BEILINSON–FLACH ELEMENTS, STARK UNITS AND p-ADIC ITERATED INTEGRALS

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Abstract. We study weight one specializations of the Euler system of Beilinson–Flach elements introduced by Kings, Loeffler and Zerbes [KLZ], with a view towards the main conjecture formulated by Darmon, Lauder and Rotger [DLR2] relating logarithms of units in suitable number fields to special values of the Hida–Rankin p-adic L-function. We show how the latter conjecture follows from expected properties of Beilinson–Flach elements, and prove the analogue of the main theorem of [CaHs] in our setting.

Contents

1. Introduction

In the last decade there has been substantial progress in the theory of Euler systems of Garrett–Rankin type associated to triples (f, g, h) of modular forms. This framework includes the original scenario of Kato's Euler system [Ka] and also encompasses the Euler systems of Beilinson–Flach elements and diagonal cycles. When f is a weight two cusp form, associated say to an elliptic curve $E/\mathbb{Q}$, and (g, h) is a pair of modular forms of weight 1, this approach yielded new results on the Birch and Swinnerton-Dyer conjecture for twists of E by an Artin representation (see [BDR2], [DR1], [DR2] and [KLZ]).

These results, together with extensive numerical computations performed with the algorithm [Lau] of A. Lauder, led to the formulation of the *elliptic Stark conjecture* in [DLR1], relating the value of a p-adic iterated integral (that may be also recast as a special value of a triple-product p-adic L-function at a point lying outside the region of interpolation) to a regulator defined in terms of logarithms of global points on E. The authors of loc. cit. proved

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their conjecture in the case where g, h are theta series attached to an imaginary quadratic field in which the prime p splits, but the general case remains open.

While no Euler systems are invoked at all in [DLR1], it was clear that they were behind the scenes, and the connection was made explicit in [DR3], where it was proved how the elliptic Stark conjecture of [DLR1] is implied by a precise (but so far unproved at the time of writing this note) recipe for the weight $(2,1,1)$ specializations of the Euler system of diagonal cycles of [DR2], [DR4].

There is a parallel story when one replaces the cusp form f and its associated abelian variety with an Eisenstein series and the multiplicative group $\mathbb{G}_m$. The article [DLR2] proposed a conjecture of the same flavor as the elliptic Stark conjecture of [DLR1], where the entries of the regulator are p-adic logarithms of Stark units in the number field cut out by the Artin representations associated to g and h. As in loc. cit., this conjecture was proved when g and h are theta series attached to an imaginary quadratic field in which the prime p splits.

One of the aims of the present article is to describe the connection between the Euler system of Beilinson–Flach elements and the arithmetic of unit groups of number fields, showing how expected properties of the former imply the main conjecture of [DLR2].

In order to state more precisely our results, let

$$g = \sum_{n\geq 1} a_n q^n \in S_1(N_g, \chi_g), \quad h = \sum_{n\geq 1} b_n q^n \in S_1(N_h, \chi_h)$$

be two normalized newforms, and let V_g and V_h denote the Artin representations attached to them by Serre and Deligne, with coefficients in a finite extension $L/\mathbb{Q}$.

Consider also the tensor product $V_{gh} := V_g \otimes V_h$, and let H be the smallest number field cut out by this representation.

Fix a prime number p which does not divide $N_g N_h$ and label the roots of the p-th Hecke polynomial of g and h as

$$X^2 - a_p(g)X + \chi_g(p) = (X-\alpha_g)(X-\beta_g) \qquad X^2 - a_p(h)X + \chi_h(p) = (X-\alpha_h)(X-\beta_h).$$

Let

$$g_\alpha(q) = g(q) - \beta_g g(q^p), \qquad h_\alpha(q) = h(q) - \beta_h h(q^p)$$

denote the p-stabilization of g (resp. h) on which the Hecke operator U_p acts with eigenvalue α_g (resp. α_h).

Let g^* denote the twist of g by the inverse of its nebentype, i.e., $g^* := g \otimes \chi_g^{-1}$. Note that the U_p-eigenvalues of g^* are $1/\alpha$ and $1/\beta$, and $(g_\alpha)^* = g^*_{1/\beta}$.

By enlarging it if necessary, assume throughout that L contains both the Fourier coefficients of g and h and the roots of their p-th Hecke polynomials. Define

$$U_{gh} = \mathcal{O}_H^\times \otimes L, \quad U_{gh}[1/p] = \mathcal{O}_H[1/p]^\times \otimes L.$$

In order to lighten notation, assume that the prime p splits completely in $L/\mathbb{Q}$, so that L is equipped with an embedding into $\mathbb{Q}_p$, which will be fixed from now on.

We assume throughout that

(H1) The reduction of V_g and V_h mod p are irreducible;
(H2) g and h are both p-distinguished, i.e. $\alpha_g \neq \beta_g \pmod p$ and $\alpha_h \neq \beta_h \pmod p$;
(H3) V_g is not induced from a character of a real quadratic field in which p splits;
(H4) $h_\alpha \neq g^*_{1/\beta}$.

Assumption (H4) splits naturally into two different settings, namely the case where $h \neq g^*$ and the case where $h = g^*$ *and* $\alpha_h = 1/\alpha_g$. Case $h_\alpha = g^*_{1/\beta}$, excluded in this note, presents remarkable differences and we refer to [RiRo] for a throughout study of this scenario.

The results of [KLZ] imply that there exists a *Beilinson–Flach class*

$$\kappa(g_\alpha, h_\alpha) \in H^1(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$$

that can be identified, via Kummer theory, with an element of

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(V_{gh}\otimes\mathbb{Q}_p, U_{gh}[1/p]) = (U_{gh}[1/p]\otimes V_{gh}^{\vee}\otimes\mathbb{Q}_p)^{G_{\mathbb{Q}}}.$$

With a slight abuse of notation, we shall still denote $\kappa(g_\alpha,h_\alpha)$ the projection of the cohomology class to the space $(U_{gh}[1/p]/p^{\mathbb{Z}}\otimes V_{gh}^{\vee}\otimes\mathbb{Q}_p)^{G_{\mathbb{Q}}}$. Note that there also exist three other classes $\kappa(g_\alpha,h_\beta)$, $\kappa(g_\beta,h_\alpha)$ and $\kappa(g_\beta,h_\beta)$ attached to the different p-stabilizations of g and h.

As in [DLR2], we impose throughout the following:

Assumption 1.1. $\dim_L(U_{gh}[1/p]/p^{\mathbb{Z}}\otimes V_{gh}^{\vee})^{G_{\mathbb{Q}}}=2$.

When $h\neq g^*$ this amounts to asking that none of the Frobenius eigenvalues of V_{gh} is equal to 1, that is to say:

$$\alpha_g\alpha_h,\ \alpha_g\beta_h,\ \beta_g\alpha_h,\ \beta_g\beta_h\neq 1.$$

Under this assumption, [DLR2, Lemma 1.1] also implies that

$$\dim_L(U_{gh}\otimes V_{gh}^{\vee})^{G_{\mathbb{Q}}}=\dim_L(U_{gh}[1/p]\otimes V_{gh}^{\vee})^{G_{\mathbb{Q}}}=2.$$

When $h_\alpha=g^*_{1/\alpha}$, the regularity assumption (H2) directly grants Assumption 1.1 and we have

$$\dim_L(U_{gh}\otimes V_{gh}^{\vee})^{G_{\mathbb{Q}}}=1,\quad \dim_L(U_{gh}[1/p]\otimes V_{gh}^{\vee})^{G_{\mathbb{Q}}}=3.$$

In either case, fix elements $\{u,v\}$ of $(U_{gh}[1/p]\otimes V_{gh}^{\vee}\otimes\mathbb{Q}_p)^{G_{\mathbb{Q}}}$ such that they project to a basis of the two-dimensional space $(U_{gh}[1/p]/p^{\mathbb{Z}}\otimes V_{gh}^{\vee}\otimes\mathbb{Q}_p)^{G_{\mathbb{Q}}}$. When $h_\alpha=g^*_{1/\alpha}$ we impose the additional condition that u spans the line $(U_{gh}\otimes V_{gh}^{\vee}\otimes\mathbb{Q}_p)^{G_{\mathbb{Q}}}$.

