phi \circ \partial = 0$. Since f is an isomorphism we can identify Φ and $\log\Phi$. It is clear that the above discussion applies to $\mathbf{D}_{[f]}$ or $\mathbf{D}_f$ when f is a modular form. Hence we will write $\log\Phi_f = e_f \circ \log\Phi$.

We will use the following notation for the branches of p-adic logarithm. We let $\log_0$ be the branch of the p-adic logarithm such that $\log_0(p) = 0$ and for every $\lambda \in F_p$ we let

$$\log_\lambda := \log_0 - \lambda\, \mathrm{ord}_p : F_p^\times \to F_p$$

be the branch of the p-adic logarithm such that $\log_\lambda(p) = -\lambda$.

Note that the definition of the monodromy module $\mathbf{D}$, as well as Φ, depends in a crucial way on the choice of a branch of the p-adic logarithm, since $\Phi^{\log}$ depends on this choice. Write $\Phi^{\log_\lambda}$,$\mathcal{L}^\lambda$, Φ^λ and $\log\Phi^\lambda$ to emphasis the dependence on this choice. The dependence on λ appears in $\mathbf{D}$ in the definition of the filtration, so that we write F_λ^{m+1}.

Proposition 3.1 *For every* $\lambda \in F_p$

$$\Phi^{\log_\lambda} = \Phi^{\log_0} - \lambda\Phi^{\mathrm{ord}} \in Hom\left(\left(\Delta^0 \otimes \mathrm{Div}^0\left(\mathcal{H}_p^{ur}\right) \otimes \mathbf{P}_n\right)_\Gamma, \mathbf{MS}^{c,\vee,w_\infty}\right).$$

Proof. We need to evaluate $\Phi^{\log_\lambda}(x - y \otimes \tau_2 - \tau_1 \otimes P)$ at $m \in \mathbf{MS}^{c,\vee,w_\infty}$ in order to prove the proposition. By definition:

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_\lambda\left(\frac{t-\tau_2}{t-\tau_1}\right) P(t)\, dm\{x \to y\}(t) =$$
$$= \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_0\left(\frac{t-\tau_2}{t-\tau_1}\right) P(t)\, dm\{x \to y\}(t) +$$
$$-\lambda \int_{\mathbb{P}^1(\mathbb{Q}_p)} \mathrm{ord}_p\left(\frac{t-\tau_2}{t-\tau_1}\right) P(t)\, dm\{x \to y\}(t).$$

Thus we need to check the formula:

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} \mathrm{ord}_p\left(\frac{t-\tau_2}{t-\tau_1}\right) P(t)\, dm\{x \to y\}(t) = \sum_{e: v_1 \to v_2} \int_{U_e} P(t)\, dm\{x \to y\}(t).$$

The proof of [BDG, Lemma 2.5] gives the claim. ■

Lemma 3.2 *For every $\lambda \in F_p$*

$$\mathcal{L}^\lambda = \mathcal{L}^0 - \lambda.$$

Proof. Proposition 3.1 implies, in light of Corollary 2.12:

$$\begin{aligned}
&\mathcal{L}^\lambda \circ \Phi^{\mathrm{ord}} \circ \partial = \Phi^{\log_\lambda} \circ \partial = \Phi^{\log_0} \circ \partial - \lambda \Phi^{\mathrm{ord}} \circ \partial = \\
&= \mathcal{L}^0 \circ \Phi^{\mathrm{ord}} \circ \partial - \lambda \Phi^{\mathrm{ord}} \circ \partial = \\
&= \left(\mathcal{L}^0 - \lambda\right) \circ \Phi^{\mathrm{ord}} \circ \partial,
\end{aligned}$$

the equality taking place in $Hom\left(H_1\left(\Gamma, \Delta^0 \otimes \mathbf{P}_n\right), \mathbf{MS}^{c,\vee,w_\infty}\right)$.

Now the claim follows from Theorem 2.10, arguing as in Corollary 2.12. ■

Suppose that in diagram (20) we have chosen the standard branch $\log_0$ of the p-adic logarithm. Choosing a different branch $\log_\lambda$ of the p-adic logarithm we find, thanks to Proposition 3.1 and Lemma 3.2:

- $f^\lambda(x, y) = -x - \left(\mathcal{L}^0 - \lambda\right) y$;
- $\log \Phi^\lambda = \Phi^{\log_\lambda} - \mathcal{L}^\lambda \Phi^{\mathrm{ord}} = \log \Phi^0$.

In particular we see that $\log \Phi^\lambda$ does not depend on the choice of a branch of the p-adic logarithm.

Assume now that f is a new modular form. We have $\mathcal{L}_f$ and $\mathcal{L}_{[f]} \in End_{\mathbb{T}_{\mathbb{Q}_p}}\left(\mathbf{MS}_{[f]}^{c,\vee,w_\infty,}\right)$ acts diagonally via the matrix $diag\left(\mathcal{L}_{f^\sigma} : \sigma\right)$ on $\mathbf{MS}_{[f]}^{c,w_\infty}$ with respect to the decomposition (15). Choosing the branch of the p-adic logarithm $\lambda = \mathcal{L}_f^0$, so that $\mathcal{L}_f^\lambda = 0$, the above expressions simplify and become:

- $f^\lambda(x, y) = -x$;
- $\log \Phi_f^0 = \log \Phi_f^\lambda = \Phi_f^{\log_\lambda}$.

Also recall that f^λ is an isomorphism.

Proposition 3.3 *Let $f \in S_{k_0}(\Gamma_0(N))$ be a new modular form. Then the $\mathbf{D}_f/F_0^{m+1}$-valued integration theory Φ_f^0 is equivalent via f^0 to the $\mathbf{MS}_f^{c,w_\infty}$-valued integration theory $\log \Phi_f^0 = \Phi_f^{\log_{\mathcal{L}_f^0}}$.*

4 Families of modular forms and families of modular symbols

Let $\mathcal{W} := Hom_{cont}\left(\mathbb{Z}_p^\times, \mathbb{G}_m\right)$ be the weight space, viewed as a rigid analytic space over $\mathbb{Q}_p$, and suppose for simplicity $p \neq 2$. The integers $\mathbb{Z}$ are embedded in $\mathcal{W}$ by sending the integer k to the function $t \mapsto t^{k-2}$. Note that this normalization follows [BD2] but not [BDI], where the integer k is sent to the function $t \mapsto t^k$. If $U \subset \mathcal{W}$ is an open affinoid defined over the local field K_p, every $\kappa \in U(K_p)$ can be uniquely written as a product $\kappa(t) = \varepsilon(t)\chi(t)\langle t\rangle^s$, where $\varepsilon : \mathbb{Z}_p^\times \to K_p^\times$ is a character of order $p-1$, $\chi : \mathbb{Z}_p^\times \to K_p^\times$ is a character of order p and $s \in \mathcal{O}_{K_p}$. We can uniquely write every element of $\mathbb{Z}_p^\times$ as a product $t = [t]\langle t\rangle$, where $[t] \in \mu_{p-1}$, the group of $p-1$-roots of unity, and $\langle t\rangle \in 1+p\mathbb{Z}_p$. With our normalization an integer $k \in U$ corresponds to the character $k(t) = [t]^{k-2}\langle t\rangle^{k-2}$, i.e. $\varepsilon(t) = [t]^{k-2}$, $\chi = 1$ and $s = k-2$. In general up to shrinking U in a neighbourhood of $k_0 \in \mathbb{Z}$, we can assume $\varepsilon(t) = [t]^{k_0-2}$ and $\chi = 1$ for every $\kappa \in U(K_p)$, so that $\kappa(t) = [t]^{k_0-2}\langle t\rangle^s$. In this case we also set $(\kappa/2)(t) := [t]^{\frac{k_0-2}{2}}\langle t\rangle^{\frac{s}{2}}$. Then we define, for every $\alpha \in \mathbb{Q}_p^{nr,\times}$,

$$\begin{aligned}
\langle\alpha\rangle^{\kappa-k_0} &:= \langle\alpha\rangle^{s-k_0+2} = \exp\left((s-k_0+2)\log_0(\alpha)\right),\\
\langle\alpha\rangle^{\frac{\kappa-k_0}{2}} &:= \langle\alpha\rangle^{\frac{s}{2}-\frac{k_0-2}{2}},\\
\langle\alpha\rangle^{\kappa-\kappa/2-1} &:= \langle\alpha\rangle^{\kappa-k_0}\left(\langle\alpha\rangle^{\frac{\kappa-k_0}{2}}\right)^{-1}\langle\alpha\rangle^{\frac{k_0-2}{2}},\\
\langle\alpha\rangle^{\kappa/2-1} &:= \langle\alpha\rangle^{\frac{\kappa-k_0}{2}}\langle\alpha\rangle^{\frac{k_0-2}{2}}.
\end{aligned}$$

Note that the first two expressions make sense for every $\kappa \in U$, since $\langle\alpha\rangle \in 1+p\mathcal{O}_{\mathbb{Q}_p^{nr}}$ and $\log_0(\alpha) \in p\mathcal{O}_{\mathbb{Q}_p^{nr}}$ (since $p \neq 2$ the exponential converges in $p\mathcal{O}_{\mathbb{C}_p}$); the subsequent two expressions are defined using the other two, i.e. $(-)^{-1}$ and $\alpha^{\frac{k_0-2}{2}}$ have the obvious meaning.

We fix the following notation to be in force for the rest of this paper. We let $W := \mathbb{Q}_p^2 - \{0\}$ be the set of non-zero vectors in $\mathbb{Q}_p^2$ and consider the natural continuous (for the p-adic topologies) projection

$$\begin{aligned}
\pi &: W \to \mathbb{P}^1(\mathbb{Q}_p)\\
\pi&((x,y)) := x/y.
\end{aligned}$$

For any $\mathbb{Z}_p$-lattice L in $\mathbb{Q}_p^2$ we denote by $L' := L - pL$ the set of primitive vectors of L and we write $|L| := p^{\mathrm{ord}_p(\det B)}$, for B any $\mathbb{Z}_p$-basis of L. Recall we let $L_* := \mathbb{Z}_p^2$ be the standard $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$ and we set $L_\infty := \mathbb{Z}_p \oplus p\mathbb{Z}_p$. Recall the Bruhat-Tits tree $\mathcal{T}$ whose set of oriented edges we denoted by $\mathcal{E} = \mathcal{E}(\mathcal{T})$. If $e \in \mathcal{E}$ let $L_{s(e)}$ and $L_{t(e)}$ be lattices whose homothety classes represent the source and the target of e, chosen in such a way that $L_{s(e)} \supset L_{t(e)}$ with index p. To the edge e are associated the open compact subsets $W_e \subset W$ and $U_e \subset \mathbb{P}^1(\mathbb{Q}_p)$ defined by the rules

$$W_e := L'_{s(e)} \cap L'_{t(e)} \text{ and } U_e := \pi(W_e).$$

We remark that W_e depends on the choices of $L_{s(e)}$ and $L_{t(e)}$, so that W_e is well defined (as a function of e) up to multiplication by elements of $\mathbb{Q}_p^\times$. On the other hand U_e is well defined and in fact it is the set of ends originating from e, when making the canonical identification $\mathcal{E}^\infty(\mathcal{T}) = \mathbb{P}^1(\mathbb{Q}_p)$ between ends of $\mathcal{T}$ and $\mathbb{P}^1(\mathbb{Q}_p)$. In particular we recall that these subsets U_e form a basis for the p-adic topology of $\mathbb{P}^1(\mathbb{Q}_p)$. We write $W_\infty = L'_* \cap L'_\infty$ to denote the set W_e obtained from the edge $e_\infty = (v_*, v_\infty)$, where $v_* = [L_*]$ and $v_\infty = [L_\infty]$.

For every open compact subset $X \subset \mathbb{Q}_p^2$ or $X \subset \mathbb{P}^1(\mathbb{Q}_p)$, write $\mathcal{A}(X)$ for the $\mathbb{Q}_p$-space of locally analytic functions on X, as defined in [BDI, Sec. 2]. Denote by $\mathcal{D}(X) := Hom_{cont}(\mathcal{A}(X), \mathbb{Q}_p)$ the continuous $\mathbb{Q}_p$-dual space, called the space of locally analytic distributions on X. As usual, for any $\mu \in \mathcal{D}(X)$ and $F \in \mathcal{A}(X)$, we write $\int_X F d\mu$ to denote the value of μ at F; then it is clear what we mean by $\int_Y F d\mu$, for any open compact subset $Y \subset X$.

We let $\mathbb{GL}_2(\mathbb{Q}_p)$ act on the left on $\mathbb{Q}_p^2$ by viewing elements of $\mathbb{Q}_p^2$ as column vectors. There is an induced action on W and $\mathcal{T}$, as well as an induced action of the subgroup $\mathbb{GL}_2(\mathbb{Z}_p)$ on L'_*; the action of the scalar matrices $\mathbb{Z}_p^\times$ on W preserves the set L' for any lattice L and will be denoted as $t(x, y) := (tx, ty)$.

It follows that $\mathcal{A}(L'_*)$ is endowed with a right $\mathbb{GL}_2(\mathbb{Z}_p)$-action and its continuous dual $\mathbb{D} := \mathcal{D}(L'_*)$ is endowed with a natural left $\mathbb{GL}_2(\mathbb{Z}_p)$-action. Denote by $R := \mathcal{D}(\mathbb{Z}_p^\times)$ the space of locally analytic distributions on $\mathbb{Z}_p^\times$.

There is a natural R-module structure on $\mathbb{D}$,

$$R \times \mathbb{D} \to \mathbb{D} \ \ (r, \mu) \mapsto r\mu,$$

defined by the formula

$$\int_{L'_*} F(x, y)\, d(r\mu)(x, y) := \int_{\mathbb{Z}_p^\times} \left(\int_{L'_*} F(tx, ty)\, d\mu(x, y) \right) dr(t).$$

Fix an integer $k \geq 0$ and let $U \subset \mathcal{W}$ be an affinoid disk such that $k \in U$, defined over a finite extension K_p of $\mathbb{Q}_p$. We can define a structure of R-algebra on the K_p-affinoid algebra $A(U)$ of U by means of the formula

$$r \mapsto \left[\kappa \mapsto \int_{\mathbb{Z}_p^\times} \kappa(t)\, dr(t) \right].$$

We denote by $\mathbb{D}_U := A(U) \widehat{\otimes}_R \mathbb{D}$ the completed tensor product over R. Now fix any $\kappa \in U$ and define, for any $\mathbb{Z}_p^\times$-stable open compact $X \subset \mathbb{Q}_p^2$:

$$\mathcal{A}^{(\kappa)}(X) := \left\{ F \in \mathcal{A}(X) : F(tx, ty) = \kappa(t) F(x, y) \ \text{ for all } t \in \mathbb{Z}_p^\times \right\}.$$

In [BDI, Section 3] it is explained how to define a continuous R-bilinear map

$$\int_X : \mathcal{A}^{(\kappa)}(X) \times \mathbb{D}_U \to K_p,$$

that we denote

$$\int_X F(x, y)\, d\mu_U.$$

For every integer $n \in \mathbb{Z}$ we will also be interested in the subspace $\mathcal{A}_n^{(\kappa)}(W) \subset \mathcal{A}^{(\kappa)}(W)$ consisting of those functions $F \in \mathcal{A}^{(\kappa)}(W)$ such that $F(px, py) = p^n F(x, y)$.

Finally note that for every homogeneous function $F \in \mathcal{A}_n^{(n)}(W)$ of degree n we can consider the locally analytic function on $\mathbb{P}^1(\mathbb{Q}_p)$ with a pole of order at most n at ∞ defined by the rule $F(t) := F(t, 1)$. Conversely given a locally analytic function on $\mathbb{P}^1(\mathbb{Q}_p)$ with a pole of order at most n at ∞ we can consider the homogeneous function of degree n defined by $F(x, y) := y^n F(x/y)$. In this way we establish a $\mathbb{GL}_2$-equivariant bijection between these spaces. The space $\mathbf{P}_n$ with the action previously considered corresponds to the space of homogeneous polynomials of degree n and we will denote again by $P = P(x, y)$ the polynomial attached to $P = P(t)$.

Lemma 4.1 *For all $\alpha \in \mathbb{Q}_p^{nr,\times}$, $\kappa \in U$ and $t \in \mathbb{Z}_p^\times$:*

- $\langle t\alpha \rangle^{\kappa - k_0} = \kappa(t)\, t^{-(k-2)} \langle \alpha \rangle^{\kappa - k_0}$;
- $\langle t\alpha \rangle^{\frac{\kappa - k_0}{2}} = (\kappa/2)(t)\, t^{-\frac{k_0 - 2}{2}} \langle \alpha \rangle^{\frac{\kappa - k_0}{2}}$;
- $\langle t\alpha \rangle^{\kappa - \kappa/2 - 1} = \kappa(t)\, (\kappa/2)^{-1}(t) \langle \alpha \rangle^{\kappa - \kappa/2 - 1}$;
- $\langle t\alpha \rangle^{\kappa/2 - 1} = (\kappa/2)(t) \langle \alpha \rangle^{\kappa/2 - 1}$.

Furthermore, for every $k \in \mathbb{Z} \cap U$ and for every $\alpha \in \mathbb{Z}_p^\times$ we have:

- $\langle \alpha \rangle^{k - k_0} = \alpha^{k - k_0}$;
- $\langle \alpha \rangle^{\frac{k - k_0}{2}} = \alpha^{\frac{k - k_0}{2}}$;
- $\langle \alpha \rangle^{k - k/2 - 1} = \alpha^{k - k/2 - 1}$;
- $\langle \alpha \rangle^{k/2 - 1} = \alpha^{k/2 - 1}$.

Suppose that $X \subset L'_*$ is an open compact subset preserved by the action of $\mathbb{Z}_p^\times$. Whenever $\mu \in \mathbb{D}_U$, $P \in \mathbf{P}_n$ and $\alpha, \beta \in \mathcal{A}(X)$ satisfy $\alpha(tx) = t\alpha(x)$ and $\beta(tx) = t\beta(x)$, it makes sense to consider the functions on U

$$\begin{aligned} \kappa &\mapsto \mu\left(P \langle \alpha \rangle^{\kappa - k_0} \chi_X\right), \\ \kappa &\mapsto \mu\left(P \langle \alpha \rangle^{\frac{\kappa - k_0}{2}} \langle \beta \rangle^{\frac{\kappa - k_0}{2}} \chi_X\right), \\ \kappa &\mapsto \mu\left(\langle \alpha \rangle^{\kappa/2 - 1} \langle \beta \rangle^{\kappa - \kappa/2 - 1} \chi_X\right). \end{aligned}$$

The above functions are analytic.