Fix a prime ideal $\wp$ of H lying above p, thus determining an embedding $H\subset H_p\subset\bar{\mathbb{Q}}_p$ of H into its completion H_p at $\wp$, and an arithmetic Frobenius $\mathrm{Fr}_p\in\mathrm{Gal}\,(H_p/\mathbb{Q}_p)$. Thanks to (H2), the $\mathrm{Gal}\,(H_p/\mathbb{Q}_p)$-modules V_g,V_h decompose as

$$V_g:=V_g^{\alpha}\oplus V_g^{\beta},\quad V_h:=V_h^{\alpha}\oplus V_h^{\beta},$$

where Fr_p acts on V_g^α with eigenvalue α_g, and similarly for the remaining summands. The tensor product V_{gh} decomposes then as $G_{\mathbb{Q}_p}$-module as the direct sum of four different lines $V_{gh}^{\alpha\alpha}:=V_g^{\alpha_g}\otimes V_h^{\alpha_h}$, ..., $V_{gh}^{\beta\beta}$. After choosing a basis, we may write this decomposition as

$$V_{gh}=L\cdot e_{\alpha\alpha}\oplus L\cdot e_{\alpha\beta}\oplus L\cdot e_{\beta\alpha}\oplus L\cdot e_{\beta\alpha},$$

where

$$\mathrm{Fr}_p(e_{\lambda\mu})=\lambda\mu\cdot e_{\lambda\mu},\quad \text{for any } \lambda\in\{\alpha_g,\beta_g\},\mu\in\{\alpha_h,\beta_h\}.$$

We denote by $\{e_{\alpha\alpha}^{\vee},e_{\alpha\beta}^{\vee},e_{\beta\alpha}^{\vee},e_{\beta\beta}^{\vee}\}$ the dual basis of $V_{gh}^{\vee}=\mathrm{Hom}(V_{gh},L)$, where

$$\mathrm{Fr}_p(e_{\alpha\alpha}^{\vee})=\chi_{gh}^{-1}(p)\beta_g\beta_h\cdot e_{\alpha\alpha}^{\vee}\,,\ \dots,\ \mathrm{Fr}_p(e_{\beta\beta}^{\vee})=\chi_{gh}^{-1}(p)\alpha_g\alpha_h\cdot e_{\beta\beta}^{\vee}.$$

Restriction to the decomposition group at p allows us to regard u and v as elements in $H^1(\mathbb{Q}_p,V_{gh}\otimes\mathbb{Q}_p(1))=(H_p^{\times}\otimes V_{gh}^{\vee}\otimes\mathbb{Q}_p)^{G_{\mathbb{Q}_p}}$, and as such we may decompose u and v as

$$u=u_{\alpha\alpha}\otimes e_{\alpha\alpha}^{\vee}+u_{\alpha\beta}\otimes e_{\alpha\beta}^{\vee}+u_{\beta\alpha}\otimes e_{\beta\alpha}^{\vee}+u_{\beta\beta}\otimes e_{\beta\beta}^{\vee},$$
$$v=v_{\alpha\alpha}\otimes e_{\alpha\alpha}^{\vee}+v_{\alpha\beta}\otimes e_{\alpha\beta}^{\vee}+v_{\beta\alpha}\otimes e_{\beta\alpha}^{\vee}+v_{\beta\beta}\otimes e_{\beta\beta}^{\vee}, \tag{1}$$

where $u_{\alpha\alpha}\in H_p^{\times}\otimes\mathbb{Q}_p$ satisfies $\mathrm{Fr}_p(u_{\alpha\alpha})=\alpha_g\alpha_h\cdot u_{\alpha\alpha}$ and similarly for the other terms.

Define the regulator

$$\mathrm{Reg}_{g_\alpha}(V_{gh})=\log_p(u_{\beta\beta})\cdot\log_p(v_{\beta\alpha})-\log_p(u_{\beta\alpha})\cdot\log_p(v_{\beta\beta}). \tag{2}$$

In the body of the paper we introduce a p-adic avatar of the second derivative of $L(g\otimes h,s)$ at $s=1$, denoted by $\mathscr{L}_p{}^{g_\alpha}(g,h)$ and which can be defined in terms of special values of the Hida–Rankin p-adic L-function; alternatively, it can also be recast as a Coleman p-adic *iterated integral*. The non-vanishing of $\mathscr{L}_p{}^{g_\alpha}(g,h)$ implies the non-vanishing of $\mathrm{Reg}_{g_\alpha}(V_{gh})$.

The following result is proved in Section 3.2. Recall that we are identifying cohomology classes with their projection to the space $(U_{gh}[1/p]/p^{\mathbb{Z}} \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$.

Theorem 1.2. *We have*

$$\kappa(g_\alpha, h_\alpha) = \Omega \cdot \Big(\log_p(v_{\beta\beta}) \cdot u - \log_p(u_{\beta\beta}) \cdot v \Big) \tag{3}$$

for some $\Omega \in H_p$. Moreover, if $\mathscr{L}_p^{g_\alpha} \neq 0$, then $\kappa(g_\alpha, h_\alpha) \neq 0$. In this case, if we additionally impose that $h \neq g^*$, the two cohomology classes

$$\kappa(g_\alpha, h_\alpha), \quad \kappa(g_\alpha, h_\beta)$$

span the whole group $(U_{gh} \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$.

The computations of [DLR2] suggest the following refinement of Theorem 1.2, which is discussed in Section 3.3. Let u_{g_α} be the Stark unit attached to g_α, as defined in [DLR1]. The choice of a basis for V_{gh} determines elements

$$\Xi_{g_\alpha} \in H_p^{\mathrm{Fr}_p = \beta_g^{-1}}, \qquad \Omega_{h_\alpha} \in H_p^{\mathrm{Fr}_p = \alpha_h^{-1}},$$

which are properly defined in [DR3, (Eq.8)] and which we later recall. From now on, we denote with the symbol $\doteq$ equality up to an element in $L^\times$.

Conjecture A. *The Beilinson–Flach class $\kappa(g_\alpha, h_\alpha)$ satisfies*

$$\kappa(g_\alpha, h_\alpha) \doteq \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot \frac{1}{\log_p(u_{g_\alpha})} \cdot \Big(\log_p(v_{\beta\beta}) \cdot u - \log_p(u_{\beta\beta}) \cdot v \Big) \pmod{L^\times}. \tag{4}$$

Theorem 1.3. *Conjecture A implies the main conjecture of* [DLR2]. *If we further assume that $\mathrm{Reg}_{g_\alpha}(V_{gh}) \neq 0$, the converse also holds.*

As an application of our results, we are able to prove the analogue of the main theorem of Castella and Hsieh [CaHs, Theorem 1] in the setting of units in number fields and Beilinson–Flach elements. Assume now that $h_\alpha = g_{1/\alpha}^*$ and recall the four global cohomology classes

$$\kappa(g_\alpha, g_{1/\alpha}^*),\ \kappa(g_\beta, g_{1/\beta}^*),\ \kappa(g_\alpha, g_{1/\beta}^*),\ \kappa(g_\beta, g_{1/\alpha}^*) \in H^1(\mathbb{Q}, V_{gg^*} \otimes \mathbb{Q}_p(1)) \tag{5}$$

arising from the various p-stabilizations of g and g^*.

Again, since these classes are unramified at primes $\ell \neq p$, they belong to the subspace which is identified with $(U_{gg^*}[1/p] \otimes V_{gg^*}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$ under the Kummer map.

It follows from [RiRo, Proposition 3.12] that $\kappa(g_\alpha, g_{1/\beta}^*) = \kappa(g_\beta, g_{1/\alpha}^*) = 0$. It is thus natural to wonder whether one can determine the remaining two classes $\kappa(g_\alpha, g_{1/\alpha}^*)$ and $\kappa(g_\beta, g_{1/\beta}^*)$.

Theorem 1.4. *Assume that $\mathrm{Reg}_{g_\alpha}(V_{gg^*}) \neq 0$. Then, $\kappa(g_\alpha, g_{1/\alpha}^*)$ and $\kappa(g_\beta, g_{1/\beta}^*)$ are non-zero and Conjecture A holds for them. Moreover,*

$$\langle \kappa(g_\alpha, g_{1/\alpha}^*) \rangle = \langle \kappa(g_\beta, g_{1/\beta}^*) \rangle.$$

Remark 1.5. When g is the theta series attached to an imaginary quadratic field where p splits or to a real quadratic field where p remains inert, we prove that $\mathrm{Reg}_{g_\alpha}(V_{gg^*}) \neq 0$ and hence the above statement holds unconditionally.

The organization of the paper is as follows. In Section 2, we recover the formulation of the main conjecture of [DLR2], both in terms of iterated integrals and of the Hida–Rankin p-adic L-function. Section 3 is devoted to prove Theorem 1.2 and Theorem 1.3, exploring some properties of Beilinson–Flach classes. Section 4 provides the proof of the analogue of the main theorem of Castella and Hsieh in the setting of Beilinson–Flach elements. Finally, Section 5 analyzes some particular cases where the representation V_{gh} is reducible, in connection with the more classical Euler systems of circular and elliptic units.

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2. The main conjecture of Darmon, Lauder and Rotger

The aim of this section is to recall briefly the elliptic Stark conjecture formulated in [DLR2] for units in number fields. We keep the same notations and assumptions of the introduction. Throughout this section we further assume

$$h \neq g^*,$$

which in particular implies, as recalled above, that

$$\alpha_g\alpha_h, \alpha_g\beta_h, \beta_g\alpha_h, \beta_g\beta_h \neq 1. \tag{6}$$

We leave the self-dual case $h_\alpha = g^*_{1/\alpha}$ for §5.

Let

$$f := E_2(1, \chi_{gh}^{-1}) \in M_2(N, \chi_{gh}^{-1})$$

be the weight two Eisenstein series for the character χ_{gh}^{-1}, and consider also

$$F := d^{-1}f = E_0^{[p]}(\chi_{gh}^{-1}, 1),$$

the overconvergent Eisenstein series of weight zero attached to the pair $(\chi_{gh}^{-1}, 1)$ of Dirichlet characters.

As shown in [DLR1], the above hypothesis ensure that any generalized overconvergent modular form associated to g_α is simply a multiple of g_α. We denote by $e_{\rm ord}$ Hida's ordinary projection on the space of overconvergent modular forms of weight one and by $e_{g^*_\alpha}$ the Hecke equivariant projection to the generalized eigenspace attached to the system of Hecke eigenvalues for the dual form g^*_α of g_α.