Proof. One has to first check the homogeneity properties of $\langle \alpha \rangle^{\kappa - k}$ and $\langle \alpha \rangle^{\frac{\kappa - k_0}{2}}$ and then use their properties to check the ones of $\langle \alpha \rangle^{\kappa - \kappa/2 - 1}$ and $\langle \alpha \rangle^{\kappa/2 - 1}$. The second statement follows from the fact that, whenever $k(t) = [t]^{k-2} \langle t \rangle^{k-2}$ is an integer in U, we can assume $[t]^{k-2} = [t]^{k_0-2}$, so that $k \equiv k_0 \bmod (p-1)$. It follows that $[\alpha]^{k-k_0} = 1$ whenever $\alpha \in \mathbb{Z}_p^\times$ and then

$$\langle\alpha\rangle^{k-k_0} := \langle\alpha\rangle^{k-2-k_0+2} = \langle\alpha\rangle^{k-k_0} = \langle\alpha\rangle^{k-k_0} [\alpha]^{k-k_0} = \alpha^{k-k_0}.$$

The claim $\langle\alpha\rangle^{\frac{k-k_0}{2}} = \alpha^{\frac{k-k_0}{2}}$ follows in a similar way and the other two equations follow from the definition of $\langle\alpha\rangle^{\kappa-\kappa/2-1}$ and $\langle\alpha\rangle^{\kappa/2-1}$.

The fact that the above functions are well defined follows because $P\alpha^{\kappa-k}$, $P\alpha^{\frac{\kappa-k_0}{2}}\beta^{\frac{\kappa-k_0}{2}}$ and $\alpha^{\kappa/2-1}\beta^{\kappa-\kappa/2-1}$ belong to $\mathcal{A}^{(\kappa)}$, so that we can apply μ. Finally, to show that they are indeed analytic, one can follow [BDI, Lemma 4.5]. ■

The following proposition will be useful for the computations of the derivatives of p-adic L-functions.

Proposition 4.2 *Let $k_0 \in \mathbb{Z}^{\geq 2}$ and $P \in \mathbf{P}_n$ with $n = k_0 - 2$. For every lattice L and every $\tau_1, \tau_2 \in \mathcal{H}_p$,*

$$\begin{aligned}
\tfrac{d}{d\kappa}\left(\int_{L'} P(x,y)\langle x-\tau_1 y\rangle^{\frac{\kappa-k}{2}}\langle x-\tau_2 y\rangle^{\frac{\kappa-k}{2}}\, dI\{r\to s\}(x,y)\right)_{\kappa=k_0} &= \\
\tfrac{1}{2}\tfrac{d}{d\kappa}\left(\int_{L'} P(x,y)\langle x-\tau_1 y\rangle^{\kappa-k}\, dI\{r\to s\}(x,y)\right)_{\kappa=k_0} &+ \\
\tfrac{1}{2}\tfrac{d}{d\kappa}\left(\int_{L'} P(x,y)\langle x-\tau_2 y\rangle^{\kappa-k}\, dI\{r\to s\}(x,y)\right)_{\kappa=k_0} &
\end{aligned}$$

Proof. Use the explicit formula of [BDI, Remark 4.7] for the derivatives appearing on the right hand side and compare it with an analogous formula for the left hand side. Note also that it makes sense to consider the derivatives in light of Lemma 4.1. ■

4.1 Families of modular symbols

We let f be a weight k_0 newform on the modular curve $X = X_0(N)$, where $N = pM$ is a factorization into prime factors and p is a prime. The Hecke operator at p acts on f with eigenvalues:

$$f \mid U_p = \pm p^{\frac{k_0-2}{2}} f.$$

A p-adic family of modular forms deforming f is the data of an affinoid disk $U \subset \mathcal{W}$ in the weight space, such that $k_0 \in U$ and a formal q-expansion

$$f_\infty = \sum\nolimits_n a_n(\kappa)\, q^n,\ a_n(\kappa) \in A(U)$$

such that:

- For every $k \in U \cap \mathbb{Z}^{\geq k_0}$ the specialization f_k is a weight k modular eigenform;

- $f_{k_0} = f$.

Since the slope of the U_p operator acting on f is strictly less than $k_0 - 1$, there exists such a family, which we assume to be an eigenfamily of modular forms of slope $(k_0 - 2)/2$, up to shrinking U. Note that whenever $k \neq k_0$ the modular form f_k is old at p. There is a unique normalized new eigenform $f_k^{\#} \in S_k(\Gamma_0(M))$ such that

$$f_k(z) = f_k^{\#}(z) - p^{k-1} a_p(k)^{-1} f_k^{\#}(pz). \tag{21}$$

Let $\widetilde{I}_k^{\#} \in \mathcal{MS}_{\Gamma_0(M)}^k(\mathbb{C})$ (resp. $\widetilde{I}_k \in \mathcal{MS}_{\Gamma_0(pM)}^k(\mathbb{C})$) be the modular symbol attached to $f_k^{\#}$ (resp. f_k) by the rule (5). Recall the periods $\Omega_k^{\#\pm} \in \mathbb{C}$ (resp. $\Omega_k^{\pm} \in \mathbb{C}$) attached to $f_k^{\#}$ (resp. f_k) by means of Proposition 2.1, allowing us to define the modular symbols:

$$\begin{aligned} I_k^{\#\pm} &: = \left(\Omega_k^{\#\pm}\right)^{-1} \widetilde{I}_k^{\#} \in \mathcal{MS}_{\Gamma_0(M)}^{k,\pm}(K_k), \\ I_k^{\pm} &: = \left(\Omega_k^{\pm}\right)^{-1} \widetilde{I}_k \in \mathcal{MS}_{\Gamma_0(M)}^{k,\pm}(K_k). \end{aligned}$$

Here K_k is a short notation for the field generated by the Fourier coefficients of $f_k^{\#}$, which is equal to the field generated by the Fourier coefficients of f_k.

We will choose from now on a sign $w_\infty \in \{\pm 1\}$, which is compatible with the same choice that was used to construct the filtered Frobenius module $\mathbf{D}$. We set

$$\begin{aligned} \Omega_k^{\#} &: = \Omega_k^{\# w_\infty}, \ \Omega_k := \Omega_k^{w_\infty}, \\ I_k^{\#} &: = I_k^{\# w_\infty} \text{ and } I_k := I_k^{w_\infty}. \end{aligned}$$

Note also that we may assume $\Omega_k^{\#} = \Omega_k$ thanks to (21). The same formula (21) translates into the following property of the modular symbol $I_k^{\#}$:

$$I_k\{r \to s\}(P) = \widetilde{I}_k^{\#}\{r \to s\}(P) - p^{k-1} a_p(k)^{-1} \widetilde{I}_k^{\#}\{r/p \to s/p\}(P(x, y/p)). \tag{22}$$

Recall the space $\mathbb{D}_U := \mathbb{D} \widehat{\otimes}_R A(U)$ previously introduced. For each $k \in \mathbb{Z}^{\geq 2} \cap U$ define a weight k specialization map

$$\begin{aligned} &\rho_k : \mathcal{MS}_{\Gamma_0(M)}(\mathbb{D}_U) \to \mathcal{MS}_{\Gamma_0(pM)}^k(\mathbb{C}_p) \\ &\rho_k(I)\{r \to s\}(P) := \textstyle\int_{W_\infty} P(x, y)\, dI\{r \to s\}(x, y). \end{aligned}$$

Theorem 4.3 *There exists $I_\infty \in \mathcal{MS}_{\Gamma_0(M)}(\mathbb{D}_U)$ such that:*

- *for every $k \in \mathbb{Z}^{\geq 2} \cap U$, $\rho_k(I_\infty) = \lambda(k) I_k$ for some $\lambda(k) \in \mathbb{C}_p^\times$;*
- $\rho_{k_0}(I_\infty) = I_{k_0}$.

By Shapiro's Lemma the modular symbol $I_\infty \in \mathcal{MS}_{\Gamma_0(M)}(\mathbb{D}_U)$ gives rise to a family of distributions $\{I_L\}_{L\subset\mathbb{Q}_p^2}$ indexed by the lattices in $\mathbb{Q}_p^2$ which is $\widetilde{\Gamma}$-invariant for the natural action of $\widetilde{\Gamma}$ on the induced module

$$\mathcal{C}\left(\mathcal{L}, \mathcal{MS}_{\Gamma_0(M)}\left(\mathcal{D}_U\left(*\right)\right)\right)$$

of maps I_* from the set $\mathcal{L}$ of lattices in $\mathbb{Q}_p^2$ to the disjoint union of the spaces $\mathcal{MS}_{\Gamma_0(M)}(\mathcal{D}_U(L'))$ with $L\in\mathcal{L}$ such that $I_L \in \mathcal{D}_U(L')$. More precisely:

Definition 4.4 *The family $\{I_L\}_{L\in\mathcal{L}}$ is defined by the rule*

$$I_L\{r\to s\}(F) := I_{L_*}\{\gamma r\to\gamma s\}\left(F\mid\gamma^{-1}\right) = \int_{L'_*}\left(F\mid\gamma^{-1}\right) I_{L_*}\{\gamma r\to\gamma s\},$$

for any locally analytic function $F\in\mathcal{A}^U(L')$, where $\gamma L = L_*$ and $\gamma\in\widetilde{\Gamma}$.

Lemma 4.5 *Let $\kappa\in U$ and let $L_2\subset L_1$ be an index p sublattice of L_1 and let $e = ([L_1],[L_2])$ be the corresponding edge. Then*

$$I_{L_2}\{r\to s\}(F) = a_p I_{L_1}\{r\to s\}(F)$$

for every locally analytic function $F\in\mathcal{A}^\kappa(W_e)$.

Proof. [BDI, Lemma 6.3]. ■

The specialization property of $I_\infty \in \mathcal{MS}_{\Gamma_0(pM)}(\mathbb{D}_U)$ can be explicitly written

$$I_\infty\{r\to s\}\left(P\chi_{W_\infty}\right) = \lambda(k)\, I_k\{r\to s\}(P) \text{ for every } P\in\mathbf{P}_{k-2}. \qquad (23)$$

The following Corollary describes the specialization in terms of the modular symbol $I_k^\#$.

Corollary 4.6 *For all $k\in\mathbb{Z}\cap U$ and all $P\in\mathbf{P}_{k-2}$*

$$I_\infty\{r\to s\}(P) = \lambda(k)\left(1-p^{k-2}a_p(k)^{-2}\right) I_k^\#\{r\to s\}(P).$$

Proof. This is proved in [BD3, Proposition 2.4] using Lemma 4.5, (22) and (21). ■

For every lattice L define a modular symbol $\pi_*(I_L)\in\mathcal{MS}\left(\mathcal{D}^{k_0-2}\left(\mathbb{P}^1(\mathbb{Q}_p)\right)\right)$ by the rule

$$\pi_*(I_L)\{r\to s\}(F) := |L|^{-\frac{k_0-2}{2}} I_L\{r\to s\}(F(x,y)), \qquad (24)$$

where F is a locally analytic function on $\mathbb{P}^1(\mathbb{Q}_p)$ with a pole of order at most k_0-2 at ∞ and $F(x,y) := y^{k_0-2}F(x/y)$. Recall the exact sequence (8). Thanks to the new assumption on f it can be used to attach to the modular symbol $I_{k_0}\in\mathcal{MS}^k_{\Gamma_0(pM)}(K_{k_0})$ an harmonic modular symbol $I_{k_0}^{har}$ belonging to

$$\mathbf{MS} := \mathcal{MS}_\Gamma\left(\mathcal{D}_n^{0,b}\left(\mathbb{P}^1(\mathbb{Q}_p)\right)\right) = \mathcal{MS}_\Gamma\left(\mathcal{C}_{har}(\mathcal{E},\mathbf{V}_n)\right),$$

where the identification is provided by Proposition 2.8.

Corollary 4.7 *For all lattices L such that [L] is even,*

$$\pi_* (I_L) = I_{k_0}^{har},$$

the modular symbol in $\mathcal{MS}_\Gamma \left(\mathcal{D}_0^{k_0-2}\left(\mathbb{P}^1(\mathbb{Q}_p)\right)\right)$ *attached to* f.

Proof. This is a consequence of Lemma 4.5 together with the specialization property (23), see [BDI, Proposition 6.4]. Our restriction to even lattices, that does not appear in [BDI], is a consequence of the fact that we are not assuming f to be a split modular form as in [BDI] (compare with [BD2, Proposition 2.12], where the analogous result is proved in the definite weight 2 setting). ■

The following definition is justified by Lemma 4.1.

Definition 4.8 *The semidefinite integral attached to* $r, s \in \mathbb{P}^1(\mathbb{Q})$, $\tau \in \mathcal{H}_p^+(\mathbb{Q}_p^{ur})$ *and* $P \in \mathbf{P}_n$ *is defined by the formula*

$$\textstyle\int_r^s \int^\tau P\omega_f := |L_\tau|^{-\frac{k_0-2}{2}} \frac{d}{d\kappa}\left(\int_{L'_\tau} P(x,y)\langle x-\tau y\rangle^{\kappa-k_0}\, d\mu_{L_\tau}\{r\to s\}(x,y)\right)_{\kappa=k_0},$$

where $[L_\tau] = red(\tau)$.

We remark that the above formula do not depend on the choice of the representative L_τ, since

$$\log_0(px - p\tau y) P(px, py) = p^{k_0-2}\log_0(x-\tau y) P(x,y).$$

Proposition 4.9 *For every* $\gamma \in \Gamma$ *and every* $\tau \in \mathcal{H}_p^+(\mathbb{Q}_p^{ur})$

$$\textstyle\int_{\gamma r}^{\gamma s}\int^{\gamma\tau} P\omega_f = \int_{\gamma r}^{\gamma s}\int^{\gamma\tau}(P \mid \gamma)\,\omega_f.$$

Proof. [BDI, Proposition 6.6]. ■

Recall the harmonic cocycle I_f.

Proposition 4.10 *For every* $\tau_1, \tau_2 \in \mathcal{H}_p^+(\mathbb{Q}_p^{ur})$

$$\textstyle\int_r^s\int^{\tau_2} P\omega_f - \int_r^s\int^{\tau_1} P\omega_f = \int_r^s\int_{\tau_1}^{\tau_2} P\omega_f^{\log_0} +$$
$$\textstyle 2p^{-\frac{k_0-2}{2}} a'_p(k_0) \sum_{e:red(\tau_1)\to red(\tau_2)} I_{k_0}^{har}(e)\{r\to s\}(P).$$

Proof. This formula is proved in the split case in [BDI, Proposition 6.7]. The methods of the proof adapt to the non-split setting as explained in [BD2, Proposition 2.19]. ■

Combining Proposition 4.9 and 4.10 with the main result of [Co] yields the following:

Theorem 4.11 *(Exceptional zero conjecture) Let f be a new modular form. Then $\mathbf{D}_f \simeq \mathbb{D}_{st}(V_f)$, the filtered Frobenius module attached to the modular form f.*

Corollary 4.12 *Choose the branch of the p-adic logarithm corresponding to $\lambda = \mathcal{L}_f^0$. Then the symbol $\int_r^s \int^\tau P\omega_f$ satisfies*

$$\int_r^s \int^{\tau_2} P\omega_f - \int_r^s \int^{\tau_1} P\omega_f = \int_r^s \int_{\tau_1}^{\tau_2} P\omega_f^{\log_\lambda}.$$

Proof. The Corollary follows by combining Proposition 4.10 and Proposition 3.1. ∎

Corollary 4.13 *We have*

$$\int_x^{\gamma_\Psi x} \int^{\tau_\Psi} P_\Psi^m \omega_f = D_K^{\frac{k_0-2}{4}} \log \Phi^{AJ}(j_\Psi)(I_f).$$

Proof. By Corollary 4.12 the $\mathbf{MS}_f^{c,\vee,w_\infty}$-valued semidefinite integration theory $\int_r^s \int^{\tau_2} P\omega_f$ lifts the integration theory $\int_r^s \int_{\tau_1}^{\tau_2} P\omega_f^{\log_{\mathcal{L}_f^0}}$. Hence the claim follows from Proposition 3.3 and Proposition 2.22, which allows us to compute the p-adic Abel-Jacobi image of the Darmon cycles using any p-adic Abel-Jacobi map. ∎

4.2 Families of modular forms on definite quaternion algebras

Let N^- be a squarefree positive integer divisible by an odd number of primes and let B be the rational definite quaternion algebra ramified at $N^-\infty$. Let $\mathcal{O}_B$ be any maximal order in B. Write $\widehat{\mathbb{Z}}$ to denote the profinite completion of $\mathbb{Z}$ and set $\widehat{B} := B \otimes \widehat{\mathbb{Z}}$. Let $\Sigma = \prod_l \Sigma_l$ be any decomposable open compact subgroup of $\widehat{B}^\times$ and let $\mathbf{V}$ be any K_p-vector space, equipped with a left action of Σ_p.

For every prime l let $\mathbb{H}_l$ be a the unique (up to isomorphism) quaternion division algebra over $\mathbb{Q}_l$. We can choose $\mathbb{Q}_l$-algebra isomorphisms $\iota_l : B \otimes \mathbb{Q}_l \overset{\simeq}{\to} \mathbb{M}_2(\mathbb{Q}_l)$ sending $\mathcal{O}_B \otimes \mathbb{Z}_l$ isomorphically onto $\mathbb{M}_2(\mathbb{Z}_l)$ for every $l \nmid N^-\infty$, as well as $\mathbb{Q}_l$-algebra isomorphisms $\iota_l : B \otimes \mathbb{Q}_l \overset{\simeq}{\to} \mathbb{H}_l$ for every $l \mid N^-\infty$, so that $\iota_l(\mathcal{O}_B \otimes \mathbb{Z}_l)$ is the unique maximal order $\mathcal{O}_{\mathbb{H}_l}$ of $\mathbb{H}_l$. Setting

$$\widetilde{\Gamma}_\Sigma := \iota_p\left(\mathcal{O}_B[1/p] \cap \prod_{l\neq p} \Sigma_l\right) \tag{25}$$

and letting Γ_Σ be the subgroup of $\widetilde{\Gamma}_\Sigma$ of elements of determinant 1, we can give the following ad hoc definition of a $\mathbf{V}$-valued p-adic automorphic form on B of level Σ (see [BDI, Sec. 1]):

Definition 4.14 *A $\mathbf{V}$-valued p-adic automorphic form on B of level Σ is a function*

$$\varphi : \mathbb{GL}_2(\mathbb{Q}_p) \to \mathbf{V} \text{ such that } \varphi(\gamma g u) = u^{-1}\varphi(g)$$
$$\text{for all } \gamma \in \widetilde{\Gamma}_\Sigma,\ g \in \mathbb{GL}_2(\mathbb{Q}_p) \text{ and } u \in \iota_p(\Sigma_p).$$

The K_p-vector space of $\mathbf{V}$-valued p-adic automorphic forms of level Σ will be denoted $S(\Sigma, V)$.

We will always assume $\iota_p(\Sigma_p) = \Gamma_0(p\mathbb{Z}_p)$ and we write Σ_∞ to denote the open compact obtained from Σ by replacing the local condition at p with the local condition $\iota_p(\Sigma_{\infty,p}) = \mathbb{GL}_2(\mathbb{Z}_p)$. When $\mathbf{V} = \mathbf{V}_{k-2}$ we will simply write $S_k(\Sigma)$. Specializing to the case $\mathbf{V} = \mathbb{D}_U$ we obtain the notion of p-adic family of automorphic forms (here again U is an affinoid disk in the weight space).

Definition 4.15 *The space of p-adic families of automorphic forms on B of level Σ parametrized by weights in U is by definition*

$$\mathbb{S}_U(\Sigma) := S(\Sigma_\infty, \mathbb{D}_U).$$

The space $\mathbb{S}_U(\Sigma)$ of p-adic families of automorphic forms on B of level Σ comes equipped with specializations maps for every $k \in U \cap \mathbb{Z}^{\geq 2}$:

$$\begin{aligned} &\rho_k : \mathbb{S}_U(\Sigma) \to S_k(\Sigma), \\ &(\rho_k(\Phi)(g))(P) := \textstyle\int_{W_\infty} P(x,y)\, d\Phi(g), \end{aligned}$$

where $P \in \mathbf{P}_{k-2}$ and $P(x,y) := y^{k-2}P(x/y)$ is the corresponding degree $k-2$ homogeneous polynomial, an element of $\mathcal{A}^{(n)}(W_\infty)$. Note that the definition of ρ_k depends on the choice of W_∞, that was only defined up to multiplication by $\mathbb{Q}_p^\times$; we choose $W_\infty := L'_* \cap L'_\infty$.

We now choose the level structure as follows. Let $N = pN^+N^-$ be a factorization into prime factors of our given integer N, where N^- corresponds to the finite primes of ramification of B. Define the open compact group Σ by the following local conditions:

$$\Sigma_l = \begin{cases} (\mathcal{O}_B \otimes \mathbb{Z}_l)^\times & l \nmid N^+p \\ \iota_l^{-1}(\Gamma_0(N\mathbb{Z}_l)) & l \mid N^+p \end{cases} \tag{26}$$

Write $\widetilde{\Gamma}' = \widetilde{\Gamma}_\Sigma$ for the corresponding group as well as $\Gamma' = \Gamma_\Sigma$.

By Jacquet-Langlands, the modular form $f = f_{k_0}$ that was fixed in the previous section corresponds to a modular form $\varphi = \varphi_{k_0}$ in the above sense for the above choice of the level. The functions f_k (resp. $f_k^{\#}$) similarly correspond to functions φ_k (resp. $\varphi_k^{\#}$), for the suitable choice of the level (26) (resp. Σ_∞). Since the stabilizer of the standard edge e_* is $\Gamma_0(p\mathbb{Z}_p)$, it is possible to attach to the modular form φ_{k_0} a cocycle $c_{k_0} \in \mathcal{C}(\mathcal{E}, \mathbf{V}_{k_0-2})^{\widetilde{\Gamma}'}$ by the rule

$$c_{k_0}(e) := \widetilde{c}_\varphi(e) := p^{-n/2\,\mathrm{ord}_p(\det(g))} g\varphi(g), \text{ if } e = ge_*.$$

We note that, since φ is new at p, the above cocyle satisfies the rules

$$\textstyle\sum_{s(e)=v} c(e) = 0,\ \sum_{t(e)=v} c(e) = 0 \text{ and } c(\overline{e}) = w_p c(e),$$

where $w_p \in \{\pm 1\}$ is the sign of the Atkin-Lehner involution at p, which is equal to -1 if c is of split multiplicative type and is equal to 1 if c is of non-split multiplicative type (see [BD3, Prop. 1.4] or [BDIS, pag. 32]). Let $\Gamma' \subset \widetilde{\Gamma}'$ be the subgroup of those elements having norm 1. To the cocycle c_{k_0} one can attach an harmonic cocycle ino $\mathcal{C}_{har}\left(\mathcal{E}, \mathbf{V}_{k_0-2}\right)^{\Gamma'}$ by the rule

$$c^{har}(e) := \begin{cases} c(e) & \text{when } e \in \mathcal{E}^{+} \\ -c(\overline{e}) & \text{when } e \in \mathcal{E}^{-} \end{cases}.$$

Let $X = X_{N^+, pN^-}$ be the Shimura curve attached to the indefinite quaternion algebra $\mathcal{B}$ ramified at the primes dividing pN^- and the choice of an Eichler order $\mathcal{R} = \mathcal{R}_{N^+, pN^-}$ of level N^+. By the Theorem of Cerednik-Drinfeld the above Shimura curve admits a rigid analytic uniformization at p. The modular form f corresponds to a rigid analytic modular form f^{rig} again by the Jacquet-Langlands correspondence and the cocycle c^{har} is precisely the cocyle attached to f^{rig} by taking the residues. As a consequence of the Theorem of Amice-Velu-Teitelbaum we may attach to the harmonic cocycle c^{har} a unique locally analytic distribution $\mu \in \mathcal{D}_n^{0,b}\left(\mathbb{P}^1\left(\mathbb{Q}_p\right)\right)^{\Gamma'}$ such that $R(\mu) = c^{har}$. This is the analogue of Proposition 2.8 in this definite setting.

Write $\mathcal{L}$ for the set of lattices in $\mathbb{Q}_p^2 - \{0\}$ and $\mathcal{L}_0$ for the set of couples (L_1, L_2) such that $L_1 \supset L_2$. Without the normalizing condition obtained by multiplying by the determinant $p^{-n/2\,\mathrm{ord}_p(\det(g))}$ it is also possible to attach to φ_{k_0} an element (that we will denote by the same symbol) $c_{k_0} \in \mathcal{C}\left(\mathcal{L}_0, \mathbf{V}_{k_0-2}\right)^{\widetilde{\Gamma}'}$. The same construction works for the modular forms φ_k (resp. $\varphi_k^{\#}$) producing elements $c_k \in \mathcal{C}\left(\mathcal{L}_0, \mathbf{V}_{k-2}\right)^{\widetilde{\Gamma}'}$ (resp. $c_k^{\#} \in \mathcal{C}\left(\mathcal{L}, \mathbf{V}_{k-2}\right)^{\widetilde{\Gamma}'}$) defined by the rule

$$\begin{aligned} &c_k\left(L_1, L_2\right) := g\varphi(g) \text{ if } \left(L_1, L_2\right) = g\left(L_*, L_\infty\right) \\ &(\text{resp. } c_k(L) := g\varphi(g) \text{ if } L = gL_*). \end{aligned}$$

We further normalize the cocyles $c_k^{\#}$ for $k \neq k_0$ by the requirement:

$$\left\langle c_k^{\#}, c_k^{\#} \right\rangle = 1, \tag{27}$$

where the inner product is the one defined in [BD2, End of Section 2.2]. Then the modular form $c_k^{\#}$ is uniquely determined up to sign. The relation (21) translates into

$$\begin{aligned} c_k\left(L_1, L_2\right) &= c_k^{\#}\left(L_2\right) - p^{k-2} a_p(k)^{-1} c_k^{\#}\left(L_1\right) = \\ &= c_k^{\#}\left(L_2\right) - a_p(k)^{-1} c_k^{\#}\left(pL_1\right) \end{aligned} \tag{28}$$

In fact, the correspondence can be merged in families:

Theorem 4.16 *There exists a family $\varphi_\infty \in \mathbb{S}_U(\Sigma)$ such that:*

- *for every $k \in \mathbb{Z}^{\geq 2} \cap U$, $\rho_k\left(\varphi_\infty\right) = \lambda_B(k)\varphi_k$ for some $\lambda_B(k) \in \mathbb{C}_p^{\times}$;*
- $\rho_{k_0}\left(\varphi_\infty\right) = \varphi_{k_0}$.

Denote by $\mathcal{C}(\mathcal{L}, \mathcal{D}(*))$ the space of maps μ_* from $\mathcal{L}$ to $\sqcup_{L\in\mathcal{L}}\mathcal{D}(L')$ such that $\mu_L \in \mathcal{D}(L')$. Define $\mathcal{C}(\mathcal{L}_0, \mathcal{D}(*))$ in a similar way, this time $\mu_{L_1,L_2} \in \mathcal{D}(W_{L_1,L_2})$, where $W_{L_1,L_2} := L_1' \cap L_2'$. The function on the lattices attached to φ_∞ obtained by Shapiro's Lemma will be denoted $\mu_* \in \mathcal{C}\left(\mathcal{L}, \mathcal{D}^U(*)\right)^{\widetilde{\Gamma}'}$.

Lemma 4.17 *Let $\kappa \in U$ and let $L_2 \subset L_1$ be an index p sublattice of L_1 and let $e = ([L_1], [L_2])$ be the corresponding edge. Then*

$$\mu_{L_2}(F) = \left(a_p \mu_{L_1}\right)(F)$$

for every locally analytic function $F \in \mathcal{A}^\kappa(W_e)$.

Proof. [BDI, Lemma 4.3] ■

The specialization property of $\varphi_\infty \in \mathbb{S}_U(\Sigma)$ can be explicitly written as

$$\varphi_\infty(g)\left(P\chi_{W_\infty}\right) = \lambda(k)\varphi_k(g)(P) \text{ for every } P \in \mathbf{P}_{k-2} \text{ and } g \in \mathbb{GL}_2(\mathbb{Q}_p).$$

In terms of μ_* and c_k this property can be restated as follows.(see [BD2, Lemma 2.10]):

$$\mu_{L_1}\left(P\chi_{L_1,L_2}\right) = \lambda(k)c_k(L_1, L_2)(P) \text{ for every } P \in \mathbf{P}_{k-2} \text{ and } L_1 \supset L_2 \quad (29)$$

The following corollary expresses the specialization in terms of $c_k^{\#}$.

Corollary 4.18 *For all $k \in \mathbb{Z} \cap U$ and all $P \in \mathbf{P}_{k-2}$,*

$$\mu_L(P) = \lambda_B(k)a_p(k)\left(1 - p^{k-2}a_p(k)^{-2}\right)c_k^{\#}(L)(P).$$

Proof. This is proved in [BD2, Proposition 2.11] using Lemma 4.17, (29) and (28). ■

For every lattice L define a locally analytic distribution $\pi_*(\mu_L)$ which belongs to $\mathcal{D}^{k_0-2}\left(\mathbb{P}^1(\mathbb{Q}_p)\right)$ by the rule

$$\pi_*(\mu_L)(F) := |L|^{-\frac{k_0-2}{2}}\mu_L(F(x,y)), \quad (30)$$

where F is a locally analytic function on $\mathbb{P}^1(\mathbb{Q}_p)$ with a pole of order at most $k_0 - 2$ at ∞ and $F(x,y) := y^{k_0-2}F(x/y)$.

Corollary 4.19 *For all lattices L such that $[L]$ is even,*

$$\pi_*(\mu_L) = \mu,$$

the measure in $\mathcal{D}_0^{k_0-2}\left(\mathbb{P}^1(\mathbb{Q}_p)\right)^{\Gamma'}$ attached to f.

Proof. This is a consequence of Lemma 4.17 together with the specialization property $\rho_{k_0}(\varphi_\infty) = \varphi_{k_0}$, see [BDI, Proposition 4.4]. Our restriction to even lattices, which does not appear in [BDI, Proposition 6.4], is again a consequence of the fact that we are not assuming φ to be split modular form as in [BDI] (compare with [BD2, Proposition 2.12], where the analogous result is proved in the weight 2 setting). ■

The following definition is justified by Lemma 4.1.

Definition 4.20 *The indefinite integral attached to* $\tau \in \mathcal{H}_p^+(\mathbb{Q}_p^{ur})$ *and* $P \in \mathbf{P}_n$ *is*

$$\int^\tau P\omega_f := |L_\tau|^{-\frac{k_0-2}{2}} \frac{d}{d\kappa}\left(\int_{L'_\tau} P(x,y)\langle x-\tau y\rangle^{\kappa-k_0} d\mu_{L_\tau}(x,y)\right)_{\kappa=k_0}$$

where $[L_\tau] = red(\tau)$.

Proposition 4.21 *For every* $\gamma \in \Gamma'$ *and every* $\tau \in \mathcal{H}_p^+$

$$\int^{\gamma\tau} P\omega_f = \int^\tau (P \mid \gamma)\,\omega_f.$$

Proof. [BDI, Proposition 4.4] ■

Proposition 4.22 *For every* $\tau_1, \tau_2 \in \mathcal{H}_p^+(\mathbb{Q}_p^{ur})$

$$\int^{\tau_2} P\omega_f - \int^{\tau_1} P\omega_f = \int_{\tau_1}^{\tau_2} P\omega_f + 2p^{-\frac{k_0-2}{2}} a'_p(k_0) \sum_{e:red(\tau_1)\to red(\tau_2)} c^{har}(e)(P).$$

Proof. This is [BDI, Proposition 4.10]. Again the restriction to even elements of $\mathcal{H}_p^+(\mathbb{Q}_p^{ur})$, which does not appear in [BDI], is a consequence of the fact that we are not assuming that φ is a split form. As explained in [BD2, Proposition 2.19] in the weight 2 setting, the non-split case can be similarly treated up to restricting to $\mathcal{H}_p^+(\mathbb{Q}_p^{ur})$ and the ideas of the proof readily adapts to the higher weight case, in order to remove the restriction appearing in [BDI]. ■

5 p-adic L-functions

5.1 The Mazur-Kitagawa p-adic L-functions

Let $g \in S_k(\Gamma_0(N))$ be an eigenform and recall the modular symbol $I_g \in \mathcal{MS}^{k,w_\infty}_{\Gamma_0(M)}(K_g)$ attached to g by means of Proposition 2.1 and the choice of a sign w_∞. Define, for our fixed g and $m \in \mathbb{N}^{>0}$, the function

$$\begin{aligned} I_{g,m}[P,a] &: \mathbf{P}_{k-2}(K_g) \times \mathbb{Z}/m\mathbb{Z} \to K_g \\ I_{g,m}[P,a] &: = I_g\{\infty \to a/m\}(P), \end{aligned}$$

where the fact that $I_{g,m}[P,a]$ depends only on the class of a in $\mathbb{Z}/m\mathbb{Z}$ follows from the invariance of I_g under the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the relation

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -a \\ 0 & m \end{pmatrix} = \begin{pmatrix} 1 & -a+m \\ 0 & m \end{pmatrix}.$$

Let now χ be any primitive Dirichlet character modulo m and consider the Gauss sum $\tau(\chi) := \sum_{a\in\mathbb{Z}/m\mathbb{Z}} \chi(a)\, e^{2\pi ia/m}$.

Proposition 5.1 *Let $1 \le j \le k-1$ be an integer and let χ be a character such that $\chi(-1) = (-1)^{k-j-1} w_\infty = (-1)^{j-1} w_\infty$ (since k is even). Then*

$$\sum\nolimits_{a\in\mathbb{Z}/m\mathbb{Z}} \chi(a)\, I_{g,m}[P_{j,a}, a] = \frac{(j-1)!\tau(\overline{\chi})}{(-2\pi i)^{j-1}\,\Omega_g} L(g,\chi,j) =: L^*(g,\chi,j),$$

where

$$P_{j,a} := \left(x - \frac{a}{m}y\right)^{j-1} y^{k-j-1}.$$

Proof. As explained in [BD2, Proposition 1.3] the above formula is a consequence of the formula of Birch and Manin expressing special values of L-series in terms of modular symbols that can be found in [MTT, Formula (8.6)], after taking into account that I_g belongs to the w_∞-eigenspace for the W_∞-action and we are assuming $\chi(-1) = (-1)^{k-j-1} w_\infty$. The assumption that χ is a quadratic character appearing in [BD2], which is done in view of the applications, is not needed. ■

To the modular symbol I_∞ we attach the symbol

$$\begin{aligned} I_{\infty,m}[F,a] &: \mathcal{A}^U(L'_*) \times \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}_p \\ I_{\infty,m}[F,a] &: = I_\infty\{\infty \to a/m\}(F). \end{aligned}$$

The following definition attaches a p-adic L-function

$$\begin{aligned} &L_p(f,\chi,\kappa,s) : U \times \mathbb{Z}_p \to \mathbb{C}_p \\ &(\kappa,s) \mapsto L_p(f,\chi,\kappa,s). \end{aligned}$$

to the data of f and a Dirichlet character.

Definition 5.2 *The Mazur-Kitagawa p-adic L-function attached to (f,χ), where $\chi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}^\times$ is a character of conductor m, is defined by the rule*

$$\begin{aligned} L_p(f,\chi,\kappa,s) \quad &: \quad = \sum\nolimits_{a\in\mathbb{Z}/m\mathbb{Z}} \chi(ap) \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \left(x - \frac{pa}{m}y\right)^{s-1} y^{\kappa-s-1} dI_\infty\left\{\infty \to \frac{pa}{m}\right\} = \\ &= \quad \sum\nolimits_{a\in\mathbb{Z}/m\mathbb{Z}} \chi(ap)\, I_{\infty,m}\left\{\infty \to \frac{pa}{m}\right\}\left(F_{s,pa}\chi_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times}\right), \end{aligned}$$

where

$$F_{s,pa} := \left(x - \frac{pa}{m}y\right)^{s-1} y^{\kappa-s-1}.$$

Note that whenever $(x,y) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ and $\chi(ap) \neq 0$ we have $(ap,m) = 1$, so that $m \in \mathbb{Z}_p^\times$ and

$$x - \frac{pa}{m} y \in \mathbb{Z}_p^\times + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times. \tag{31}$$

We have defined the above Mazur-Kitagawa p-adic L-function as a two variable function and it is indeed analytic in both variables. For the applications we have in mind it is sufficient to consider the restriction of this function to the critical line $(\kappa, \kappa/2)$. In this case we are fully justified by Lemma 4.1, since the function of the κ-variable we have defined is the linear combination of functions of the form:

$$\kappa \mapsto \mu\left(\alpha^{\kappa/2-1}\beta^{\kappa-\kappa/2-1}\chi_X\right) = \mu\left(\langle\alpha\rangle^{\kappa/2-1}\langle\beta\rangle^{\kappa-\kappa/2-1}\chi_X\right).$$

Here the equality follows from (31) and again Lemma 4.1 when $\kappa = k \in \mathbb{Z} \cap U$. Hence the right hand side can be taken as a definition, while the notation on the left hand side for more general $\kappa \in U$ suggests what is the value at the integers. This is indeed needed in order to investigate the interpolation properties of the Mazur-Kitagawa p-adic L-function, as it is done in the subsequent theorem.