We attach to g_α a two-dimensional subspace of the representation V_{gh}, namely

$$V_{gh}^\beta := V_g^\beta \otimes V_h.$$

Remark 2.1. In the Eisenstein case the conjecture also makes sense, as it is emphasized in [DLR2]. In this case, if $g = E_1(\chi^+, \chi^-)$, the classicality assumption asserts that $\chi^+(p) = \chi^-(p)$ and the role of the two dimensional space V_{gh}^β is played by $W \otimes V_h$, where W is any line in V_g which is not stable under $G_{\mathbb{Q}}$.

Recall the unit u_{g_α} in $\mathcal{O}_H^\times \otimes L$ attached to the p-stabilized eigenform g_α, as it is defined in [DLR1, 1.2]. It belongs to the ${\rm Ad}^0(g)$-isotypic part of $\mathcal{O}_H^\times \otimes L$ and is an eigenvector for ${\rm Fr}_p$ with eigenvalue β_g/α_g. For a Dirichlet character of conductor m, we can consider

$$\mathfrak{g}(\chi) := \sum_{a=1}^{m} \chi^{-1}(a) e^{2\pi i a/m}, \tag{7}$$

the usual Gauss sum, on which $G_{\mathbb{Q}}$ acts through χ and thus ${\rm Fr}_p$ acts with eigenvalue $\chi_{gh}(p)$.

Since g_α is new at level Np, the L-dual space $(S_1(Np, \chi_g^{-1})_L^\vee[g_\alpha^*])^\vee$ is one-dimensional and we may fix a basis, say γ_α. As before, we consider a regulator given in terms of the p-units u and v, defined at the Introduction and which admits a Frobenius decomposition as in (1):

$${\rm Reg}_{g_\alpha}(V_{gh}) = \log_p(u_{\beta\beta}) \cdot \log_p(v_{\beta\alpha}) - \log_p(u_{\beta\alpha}) \cdot \log_p(v_{\beta\beta}).$$

The following question is the main conjecture of [DLR2].

Conjecture 2.2. *It holds that*

$$\gamma_\alpha(e_{g_\alpha^*} e_{\mathrm{ord}}(Fh)) \doteq \frac{\mathrm{Reg}_{g_\alpha}(V_{gh})}{\mathfrak{g}(\chi_{gh}) \cdot \log_p(u_{g_\alpha})} \pmod{L^\times}. \tag{8}$$

We can reformulate the previous conjecture in the language of special values of p-adic L-functions. Let $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$ and $\mathbf{h} \in \Lambda_{\mathbf{h}}[[q]]$ be Hida families through g_α and h_α with coefficients in finite flat extensions $\Lambda_{\mathbf{g}}$, $\Lambda_{\mathbf{h}}$ of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, respectively. Write $\mathcal{W} = \mathrm{Spf}(\Lambda)$, $\mathcal{W}_{\mathbf{g}} = \mathrm{Spf}(\Lambda_{\mathbf{g}})$ and $\mathcal{W}_{\mathbf{h}} = \mathrm{Spf}(\Lambda_{\mathbf{h}})$ for the associated weight spaces. Let y_0 and z_0 be weight one points of $\mathcal{W}_{\mathbf{g}}$ and $\mathcal{W}_{\mathbf{h}}$ such that $\mathbf{g}_{y_0} = g_\alpha$ and $\mathbf{h}_{z_0} = h_\alpha$. Associated to the two cuspidal Hida families $\mathbf{g}$ and $\mathbf{h}$, Hida constructed in [Hi1] and [Hi2] a three-variable p-adic Rankin L-function $L_p(\mathbf{g},\mathbf{h})$ on $\mathcal{W}_{\mathbf{gh}} := \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}} \times \mathcal{W}$ interpolating the algebraic parts of the critical values $L(g_y, h_z, s)$. See [RiRo, §2.2] for more details on the notations and normalizations we adopt. The next result follows from [DLR2, Lemma 4.2].

Proposition 2.3. *Up to multiplication by a scalar in $L^\times$, we have*

$$L_p(\mathbf{g},\mathbf{h})(y_0, z_0, 1) \doteq \mathfrak{g}(\chi_{gh}) \times \gamma_\alpha(e_{g_\alpha^*} e_{\mathrm{ord}}(Fh)). \tag{9}$$

Hence, as pointed out already in [DLR2], the above conjecture may be recast as

$$L_p(\mathbf{g},\mathbf{h})(y_0, z_0, 1) \overset{?}{\doteq} \frac{\mathrm{Reg}_{g_\alpha}(V_{gh})}{\log_p(u_{g_\alpha})} \pmod{L^\times}. \tag{10}$$

3. Beilinson–Flach elements and the main conjecture

3.1. **The Euler system of Beilinson–Flach elements.** We begin this section with a quick review of the main results of [KLZ], which are crucially used to study the conjecture we have discussed along the previous section. We also refer the reader to [RiRo, §2.1, §3.1] for a expanded description of the results of [KLZ] with the same notations and normalizations adopted here.

Let

$$\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]], \quad \mathbf{h} \in \Lambda_{\mathbf{h}}[[q]]$$

be Hida families of tame level N and tame characters $\chi_{\mathbf{g}}$ and $\chi_{\mathbf{h}}$ respectively, and let $\Lambda_{\mathbf{gh}} := \Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{h}} \hat{\otimes} \Lambda$. Let also $\underline{\varepsilon}_{\mathrm{cyc}}$ denote the Λ-adic cyclotomic character. Let $\mathbb{V}_{\mathbf{g}}$ and $\mathbb{V}_{\mathbf{h}}$ stand for the Λ-adic representations attached to $\mathbf{g}$ and $\mathbf{h}$, respectively, endowed with the filtration described in [DR3, §2],

$$0 \to \mathbb{V}_{\mathbf{g}}^+ \to \mathbb{V}_{\mathbf{g}} \to \mathbb{V}_{\mathbf{g}}^- \to 0,$$

and similarly for $\mathbf{h}$. We also consider the canonical differentials $\eta_{\mathbf{g}}$ and $\omega_{\mathbf{g}}$ as introduced in [Oh] and [KLZ, §10.1], and denote by η_{g_y}, ω_{g_y} the corresponding specializations at weight y. As it has been extensively discussed in loc. cit. and recalled in [RiRo, §2.1], this induces homomorphisms of $\Lambda_{\mathbf{gh}}$-modules given by the pairings with these differentials; these pairings are denoted as $\langle \cdot, \cdot \rangle$. Finally, let α_g and β_g stand for the roots of the p-th Hecke polynomial of g, ordered in such a way that $\mathrm{ord}_p(\alpha_g) \leq \mathrm{ord}_p(\beta_g)$. We also consider the same objects for the family $\mathbf{h}$.

We say that a weight $y \in \mathcal{W}_{\mathbf{g}}$ is crystalline when there exists an eigenform g_y° of level N such that g_y is the ordinary p-stabilization of g_y°. We denote by $\mathcal{W}_{\mathbf{g}}^\circ$ (resp. $\mathcal{W}_{\mathbf{h}}^\circ$, $\mathcal{W}_{\mathbf{gh}}^\circ$) the set of crystalline points of $\mathcal{W}_{\mathbf{g}}$ (resp. $\mathcal{W}_{\mathbf{h}}$, $\mathcal{W}_{\mathbf{gh}}$). A point in the latter space is identified with a triple (y, z, s), where the weights are referred to as (ℓ, m, s). Note that for a matter of simplicity we are just assuming that the points corresponding to the third variable have trivial nebentype.

The following result establishes the existence of a Perrin-Riou map (also referred in the literature as *big regulator*) interpolating both the Bloch–Kato logarithm and the dual exponential map. Its relevance will be clear throughout the text.

Proposition 3.1. *There exists an injective homomorphism of $\Lambda_{\mathbf{gh}}$-modules*

$$\mathcal{L}_{\mathbf{gh}}^{-+} : H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^{-}\hat{\otimes}\mathbb{V}_{\mathbf{h}}^{+}\hat{\otimes}\Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1})) \to \Lambda_{\mathbf{gh}}$$

satisfying the following interpolation property: for every $(y,z,s) \in \mathcal{W}_{\mathbf{gh}}^{\circ}$, set $g := g_y^{\circ}$ and $h = h_z^{\circ}$. Then, the specialization of $\mathcal{L}_{\mathbf{gh}}^{-+}$ at (y,z,s) is the homomorphism

$$\mathcal{L}_{\mathbf{gh}}^{-+}(y,z,s) : H^1(\mathbb{Q}_p, V_g^{-} \otimes V_h^{+}(1-s)) \to \mathbb{C}_p$$

given by

$$\mathcal{L}_{\mathbf{gh}}^{-+}(y,z,s) = \frac{(1-p^{s-1}\alpha_g^{-1}\beta_h^{-1})(1-\alpha_h^{-1}\beta_h)}{(1-p^{-s}\alpha_g\beta_h)(1-p^{-1}\alpha_g^{-1}\beta_g)} \times \begin{cases} \frac{(-1)^{m-s-1}}{(m-s-1)!} \times \langle \log_{\mathrm{BK}}, \eta_g \otimes \omega_h \rangle & \text{if } s < m \\ (s-m)! \times \langle \exp^*_{\mathrm{BK}}, \eta_g \otimes \omega_h \rangle & \text{if } s \geq m, \end{cases}$$

where $\log_{\mathrm{BK}}$ is the Bloch-Kato logarithm and $\exp^*_{\mathrm{BK}}$, the dual exponential map (see [BK] *for proper definitions of these morphisms).*

Proof. This follows from [KLZ, Theorem 8.2.8, Proposition 10.1.1] and the relations

$$\eta_{g_y} = \Big(1 - \frac{\beta_{g_y^{\circ}}}{\alpha_{g_y^{\circ}}}\Big)\eta_{g_y^{\circ}}, \quad \omega_{h_z} = \Big(1 - \frac{\beta_{h_z^{\circ}}}{\alpha_{h_z^{\circ}}}\Big)\omega_{h_z}^{\circ}.$$

□

We can now formulate the main results of [KLZ], which assert that there exists a family of cohomology classes indexed by points of $\mathcal{W}_{\mathbf{gh}}$ and whose image under the previous Perrin-Riou map agrees with the Hida–Rankin p-adic L-function.