Theorem 5.3 *Assume χ is a primitive character, $k \in U \cap \mathbb{Z}$ and $1 \leq j \leq k-1$ satisfies $\chi(-1) = (-1)^{j-1} w_\infty$. Then*

$$L_p(f,\chi,k,j) = \lambda(k)\left(1 - \chi(p)p^{j-1}a_p(k)^{-1}\right)L^*(f_k,\chi,j).$$

Proof. In light of the preceding remarks we can appeal to the proof of [BD2, Theorem 1.12], which uses Proposition 5.1 after a direct calculation (again the assumption that χ is a quadratic character is not needed). Note that strictly speaking we are only allowed to move along the line $(k,k/2)$, since we have not defined $L_p(f,\chi,\kappa,s)$ out of the line $(\kappa,\kappa/2)$ but rather remarked that it could be done; in other words following the proof of [BD2, Theorem 1.12] with $j = k/2$ we can prove the subsequent Corollary 5.4. ■

What really matters is the following corollary, which specializes to $j = k/2$ Theorem 5.3 and expresses the interpolation property in terms of the modular form $f_k^\#$ using the relation

$$L^*(f_k,\chi,j) = \left(1 - \chi(p)p^{k-j-1}a_p(k)^{-1}\right)L^*\left(f_k^\#,\chi,j\right),$$

which follows from (21).

Corollary 5.4 *Assume $\chi(-1) = (-1)^{\frac{k-2}{2}} w_\infty$. Then*

$$L_p(f,\chi,k,k/2) = \lambda(k)\left(1 - \chi(p)p^{\frac{k-2}{2}}a_p(k)^{-1}\right)^2 L^*\left(f_k^\#,\chi,k/2\right),$$

where

$$L^*\left(f_k^\#,\chi,k/2\right) := \frac{(k/2-1)!\tau(\chi)}{(-2\pi i)^{k/2-1}\Omega_k} L\left(f_k^\#,\chi,k/2\right).$$

5.2 p-adic L-functions attached to real quadratic fields

In this subsection we let $K/\mathbb{Q}$ be a real quadratic field such that:

- p is inert in K;
- all the prime factors of M split in K.

Let $\Psi \in \mathcal{E}mb^+(\mathcal{O}, \mathcal{R})$ be an optimal embedding of conductor c prime to D_K, the discriminant of $K/\mathbb{Q}$, and N. Denote by $G_{H^+_{\mathcal{O}}/K}$ the Galois group of the corresponding narrow ring class field. Recall the data $(\tau_\Psi, P_\Psi, \gamma_\Psi)$ attached to it and further consider a $\mathbb{Z}_p$-lattice L_Ψ such that $[L_\Psi] = v_\Psi$. The following definition attaches a p-adic L-function

$$\begin{aligned} \mathcal{L}_p(f/K, \Psi, -) &: U \to \mathbb{C}_p \\ \kappa &\mapsto \mathcal{L}_p(f/K, \Psi, \kappa) \end{aligned}$$

to the above data. It is easily checked that the definition below does not depend on the choice of L_Ψ.

Definition 5.5 *Let $r \in \mathbb{P}^1(\mathbb{Q})$ be any base point. The partial p-adic L-function attached to $(f/K, \Psi)$ is*

$$\mathcal{L}_p(f/K, \Psi, \kappa) := |L_\Psi|^{-\frac{k_0-2}{2}} \int_{L'_\Psi} \langle P_\Psi(x,y)\rangle^{\frac{\kappa-k_0}{2}} P_\Psi^m(x,y)\, dI_{L_\Psi}\{r \to \gamma_\Psi r\}.$$

The partial p-adic L-function attached to $(f/K, \chi)$, where $\chi : G_{H^+_{\mathcal{O}}/K} \to \mathbb{C}^\times$ is a character, is

$$\mathcal{L}_p(f/K, \chi, \kappa) := \sum_{\sigma \in G_{H^+_{\mathcal{O}}/K}} \chi^{-1}(\sigma)\, \mathcal{L}_p(f/K, \sigma\Psi, \kappa).$$

The p-adic L-function attached to $(f/K, \chi)$, where $\chi : G_{H^+_{\mathcal{O}}/K} \to \mathbb{C}^\times$ is a character, is

$$L_p(f/K, \chi, \kappa) := \mathcal{L}_p(f/K, \chi, \kappa)^2.$$

In order to justify the above definition and the fact that the above p-adic L-functions are analytic we can appeal to Lemma 4.1 after noticing that they are built from functions of the form $\kappa \mapsto \mu\left(P\langle\alpha\rangle^{\frac{\kappa-k_0}{2}}\langle\beta\rangle^{\frac{\kappa-k_0}{2}}\chi_X\right)$.

Remark 5.6 *The above p-adic L-functions depend, of course, on the choice of the modular symbol I_∞ that was used to define the family $\{I_L\}_{L\subset\mathbb{Q}_p^2}$. It can be shown that the definition depends only on the class of Ψ in $\Gamma\backslash\mathcal{E}mb^+(\mathcal{O}, \mathcal{R})$. It turns out that many of the properties of these p-adic L-functions actually depend only on f/K.*

We note that a suitable choice of the lattice L_Ψ can be made as follows. Since the group Γ acts transitively on the positive vertices $\mathcal{V}^+$ we can choose $\gamma \in \Gamma$ such that $\gamma v_\Psi = v_*$. Hence $v_* = v_{\gamma\Psi\gamma^{-1}}$ and $L_* = \gamma L_\Psi = L_{\gamma\Psi\gamma^{-1}}$ is associated to the embedding $\gamma\Psi\gamma^{-1} \in [\Psi]$. It is clear that this choice is the natural one in investigating the relations with the Mazur-Kitagawa p-adic L-function, whose definition was given in terms of $I_\infty = I_{L_*}$.

Note the following vanishing property of the above p-adic L-functions.

Proposition 5.7 *The p-adic L-functions vanish at k_0:*

$$\mathcal{L}_p(f/K,\Psi,k_0)=\mathcal{L}_p(f/K,\chi,k_0)=L_p(f/K,\chi,k_0)=0.$$

Furthermore,

$$\frac{d}{d\kappa}\left[L_p(f/K,\chi,\kappa)\right]_{\kappa=k_0}=0.$$

Proof. By definition and (24)

$$\begin{aligned}\mathcal{L}_p(f/K,\Psi,k_0) \quad : &= |L_\Psi|^{-\frac{k_0-2}{2}}\int_{L'_\Psi}P_\Psi^m(x,y)\,dI_{L_\Psi}\{r\to\gamma_\Psi r\}=\\ &= \int_{\mathbb{P}^1(\mathbb{Q}_p)}P_\Psi^m d\pi_*(I_{L_\Psi})\{r\to\gamma_\Psi r\}.\end{aligned}$$

The claim now follows from Corollary 4.7. By definition the same vanishing property holds for the other p-adic L-functions and the defining relation $L_p=\mathcal{L}_p^2$ yields the last assertion. ■

5.2.1 Interpolation properties of the p-adic L-functions attached to real quadratic fields and functional equation

The following theorem encodes the main interpolation property of the p-adic L-function $L_p(f/K,\chi,\kappa)$.

Theorem 5.8 *For all $k\in\mathbb{Z}^{\geq 2,\neq k_0}\cap U$*

$$L_p(f/K,\chi,\kappa)=\lambda(k)^2\left(1-p^{k-2}a_p(k)^{-2}\right)^2 D_K^{\frac{k-2}{2}}L^*\left(f_k^\#/K,\chi,k/2\right),$$

where

$$L^*\left(f_k^\#/K,\chi,k/2\right):=\frac{\left(\frac{k-2}{2}\right)!^2\sqrt{D_K}}{(2\pi i)^{k-2}\,\Omega_k^2}L\left(f_k^\#/K,\chi,k/2\right).$$

Proof. The proof of [BD3, Theorem 3.5] readily adapts to our higher weight setting. As explained in [BD3] the proof is reduced to Popa's formula [Po, Theorem 6.3.1]. ■

Recall that a genus character of G_K is a quadratic unramified character of G_K. Such a character corresponds to a biquadratic (or quadratic when $\chi=1$) extension of $\mathbb{Q}$ which is explicitly given by

$$H_\chi=\mathbb{Q}\left(\sqrt{D_1},\sqrt{D_2}\right)\supset\mathbb{Q}\left(\sqrt{D}\right)=K,$$

where $D_K=:D=D_1D_2$ is a factorization of the fundamental discriminant D into factors D_i prime each other.

Let χ_i (resp. ε_K) be the Dirichlet character attached to $\mathbb{Q}\left(\sqrt{D_i}\right)/\mathbb{Q}$ (resp. $K/\mathbb{Q}$). Then $\varepsilon_K = \chi_1\chi_2$. We say that χ is real (resp. imaginary) whenever H_χ/K is totally real (resp. imaginary). Note that

$$1 = \varepsilon_K(-1) = \chi_1(-1)\,\chi_2(-1)\,,$$

so that

$$\chi_1(-1) = \chi_2(-1)\,, \tag{32}$$

depending of whether $\mathbb{Q}\left(\sqrt{D_i}\right)/\mathbb{Q}$ are imaginary or real. Note that $D_K \in \mathbb{Z}_p^\times$, since by assumption p is prime to D_K. In particular $D_K^{\frac{k-2}{2}}$ extends on U to an analytic function $D_K^{\frac{\kappa-2}{2}} := \langle D_K\rangle^{\frac{\kappa-2}{2}}$, thanks to Lemma 4.1.

Theorem 5.9 *Let χ be a genus character such that $\chi(-1) = (-1)^{\frac{k_0-2}{2}} w_\infty$. Then*

$$L_p(f/K,\chi,\kappa) = D_K^{\frac{\kappa-2}{2}} L_p(f,\chi_1,\kappa,\kappa/2)\, L_p(f,\chi_2,\kappa,\kappa/2)\,,$$

where (χ_1,χ_2) is the pair of Dirichlet characters attached to χ.

Proof. The proof of [BD3, Theorem 3.5] adapts. ■

Remark 5.10 *Let (χ_1,χ_2) be the pair attached to χ. Since the primes dividing M are split in K, it follows from (32) that $\chi_1(-M) = \chi_2(-M)$. Hence we shall simply write $\chi_i(-M)$.*

5.2.2 Derivatives of p-adic L-functions attached to real quadratic fields

Theorem 5.11 *Let $\Psi \in \mathcal{E}mb^+(\mathcal{O},\mathcal{R})$. Then*

$$\frac{d}{d\kappa}\left[\mathcal{L}_p(f/K,\Psi,\kappa)\right]_{\kappa=k_0} = \frac{1}{2} D_K^{\frac{k_0-2}{4}} \cdot \\ \left(\log\Phi^{AJ}(j_\Psi)(I_f) + (-1)^{m+1}\log\Phi^{AJ}(j_{\overline{\Psi}})(I_f)\right).$$

Proof. Consider the factorization

$$P_\Psi(x,y) = A(x-\tau_\Psi y)(x-\overline{\tau}_\Psi y)$$

and write

$$\mathcal{L}_p(f/K,\Psi,\kappa) := |L_\Psi|^{-\frac{k_0-2}{2}} \langle A\rangle^{\frac{\kappa-k_0}{2}} \cdot \\ \int_{L'_\Psi} P_\Psi^m(x,y)\langle x-\tau_\Psi y\rangle^{\frac{\kappa-k_0}{2}} \langle x-\overline{\tau}_\Psi y\rangle^{\frac{\kappa-k_0}{2}} dI_{L_\Psi}\{r\to\gamma_\Psi r\}\,.$$

In light of Proposition 5.7, the usual formula for the derivatives of the product of two functions yields

$$\frac{d}{d\kappa}\left[\mathcal{L}_p(f/K,\Psi,\kappa)\right]_{\kappa=k_0} = |L_\Psi|^{-\frac{k_0-2}{2}} \cdot \\ \frac{d}{d\kappa}\left[\int_{L'_\Psi} P_\Psi^m(x,y)\langle x-\tau_\Psi y\rangle^{\frac{\kappa-k_0}{2}} \langle x-\overline{\tau}_\Psi y\rangle^{\frac{\kappa-k_0}{2}} dI_{L_\Psi}\{r\to\gamma_\Psi r\}\right]_{\kappa=k_0}.$$

By Proposition 4.2,

$$\begin{aligned}&|L_\Psi|^{-\frac{k_0-2}{2}}\frac{d}{d\kappa}\left[\int_{L'_\Psi}P^m_\Psi(x,y)\langle x-\tau_\Psi y\rangle^{\frac{\kappa-k_0}{2}}\langle x-\overline{\tau}_\Psi y\rangle^{\frac{\kappa-k_0}{2}}dI_{L_\Psi}\{r\to\gamma_\Psi r\}\right]_{\kappa=k_0}\\ &=\frac{1}{2}|L_\Psi|^{-\frac{k_0-2}{2}}\frac{d}{d\kappa}\left(\int_{L'}P^m_\Psi(x,y)\langle x-\tau_\Psi y\rangle^{\kappa-k}dI_{L_\Psi}\{r\to\gamma_\Psi r\}\right)_{\kappa=k}+\\ &\quad\frac{1}{2}|L_\Psi|^{-\frac{k_0-2}{2}}\frac{d}{d\kappa}\left(\int_{L'}P^m_\Psi(x,y)\langle x-\overline{\tau}_\Psi y\rangle^{\kappa-k}dI_{L_\Psi}\{r\to\gamma_\Psi r\}\right)_{\kappa=k}.\end{aligned}$$

Note now that $L_\Psi=L_{\overline{\Psi}}$ and, by Remark 2.20, $\overline{\tau}_\Psi=\tau_{\overline{\Psi}}$, $P^m_\Psi=(-1)^m P^m_{\overline{\Psi}}$ and $\gamma_\Psi=\gamma^{-1}_{\overline{\Psi}}$. It follows that the last expression is equal to

$$\frac{1}{2}\left(\int_x^{\gamma_\Psi x}\int^{\tau_\Psi}P^m_\Psi\omega_f+(-1)^m\int_x^{\gamma^{-1}_{\overline{\Psi}}x}\int^{\tau_{\overline{\Psi}}}P^m_{\overline{\Psi}}\omega_f\right).$$

By Lemma 2.19, replacing x by $\gamma_{\overline{\Psi}}x$ to compute the integral gives

$$\int_x^{\gamma^{-1}_{\overline{\Psi}}x}\int^{\tau_{\overline{\Psi}}}P^m_{\overline{\Psi}}\omega_f=\int_{\gamma_{\overline{\Psi}}x}^{x}\int^{\tau_{\overline{\Psi}}}P^m_{\overline{\Psi}}\omega_f=-\int_x^{\gamma_{\overline{\Psi}}x}\int^{\tau_{\overline{\Psi}}}P^m_{\overline{\Psi}}\omega_f.$$

The claim now follows from Corollary 4.13. ■

Recall the linear combination j^χ introduced in (19) and set

$$\overline{j}^\chi:=\sum_{\sigma\in G_{H^+_{\mathcal{O}}/K}}\chi^{-1}(\sigma)\,j_{\overline{\sigma\Psi}}.$$

Corollary 5.12 *Let $\chi:G_{H^+_{\mathcal{O}}/K}\to\mathbb{C}^\times$ be a character. Then*

$$\frac{d^2}{d\kappa^2}\left[L_p(f/K,\chi,\kappa)\right]_{\kappa=k}=\frac{1}{2}D_K^{\frac{k_0-2}{2}}\cdot\left(\log\Phi^{AJ}(j^\chi)(I_f)+(-1)^{m+1}\log\Phi^{AJ}\left(\overline{j}^\chi\right)(I_f)\right)^2.$$

Proof. This is a consequence of Theorem 5.11, in light of Proposition 5.7. ■

Let now χ be a genus character attached to the pair (χ_1,χ_2) and let H^+ be the narrow Hilbert ring class field. Recall that by Remark 5.10 $\chi_i(-M)$ does not depend on $i=1,2$.

Corollary 5.13 *Let $\chi:G_{H^+/K}\to\mathbb{C}^\times$ be a genus character. Then*

$$\frac{d^2}{d\kappa^2}\left[L_p(f/K,\chi,\kappa)\right]_{\kappa=k}=\frac{1}{2}D_K^{\frac{k_0-2}{2}}\cdot\left(1+(-1)^{m+1}w_M\chi_i(-M)\right)^2\log\Phi^{AJ}(j^\chi)(I_f)^2.$$

Proof. First of all note that, since I_f is an eigenform for the Atkin-Lehner involution W_M with eigenvalue w_M, we may write

$$w_M\log\Phi^{AJ}(j_{\overline{\sigma\Psi}})(I_f)=\log\Phi^{AJ}(j_{\overline{\sigma\Psi}})(I_f\mid W_M)=\log\Phi^{AJ}\left(j_{\alpha_M\overline{\sigma\Psi}\alpha_M^{-1}}\right)(I_f).$$

Let $\sigma\Psi \in \Gamma\backslash\mathcal{E}mb^{+\mathfrak{d}}(\mathcal{O},\mathcal{R})$ be any oriented embedding as explained in Remark 2.24 and note that $\overline{\sigma\Psi}$ has the same orientation at p of Ψ but for every prime $l^e \parallel M$ the orientation of $\overline{\sigma\Psi}$ is opposite to the orientation of $\sigma\Psi$. As explained in Remark 2.24 the Atkin-Lehner involution W_M exchanges these orientations, so that $\alpha_M\overline{\sigma\Psi}\alpha_M^{-1} \in \Gamma\backslash\mathcal{E}mb^{+\mathfrak{d}}(\mathcal{O},\mathcal{R})$. Since we noted in Remark 2.24 that $\Gamma\backslash\mathcal{E}mb^{+\mathfrak{d}}(\mathcal{O},\mathcal{R})$ is a torsor under the $G_{H^+/K}$-action, there exists a unique $\delta_{\sigma\Psi} \in G_{H^+/K}$ such that $\alpha_M\overline{\sigma\Psi}\alpha_M^{-1} = \delta_{\sigma\Psi}\sigma\Psi$. According to [BD3, (17)] $\delta_{\sigma\Psi} = \delta_\Psi\sigma^{-2}$, so that we find (we have $\chi^2 = 1$):

$$\begin{aligned}
&\textstyle\sum_{\sigma\in G_{H_{\mathcal{O}}^+/K}} \chi(\sigma) \log \Phi^{AJ}(j_{\overline{\sigma\Psi}})(I_f) = \\
&= w_M \textstyle\sum_{\sigma\in G_{H_{\mathcal{O}}^+/K}} \chi(\sigma) \log \Phi^{AJ}(j_{\delta_\Psi\sigma^{-1}\Psi})(I_f) = \\
&= w_M \chi(\delta_\Psi) \textstyle\sum_{\sigma\in G_{H_{\mathcal{O}}^+/K}} \chi\left(\delta_\Psi\sigma^{-1}\right) \log \Phi^{AJ}(j_{\delta_\Psi\sigma^{-1}\Psi})(I_f) = \\
&= w_M \chi(\delta_\Psi) \textstyle\sum_{\sigma\in G_{H_{\mathcal{O}}^+/K}} \chi(\sigma) \log \Phi^{AJ}(j_{\sigma\Psi})(I_f).
\end{aligned}$$

But since χ is a genus character, [BD3, Proposition 1.8] tells us that $\chi(\delta_\Psi) = \chi_i(-M)$. The claim now follows from 5.12. ■

Corollary 5.14 *Let $\chi : G_{H^+/K} \to \mathbb{C}^\times$ be a genus character. Then:*

$$\frac{d^2}{d\kappa^2}\left[L_p(f/K,\chi,\kappa)\right]_{\kappa=k} = \begin{cases} 2D_K^{\frac{k_0-2}{2}} \log \Phi^{AJ}(j^\chi)(I_f)^2 & \text{if } \chi_i(-M) = (-1)^{m+1} w_M \\ 0 & \text{if } \chi_i(-M) = (-1)^m w_M. \end{cases}$$

5.3 p-adic L-functions attached to imaginary quadratic fields

In this subsection we let $K'/\mathbb{Q}$ be an imaginary quadratic field of discriminant $D_{K'}$ and we consider a factorization $N = pN^+N^-$ such that:

- p is inert or split in K';
- all the prime factors of N^+ split in K';
- N^- is the squarefree product of an odd number of primes which remain inert in K'.

Recall the definite quaternion algebra B of discriminant $N^-\infty$ and fix an identification $B_p = \mathbb{M}_2(\mathbb{Q}_p)$, so that B_p acts on the p-adic upper halfplane as well as on the Bruhat-Tits three and on the sets $\mathcal{L}$ and $\mathcal{L}_0$. As in the Darmon setting it is possible to define the set of optimal embeddings of level N^+ and pN^+ of a $\mathbb{Z}[1/p]$-order $\mathcal{O}$ of conductor c, prime to $D_{K'}$ and N, into the corresponding Eichler order R. More precisely the definition in [BD2, Definition 3.2] is given in terms of optimal embeddings of a $\mathbb{Z}$-order $\mathcal{O}_\mathbb{Z}$ into an Eichler order $R_\mathbb{Z}$, this last of level N^+ or pN^+, such that $\mathcal{O} = \mathbb{Z}[1/p] \otimes \mathcal{O}_\mathbb{Z}$ and $R = \mathbb{Z}[1/p] \otimes R_\mathbb{Z}$. By [BD1, Lemma 2.1] the set of optimal embeddings of level pN^+ is non-empty only when p is split, so that this assumption will be implicit when considering embeddings of level pN^+. As explained in [BD2, Section 3.1], by the strong approximation theorem, these sets can be realized as subsets of

$$\begin{array}{ll} R^\times \backslash (\mathcal{E}mb\,(\mathcal{O}, R) \times \mathcal{L}) & \text{when the level is } N^+, \\ R^\times \backslash (\mathcal{E}mb\,(\mathcal{O}, R) \times \mathcal{L}_0) & \text{when the level is } pN^+. \end{array}$$

More precisely, the elements of the first set are those represented by the couples $[\Psi, L_\Psi]$, where L_Ψ is preserved by the action of $\Psi(\mathcal{O})$, while the elements of the second set are those represented by the triples $\left[\Psi, L^1_\Psi, L^2_\Psi\right]$ such that L^1_Ψ and L^2_Ψ are both preserved under the action of $\Psi(\mathcal{O})$ (when p is split).

There are the following data attached to the optimal embeddings of level N^+, say represented by the couple $[\Psi, L_\Psi]$:

- the two fixed points $\tau_\Psi, \overline{\tau}_\Psi \in \mathcal{H}_p$ for the action of $\Psi(K'^\times)$ on $\mathcal{H}_p(K')$, ordered in such a way that the action of $K'^\times$ on the tangent space at τ_Ψ is through the character $z \mapsto z/\overline{z}$, when p is inert;
- the unique fixed vertex $v_\Psi \in \mathcal{V}$ for the action of $\Psi(K'^\times)$ on $\mathcal{V}$, which is nothing but the reduction $red\,(\tau_\Psi) = red\,(\overline{\tau}_\Psi)$, when p is inert;
- the lattice L_Ψ such that $[L_\Psi] = v_\Psi$, when p is inert, and the lattice L_Ψ which is fixed by the action of the split quadratic algebra $\Psi(\mathcal{O} \otimes \mathbb{Z}_p)$ and hence admits a $\mathbb{Z}_p$-basis $\{x_\Psi, y_\Psi\}$ of eigenvectors for this action, when p is split;
- the unique polynomial up to sign P_Ψ in $\mathbf{P}_2$ which is fixed by the action of $\Psi(K'^\times)$ on $\mathbf{P}_2 \otimes \det^{-1}$ and satisfies $\langle P_\Psi, P_\Psi\rangle_{\mathbf{P}_2} = -D_{K'}$ (the pairing being defined as in [BDIS]), which we fix by the choice

$$P_\Psi := \mathrm{Tr}\left(\Psi\left(\sqrt{D_{K'}}\right) \cdot \begin{pmatrix} X & -X^2 \\ 1 & -X \end{pmatrix}\right) \in \mathbf{P}_2,$$

  the other one being obtained by replacing $\sqrt{D_{K'}}$ with $-\sqrt{D_{K'}}$; note that P_Ψ is either irreducible over $\mathbb{Q}_p$ or it splits into two linear forms corresponding to the basis $\{x_\Psi, y_\Psi\}$, according to whether p is inert or split.

Define

$$L''_\Psi := \begin{cases} L'_\Psi & \text{when } p \text{ is inert} \\ \mathbb{Z}_p^\times x_\Psi \oplus \mathbb{Z}_p^\times y_\Psi & \text{when } p \text{ is split} \end{cases}$$

Recall the family μ_* that was attached to φ by means of Theorem 4.16. The following definition attaches a p-adic L-function

$$\begin{aligned} \mathcal{L}_p\,(f/K', \Psi, -) &: U \to \mathbb{C}_p \\ \kappa &\mapsto \mathcal{L}_p\,(f/K', \Psi, \kappa) \end{aligned}$$

to the data of the embedding $[\Psi, L_\Psi]$ of level N^+. It is easily checked that the subsequent definitions do not depend on the choice of L_Ψ such that $[L_\Psi] = v_\Psi$ when p is inert.

Definition 5.15 *The partial p-adic L-function attached to* $(f/K', \Psi)$ *is*

$$\mathcal{L}_p\left(f/K', \Psi, \kappa\right) := \left|L_\Psi\right|^{-\frac{k_0-2}{2}} \int_{L''_\Psi} \left\langle P_\Psi\left(x, y\right)\right\rangle^{\frac{\kappa-k_0}{2}} P_\Psi^m d\mu_{L_\Psi}.$$

The partial p-adic L-function attached to $(f/K', \chi)$*, where* $\chi : G_{H_\mathcal{O}/K'} \to \mathbb{C}^\times$ *is a character, is*

$$\mathcal{L}_p\left(f/K', \chi, \kappa\right) := \sum_{\sigma \in G_{H_\mathcal{O}/K'}} \chi^{-1}\left(\sigma\right) \mathcal{L}_p\left(f/K', \sigma\Psi, \kappa\right).$$

The p-adic L-function attached to $(f/K', \chi)$*, where* $\chi : G_{H_\mathcal{O}/K'} \to \mathbb{C}^\times$ *is a character, is*

$$L_p\left(f/K', \chi, \kappa\right) := \mathcal{L}_p\left(f/K', \chi, \kappa\right) \mathcal{L}_p\left(f/K', \chi^{-1}, \kappa\right).$$

As before, in order to justify the above definition and the fact that the above p-adic L-functions are analytic we can appeal to Lemma 4.1 after noticing that they are built from functions of the form $\kappa \mapsto \mu\left(P\left\langle\alpha\right\rangle^{\frac{\kappa-k_0}{2}}\left\langle\beta\right\rangle^{\frac{\kappa-k_0}{2}}\chi_X\right)$.

Of course Remark 5.6 also holds in this setting. Note the following vanishing property of the p-adic L-functions, which is proved exactly as in Proposition 5.7 using (30) and Corollary 4.19.

Proposition 5.16 *Assume p is inert. The p-adic L-functions vanishes at* k_0*:*

$$\mathcal{L}_p\left(f/K', \Psi, k_0\right) = \mathcal{L}_p\left(f/K', \chi, k_0\right) = L_p\left(f/K', \chi, k_0\right) = 0.$$

Furthermore

$$\frac{d}{d\kappa}\left[L_p\left(f/K', \chi, \kappa\right)\right]_{\kappa=k_0} = 0.$$

5.3.1 Interpolation properties of the p-adic L-functions attached to imaginary quadratic fields and functional equation

The next two theorems collect the interpolation properties of these p-adic L-functions.

Theorem 5.17 *Assume p is inert. Then, for all* $k \in \mathbb{Z}^{\geq 2, \neq k_0} \cap U$*,*

$$L_p\left(f/K', \chi, k\right) = \lambda_B\left(k\right)^2 a_p\left(k\right)^2 \left(1 - p^{k-2} a_p\left(k\right)^{-2}\right)^2 L^*\left(f_k^\#/K', \chi, k/2\right),$$

where

$$L^*\left(f_k^\#/K', \chi, k/2\right) := \frac{\left(\frac{k-2}{2}\right)!^2 D_K^{\frac{k-1}{2}}}{\left(2\pi\right)^{k-2}\left\langle f_k^\#, f_k^\#\right\rangle} L\left(f_k^\#/K', \chi, k/2\right).$$

Proof. The proof proceeds precisely along the same lines as in [BD2, Theorem 3.8] and we repeat the ideas for the convenience of the reader. Using [BD2, Lemma 3.7] and noticing that $[P_\Psi(x,y)]^{\frac{k-k_0}{2}} = 1$ by Lemma 4.1, one can show that $P_\Psi(x,y)^{\frac{k-k_0}{2}} = |L_\Psi|^{\frac{k-k_0}{2}} \langle P_\Psi(x,y)\rangle^{\frac{k-k_0}{2}}$ on $L''_\Psi = L'_\Psi$. Hence,

$$\mathcal{L}_p(f/K', \Psi, k) = |L_\Psi|^{-\frac{k-2}{2}} \int_{L''_\Psi} P_\Psi^{\frac{k-2}{2}} d\mu_{L_\Psi}.$$

Now we simply need to replace the use of [BD2, Proposition 2.11] with the more general, but formally identical, Corollary 4.18, which gives

$$\begin{aligned} &|L_\Psi|^{-\frac{k-2}{2}} \int_{L''_\Psi} P_\Psi^{\frac{k-2}{2}} d\mu_{L_\Psi} = \\ &= \lambda_B(k)\, a_p(k) \left(1 - p^{k-2} a_p(k)^{-2}\right) |L_\Psi|^{-\frac{k-2}{2}} c_k^{\#}(L_\Psi)\left(P_\Psi^{\frac{k-2}{2}}\right). \end{aligned}$$

Summing together, multiplying $\mathcal{L}_p(f/K', \chi, \kappa)$ with $\mathcal{L}_p\left(f/K', \chi^{-1}, \kappa\right)$ and applying Hatcher-Hui Xue's formula

$$\begin{aligned} &\left(\sum\nolimits_{\sigma\in G_{H_{\mathcal{O}}/K'}} \chi^{-1}(\sigma)\, |L_{\sigma\Psi}|^{-\frac{k-2}{2}} c_k^{\#}(L_{\sigma\Psi})\left(P_{\sigma\Psi}^{\frac{k-2}{2}}\right)\right)\cdot \\ &\left(\sum\nolimits_{\sigma\in G_{H_{\mathcal{O}}/K'}} \chi(\sigma)\, |L_{\sigma\Psi}|^{-\frac{k-2}{2}} c_k^{\#}(L_{\sigma\Psi})\left(P_{\sigma\Psi}^{\frac{k-2}{2}}\right)\right) = \\ &= \left\langle c_k^{\#}, c_k^{\#}\right\rangle L^*\left(f_k^{\#}/K', \chi, k/2\right) \end{aligned} \tag{33}$$

as reformulated in [BD2, Proposition 3.3] yields the result, in light of the normalization (27). ■

Theorem 5.18 *Assume p split in K' and let $\mathfrak{p} \mid p$ be a prime of K' above p. Then*

$$\begin{aligned} L_p(f/K', \chi, k_0) &= \left(1 - \chi(\mathfrak{p})\, p^{\frac{k_0-2}{2}} a_p(k_0)^{-1}\right)\left(1 - \chi^{-1}(\mathfrak{p})\, p^{\frac{k_0-2}{2}} a_p(k_0)^{-1}\right) \\ &\quad \cdot \langle c_{k_0}, c_{k_0}\rangle L^*(f_{k_0}/K', \chi, k_0/2) \end{aligned}$$

and for all $k \in \mathbb{Z}^{\geq 2, \neq k_0} \cap U$

$$\begin{aligned} L_p(f/K', \chi, k) &= \lambda_B^2(k)\left(a_p(k) + p^{k-2} a_p(k)^{-1} - p^{\frac{k-2}{2}}\chi(\mathfrak{p}) - p^{\frac{k-2}{2}}\chi^{-1}(\mathfrak{p})\right)^2 \\ &\quad \cdot L^*\left(f_k^{\#}/K', \chi, k/2\right). \end{aligned}$$

Proof. Again the proof of [BD2, Theorem 3.12] works in this setting (even with more general characters). We do not recall which are the main ingredients, since the computation is more involved than Theorem 5.17. As explained in [BD2, Proposition 3.4], the appearance of the factor $\langle c_{k_0}, c_{k_0}\rangle$ at $k = k_0$ is due to the fact that no normalization condition was imposed on the modular form c_{k_0}, so that in the Hatcher-Hui Xue's formula (33) this factor needs to be considered. ■

We now specialize the above theorem to a genus character χ of the imaginary quadratic field K', say attached to the pair of Dirichlet characters (χ_1, χ_2). Note that, since p is split, $\chi(\mathfrak{p}) = \chi_i(p)$ does not depend on i. Furthermore, since $\chi(\mathfrak{p})^2 = 1$, the Euler factor appearing in Theorem 5.18 can be rewritten and one deduces the following corollary:

Corollary 5.19 *Assume p split in K', let $\mathfrak{p} \mid p$ be a prime of K' above p and let χ be the genus character attached to the pair of Dirichlet characters (χ_1, χ_2). Then*

$$L_p(f/K', \chi, k_0) = \left(1 - \chi_i(p)\, p^{\frac{k_0-2}{2}} a_p(k_0)^{-1}\right)^2 \langle c_{k_0}, c_{k_0}\rangle\, L^*(f_{k_0}/K', \chi, k/2)$$

and for all $k \in \mathbb{Z}^{\geq 2, \neq k_0} \cap U$

$$L_p(f/K', \chi, k) = \lambda_B(k)^2 a_p(k)^2 \left(1 - \chi_i(p)\, p^{\frac{k-2}{2}} a_p(k)^{-1}\right)^4 L^*\left(f_k^{\#}/K', \chi, k/2\right).$$

Definition 5.20 *$\eta : \mathbb{Z}^{\geq 2} \to \mathbb{C}_p$ is the function*

$$\eta(k) := \begin{cases} \frac{\lambda_B(k)^2 a_p(k)^2}{\lambda^+(k)\lambda^-(k)} D_{K'}^{\frac{k-2}{2}} i^{k-2} & \text{for } k \neq k_0 \\ \langle c_{k_0}, c_{k_0}\rangle D_{K'}^{\frac{k_0-2}{2}} i^{k_0-2} & \text{for } k = k_0 \end{cases}$$

Theorem 5.21 *The function $\eta(k)$ uniquely extends to an analytic function such that $\eta(\kappa) \neq 0$ on U (up to shrinking it). Moreover, for every genus character χ, say attached to the couple of Dirichlet characters (χ_1, χ_2),*

$$L_p(f/K', \chi, \kappa) = \eta(\kappa)\, L_p(f, \chi_1, \kappa, \kappa/2)\, L_p(f, \chi_2, \kappa, \kappa/2)$$

on U.

Proof. The proof is the same of [BD2, Corollary 5.3], after noticing that the main result of [MM] extends to our higher weight setting. ■

5.3.2 Derivatives of p-adic L-functions attached to imaginary quadratic fields

Assume until the end of this section that p is inert in K'. Let $X = X_{N^+, pN^-}$ be the Shimura curve attached to the indefinite quaternion algebra ramified at the primes pN^-. As explained in [BD1, Section 1.5], the Shimura curve X is endowed with Hecke operators T_l for $l \nmid N$ as well as Atkin-Lehner involutions W_l^+ for $l \mid N^+$ and Atkin-Lehner involutions W_l^- for $l \mid pN^-$. The Atkin-Lehner involution W_p^- will be of particular interest for us. Write $X_{W_p^-}$ to denote the twist of the Shimura curve X by the cocycle in $H^1\left(G_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}, Aut(X)\right)$ which maps the non trivial element $Frob_p \in G_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}$ to W_p^-. Recall the group $\widetilde{\Gamma}'$

defined by (26) and denote by Γ' the subgroup of norm one elements. By the Cerednik-Drinfeld Theorem, $X_{W_p^-}$ admits a rigid analytic uniformization over $\mathbb{Q}_p$:

$$\Gamma' \backslash \mathcal{H}_p = X_{W_p^-}^{an},$$

where $X_{W_p^-}^{an}$ is the analytification of $X_{W_p^-}$. We will make an abuse of notation by writing $X = X^{an}$ or $X_{W_p^-} = X_{W_p^-}^{an}$.

The optimal embeddings of conductor N^+ admit a particular simple description. More precisely, fix an embedding $\sigma_p : H = H_{\mathcal{O}} \hookrightarrow \mathbb{Q}_{p^2}$ (this is possible since p is inert in K' and hence it splits completely in H). The p-adic uniformization allow us to view $\Gamma' \backslash \mathcal{E}mb(\mathcal{O}, R)$ as a subset of $X(\mathbb{Q}_{p^2})$:

$$\Gamma' \backslash \mathcal{E}mb(\mathcal{O}, R) \hookrightarrow \Gamma' \backslash \mathcal{E}mb(\mathbb{Q}_{p^2}, B_p) = \Gamma' \backslash \mathcal{E}mb(\mathbb{Q}_{p^2}, \mathbb{M}_2(\mathbb{Q}_p)) = X(\mathbb{Q}_{p^2}).$$

In this way we shall identify (the class of) Ψ with its image in $X(\mathbb{Q}_{p^2})$.