Theorem 3.2. *Fix an integer $c > 1$ relatively prime to $6pN$. Then, there exists a global cohomology class*

$${}_c\kappa(\mathbf{g},\mathbf{h}) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{g}}\hat{\otimes}\mathbb{V}_{\mathbf{h}}\hat{\otimes}\Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}))$$

such that:

(1) *The projection of the local class $\mathrm{res}_p({}_c\kappa(\mathbf{g},\mathbf{h}))$ to $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^{-}\hat{\otimes}\mathbb{V}_{\mathbf{h}}(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}))$ lands in*

$$H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^{-}\hat{\otimes}\mathbb{V}_{\mathbf{h}}^{+}(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1})).$$

(2) *Letting ${}_c\kappa_p^{-+}(\mathbf{g},\mathbf{h})$ denote the local cohomology class in the above space, we have*

$$\mathcal{L}_{\mathbf{gh}}^{-+}({}_c\kappa_p^{-+}(\mathbf{g},\mathbf{h})) = \frac{(-1)^s}{\lambda_{\mathbf{g}}} \cdot (c^2 - c^{2s-l-m+2}) \times L_p(\mathbf{g},\mathbf{h}),$$

where $\lambda_{\mathbf{g}}$ denotes the pseudo-eigenvalue of $\mathbf{g}$, an Iwasawa function interpolating the pseudo-eigenvalue at N of the crystalline classical specializations of $\mathbf{g}$.

Proof. The global cohomology class ${}_c\kappa(\mathbf{g},\mathbf{h})$ is introduced in [KLZ, Definition 8.1.1]. The first part of the result is just [KLZ, Proposition 8.1.7], while the second part is Theorem B of [KLZ]. □

Since c is fixed throughout, we may sometimes drop it from the notation. The constant does make an appearance in fudge factors accounting for the interpolation properties satisfied by the Euler system, but in the case we are interested in these fudge factors do not vanish. We typically refer to this class as the *Beilinson–Flach class* or the *Beilinson–Flach element attached to* $\mathbf{g}$ *and* $\mathbf{h}$.

3.2. **An explicit description of the cohomology classes.** From now on, we retain the setting of §2, where $g \in S_1(N, \chi_g)$ and $h \in S_1(N, \chi_h)$ are two cuspidal eigenforms satisfying hypotheses (H1)-(H3) and $h \neq g^*$. Let L_p denote the completion of L in $\bar{\mathbb{Q}}_p$ under the embedding $L \subset \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ fixed at the outset. Under our running hypothesis, we actually have $L_p = \mathbb{Q}_p$, although recall this was only assumed for simplicity of exposition.

Definition 3.3. Let $\mathbf{g}$ and $\mathbf{h}$ be Hida families passing through p-stabilizations g_α, h_α of g, h at some point $(y_0, z_0) \in \mathcal{W}_{\mathbf{g}}^\circ \times \mathcal{W}_{\mathbf{h}}^\circ$ of weights $(1,1)$. Define

$$\kappa(g_\alpha, h_\alpha) := \kappa(\mathbf{g}, \mathbf{h})(y_0, z_0, 0) \in H^1(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$$

as the specialization of $\kappa(\mathbf{g}, \mathbf{h})$ at the point $(y_0, z_0, 0)$.

This procedure yields four a priori different global cohomology classes:

$$\kappa(g_\alpha, h_\alpha), \quad \kappa(g_\alpha, h_\beta), \quad \kappa(g_\beta, h_\alpha), \quad \kappa(g_\beta, h_\beta), \tag{11}$$

one for each choice of pair of roots of the p-th Hecke polynomials of g and h.

Let $H^1_{\mathrm{f}}(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$ denote the *finite* Bloch-Kato Selmer group, which is the subspace of $H^1(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$ which consists on those classes which are crystalline at p and unramifed at all $\ell \neq p$

Proposition 3.4. *The cohomology classes in* (11) *belong in fact to* $H^1_{\mathrm{f}}(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$.

Proof. The two cohomology spaces $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1))$ and $H^1(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1))$ are equal according to the discussion in [DR4, §1.4] combined with the results of [Bel, 2.8 and 2.21]. Then, the restrictions to $\mathbb{Q}_\ell$, for $\ell \neq p$, are unramified because of the results established in [Nek, 2.4]. □

By standard results in Kummer theory (see for example [Bel, Prop.2.12]), there exists an isomorphism between $H^1_{\mathrm{f}}(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$ and $(U_{gh} \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$.

As we have already mentioned, the units $u, v \in (U_{gh} \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$ introduced in §1 can be also identified with elements in $\mathrm{Hom}(V_{gh} \otimes \mathbb{Q}_p, U_{gh})^{G_\mathbb{Q}}$. In the lemma below, we regard the local class $\kappa_p(g_\alpha, h_\alpha)$ as a homomorphism in $\mathrm{Hom}(V_{gh} \otimes \mathbb{Q}_p, H_p^\times \otimes L)^{G_\mathbb{Q}}$.

Lemma 3.5. $\kappa_p(g_\alpha, h_\alpha)(e_{\beta\beta}) = 0$.

Proof. This follows after specializing the content of [KLZ, Proposition 8.1.7] (also rephrased here in the first part of Theorem 3.2), at the point $(y_0, z_0, 0)$. It asserts that the component of the Beilinson–Flach class corresponding to the projection in the quotient $\mathbb{V}_{\mathbf{g}}^- \hat{\otimes} \mathbb{V}_{\mathbf{h}}^-$ vanishes. Combining the natural dualities with the above referred identifications between the spaces of homomorphisms and the cohomology groups, the result follows. □

Consider again the special p-adic L-value $\mathscr{L}_p{}^{g_\alpha} = L_p(\mathbf{g}, \mathbf{h})(y_0, z_0, 1)$. Recall that this value is the same if $\mathbf{h}$ is chosen to be the Hida family through h_β. This value can be understood as a p-adic avatar of the second derivative of the classical Hida–Rankin L-function, because of the following result.

Proposition 3.6. *The order of vanishing of* $L(V_{gh}, s)$ *at* $s = 1$ *is two.*

Proof. According to [Das1, §3.2], we know that

$$\mathrm{ord}_{s=0} L(V_{gh}, s) = 2 - \dim_L (V_{gh})^{G_\mathbb{Q}},$$

and the order of vanishing at $s = 1$ can be derived via a functional equation relating the values at $s = 0$ and $s = 1$, where some gamma factors arise.

Besides, the assumptions we have fixed imply that $\dim_L (V_{gh})^{G_\mathbb{Q}} = 0$, and since the functional equation introduces no extra zero or pole at $s = 1$ (see [Das2]), we conclude that the order of vanishing at $s = 1$ is also 2. □

The following result was stated in the Introduction as Theorem 1.2.

Theorem 3.7. *There exists a period* $\Omega \in H_p$ *such that*

$$\kappa(g_\alpha, h_\alpha) = \Omega \cdot (\log_p(v_{\beta\beta}) \cdot u - \log_p(u_{\beta\beta}) \cdot v).$$

Moreover, if $\mathscr{L}_p{}^{g_\alpha} \neq 0$, *then* $\kappa(g_\alpha, h_\alpha) \neq 0$ *and the two global classes*

$$\kappa(g_\alpha, h_\alpha),\ \kappa(g_\alpha, h_\beta)$$

are linearly independent in the Selmer group $H^1(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$.

Proof. The running assumptions imply that $\alpha_g\alpha_h \neq 1$ and then $H^1(\mathbb{Q}_p, V_{gh}^{\alpha\alpha} \otimes \mathbb{Q}_p(1))$ is one-dimensional. Observe that $V_{gh}^{\alpha\alpha} \simeq \mathbb{Q}_p(\alpha_g\alpha_h)$ as $G_{\mathbb{Q}_p}$-modules, where the latter stands for the unramified character sending Fr_p to $\alpha_g\alpha_h$. Hence, the Bloch-Kato logarithm associated to this p-adic representation gives rise to an isomorphism

$$\log_{\mathrm{BK}} : H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{gh}^{\alpha\alpha} \otimes \mathbb{Q}_p(1)) \xrightarrow{\sim} H_p.$$

Since $\{u, v\}$ forms a basis of $H^1(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$, we may write

$$\kappa(g_\alpha, h_\alpha) = \lambda u + \mu v,$$

with $\lambda, \mu \in \mathbb{Q}_p$. The preceding lemma implies that

$$0 = \lambda \cdot u_{\beta\beta} \otimes e_{\beta\beta}^\vee + \mu \cdot v_{\beta\beta} \otimes e_{\beta\beta}^\vee,$$

and taking logarithms, we conclude that (λ, μ) is a scalar multiple of $(\log_p(v_{\beta\beta}), -\log_p(u_{\beta\beta}))$. In particular,

$$\kappa(g_\alpha, h_\alpha) = \Omega \cdot (\log_p(v_{\beta\beta}) \cdot u - \log_p(u_{\beta\beta}) \cdot v),$$

for some $\Omega \in H_p$.

For the second part of the statement, observe that since $\kappa_p(g_\alpha, h_\alpha)$ and $\kappa_p(g_\alpha, h_\beta)$ may be regarded as elements in $\mathrm{Hom}(V_{gh} \otimes \mathbb{Q}_p, U_{gh})^{G_{\mathbb{Q}}}$, it suffices to prove that the action over two different vectors of V_{gh} gives rise to an invertible matrix. By Lemma 3.5 we have

$$\begin{pmatrix} \kappa_p(g_\alpha, h_\alpha)(e_{\beta\beta}) & \kappa_p(g_\alpha, h_\alpha)(e_{\beta\alpha}) \\ \kappa_p(g_\alpha, h_\beta)(e_{\beta\beta}) & \kappa_p(g_\alpha, h_\beta)(e_{\beta\alpha}) \end{pmatrix} = \begin{pmatrix} 0 & ? \\ ? & 0 \end{pmatrix}$$

and hence we must show that the two entries off the diagonal are non-zero. Combining the injectivity of the Perrin-Riou map introduced in Proposition 3.1 with the reciprocity law for the Beilinson–Flach classes as recalled in the second part of Theorem 3.2, this is equivalent to the non-vanishing of $\mathscr{L}_p{}^{g_\alpha}(g, h)$, as claimed. ☐

Remark 3.8. The assumption $h \neq g^*$ is necessary, since we need to guarantee that the Euler-like factors at p appearing in the description of the Perrin-Riou map of [KLZ, Theorem 10.2.2] do not vanish. In fact, when $h_\alpha = g^*_{1/\beta}$ we have $\kappa(g_\alpha, g^*_{1/\beta}) = 0$ (see [RiRo, §3.2]). Similarly, if $h_\alpha = g^*_{1/\alpha}$ then $\kappa(g_\alpha, g^*_{1/\alpha}) = 0$.

3.3. **A conjecture in terms of Beilinson–Flach elements.** We now come back to the main question of [DLR2].

Fix eigenbasis $\{e_g^\alpha, e_g^\beta\}$ and $\{e_h^\alpha, e_h^\beta\}$ of V_g and V_h respectively, which are compatible with the choice of the basis for V_{gh}, i.e.,

$$e_{\alpha\alpha} = e_g^\alpha \otimes e_h^\alpha, \quad e_{\alpha\beta} = e_g^\alpha \otimes e_h^\beta, \quad e_{\beta\alpha} = e_g^\beta \otimes e_h^\alpha, \quad e_{\beta\beta} = e_g^\beta \otimes e_h^\beta.$$

As before, let $\eta_{g_\alpha} \in (H_p \otimes V_g^\beta)^{G_{\mathbb{Q}_p}}$ and $\omega_{h_\alpha} \in (H_p \otimes V_h^\alpha)^{G_{\mathbb{Q}_p}}$ stand for the canonical periods arising as the weight one specialization of the Λ-adic periods $\eta_{\mathbf{g}}$ and $\omega_{\mathbf{h}}$.

We can now follow [DR3] and define p-adic periods $\Xi_{g_\alpha} \in H_p^{\mathrm{Fr}_p = \beta_g^{-1}}$ and $\Omega_{h_\alpha} \in H_p^{\mathrm{Fr}_p = \alpha_h^{-1}}$ by setting

$$\Xi_{g_\alpha} \otimes e_g^\beta = \eta_{g_\alpha}, \qquad \Omega_{h_\alpha} \otimes e_h^\alpha = \omega_{h_\alpha}.$$

Hence, we have that

$$\Xi_{g_\alpha} \cdot \Omega_{h_\alpha} \otimes e_{\beta\alpha} = \eta_{g_\alpha} \otimes \omega_{h_\alpha}.$$

We now apply the Perrin-Riou big logarithm described in Proposition 3.1 to the *local* cohomology class $\kappa_p(g_\alpha, h_\alpha \in H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1)) \simeq (H_p^\times \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_{\mathbb{Q}_p}}$.

Indeed, let

$$\log^{-+} : H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1)) \xrightarrow{\mathrm{pr}^{-+}} H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{gh}^{\alpha\beta} \otimes \mathbb{Q}_p(1)) \xrightarrow{\mathcal{L}^{-+}} \mathbb{Q}_p,$$

where here $\mathcal{L}^{-+}$ must be understood as the composition of the map of [KLZ, Theorem 8.2.8] specialized at $(y_0, z_0, 0)$ with the pairing with the differentials $\eta_{g_\alpha} \otimes \omega_{h_\alpha}$.

Under the identification of $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{gh}^{\alpha\beta} \otimes \mathbb{Q}_p(1)) \simeq H_p^\times \otimes e_{\beta\alpha}^\vee$, the map $\mathcal{L}^{-+}$ corresponds to the usual p-adic logarithm in $H_p^\times$, followed by the pairing with $\Xi_{g_\alpha}\Omega_{h_\alpha} \otimes e_{\beta\alpha}$. Alternatively, and via the identification of $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1))$ with $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(V_{gh} \otimes \mathbb{Q}_p, H_p^\times \otimes L)$, the map $\log^{-+}$ is $\phi \mapsto \Xi_{g_\alpha} \cdot \Omega_{h_\alpha} \cdot \log_p(\phi(e_{\beta\alpha}))$. Then, combining the reciprocity law of Theorem 3.2 with Theorem 3.7, we have that

$$L_p(\mathbf{g}, \mathbf{h})(y_0, z_0, 1) = \Omega \cdot \Xi_{g_\alpha} \cdot \Omega_{h_\alpha} \cdot (\log_p(v_{\beta\beta}) \cdot \log_p(u_{\beta\alpha}) - \log_p(u_{\beta\beta}) \cdot \log_p(v_{\beta\alpha})). \tag{12}$$

Hence, Conjecture 2.2 in the form given in (10) suggests that

$$\Omega \doteq \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot \frac{1}{\log_p(u_{g_\alpha})} \pmod{L^\times}. \tag{13}$$

Moreover, if we assume [DR3, Conjecture 2.1] this reduces to

$$\Omega \doteq \frac{1}{\Omega_{g_\alpha} \cdot \Omega_{h_\alpha}} \pmod{L^\times}. \tag{14}$$

Remark 3.9. These periods we have described are completely non-canonical and depend on the choice of an L-basis. It is possible to formulate an analogue conjecture to [DR3, Conjecture 3.12], which only involves the so-called *enhanced* regulator as well as the differentials $\omega_{g_\alpha}, \omega_{h_\alpha}$, and the Beilinson–Flach class. This formulation has the advantage that it overcomes the period dependence by giving an equality in $D(V_{gh}^{\alpha\alpha}) \otimes (U_{gh} \otimes V_{gh} \otimes \mathbb{Q}_p)^{G_{\mathbb{Q}}}$ up to multiplication in $L^\times$; in particular, it would state that

$$\omega_{g_\alpha}\omega_{h_\alpha} \otimes \kappa(g_\alpha, h_\alpha) \doteq (\log(v_{\beta\beta}) \cdot u - \log(u_{\beta\beta}) \cdot v) \otimes e_{\alpha\alpha} \pmod{L^\times}. \tag{15}$$

This conjecture itself is not directly equivalent to the main conjecture of [DLR2] and also relies on [DR3, Conjecture 2.1].

In any case, and under the assumptions of the introduction on g and h, we can formulate the following conjecture (Conjecture A at the Introduction).

Conjecture 3.10. *The Belinson-Flach element $\kappa(g_\alpha, h_\alpha)$ satisfies the following equality in $(U_{gh} \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_{\mathbb{Q}}}$.*

$$\kappa(g_\alpha, h_\alpha) \doteq \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot \frac{1}{\log_p(u_{g_\alpha})} \cdot \Big(\log_p(v_{\beta\beta}) \cdot u - \log_p(u_{\beta\beta}) \cdot v\Big) \pmod{L^\times}. \tag{16}$$

Under a quite general non-vanishing assumption, it turns out that the previous conjecture is equivalent to Conjecture 2.2. The following is what we anticipated as Theorem 1.3.

Proposition 3.11. *Conjecture 3.10 implies Conjecture 2.2. If we assume that* $\mathrm{Reg}_{g_\alpha}(V_{gh}) \neq 0$*, then the converse also holds.*

Proof. As before, let

$$\log^{-+} : H^1(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1)) \xrightarrow{\mathrm{pr}^{-+}} H^1(\mathbb{Q}_p, V_{gh}^{\alpha\beta} \otimes \mathbb{Q}_p(1)) \xrightarrow{\mathcal{L}^{-+}} \mathbb{Q}_p.$$

Applying this map to both sides of (16) and using the reciprocity law which relates the Beilinson-Flach class with the Hida–Rankin p-adic L-function, as recalled in Theorem 3.2, we get the main result of [DLR2] as stated in (10).