Remark 5.22 *In view of the above twist that enter in the rigid analytic parametrization, the optimal embedding $\overline{\Psi}$ corresponds in $X(\mathbb{Q}_{p^2})$ to the optimal embedding $W_p^- Frob_p \Psi$, regarded like a point of X.*

Recall the rigid analytic modular form f^{rig} that was attached to the modular form f by means of the Jacquet-Langlands correspondence. It satisfies the following relation with respect to the action of the Atkin-Lehner involution W_p^- (see [BD1, Theorem 1.2]):

$$f^{rig} \mid W_p^- = -w_p f^{rig}, \tag{34}$$

where w_p is the sign of the Atkin-Lehner involution W_p acting on f.

Let $\mathcal{M}_n$ be the Chow motive (over $\mathbb{Q}$) of weight k_0 modular forms constructed in [IS, Appendix]. As explained in [IS, Appendix] one can attach to an optimal embedding Ψ an element $y_\Psi^{(n)} \in CH^{m+1}(\mathcal{M}_{n,H})$, the Chow group of codimension $m+1$ cycles of $\mathcal{M}_n$ base changed to $H := H_{\mathcal{O}}$. The p-adic realization $V(m+1) := H_p\left(\mathcal{M}_{n,\overline{\mathbb{Q}}}, \mathbb{Q}_p(m+1)\right)$ of the motive $\mathcal{M}_n$ affords representations attached to cusp forms that are new at pN^-. Consider the p-adic Abel-Jacobi map

$$cl_0^{m+1} : CH^{m+1}(\mathcal{M}_{n,H}) \to Ext^1_{G_H}(\mathbb{Q}_p, V(m+1)).$$

After a base change from H to $F_p \supset \mathbb{Q}_{p^2}$, the p-adic Abel-Jacobi map can be identified with

$$\Phi^{AJ} : CH^{m+1}(\mathcal{M}_{n,F_p}) \to Ext^1_{MF}(F_p, \mathbb{D}(m+1)) = \frac{\mathbb{D}_{F_p}}{F^{m+1}\mathbb{D}_{F_p}},$$

where we write $\mathbb{D} := \mathbb{D}_{st}(V)$. Here $\mathbb{D}_{st}$ is the Fontaine functor attaching to a Galois representation of G_{F_p} a filtered Frobenius module over F_p. As recalled in the introduction, the above ext group is explicitly computed in [IS, (49)] and the p-adic étale Abel-Jacobi map can be can be interpreted as

$$\log \Phi^{AJ} : CH^{m+1}\left(\mathcal{M}_{n,F_p}\right) \to M_k\left(\Gamma', F_p\right)^{\vee},$$

where $M_k\left(\Gamma', F_p\right)^{\vee}$ denotes the F_p-dual space.

Note that the Frobenius $Frob_p$ introduced in Remark 5.22 also acts on the Chow group $CH^{m+1}\left(\mathcal{M}_{n,\mathbb{Q}_{p^2}}\right)$.

Lemma 5.23 $y^{(n)}_{Frob_p\Psi} = (-1)^m Frob_p y^{(n)}_{\Psi}$.

Proof. Let W be the group generated by the Atkin-Lehner involutions $W_l^{\pm}$ for $l \mid N$. The proof is easily reduced to the case $m = 1$, i.e. weight $k_0 = 4$. In this case the Heegner cycles $y^{(2)}_{\Psi}$ are defined by fixing $y^{(2)}_{\Psi_0}$ for some Ψ_0 and then exploiting the simply transitive action of $G_{H/K'} \times W$ on the optimal embeddings in order to make these cycles compatible with the action of this group. Indeed, the elements $y^{(2)}_{\Psi}$ are only canonical up to sign. More precisely, they correspond to $z^{(2)}_{\Psi} \in End_{\mathcal{R}}\left(A_{\Psi}\right) = \mathcal{O}$ that are only defined up to sign. Since $A_{\Psi} = E^2_{\Psi}$, where E_{Ψ} is an elliptic curve such that $End\left(E_{\Psi}\right) = End_{\mathcal{R}}\left(A_{\Psi}\right)$, we can reduce to consider elliptic curves. In this case we can fix an isomorphism $[-]_{\Psi} : \mathcal{O} \simeq End\left(E_{\Psi}\right)$ with the property that for every $\sigma \in Aut\left(\mathbb{C}\right)$ and $\alpha \in \mathcal{O}$ we have $\sigma\left[\alpha\right]_{\Psi} = \left[\alpha^{\sigma}\right]_{\sigma\Psi}$ and define the element $z^{(2)}_{\Psi}$ by making it correspond to the choice of a fixed root $\sqrt{D_{K'}} \in K'$, the other choice $-\sqrt{D_{K'}}$ giving rise to the element $-z^{(2)}_{\Psi}$. With this choice the elements $z^{(2)}_{\Psi}$ are compatible for the action of the group $G_{H/K'} \times W$. Furthermore, since $Frob_p$ is induced by the complex conjugation τ (because p is inert) we find that $\tau\left[\sqrt{D_{K'}}\right]_{\Psi} = \left[-\sqrt{D_{K'}}\right]_{\tau\Psi}$, which gives $Frob_p z^{(2)}_{\Psi} = -z^{(n)}_{Frob_p\Psi}$. ■

Theorem 5.24 *Let* $\Psi \in \mathcal{E}mb^{+}\left(\mathcal{O}, \mathcal{R}\right)$. *Then*

$$\frac{d}{d\kappa}\left[\mathcal{L}_p\left(f/K', \Psi, \kappa\right)\right]_{\kappa=k_0} = \frac{1}{2}\left(\log \Phi^{AJ}\left(y^{(n)}_{\Psi}\right)\left(f^{rig}\right) - w_p \log \Phi^{AJ}\left(Frob_p y^{(n)}_{\Psi}\right)\left(f^{rig}\right)\right).$$

Proof. By the main result of [Se]:

$$\frac{d}{d\kappa}\left[\mathcal{L}_p\left(f/K', \Psi, \kappa\right)\right]_{\kappa=k_0} = \frac{1}{2}\left(\log \Phi^{AJ}\left(y^{(n)}_{\Psi}\right)\left(f^{rig}\right) + (-1)^m \log \Phi^{AJ}\left(y^{(n)}_{\overline{\Psi}}\right)\left(f^{rig}\right)\right).$$

By Remark 5.22 and Lemma 5.23

$$y^{(n)}_{\overline{\Psi}} = y^{(n)}_{W_p^- Frob_p\Psi} = W_p^- y^{(n)}_{Frob_p\Psi} = (-1)^m W_p^- Frob_p y^{(n)}_{\Psi}.$$

Now the claim follows from (34). ■

Whenever F is a field, let us write $MW_f(F)$ to denote the image of the Chow group over F in $Ext^1_{G_F}\left(\mathbb{Q}_p, V_{[f]}(m+1)\right)$, i.e. the image obtained by $cl^{m+1}_{0,f} := e_{[f]} \circ cl^{m+1}_0$. By the theory of complex multiplication,

$$y^\chi := \sum_{\sigma \in G_{H_{\mathcal{O}}/K'}} \chi^{-1}(\sigma)\, y^{(n)}_{\sigma\Psi} \in CH^{m+1}\left(\mathcal{M}_{n,H_\chi}\right)^\chi,$$

where H_χ/K' is the subextension of H/K' that corresponds to the kernel of χ. Hence $cl^{m+1}_{0,f}(y^\chi) \in MW_f(H_\chi)^\chi$.

Corollary 5.25 *Let $\chi : G_{H_{\mathcal{O}}/K'} \to \mathbb{C}^\times$ be a character. Then*

$$\begin{aligned} &\frac{d^2}{d\kappa^2}\left[L_p(f/K',\chi,\kappa)\right]_{\kappa=k} = \\ &\quad \frac{1}{2}\left(\log\Phi^{AJ}(y^\chi)\left(f^{rig}\right) - w_p \log\Phi^{AJ}(Frob_p y^\chi)\left(f^{rig}\right)\right)\cdot \\ &\quad \left(\log\Phi^{AJ}\left(y^{\chi^{-1}}\right)\left(f^{rig}\right) - w_p \log\Phi^{AJ}\left(Frob_p y^{\chi^{-1}}\right)\left(f^{rig}\right)\right). \end{aligned}$$

Proof. This is a consequence of Theorem 5.24, in light of Proposition 5.16. ■

Let us now focus for the remainder of this section on a genus character χ attached to the couple (χ_1, χ_2). We note that the signs of the twisted L-functions $L(f, \chi_i, s)$ are given by (see [Sh, Theorem 3.66]):

$$(-1)^{\frac{k_0}{2}}\, w_N \chi_i(-N). \tag{35}$$

Furthermore, since the number of the inert primes pN^- dividing N is even and $\varepsilon_{K'} = \chi_1\chi_2$, where $\varepsilon_{K'}$ is the Dirichlet character attached to the imaginary quadratic extension $K'/\mathbb{Q}$,

$$\chi_1(-N)\,\chi_2(-N) = \varepsilon_{K'}(-1) = -1.$$

Hence the signs of the twisted complex L-functions $L(f, \chi_i, s)$ are opposite to each other. The genus character χ cuts out a biquadratic extension of $\mathbb{Q}$. Write $\mathbb{Q}_{\chi_i}$ to denote the quadratic extension that corresponds to the Dirichlet character χ_i.

Whenever V is a $\mathbb{Q}_p\left[G_{H_\chi/\mathbb{Q}}\right]$-module let us write $V^\pm$ to denote the subspace on which the complex conjugation τ acts as $\pm$, so that $V = V^+ \oplus V^-$. Since $\mathrm{Ind}^{G_\mathbb{Q}}_{G_{K'}}(\chi) = \chi_1 \oplus \chi_2$, we also have $V^\chi = V^{\chi_1} \oplus V^{\chi_2}$, where the left hand side is viewed as a $G_{H_\chi/K'}$-module and the right hand side as a $G_{H_\chi/\mathbb{Q}}$-module. Since $\chi_2(-1) = -\chi_1(-1)$ we may order (χ_1, χ_2) in such a way that $\mathbb{Q}_{\chi_1}/\mathbb{Q}$ is a real field. Then we have $V^{\chi_1} \subset V^{\chi,+}$ and $V^{\chi_2} \subset V^{\chi,-}$, so that $V^{\chi_1} = V^{\chi,+}$ and $V^{\chi_1} = V^{\chi,-}$. This remark applied to $V = CH^{m+1}\left(\mathcal{M}_{n,H_\chi}\right)$ and $V =$

$Ext^1_{G_{H_\chi}}(\mathbb{Q}_p, V_f)$ implies that $CH^{m+1}\left(\mathcal{M}_{n,H_\chi}\right)^\chi$ and $MW_f(H_\chi)^\chi$ both have a direct sum decomposition with

$$\begin{aligned} CH^{m+1}\left(\mathcal{M}_{n,H_\chi}\right)^{\chi,+} &= CH^{m+1}\left(\mathcal{M}_{n,\mathbb{Q}_{\chi_1}}\right)^{\chi_1}, \\ CH^{m+1}\left(\mathcal{M}_{n,H_\chi}\right)^{\chi,-} &= CH^{m+1}\left(\mathcal{M}_{n,\mathbb{Q}_{\chi_2}}\right)^{\chi_2}, \\ MW_f(H_\chi)^{\chi,+} &= MW_f\left(\mathbb{Q}_{\chi_1}\right)^{\chi_1} \text{ and} \\ MW_f(H_\chi)^{\chi,-} &= MW_f\left(\mathbb{Q}_{\chi_2}\right)^{\chi_2}. \end{aligned} \tag{36}$$

Whenever Ψ is an oriented optimal embedding of level N^+, viewed as an element of $X(H)$, $\tau\Psi$ is an optimal embedding whose orientations at the primes dividing N^+ have been reversed. Define $W_N := \prod_{l|N^+} W_l^+ \prod_{l|pN^-} W_l^-$. Since $\#\{l : l \mid pN^-\}$ is even, it follows from the analogous of (34) at the primes dividing pN^-, that $W_N f^{rig} = w_N f^{rig}$, where w_N is the sign of the Atkin-Lehner involution acting on f. Since $W_N\Psi$ reverses all the orientations too, we have

$$\tau\Psi = W_N\delta\Psi, \text{ for some } \delta \in G_{H/K'}. \tag{37}$$

It is easily checked from Lemma 5.23 and (37) that $cl^{m+1}_{0,f}(y^\chi) \in MW_f(H_\chi)^{\chi,\pm}$ for a suitable choice of a sign. Let H be the Hilbert ring class field.

Corollary 5.26 *Let $\chi : G_{H/K} \to \mathbb{C}^\times$ be a genus character. If $cl^{m+1}_{0,f}(y^\chi) \in MW_f\left(\mathbb{Q}_{\chi_i}\right)^{\chi_i}$ then*

$$\frac{d^2}{d\kappa^2}\left[L_p(f/K', \chi, \kappa)\right]_{\kappa=k} = \frac{1}{2}\left(1 + a_p p^{-\frac{k_0-2}{2}}\chi_i(p)\right)^2 \log\Phi^{AJ}(y^\chi)\left(f^{rig}\right)^2.$$

Proof. Since $cl^{m+1}_{0,f}(y^\chi) \in MW_f\left(\mathbb{Q}_{\chi_i}\right)^{\chi_i}$ we have

$$cl^{m+1}_{0,f}(Frob_p y^\chi) = Frob_p cl^{m+1}_{0,f}(y^\chi) = \chi_i(p)\, cl^{m+1}_{0,f}(y^\chi).$$

The p-adic Abel-Jacobi map $\Phi^{AJ}(-)\left(f^{rig}\right)$ factors through $cl^{m+1}_{0,f}$ by definition, so that we find

$$\log\Phi^{AJ}(Frob_p y^\chi)\left(f^{rig}\right) = \chi_i(p)\log\Phi^{AJ}(y^\chi)\left(f^{rig}\right).$$

The claim follows from Corollary 5.25, since we have $\chi = \chi^{-1}$ and $-w_p = a_p p^{-\frac{k_0-2}{2}}$. ■

Corollary 5.27 *Let $\chi : G_{H/K} \to \mathbb{C}^\times$ be a genus character. If $cl^{m+1}_{0,f}(y^\chi) \in MW_f\left(\mathbb{Q}_{\chi_i}\right)^{\chi_i}$ then*

$$\frac{d^2}{d\kappa^2}\left[L_p(f/K', \chi, \kappa)\right]_{\kappa=k} = \begin{cases} 2\log\Phi^{AJ}(y^\chi)\left(f^{rig}\right)^2 & \text{if } \chi_i(p) = a_p p^{-\frac{k_0-2}{2}} = -w_p \\ 0 & \text{if } \chi_i(p) = -a_p p^{-\frac{k_0-2}{2}} = w_p. \end{cases}$$

We will also need the following deep result of Kato.

Lemma 5.28 *If* $\chi_i(-N) = (-1)^{\frac{k_0-2}{2}} w_N$ *and* $L(f, \chi_j, k_0/2) \neq 0$ *with* $i \neq j$ *or if* $\chi_i(-N) = (-1)^{\frac{k_0-2}{2}} w_N$ *and* $\chi_i(pN^-) = 1$ *then* $cl_{0,f}^{m+1}(y^\chi) \in MW_f(\mathbb{Q}_{\chi_i})^{\chi_i}$.

Proof. We only prove the first statement, which is the one we need in the subsequent section. If $\chi_i(-N) = (-1)^{\frac{k_0-2}{2}} w_N$ the sign of $L(f, \chi_i, s)$ is negative and we assume that $L(f, \chi_j, k_0/2) \neq 0$. Then $MW_f\left(\mathbb{Q}_{\chi_j}\right)^{\chi_j} = 0$ by [K, Theorem 14.2 (2)]. ■

Remark 5.29 *Suppose that* $0 \neq cl_{0,f}^{m+1}(y_{\mathfrak{p}}^\chi) \in MW_{f,\mathfrak{p}}(\mathbb{Q}_{\chi_i})^{\chi_i}$. *As an application of Kolyvagin methods developed in [Ne1] and [Ne3] one can show that* $K_{f,\mathfrak{p}} cl_{0,f}^{m+1}(y_{\mathfrak{p}}^\chi) = MW_{f,\mathfrak{p}}(\mathbb{Q}_{\chi_i})^{\chi_i}$.

6 Proof of the main results

Recall our factorization $N = pN^+N^- = pM$ into factors prime each other, where N^- is squarefree and divisible by an odd number of prime factors. In the following theorem we will assume the existence of a prime $q \parallel M$ and the consideration of a factorization with $q \mid N^-$ will be implicit in order to apply the results of the previous section. Recall the harmonic cocycle $c^{har} = c_f^{har}$ that was associated to f in subsection 4.2. We may assume that $c^{har} \in \mathcal{C}_{har}(\mathcal{E}, \mathbf{V}_n(K_f))^{\Gamma'}$, so that $\langle c_{k_0}, c_{k_0} \rangle \in K_f^\times$.