Conversely, assuming Conjecture 2.2 and using again the reciprocity law presented as Theorem 3.2, we have that

$$\log^{-+}(\kappa_p(g_\alpha, h_\alpha)) \doteq \frac{\log_p(v_{\beta\beta}) \cdot \log_p(u_{\beta\alpha}) - \log_p(u_{\beta\beta}) \cdot \log_p(v_{\beta\alpha})}{\log_p(u_{g_\alpha})} \pmod{L^\times}.$$

The class

$$\kappa_\circ \doteq \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot \frac{1}{\log_p(u_{g_\alpha})} \cdot \Big(\log_p(v_{\beta\beta}) \cdot u - \log_p(u_{\beta\beta}) \cdot v \Big) \pmod{L^\times}$$

satisfies

$$\log^{-+}(\kappa_p(g_\alpha, h_\alpha)) = \log^{-+}(\mathrm{res}_p(\kappa_\circ)).$$

We may write the cohomology class $\kappa(g_\alpha, h_\alpha)$ as a linear combination

$$\kappa(g_\alpha, h_\alpha) = \kappa_\circ + a \cdot u + b \cdot v,$$

where $a, b \in \mathbb{Q}_p$. Since $\kappa_p(g_\alpha, h_\alpha) - \mathrm{res}_p(\kappa_\circ)$ lies in the kernel of $\log^{-+}$, one must have

$$a \cdot \log_p(u_{\beta\alpha}) + b \cdot \log_p(v_{\beta\alpha}) = 0. \tag{17}$$

Consider also the map

$$\log^{--} : H^1(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1)) \xrightarrow{\mathrm{pr}^{--}} H^1(\mathbb{Q}_p, V_{gh}^{\alpha\alpha} \otimes \mathbb{Q}_p(1)) \xrightarrow{\log_{\mathrm{BK}}} \mathbb{Q}_p.$$

The class $\kappa_p(g_\alpha, h_\alpha) - \mathrm{res}_p(\kappa_\circ)$ is also in the kernel of this map according to the first part of Theorem 3.2, and hence

$$a \cdot \log_p(u_{\beta\beta}) + b \cdot \log_p(v_{\beta\beta}) = 0. \tag{18}$$

However, if both (17) and (18) are satisfied and $(a, b) \neq (0, 0)$, then $\mathrm{Reg}_{g_\alpha}(V_{gh})$ is zero, contradicting the hypothesis. □

Remark 3.12. We expect that the non-vanishing of the regulator $\mathrm{Reg}_{g_\alpha}(V_{gh})$ could follow from results on transcendental number theory.

4. The self-dual case

Throughout this section we assume that $h = g^*$ and $\alpha_h = 1/\alpha_g$, which amounts to saying that $h_\alpha = g^*_{1/\alpha}$.

As usual, there exist four global cohomology classes, that we denote

$$\kappa(g_\alpha, g^*_{1/\alpha}), \quad \kappa(g_\beta, g^*_{1/\beta}), \quad \kappa(g_\alpha, g^*_{1/\beta}), \quad \kappa(g_\beta, g^*_{1/\alpha}) \in H^1(\mathbb{Q}, V_{gg^*} \otimes \mathbb{Q}_p(1)). \tag{19}$$

It was proved in [RiRo, §3.2] that $\kappa(g_\alpha, g^*_{1/\beta}) = \kappa(g_\beta, g^*_{1/\alpha}) = 0$, using an argument involving the *exceptional* vanishing of some Euler factors. The aim of this section is to describe the two remaining classes.

4.1. **An explicit description of the cohomology classes.** Under our running assumptions, $(U_{gg^*}\otimes V_{gg^*}^\vee\otimes\mathbb{Q}_p)^{G_\mathbb{Q}}$ has dimension 1, while $(U_{gg^*}[1/p]\otimes V_{gg^*}^\vee\otimes\mathbb{Q}_p)^{G_\mathbb{Q}}$ is a 3-dimensional space. Fix now a basis $\{u,v,p\}$ of the latter, with the element u spanning the line $(U_{gh}\otimes V_{gg^*}^\vee\otimes\mathbb{Q}_p)^{G_\mathbb{Q}}$. Write the Frobenius decomposition of these units as

$$u = u_{\alpha,1/\beta}\otimes e^\vee_{\alpha,1/\beta}+u_{\alpha,1/\alpha}\otimes e^\vee_{\alpha,1/\alpha}+u_{\beta,1/\beta}\otimes e^\vee_{\beta,1/\beta}+u_{\beta,1/\alpha}\otimes e^\vee_{\beta,1/\alpha} \tag{20}$$

$$v = v_{\alpha,1/\beta}\otimes e^\vee_{\alpha,1/\beta}+v_{\alpha,1/\alpha}\otimes e^\vee_{\alpha,1/\alpha}+v_{\beta,1/\beta}\otimes e^\vee_{\beta,1/\beta}+v_{\beta,1/\alpha}\otimes e^\vee_{\beta,1/\alpha}. \tag{21}$$

Observe that the above cohomology classes are no longer crystalline at p, and according to the discussion of Proposition 3.4, they belong to $H^1_{\mathrm{f},p}(\mathbb{Q},V_{gg^*}\otimes\mathbb{Q}_p(1))$, the subspace of $H^1(\mathbb{Q},V_{gg^*}\otimes\mathbb{Q}_p(1))$ formed by those classes which are unramified at any prime $\ell\neq p$ and de Rham at p. The one-dimensional subspace $H^1_\mathrm{f}(\mathbb{Q},V_{gg^*}\otimes\mathbb{Q}_p(1))$ is spanned by the unit u.

The representation $V_{gg^*}^\vee$ is no longer irreducible, as

$$V_{gg^*}^\vee\simeq \mathrm{ad}^0(V_g^\vee)\oplus \mathrm{Id}, \tag{22}$$

where $\mathrm{ad}^0(V_g)$ stands for the adjoint representation of g and Id for the trivial representation.

The isomorphism (22) can be explicitly described as follows: fixing a basis $\{e_1^\vee,e^\vee_{\alpha/\beta},e^\vee_{\beta/\alpha}\}$ of $\mathrm{ad}^0(V_g^\vee)$ and also a basis $\{e^\vee_{\mathrm{Id}}\}$ for Id, it is given by the rule

$$\begin{gathered} e^\vee_{\alpha,1/\alpha}+e^\vee_{\beta,1/\beta}\mapsto(0,e^\vee_{\mathrm{Id}}),\quad e^\vee_{\alpha,1/\alpha}-e^\vee_{\beta,1/\beta}\mapsto(e^\vee_1,0),\\ e_{\alpha,1/\beta}\mapsto(e^\vee_{\alpha/\beta},0),\quad e_{\beta,1/\alpha}\mapsto(e^\vee_{\beta/\alpha},0).\end{gathered} \tag{23}$$

Considering the decomposition

$$H^1_{\mathrm{f},p}(\mathbb{Q},V^\vee_{gg^*}\otimes\mathbb{Q}_p(1))=H^1_{\mathrm{f},p}(\mathbb{Q},\mathrm{ad}^0(V_g^\vee)\otimes\mathbb{Q}_p(1))\oplus H^1_{\mathrm{f},p}(\mathbb{Q},\mathbb{Q}_p(1)), \tag{24}$$

we observe that according to [Bel, Prop. 2.12], the space $H^1_{\mathrm{f},p}(\mathbb{Q},\mathbb{Q}_p(1))\simeq(\mathbb{Z}[1/p]^\times)\otimes\mathbb{Q}_p$ has dimension 1 and it is spanned by p, while there is a canonical identification

$$H^1_{\mathrm{f},p}(\mathbb{Q},\mathrm{ad}^0(V_g^\vee)\otimes\mathbb{Q}_p(1))=(U_{gh}[1/p]\otimes\mathrm{ad}^0(V_g^\vee)\otimes\mathbb{Q}_p)^{G_\mathbb{Q}}. \tag{25}$$

Since $u,v\in(U_{gg^*}[1/p]\otimes\mathrm{ad}^0(V_g^\vee))^{G_\mathbb{Q}}$, it follows from the first equation of (23) that

$$u_{\alpha,1/\alpha}\cdot u_{\beta,1/\beta}=v_{\alpha,1/\alpha}\cdot v_{\beta,1/\beta}=1.$$

Set $u_1:=u_{\alpha,1/\alpha}=u^{-1}_{\beta,1/\beta}$ and $v_1:=v_{\alpha,1/\alpha}=v^{-1}_{\beta,1/\beta}$. Making a slight abuse of notation, we still denote by $\kappa(g_\alpha,g^*_{1/\alpha})$ the projection of the cohomology class to the space $H^1_{\mathrm{f},p}(\mathbb{Q},\mathrm{ad}^0(V_g^\vee)/p^\mathbb{Z}\otimes\mathbb{Q}_p(1))$. The following result corresponds to Theorem 1.2 when $h_\alpha=g^*_{1/\alpha}$.

Proposition 4.1. *There exists a period $\Omega\in H_p$ such that the equality*

$$\kappa(g_\alpha,g^*_{1/\alpha})\doteq\Omega\cdot(\log_p(u_1)\cdot v-\log_p(v_1)\cdot u)\pmod{L^\times} \tag{26}$$

holds in $H^1_{\mathrm{f},p}(\mathbb{Q},\mathrm{ad}^0(V_g^\vee)/p^\mathbb{Z}\otimes\mathbb{Q}_p(1))$. Moreover, if

$$\Omega\doteq\frac{1}{\Xi_{g_\alpha}\cdot\Omega_{g^*_{1/\alpha}}}\cdot\frac{1}{\log_p(u_{\beta/\alpha})},$$

Conjecture 2.2 is true, and under the assumption that $\mathrm{Reg}_{g_\alpha}(V_{gg^*})\neq0$ the converse also holds.

Proof. The first part of the statement follows the same argument used in the proof of Theorem 3.7. Now, we may write

$$\kappa(g_\alpha,g^*_{1/\alpha})=\lambda u+\mu v+\nu p$$

for some p-adic scalars λ,μ and ν. Next, we project to $H^1(\mathbb{Q}_p,V_g^\alpha\otimes V_{g^*}^{1/\alpha}\otimes\mathbb{Q}_p(1))$, which is no longer one-dimensional, but isomorphic to the two-dimensional space $H^1(\mathbb{Q}_p,\mathbb{Q}_p(1))\simeq$

$\mathbb{Q}_p^\times \hat{\otimes} \mathbb{Q}_p \simeq \mathbb{Q}_p \oplus \mathbb{Q}_p$. This amounts to saying that both the p-adic valuation and p-adic logarithm are zero. In particular,

$$\lambda \log_p(u_1) + \mu \log_p(v_1) = 0,$$

and the equality in (26) follows.

In the same way, Proposition 3.11 is equally valid once we have considered the quotient by the trivial representation and we can write the cohomology class as a combination of the units u and v. □

Remark 4.2. We have previously proven that when $h \neq g^*$, the non-vanishing of the special value $\mathscr{L}_p{}^{g_\alpha}$ allows us to conclude that the classes $\kappa(g_\alpha, h_\alpha)$ and $\kappa(g_\alpha, h_\beta)$ are linearly independent. The same argument implies that, under the same non-vanishing hypothesis, the class $\kappa(g_\alpha, g^*_{1/\alpha})$ and the derived class $\kappa'(g_\alpha, g^*_{1/\beta})$ constructed in [RiRo] are linearly independent.

Assuming again that we know that the regulator does not vanish, we can prove that $\Omega \neq 0$ in (26) and can provide a formula for Ω in $H_p^\times / L^\times$. Furthermore, it clearly follows from Proposition 4.1 that the two classes $\kappa(g_\alpha, g^*_{1/\alpha})$ and $\kappa(g_\beta, g^*_{1/\beta})$ are linearly dependent.

4.2. **Proof of Theorem 1.4.** We now move to the proof of Theorem 1.4 of the introduction. Although we stated the result in terms of the unit group $(U_{gh}[1/p]/p^{\mathbb{Z}} \otimes V_{gh}^\vee)^{G_\mathbb{Q}}$, the identifications of Kummer theory allow us to consider an equivalent formulation in terms of cohomology groups.

Proposition 4.3. *If we assume that* $\mathrm{Reg}_{g_\alpha}(V_{gg^*}) \neq 0$, *the following equalities hold in the space* $H^1_{\mathrm{f},p}(\mathbb{Q}, \mathrm{ad}^0(V_g^\vee)/p^{\mathbb{Z}} \otimes \mathbb{Q}_p(1))$, *modulo* $L^\times$*:*

$$\kappa(g_\alpha, g^*_{1/\alpha}) \doteq \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\alpha}}} \cdot \frac{\log_p(u_1) \cdot v - \log_p(v_1) \cdot u}{\log_p(u_{\beta/\alpha})}, \tag{27}$$

$$\kappa(g_\beta, g^*_{1/\beta}) \doteq \frac{1}{\Xi_{g_\beta} \cdot \Omega_{g^*_{1/\beta}}} \cdot \frac{\log_p(u_1) \cdot v - \log_p(v_1) \cdot u}{\log_p(u_{\alpha/\beta})}. \tag{28}$$

Proof. According to Proposition 4.1, this is equivalent to Conjecture 2.2, concerning the value of $L_p(\mathbf{g}, \mathbf{h})$ at $(y_0, y_0, 1)$. However, observe that this value does not depend on the p-stablization of h, and hence this follows from [RiRo, Theorem A], where it was proved that

$$L_p(\mathbf{g}, \mathbf{h})(y_0, y_0, 1) \doteq \frac{\log_p(u_1) \cdot v - \log_p(v_1) \cdot u}{\log_p(u_{\beta/\alpha})} \pmod{L^\times}.$$

□

In particular, the non-vanishing of the regulator implies that these classes are non-zero, since the vanishing of both $\log_p(u_1)$ and $\log_p(v_1)$ would automatically imply that $\mathrm{Reg}_{g_\alpha}(V_{gg^*}) = 0$. Furthermore, it is clear from this description that the two classes span a one-dimensional subspace, as asserted in Theorem 1.4.

4.3. **Theta series of quadratic fields.** When g is the theta series of an imaginary quadratic field where p splits, or the theta series of a real quadratic field where p remains inert, we can give a more explicit description of the classes. Furthermore, in these cases we know that the regulator does not vanish.

Quadratic imaginary case, with p split. Let $\psi : G_K \to \mathbb{C}^\times$ be a ring class character of conductor prime to p, and write $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K. Set $\alpha := \psi(\bar{\mathfrak{p}})$ and $\beta := \psi(\mathfrak{p})$. Let g and h be the weight 1 theta series of ψ and ψ^{-1}, respectively.

In this case,

$$V_{gh} \simeq \mathrm{Ind}_K^\mathbb{Q} \psi_{\mathrm{triv}} \oplus \mathrm{Ind}_K^\mathbb{Q} \psi^2,$$

where ψ_{triv} stands for the trivial character of K. The theory of elliptic units allows us to attach a canonical unit u_{ψ^2} (resp. $u_{\psi^{-2}}$) to the character ψ^2 (resp. ψ^{-2}), where Fr_p acts with eigenvalue α/β (resp. β/α). Let $u = u_{\psi^2} \otimes e^\vee_{\alpha/\beta} + u_{\psi^{-2}} \otimes e^\vee_{\beta/\alpha}$.

Let h_K denote the class number of K, and write $v_{\mathfrak{p}} \in K^\times$ for any p-unit satisfying $(v_{\mathfrak{p}}) = \mathfrak{p}^{h_K}$. Let $v = v_{\mathfrak{p}} \otimes e_1^\vee$.

From the description of u and v we see that $u_1 = 0$ and hence we have that

$$\kappa(g_\alpha, g^*_{1/\alpha}) \doteq \tilde{\Omega}_1 \cdot u, \quad \kappa(g_\beta, g^*_{1/\beta}) \doteq \tilde{\Omega}_2 \cdot u, \tag{29}$$

where $\tilde{\Omega}_1, \tilde{\Omega}_2 \in H_p$. We must prove that these numbers are both non-zero. Projecting to the (β,α)-component of V_{gh} and applying the Perrin-Riou map described in Proposition 3.1, the explicit reciprocity law in the form of Theorem 3.2 gives that

$$\log^{-+}(\kappa(g_\alpha, g^*_{1/\alpha})) \doteq \tilde{\Omega}_1 \cdot \log(u_{\psi^{-2}}) \doteq L_p(\mathbf{g},\mathbf{h})(y_0,y_0,1) \pmod{L^\times}.$$

Although $L_p(\mathbf{g},\mathbf{h})(y_0,y_0,1)$ depends on the chosen of a p-stabilization for g, this is not the case for h. Moreover, according to [Theorem 4.2, DLR2],

$$L_p(\mathbf{g},\mathbf{h})(y_0,y_0,1) \doteq \frac{\log_p(v_{\mathfrak{p}}) \cdot \log_p(u_{\psi^{-2}})}{\log_p(u_{\psi^{-2}})} \doteq \log_p(v_{\mathfrak{p}}) \pmod{L^\times},$$

and in particular

$$\tilde{\Omega}_1 \doteq \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\alpha}}} \cdot \frac{\log_p(v_{\mathfrak{p}})}{\log_p(u_{\psi^{-2}})}, \qquad \tilde{\Omega}_2 \doteq \frac{1}{\Xi_{g_\beta} \cdot \Omega_{g^*_{1/\beta}}} \cdot \frac{\log_p(v_{\mathfrak{p}})}{\log_p(u_{\psi^{2}})} \pmod{L^\times}.$$

Real quadratic case, with p inert. In this case, u is the fundamental unit ϵ_K attached to K and v is a p-unit in the field H cut out by the character. Writing v^+ for the norm of v and keeping the same notations as in the previous case, we get that the periods $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are given by

$$\tilde{\Omega}_1 \doteq \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\alpha}}} \cdot \frac{\log_p(v^+)}{\log_p(\epsilon_K)}, \quad \tilde{\Omega}_2 \doteq \frac{1}{\Xi_{g_\beta} \cdot \Omega_{g^*_{1/\beta}}} \cdot \frac{\log_p(v^+)}{\log_p(\epsilon_K)} \pmod{L^\times}.$$

As a consequence of this discussion, the following theorem is proved.

Theorem 4.4. *Let g be a theta series of an imaginary (resp. real) quadratic field K where p splits (remains inert). Then, the cohomology classes of* (19) *span a line in the space $H^1_{\mathrm{f},p}(\mathbb{Q}, \mathrm{ad}^0(V_g^\vee)/p^{\mathbb{Z}} \otimes \mathbb{Q}_p(1))$. To be more precise, $\kappa(g_\alpha, g^*_{1/\beta}) = \kappa(g_\beta, g^*_{1/\alpha}) = 0$, and the remaining classes can be described as follows:*

(a) *In the imaginary quadratic case, with p a prime which splits in K,*

$$\kappa(g_\alpha, g^*_{1/\alpha}) \doteq \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\alpha}}} \cdot \frac{\log_p(v_{\mathfrak{p}})}{\log_p(u_{\psi^{-2}})} \cdot u, \quad \kappa(g_\beta, g^*_{1/\beta}) \doteq \frac{1}{\Xi_{g_\beta} \cdot \Omega_{g^*_{1/\beta}}} \cdot \frac{\log_p(v_{\mathfrak{p}})}{\log_p(u_{\psi^{2}})} \cdot u \pmod{L^\times}.$$

(b) *In the real quadratic case, with p an inert prime,*

$$\kappa(g_\alpha, g^*_{1/\alpha}) \doteq \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\alpha}}} \cdot \frac{\log_p(v^+)}{\log_p(\epsilon_K)} \cdot u, \quad \kappa(g_\beta, g^*_{1/\beta}) \doteq \frac{1}{\Xi_{g_\beta} \cdot \Omega_{g^*_{1/\beta}}} \cdot \frac{\log_p(v^+)}{\log_p(\epsilon_K)} \cdot u \pmod{L^\times}.$$

Remark 4.5. The proof of our result is ostensibly easier than that of Castella and Hsieh. The main reason is that, while the order of vanishing of the theta element $\Theta_{f/K}(T)$ is unknown, we do know that the Katz's two-variable p-adic L-function $L_p^{\mathrm{Katz}}(K)$ attached to the trivial character vanishes at order zero at $s = 0$. To overcome this difficulty, Castella and Hsieh use a bound for this order of vanishing coming from Iwasawa theory and later develop the theory of derived heights in the case of elliptic curves. Although this treatment would make sense here, it would not yield to any new result.