Theorem 6.1 *Suppose there exists* $q \parallel M$. *Let* ω *be a quadratic Dirichlet character, of conductor prime to* N *such that*

$$\omega(-N) = (-1)^{\frac{k_0-2}{2}} w_N \text{ and } \omega(p) = a_p p^{-\frac{k_0-2}{2}} = -w_p.$$

Then:

1. *the* p*-adic* L*-function* $L_p(f, \omega, \kappa, \kappa/2)$ *vanishes to order*
$$\mathrm{ord}_{\kappa=k_0} L_p(f, \omega, \kappa, \kappa/2) \geq 2;$$
2. *there exists* $y^\omega \in CH^{m+1}(\mathcal{M}_{n,\mathbb{Q}_\omega})^\omega$ *and* $t_f \in K_f^\times$ *such that*
$$\frac{d^2}{d\kappa^2}\left[L_p(f/K, \omega, \kappa, \kappa/2)\right]_{\kappa=k_0} = t_f \cdot \log \Phi^{AJ}(y^\omega)\left(f^{rig}\right)^2;$$
3. *If* $cl_{0,f}^{m+1}(y_{\mathfrak{p}}^\omega) \neq 0$ *then* $MW_{f,\mathfrak{p}}(\mathbb{Q}_\omega)^\omega = K_{f,\mathfrak{p}} cl_{0,f}^{m+1}(y_{\mathfrak{p}}^\omega)$.
4. *we have*
$$t_f/2 \equiv L^*(f, \psi, 1) \text{ in } K_f^\times / K_f^{\times 2},$$
for any quadratic Dirichlet character ψ *such that*
$$\psi(l) = \omega(l) \text{ for every } l \mid M := N/p,$$
$$\psi(p) = -\omega(p) \text{ and}$$
$$L(f, \psi, 1) \neq 0.$$

Proof. Set $\omega = \chi_1$ and choose an auxiliary quadratic Dirichlet character χ_2 of conductor prime to the conductor of χ_1 such that:

(a) $\chi_2(l) = \chi_1(l)$ for all $l \mid N^+$

(b) $\chi_2(l) = -\chi_1(l)$ for all $l \mid pN^-$ and $\chi_2(-1) = -\chi_1(-1)$

(c) $L(f, \chi_2, k_0/2) \neq 0$

This is possible since the main result of [MM] generalizes to higher weight modular forms. The Dirichlet character $\varepsilon_{K'} := \chi_1\chi_2$ cuts out an imaginary quadratic extension $K'/\mathbb{Q}$ and there is a genus character χ attached to the pair (χ_1, χ_2). Furthermore, note that the sign of $L(f, \chi_1, s)$ is -1 in light of the assumption $\chi_1(-N) = (-1)^{\frac{k_0-2}{2}} w_N$ (by (35)), while the sign of $L(f, \chi_2, s)$ is 1. Note that, thanks to (c) and Lemma 5.28, we can apply Corollary 5.27 with $\chi_i = \chi_1$.

By Theorem 5.21,

$$L_p(f/K', \chi, \kappa) = \eta(\kappa) L_p(f, \chi_1, \kappa, \kappa/2) L_p(f, \chi_2, \kappa, \kappa/2). \tag{38}$$

The factor $\eta(\kappa) L_p(f, \chi_2, \kappa, \kappa/2)$ does not vanish at the critical point $\kappa = k_0$, since $\eta(\kappa) \neq 0$ on U and we have

$$\begin{aligned} L_p(f, \chi_2, k_0, k_0/2) &= \left(1 - \chi_2(p) p^{\frac{k_0-2}{2}} a_p^{-1}\right)^2 L^*(f, \chi_2, k_0/2) = \\ &= 4L^*(f, \chi_2, k_0/2) \neq 0. \end{aligned} \tag{39}$$

Indeed the first equality follows by Corollary 5.4, the second one follows by the assumption $\chi_1(p) = p^{-\frac{k_0-2}{2}} a_p$, together with (b) assuring us that $\chi_2(p) = -\chi_1(p)$, and the non-vanishing is a consequence of (c).

On the other hand, the factor $L_p(f, \chi_1, \kappa, \kappa/2)$ vanishes at the critical point $\kappa = k_0$, again by Corollary 5.4 and the assumption $\chi_1(p) = p^{-\frac{k_0-2}{2}} a_p$, or thanks to the fact that $L^*\left(f_k^{\#}, \chi_1, k_0/2\right) = 0$ by the above considerations on the complex L-functions. Hence

$$L_p(f, \chi_1, k_0, k_0/2) = 0. \tag{40}$$

This preliminary discussion has the effect of avoiding appealing to [BD2, Remark 1.13], since we have not exploited the Mazur-Kitagawa p-adic L-function as a two variable function.

1. A formal computation using (38) and (40) yields

$$\frac{d}{d\kappa}\left[L_p(f/K', \chi, \kappa)\right]_{\kappa=k_0} = \frac{d}{d\kappa}\left[L_p(f, \chi_1, \kappa, \kappa/2)\right]_{\kappa=k_0} \eta(k_0) L_p(f, \chi_2, k_0, k_0/2).$$

Note that $\varepsilon_K(p) = -1$ so that we are in the inert case and the left hand side vanishes by Proposition 5.16. Now (39) implies that

$$\frac{d}{d\kappa}\left[L_p(f, \chi_1, \kappa, \kappa/2)\right]_{\kappa=k_0} = 0, \tag{41}$$

so that the claim 1. follows. Note that that the same sign considerations of [BD2, Theorem 5.4] apply in order to deduce the order two vanishing along the line (k_0, s) of $L_p(f, \chi_1, \kappa, s)$ and hence the order two vanishing of the two variable p-adic L-function.

2. A formal computation using (38), (40) and (41) yields

$$\frac{d^2}{d\kappa^2}\left[L_p\left(f/K', \chi, \kappa\right)\right]_{\kappa=k_0} = \frac{d^2}{d\kappa^2}\left[L_p\left(f, \chi_1, \kappa, \kappa/2\right)\right]_{\kappa=k_0} \eta\left(k_0\right) L_p\left(f, \chi_2, k_0, k_0/2\right) \tag{42}$$

By (39) and Corollary 5.27 we can write

$$\begin{aligned}\frac{d^2}{d\kappa^2}\left[L_p\left(f, \chi_1, \kappa, \kappa/2\right)\right]_{\kappa=k_0} &= \frac{1}{2}\left\langle c_{k_0}, c_{k_0}\right\rangle^{-1} D_K^{\frac{2-k_0}{2}} i^{2-k_0} \cdot \\ &\quad L^*\left(f, \chi_2, k_0/2\right)^{-1} \log \Phi^{AJ}\left(y^\chi\right)\left(f^{rig}\right)^2\end{aligned}$$

where $y^\chi \in CH^{m+1}\left(\mathcal{M}_{n,\mathbb{Q}_\omega}\right)^\omega$. Now the claim 2. follows since we have, writing $D_{K'} = -D \in \mathbb{Z}$ with $D > 0$ and $k_0 = 2h$,

$$D_{K'}^{\frac{2-k_0}{2}} i^{2-k_0} = (-1)^{2-k_0} D^{1-h} \in \mathbb{Z}.$$

3. This is a consequence of Remark 5.29.

4. Let $\psi := \chi_1'$ be any Dirichlet character satisfying the conditions of 4. and consider the Dirichlet character $\varepsilon_{K''} := \chi_1'\chi_2$. It cuts out an imaginary quadratic field $K''/\mathbb{Q}$. There is a genus character χ' attached to the couple (χ_1', χ_2), but now p is split in K''. In particular $\chi_i(p) = \chi'(\mathfrak{p})$ for any $\mathfrak{p} \mid p$ and Corollary 5.19 yields, in light of the fact that $\chi_2(p) = -p^{-\frac{k_0-2}{2}} a_p$:

$$\begin{aligned}L_p\left(f/K'', \chi', k_0\right) &= \left(1 - p^{\frac{k_0-2}{2}} a_p^{-1} \chi_2(p)\right)^2 \left\langle c_{k_0}, c_{k_0}\right\rangle L^*\left(f_{k_0}/K'', \chi', k_0/2\right) = \\ &= 4\left\langle c_{k_0}, c_{k_0}\right\rangle L^*\left(f_{k_0}/K'', \chi', k_0/2\right).\end{aligned}$$

By (38) relative to (χ_1', χ_2) together with (39) relative to χ_1'

$$\left\langle c_{k_0}, c_{k_0}\right\rangle L^*\left(f_{k_0}/K'', \chi', k_0/2\right) = L^*\left(f, \chi_1', k_0/2\right) \eta\left(k_0\right) L_p\left(f, \chi_2, k_0, k_0/2\right).$$

Besides, thanks to (42), $t_f/2 = \eta\left(k_0\right)^{-1} L_p\left(f, \chi_2, k_0, k_0/2\right)^{-1}$, so that

$$\left\langle c_{k_0}, c_{k_0}\right\rangle L^*\left(f_{k_0}/K'', \chi', k_0/2\right) \equiv t_f/2 L^*\left(f, \chi_1', k_0/2\right) \mod K_f^{\times 2}.$$

But the left hand side is a square by the Hatcher-Hui Xue formula applied to the newform c_{k_0} of level pN^+, that can be reformulated in a similar way as it is done in [BD2, Proposition 3.4] when $k_0 = 2$, thus getting a formula having the same shape of (33) but involving optimal embeddings of level pN^+. ∎

We now turn to the case where $K/\mathbb{Q}$ is a real quadratic field and χ is a genus character attached to (χ_1, χ_2). Recall that, by Remark 5.10, $\chi_1(-M) = \chi_2(-M)$. Assume again $c^{har} \in \mathcal{C}_{har}\left(\mathcal{E}, \mathbf{V}_n\left(K_f\right)\right)^{\Gamma'}$.

Theorem 6.2 *Suppose $N = pM$, that there exists $q \parallel M$ and that*

$$\chi_i(-M) = (-1)^{\frac{k_0}{2}} w_M.$$

Then:

1. *there exist $y^\chi \in CH^{m+1}\left(\mathcal{M}_{n,H_\chi}\right)^\chi$ and $s_f \in K_f^\times$ such that*

$$\log \Phi^{AJ}(j^\chi)(I_f) = s_f \cdot \log \Phi^{AJ}(y^\chi)\left(f^{rig}\right);$$

2. *$\chi_2(-N) = -\chi_1(-N)$, $cl_{0,f}^{m+1}(y^\chi) \in MW_{f,\mathfrak{p}}\left(\mathbb{Q}_{\chi_i}\right)^{\chi_i}$ where $\chi_i(-N) = (-1)^{\frac{k_0-2}{2}} w_N$ and, if $cl_{0,f}^{m+1}\left(y_\mathfrak{p}^\chi\right) \neq 0$,*

$$MW_{f,\mathfrak{p}}(H_\chi)^\chi = MW_{f,\mathfrak{p}}\left(\mathbb{Q}_{\chi_i}\right)^{\chi_i} = K_{f,\mathfrak{p}} cl_{0,f}^{m+1}(y_{\chi,\mathfrak{p}}).$$

Proof. Consider the functional equation given by Theorem 5.9:

$$L_p(f/K,\chi,\kappa) = D_K^{\frac{\kappa-2}{2}} L_p(f,\chi_1,\kappa,\kappa/2)\, L_p(f,\chi_2,\kappa,\kappa/2). \tag{43}$$

In light of the assumption $\chi_i(-M) = (-1)^{\frac{k_0}{2}} w_M$ we find

$$\chi_i(-N) = (-1)^{\frac{k_0}{2}} w_M \chi_i(p) \tag{44}$$

Furthermore, since p is inert in K, we have $\chi_1(p) = -\chi_2(p)$. In particular, one of the two Dirichlet characters, say χ_1, will be such that $\chi_1(p) = -w_p$; then (44) tells us that

$$\chi_1(-N) = (-1)^{\frac{k_0-2}{2}} w_N. \tag{45}$$

It follows that the Dirichlet character $\chi_1 = \omega$ satisfies the assumption of Theorem 6.1. By 1. we know that the order of vanishing of the p-adic L-function $L_p(f,\chi_1,\kappa,\kappa/2)$ is at least 2. A formal computation using this information and (43) gives

$$\frac{d^2}{d\kappa^2}\left[L_p(f/K,\chi,\kappa)\right]_{\kappa=k_0} = \frac{d^2}{d\kappa^2}\left[L_p(f,\chi_1,\kappa,\kappa/2)\right]_{\kappa=k_0} D_K^{\frac{k_0-2}{2}} L_p(f,\chi_2,k_0,k_0/2). \tag{46}$$

By Corollary 5.14 (again use $\chi_i(-M) = (-1)^{\frac{k_0}{2}} w_M$), Theorem 6.1 2. and (46)

$$\log \Phi^{AJ}(j^\chi)(I_f)^2 = t_f/2 \cdot \log \Phi^{AJ}(y^\omega)\left(f^{rig}\right)^2 L_p(f,\chi_2,k_0,k_0/2). \tag{47}$$

Again note that $\chi_2(p) = -\chi_1(p)$, so that (44) and (45) tells us that $\chi_2(p) = w_p = -p^{\frac{k_0-2}{2}} a_p^{-1}$. Hence thanks to (39) we can rewrite (47) as

$$\log \Phi^{AJ}(j^\chi)(I_f)^2 = 4t_f/2 \cdot \log \Phi^{AJ}(y^\omega)\left(f^{rig}\right)^2 L^*(f,\chi_2,k_0/2).$$

If $L^*(f,\chi_2,k_0/2) = 0$, we deduce that $\log \Phi^{AJ}(j^\chi)(I_f) = 0$ and the first part of the Theorem is trivially true by setting $y_\chi = 0$. Hence suppose $L^*(f,\chi_2,k_0/2) \neq$

0. In this case note that χ_2 satisfies the assumption that was made on ψ in Theorem 6.1 4.. Hence we know that $t_f/2L^*(f,\chi_2,k_0/2)$ is a square in $K_f^\times$, as well as all the remaining factors. The first claim follows by setting $y^\chi = y^\omega$ and extracting the square roots.

For the second statement we already proved that $\chi_2(-N) = -\chi_1(-N)$, we are assuming that $\chi_1(-N) = (-1)^{\frac{k_0-2}{2}} w_N$ and we know that y^ω belongs to $MW_f(\mathbb{Q}_{\chi_1})^{\chi_1}$, so that y^χ belongs to $MW_f(\mathbb{Q}_{\chi_1})^{\chi_1}$ by construction. Since $\mathrm{Ind}_{G_K}^{G_\mathbb{Q}}(\chi) = \chi_1 \oplus \chi_2$ we can write

$$MW_f(H_\chi)^\chi = MW_f(\mathbb{Q}_{\chi_1})^{\chi_1} \oplus MW_f(\mathbb{Q}_{\chi_2})^{\chi_2}.$$

When $cl_{0,f}^{m+1}(y_\mathfrak{p}^\chi) \neq 0$ we are in the case $L^*(f,\chi_2,k_0/2) \neq 0$ and then $y^\chi = y^\omega$. It follows from Theorem 6.1 3. that we have $MW_{f,\mathfrak{p}}(\mathbb{Q}_{\chi_1})^{\chi_1} = K_{f,\mathfrak{p}} cl_{0,f}^{m+1}(y_\mathfrak{p}^\chi)$ and, since $L^*(f,\chi_2,k_0/2) \neq 0$, [K, Theorem 14.2 (2)] implies $MW_{f,\mathfrak{p}}(\mathbb{Q}_{\chi_2})^{\chi_2} = 0$. ∎

Remark 6.3 *Let $\sigma(f)$ be the companion form of f obtained by applying the automorphism σ to the Fourier coefficients of f. If we choose $c_{\sigma(f)}^{har} := \sigma\left(c_f^{har}\right)$, the quantities s_f appearing in the statement of Theorem 6.2 satisfies the relation $\sigma(s_f) = s_{\sigma(f)}$. It follows that there is $s \in K_f \otimes_\mathbb{Q} F_p$ inducing $s_{\sigma(f)}$ on the $\sigma(f)$-component. Recall that $F_p/\mathbb{Q}_p$ denotes an extension such that $K_{[f]} \subset F_p$, where $K_{[f]}$ is the field generated by the Fourier coefficients of f and its companion forms.*

Let us now prove the main result Theorem 1.1. Let $V_{[f]}$ be the p-adic representation attached to the new modular form f, with associated filtered Frobenius module $\mathbb{D}_{[f]}$. Note that $MW_f(H_\chi)^\chi$ is naturally a K_f-vector space, since the Hecke correspondences act on the rational Chow groups through the idempotent $e_{[f]}$ corresponding to the f-isotypic component. Let the assumptions be as in Theorem 6.2. Fix an isomorphism of monodromy modules $\mathbf{D}_{[f]} \simeq \mathbb{D}_{[f]}$ over $\mathbb{Q}_p$ as granted by Theorem 4.11.

The identification $\varphi : \mathbf{D}_{[f]} \simeq \mathbb{D}_{[f]}$ in $MF_{\mathbb{Q}_p}(\phi, N)$ allow us to identify the tangent spaces:

$$\begin{array}{rcccc} \alpha : \mathbf{MS}_{[f]}^{c,\vee,w_\infty}(F_p) & = & \frac{\mathbf{D}_{[f],F_p}}{F^{m+1}\mathbf{D}_{[f],F_p}} & \overset{\varphi}{\simeq} & \frac{\mathbb{D}_{[f],F_p}}{F^{m+1}\mathbb{D}_{[f],F_p}} \\ & \overset{\exp}{\simeq} & H_{st}^1\left(K, V_{[f]}(m+1)\right) & = & Ext_{MF}^1\left(F_p, \mathbb{D}_{[f]}(m+1)\right) \\ & = & \frac{\mathbb{D}_{[f],F_p}}{F^{m+1}\mathbb{D}_{[f],F_p}} & = & \left(F^{m+1}\mathbb{D}_{[f],F_p}\right)^\vee \\ & = & e_{[f]} M_k(X, F_p)^\vee & = & e_{[f]} M_k(\Gamma', F_p)^\vee . \end{array} \tag{48}$$

The above identifications hold over any complete field extension $F_p/\mathbb{Q}_p$, with the only possible exception of the last identification, that holds assuming $F_p \supset \mathbb{Q}_{p^2}$.

The first identification is the morphism

$$f^0 : \frac{\mathbf{D}_{[f],F_p}}{F^{m+1}\mathbf{D}_{[f],F_p}} \to \mathbf{MS}_{[f]}^{c,\vee,w_\infty}(F_p)$$

that was considered in section 3. The last five identifications are given by

$$IS : H^1_{st}\left(K, V_{[f]}(m+1)\right) \to e_{[f]} M_k \left(\Gamma', F_p\right)^{\vee}.$$

We have the following commutative diagram

$$\begin{array}{ccc} \frac{\mathbf{D}_{[f],F_p}}{F^{m+1}\mathbf{D}_{[f],F_p}} & \overset{\exp\circ\varphi}{\to} & H^1_{st}\left(K, V_{[f]}(m+1)\right) \\ f^0 \downarrow & & \downarrow IS \\ \mathbf{MS}^{c,\vee,w_\infty}_{[f]}\left(F_p\right) & \overset{\alpha}{\to} & e_{[f]} M_k \left(\Gamma', F_p\right)^{\vee}. \end{array} \tag{49}$$

It will be convenient to give an explicit description of the monodromy module $\mathbb{D}$ by means of Teitelbaum's p-adic integration theory (as developed for example in [Te]). More explicitly let Γ' be the arithmetic group defined in subsection 5.3.2. As it is well known there is an analogue of Proposition 2.8 in this definite setting: the morphism

$$R : \mathcal{D}^0_n\left(\mathbb{P}^1\left(\mathbb{Q}_p\right)\right) \to \mathcal{C}_{har}\left(\mathcal{E}, \mathbf{V}_n\right)$$

that was considered in subsection 2.2 induces an isomorphism $\mathcal{D}^0_n\left(\mathbb{P}^1\left(\mathbb{Q}_p\right)\right)^{\Gamma'} = \mathcal{C}_{har}\left(\mathcal{E}, \mathbf{V}_n\right)^{\Gamma'} =: \mathbf{C}_{har}\left(F_p\right)$ (over any complete local field $F_p/\mathbb{Q}_p$). Define $\mathbf{D}^T$ over $\mathbb{Q}_p$ as follows:

$$\mathbf{D}^T := \mathbf{C}_{har}\left(\mathbb{Q}_p\right)^{\vee} \oplus \mathbf{C}_{har}\left(\mathbb{Q}_p\right)^{\vee},$$

with filtration, monodromy operator and Frobenius formally defined exactly as in 2.3, Teitelbaum's $\mathcal{L}$-invariant replacing Orton's $\mathcal{L}$-invariant. Similarly as in section 3 there is an identification obtained by means of $f(x,y) = -x - \mathcal{L}y$:

$$f : \frac{\mathbf{D}^T_{F_p}}{F^{m+1}\mathbf{D}^T_{F_p}} \overset{\simeq}{\to} \mathbf{C}_{har}\left(F_p\right)^{\vee}. \tag{50}$$

By [IS] Teitelbaum's $\mathcal{L}$-invariant equals the $\mathcal{L}$-invariant of the monodromy module $\mathbb{D}$ and there is an identification $\mathbf{D}^T \simeq \mathbb{D}$ in $MF_{\mathbb{Q}_p}(\phi, N)$. As it follows from [RoSe, proof of Lemma 4.4], in order to give an explicit identification $\mathbf{D}^T \simeq \mathbb{D}$, we can simply identify the m-isotypic components as Hecke modules. Furthermore, we can identify $\mathbf{D}^T \simeq \mathbb{D}$ in $MF_{\mathbb{Q}_{p^2}}(\phi, N)$, since we have $Hom_{MF_{\mathbb{Q}_p}(\phi,N)}\left(D_1, D_2\right) = Hom_{MF_{\mathbb{Q}_{p^2}}(\phi,N)}\left(D_{1,\mathbb{Q}_{p^2}}, D_{2,\mathbb{Q}_{p^2}}\right)$, whenever $D_i \in MF_{\mathbb{Q}_p}(\phi, N)$ (see [RoSe, proof of Lemma 4.4]). As in [IS], let $\mathcal{V}_n$ be the coherent sheaf on the Shimura curve X over $\mathbb{Q}_{p^2}$ associated to the representation $\mathbf{V}_n$, so that $\mathbb{D}_{\mathbb{Q}_{p^2}} = H^1\left(X, \mathcal{V}_n\right)$. As it follows from [IS], the m-isotypic component of $H^1\left(X, \mathcal{V}_n\right)$ is $H^1\left(X, \mathcal{V}_n\right)^m = \iota\left(H^1\left(\Gamma', \mathbf{V}_n\left(\mathbb{Q}_{p^2}\right)\right)\right)$, where ι is the injection [IS, (76)]. Let

$$\langle -, - \rangle_{\Gamma'} : \mathbf{C}_{har}\left(\mathbb{Q}_{p^2}\right) \otimes H^1\left(\Gamma', \mathbf{V}_n\left(\mathbb{Q}_{p^2}\right)\right) \to \mathbb{Q}_{p^2}$$

be the perfect pairing [IS, (75)]. It induces an isomorphism

$$H^1\left(\Gamma', \mathbf{V}_n\left(\mathbb{Q}_{p^2}\right)\right) \overset{\simeq}{\to} \mathbf{C}_{har}\left(\mathbb{Q}_{p^2}\right)^{\vee} \tag{51}$$

that we use to identify the m-isotypic components. Let $I : M_k\left(\Gamma', \mathbb{Q}_{p^2}\right) \stackrel{\simeq}{\to} \mathbf{C}_{har}\left(\mathbb{Q}_{p^2}\right)$ be the residue map, thus inducing a map $I^\vee : \mathbf{C}_{har}\left(\mathbb{Q}_{p^2}\right)^\vee \to M_k\left(\Gamma', \mathbb{Q}_{p^2}\right)^\vee$.

Lemma 6.4 *The isomorphism (51) induces an identification $\psi : \mathbf{D}^T \simeq \mathbb{D}$ making the following diagram commutative:*

$$\begin{array}{ccc} \frac{\mathbf{D}^T_{\mathbb{Q}_{p^2}}}{F^{m+1}\mathbf{D}^T_{\mathbb{Q}_{p^2}}} & \stackrel{f}{=} & \mathbf{C}_{har}\left(\mathbb{Q}_{p^2}\right)^\vee \\ \psi \Vert\downarrow & & \Vert\downarrow I^\vee \\ \frac{\mathbb{D}_{\mathbb{Q}_{p^2}}}{F^{m+1}\mathbb{D}_{\mathbb{Q}_{p^2}}} & = & M_k\left(\Gamma', \mathbb{Q}_{p^2}\right)^\vee . \end{array}$$

Here the lower horizontal arrow is the composition of the last three identifications in the definition of α.

Proof. The morphism f maps the class $\mathbf{d} = [x, y] \in \mathbf{D}^T_{\mathbb{Q}_{p^2}}/F^{m+1}$ to the coordinate $f(\mathbf{d})$ in $\mathbf{C}_{har}\left(\mathbb{Q}_{p^2}\right)^\vee$ of the opposite of the unique element $-(f(\mathbf{d}), 0)$ in $\ker N = \mathbf{D}^{T,m}$ representing $\mathbf{d}$. Let $\psi(\mathbf{d}) \in \mathbb{D}_{\mathbb{Q}_{p^2}}/F^{m+1}$ be the corresponding element and denote by d the unique element of $\ker N = \mathbb{D}^{T,m}_{\mathbb{Q}_{p^2}}$ representing $\psi(\mathbf{d})$. By unicity we have $\psi((f(\mathbf{d}), 0)) = -d$. Let

$$P : H^1(X, \mathcal{V}_n) \to H^1\left(\Gamma', \mathbf{V}_n\left(\mathbb{Q}_{p^2}\right)\right)$$

be the left inverse of ι as defined in [IS, (15)]. If we write $x \in \mathbb{D}_{\mathbb{Q}_{p^2}} = H^1(X, \mathcal{V}_n)$ as $x = x^m + x^{m+1}$ according to its slope decomposition we have $x^m = \iota(P(x))$ (see [IS]). In particular we have $d = d_m = \iota(P(d))$. Since ψ is induced by (51) we deduce, from the equality $\psi((f(\mathbf{d}), 0)) = -d$, that $f(\mathbf{d}) = -\langle -, P(d)\rangle_{\Gamma'}$.

Besides, the identification $\mathbb{D}_{\mathbb{Q}_{p^2}}/F^{m+1} = \left(F^{m+1}\mathbb{D}_{\mathbb{Q}_{p^2}}\right)^\vee$ arises from Serre duality induced by cup product and the canonical identification $F^{m+1}\mathbb{D}_{\mathbb{Q}_{p^2}} = M_k\left(\Gamma', \mathbb{Q}_{p^2}\right)$ (see [IS, Proposition 6.1]). Let d be as above, so that $d \in \ker N = \ker I$ and d corresponds to $\langle -, d\rangle_X \in M_k\left(\Gamma', \mathbb{Q}_{p^2}\right)^\vee$, where $\langle -, -\rangle_X$ is the cup product. The reciprocity law [IS, Theorem 10.3] implies

$$\langle -, d\rangle_X = -\langle I(-), P(d)\rangle_X = I^\vee(f(\mathbf{d})),$$

which is the claim. ■

Lemma 6.5 *We may choose $\varphi : \mathbf{D}_{[f]} \simeq \mathbb{D}_{[f]}$ in such a way that $\alpha^\vee\left(ev_{\sigma(f)^{rig}}\right) = ev_{I_{\sigma(f)}}$ and Remark 6.3 holds.*

Proof. Let us write $\mathbf{D}^T_{[f]}$ (resp. $\mathbf{C}_{har,[f]}$) to denote the space obtained by taking the f-isotypic component by means of the idempotent $e_{[f]}$. According to Lemma 6.4 there is a commutative diagram:

$$\begin{array}{ccccc} & & \frac{\mathbf{D}^T_{[f]}}{F^{m+1}\mathbf{D}^T_{[f]}} & \stackrel{f}{=} & \mathbf{C}_{har,[f]}(F_p)^\vee \\ & & \psi \Vert\downarrow & & \Vert\downarrow I^\vee \\ \mathbf{MS}^{c,\vee,w_\infty}_{[f]}(F_p) & = & \frac{\mathbb{D}_{F_p}}{F^{m+1}\mathbb{D}_{F_p}} & = & M_k(\Gamma', F_p)^\vee . \end{array}$$

Here the lower row comes from (48).

The above identifications holds even with $F_p = \mathbb{Q}_p$, the only possible exception being the last one appearing in the lower row. Denote by β the arrow from $\mathbf{MS}_{[f]}^{c,\vee,w_\infty}(\mathbb{Q}_p)$ to $\mathbf{C}_{har,[f]}(\mathbb{Q}_p)^\vee$, so that we have

$$\beta^\vee : \mathbf{C}_{har,[f]}(\mathbb{Q}_p)^{\vee\vee} \to \mathbf{MS}_{[f]}^{c,\vee\vee,w_\infty}(\mathbb{Q}_p).$$

$\mathbf{MS}_{[f]}^{c,w_\infty}(\mathbb{Q}_p)$ (resp. $\mathbf{C}_{har,[f]}(\mathbb{Q}_p)$) is naturally endowed with the $\mathbb{Q}$-structure $\mathbf{MS}_{[f]}^{c,w_\infty}(\mathbb{Q})$ (resp. $\mathbf{C}_{har,[f]}(\mathbb{Q})$) and they are both rank one K_f-modules. Fix an isomorphism $b : \mathbf{C}_{har,[f]}(\mathbb{Q}) \simeq \mathbf{MS}_{[f]}^{c,w_\infty}(\mathbb{Q})$ of K_f-modules, thus inducing an isomorphism b of $K_f \otimes L$-modules $\mathbf{C}_{har,[f]}(L) \simeq \mathbf{MS}_{[f]}^{c,w_\infty}(L)$ over any field extension. Once we fix $I_f = I_f^{w_\infty} \in \mathbf{MS}_{[f]}^{c,w_\infty}(K_f)$, we may choose $I_{\sigma(f)} := \sigma(I_f) \in \mathbf{MS}_{[f]}^{c,w_\infty}\left(K_{\sigma(f)}\right)$, the quantity $\Omega_{\sigma(f)}^{w_\infty}$ appearing in Proposition 2.1 being well defined only up to multiplication by an element in $K_{\sigma(f)}^\times$. Setting $c_{\sigma(f)}^{har} := b^{-1}\left(I_{\sigma(f)}\right) \in \mathbf{C}_{har,[f]}\left(K_{\sigma(f)}\right)$, the relation $c_{\sigma(f)}^{har} = \sigma\left(c_f^{har}\right)$ in Remark 6.3 is satisfied and Theorem 6.2 is in force. By biduality we find the morphism

$$b^{\vee\vee} : \mathbf{C}_{har,[f]}(\mathbb{Q}_p)^{\vee\vee} \to \mathbf{MS}_{[f]}^{c,\vee\vee,w_\infty}(\mathbb{Q}_p)$$

such that $b^{\vee\vee}\left(ev_{c_{\sigma(f)}^{har}}\right) = ev_{I_{\sigma(f)}}$ (after extending the scalars to $F_p \supset K_{[f]}$). Since $\mathbf{MS}_{[f]}^{c,\vee\vee,w_\infty}(\mathbb{Q}_p) \simeq K_f \otimes \mathbb{Q}_p$ there exists $t \in (K_f \otimes \mathbb{Q}_p)^\times$ such that $b^{\vee\vee} = t \circ \beta^\vee$. By [RoSe, Lemma 4.4] $End_{MF_{\mathbb{Q}_p}(\phi,N)}\left(\mathbb{D}_{[f]}\right) = K_f \otimes \mathbb{Q}_p$. Replacing φ by $t \circ \varphi$, the morphism $\beta^\vee$ turns into $t \circ \beta^\vee = b^{\vee\vee}$, because the above morphisms are Hecke equivariant. Hence we may assume that $\beta^\vee\left(ev_{c_{\sigma(f)}^{har}}\right) = ev_{I_{\sigma(f)}}$. Recall that the rigid analytic modular form $\sigma(f)^{rig}$ was obtained as $I\left(\sigma(f)^{rig}\right) = c_{\sigma(f)}^{har}$, so that $I^{\vee\vee}\left(ev_{\sigma(f)^{rig}}\right) = ev_{c_{\sigma(f)}^{har}}$. We have $\alpha = I^\vee \circ \beta$, hence $\alpha^\vee = \beta^\vee \circ I^{\vee\vee}$ satisfies $\alpha^\vee\left(ev_{\sigma(f)^{rig}}\right) = \beta^\vee\left(ev_{c_{\sigma(f)}^{har}}\right) = ev_{I_{\sigma(f)}}$. ∎

By Lemma 6.5 we have $\alpha^\vee\left(ev_{\sigma(f)^{rig}}\right) = ev_{I_{\sigma(f)}}$. Hence, by Remark 6.3 (which is in force in light of Lemma 6.5), Theorem 6.2 implies:

$$\alpha\left(\log \Phi_{[f]}^{AJ}(j^\chi)\right) = \log \Phi_{[f]}^{AJ}(sy^\chi) = IS\left(cl_{0,f}^{m+1}(sy^\chi)\right). \tag{52}$$

Then we have $\exp\left(\varphi\left(\Phi_{[f]}^{AJ}(j^\chi)\right)\right) = cl_{0,f}^{m+1}(sy^\chi)$ if and only if we have $IS\left(\exp\left(\varphi\left(\Phi_{[f]}^{AJ}(j^\chi)\right)\right)\right) = IS\left(cl_{0,f}^{m+1}(sy^\chi)\right)$. This is true since the left hand side is $\alpha\left(f^0\left(\Phi_{[f]}^{AJ}(j^\chi)\right)\right) = \alpha\left(\log \Phi_{[f]}^{AJ}(j^\chi)\right)$, thanks to the commutativity of (49). The claim follows from (52).

References

[BD1] Bertolini M., Darmon H.: Heegner points on Mumford-Tate curves. Invent. Math. 126, 413-456 (1996).

[BD2] Bertolini M., Darmon H.: Hida families and rational points on elliptic curves. Invent. Math. 168, no. 2, 371-431 (2007).

[BD3] Bertolini M., Darmon H.: The rationality of Stark-Heegner points over genus fields of real quadratic fields. Annals of Math. 170, 343-369 (2009).

[BDI] Bertolini M., Darmon H., Iovita A.: Families of automorphic forms on definite quaternion algebras and Teitelbaum's conjecture. Astérisque, to appear.

[BDIS] Bertolini M., Darmon H., Iovita A., Spieß M.: Teitelbaum's exceptional zero conjecture in the anticyclotomic setting. Am. J. Math. 124, 411–449 (2002).

[BDG] Bertolini M., Darmon H., P. Green: Periods and points attached to quadratic algebras. Proceedings of an MSRI workshop on special values of L-series, H. Darmon and S. Zhang eds., 323-382 (November 2004).

[Be] Beilinson A. A.: Heights pairing between algebraic cycles. Comtemp. Math. 67, 1-24 (1987).

[BK] Bloch S., Kato K. , L-functions and Tamagawa numbers of motives, in: The Grothendieck Festschrift (Vol. I), Prog. in Math. 86, Birkhauser, Boston (1990), 333-400, P. Cartier, et al., eds.

[Co] Colmez P.: Invariants $\mathcal{L}$ et dérivées de valeurs propres de Frobenius. Astérisque 331, 13–28 (2010).

[Da] Darmon H.: Integration on $\mathcal{H}_p \times \mathcal{H}$ and arithmetic applications. Annals of Math. (2) 154 No. 3, 589-639 (2001).

[Das] Dasgupta S.: Stark-Heegner points on modular Jacobians. Ann. Scient. Ec. Norm. Sup., 4e ser., 38, 427-469 (2005).

[DT] Dasgupta S., Teitelbaum J.: The p-adic upper half plane, in p-adic Geometry. Lectures from the 2007 Arizona Winter School, ed. David Savitt, Dinesh Thakur. University Lecture Series 45. American Mathematical Society, Providence, RI, 2008.

[Gr] Greenberg M.: Stark-Heegner points and the cohomology of quaternionic Shimura varieties. Duke Math. J. 147 No. 3, 541-575 (2009).

[Hi] Hida H.: Elementary theory of L-functions and Eisenstein series. Cambridge University Press (1993).

[IS] Iovita A., Spieß M.: Derivatives of p-adic L-functions, Heegner cycles and monodromy modules attached to modular forms. Invent. Math. 154, 333-384 (2003).

[K] Kato K.: p-adic Hodge theory and values of zeta functions of modular forms. Astérisque 295, 117–290 (2004).

[Ne1] Nekovář J.: Kolyvagin's method for Chow groups of Kuga-Sato varieties. Invent. Math. 107, 99–125 (1992).

[Ne2] Nekovář J.: On p-adic height pairings. Séminaire de Théorie des Nombres, Paris, 1990-91, Progr. Math. 108, Birkhäuser, Boston (1993), 127-202.

[Ne3] Nekovář J.: The Euler system method for CM-points on Shimura curves. In: L-functions and Galois representations, (Durham July 2004), LMS Lecture Note Series 320, Cambridge Univ. Press, 471-547 (2007).

[Or] Orton L.: An elementary proof of a weak exceptional zero conjecture. Canad. J. Math. 56, no. 2, 373–405 (2004).

[Po] Popa A.: Central values of Rankin L-series over real quadratic fields. Compositio Mathematica 142, 811-866 (2006).

[MTT] Mazur B., Tate J., Teitelbaum J.: On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer. Invent. Math. 84 (1), 1–48 (1986).

[MM] Murty M.R., Murty K.V.: Non-vanishing of L-functions and applications. Prog. Math., vol. 157. Birkhäuser, Basel (1997).

[RoSe] Rotger V., Seveso M. A.: $\mathcal{L}$-invariants and Darmon cycles attached to modular forms. Submitted.

[Sc] Scholl A. J.: Motives for modular forms. Invent. Math. 100, 419–430 (1990).

[Se] Seveso M. A.: Heegner cycles and derivatives of two variable p-adic L-functions. Submitted.

[Sh] Shimura G.: Introduction to the arithmetic theory of automorphic functions. Princeton University Press (1971).

[Te] Teitelbaum J.: Values of p-adic L-functions and a p-adic Poisson kernel. Invent. Math. 101, 395-410 (1990).