5. Particular cases of the conjecture

In this section we discuss in more detail some of the reducible cases that are considered in [DLR2]. These are scenarios where the Beilinson–Flach classes appearing in the main body of the paper can be recast in terms of circular units, elliptic units or Beilinson–Kato elements, and our statements admit an ostensibly simpler formulation.

As in §3, and to ease the exposition, we assume that $h \neq g^*$ throughout this section.

5.1. **Eisenstein series and circular units.** Let

$$g = E_1(\chi_g^+, \chi_g^-), \quad h = E_1(\chi_h^+, \chi_h^-)$$

denote two Eisenstein series attached to pairs of Dirichlet characters, with the assumption that χ_g^+ is even and χ_g^- is odd, and likewise for χ_h^+ and χ_h^-. In the Eisenstein case, as discussed in [DLR1], the classicality hypothesis on g reads as

$$\chi_g^+(p) = \chi_g^-(p).$$

Let $\chi_g := \chi_g^+\chi_g^-$ and $\chi_h := \chi_h^+\chi_h^-$. The representation V_{gh} decomposes as a direct sum of four one-dimensional characters,

$$V_{gh} = \chi_{gh}^{++} \oplus \chi_{gh}^{--} \oplus \chi_{gh}^{+-} \oplus \chi_{gh}^{-+}.$$

Given an even character χ factoring through a finite abelian extension H_χ of $\mathbb{Q}$ and taking values in L, denote by $u(\chi) \in L \otimes \mathcal{O}_{H_\chi}^\times$ the fundamental unit in the χ-eigenspace for the $G_\mathbb{Q}$-action (this could be seen as the circular unit attached to χ, following the terminology of [DLR2, §3]). The pair (χ_g^+, χ_g^-) corresponds to a genus character ψ_g of the imaginary quadratic field K cut out by the odd Dirichlet character $\chi_g := \chi_g^+\chi_g^-$. The weight one Eiseinstein series $E_1(\chi_g^+, \chi_g^-)$ is equal to the theta series $\theta_K(\psi_g)$, and hence the modular form g admits three natural ordinary deformations, since

$$E_1(\chi_g^+, \chi_g^-) = E_1(\chi_g^-, \chi_g^+) = \theta_K(\psi_g).$$

Let $\mathbf{g}$ (resp. $\mathbf{h}$) denote the cuspidal family passing through the p-stabilization of g (resp. h).

Instead of going through all the possible cases we just focus on the generic situation where χ_{gh}^{++} and χ_{gh}^{--} are both non-trivial and $\chi_{gh}^{\bullet\circ} \neq 1$ for $\bullet, \circ \in \{\pm\}$.

Then, [DLR2, Theorem 3.1] asserts that

$$L_p(\mathbf{g}, \mathbf{h})(y_0, z_0, 1) \doteq \log^{-+}(\kappa_p(g_\alpha, h_\alpha)) \doteq \frac{\log_p(u(\chi_{gh}^{++}))\log_p(u(\chi_{gh}^{--}))}{\log_p(u_{g_\alpha})} \pmod{L^\times}. \tag{30}$$

From here, we see that

$$\kappa(g_\alpha, h_\alpha) \doteq \mathfrak{C} \cdot u \pmod{L^\times},$$

where u is the unit $u = u(\chi_{gh}^{--}) \otimes e_{\beta\alpha}^\vee + u(\chi_{gh}^{-+}) \otimes e_{\alpha\alpha}^\vee$, whose (β, α)-component agrees with $u(\chi_{gh}^{--})$, and $\mathfrak{C}$ is an explicit constant in terms of the periods we have previously defined, namely

$$\mathfrak{C} \doteq \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot \frac{\log_p(u(\chi_{gh}^{++}))}{\log_p(u(g_\alpha))} \pmod{L^\times}. \tag{31}$$

Remark 5.1. When some of the characters χ_{gh}^{++} or χ_{gh}^{--} is trivial, the regulator also involves the fundamental p-units in the χ-eigenspace for the Galois action, for an appropriate χ. See Case 2 and Case 3 of [DLR2, Section 3] for more details.

5.2. **Theta series attached to imaginary quadratic fields and elliptic units.** Let

$$g = \theta_{\psi_g} \in M_1(N_g, \chi_K \chi_g), \quad h = \theta_{\psi_h} \in M_1(N_h, \chi_K \chi_h)$$

be the theta series associated to two arbitrary finite order characters ψ_g, ψ_h of the imaginary quadratic field K. Let $\psi_1 = \psi_g \psi_h$ and $\psi_2 = \psi_g \psi_h'$, where ψ' denote the character given by $\psi'(\sigma) = \psi(\sigma_0 \sigma \sigma_0^{-1})$, for any choice of $\sigma_0 \in G_{\mathbb{Q}} \backslash G_K$. Let $V_g = \mathrm{Ind}_K^{\mathbb{Q}}(\psi_g)$ and $V_h = \mathrm{Ind}_K^{\mathbb{Q}}(\psi_h)$ denote the two-dimensional induced representations of ψ_g and ψ_h. Then,

$$V_{gh} = V_{\psi_g} \otimes V_{\psi_h} \simeq V_{\psi_1} \oplus V_{\psi_2}.$$

For any character ψ, we define u_ψ as the corresponding elliptic unit attached to it, as recalled in [DLR2, Section 4].

Following the same argument as in [RiRo, Theorem 6.2], we have that for any $s \in \mathbb{Z}_p$,

$$L_p(g,h,s) \doteq \frac{1}{\log_p(u_{\psi_{\mathrm{ad}}})} \cdot L_p^{\mathrm{Katz}}(\psi_{gh}^{-1}, s) \cdot L_p^{\mathrm{Katz}}(\psi_{gh'}^{-1}, s) \pmod{L^\times}, \tag{32}$$

where $\psi_{gh} = \psi_g \cdot \psi_h$, $\psi_{gh'} = \psi_g \cdot \psi_{h'}$, and $L_p^{\mathrm{Katz}}(\psi, s)$ stands for the evaluation of the two-variable Katz p-adic L-function attached to ψ at the character $\mathbb{N}^s$. In particular,

$$\log^{-+}(\kappa(g_\alpha, h_\alpha)) \doteq \frac{\log_p(u_{\psi_1'}) \log_p(u_{\psi_2'})}{\log_p(u_{g_\alpha})} \pmod{L^\times}. \tag{33}$$

As we have done before, we can express $\kappa(g_\alpha, h_\alpha)$ as an explicit constant multiplied by the elliptic unit $u := u_{\psi_2} \otimes e_{\alpha\beta}^\vee + u_{\psi_2'} \otimes e_{\beta\alpha}^\vee$, making use of the periods we have introduced before, as

$$\kappa(g_\alpha, h_\alpha) \doteq \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot \frac{\log_p(u_{\psi_1'})}{\log_p(u_{g_\alpha})} \cdot u \pmod{L^\times}$$

5.3. **Eisenstein series and Beilinson–Kato elements.** In the case where exactly one of the modular forms is Eisenstein, the representation V_{gh} also decomposes as a sum of two irreducible representations of dimension two. Let $g \in S_1(N_g, \chi_g)$ and $h = E_1(\chi_h^+, \chi_h^-)$; then,

$$V_{gh} = V_g(\chi_h^+) \oplus V_g(\chi_h^-),$$

which gives via Artin formalism a decomposition

$$L(V_{gh}, s) = L(g, \chi_h^+, s) \cdot L(g, \chi_h^-, s),$$

where $L(g, \chi_h^\bullet, s)$ is the L-function attached to g twisted by the finite order character $\chi_h^\bullet$.

Let $L_p(\mathbf{g}, E_1(\chi_h^+, \chi_h^-))$ denote the two-variable p-adic L-function of [BDR1, Section 2.2.2] (when the second modular form does not vary in a Hida family it is allowed to be Eisenstein). The following theorem is a *weak* analogue of [RiRo, eq.(62)] in this setting. Let χ be a Dirichlet character, and consider $L_p(\mathbf{g}, \chi)$, the two-variable Mazur–Kitagawa p-adic L-function attached to $\mathbf{g}$ and χ. With the notations of [BD], let $\lambda^\pm(\ell) \in \mathbb{C}_p$ stand for the canonical periods involved in the construction of the Mazur–Kitagawa p-adic L-function. We adopt as in loc. cit. the normalization

$$L_p(\mathbf{g}, \chi)(y,s) = \lambda^\pm(\ell) \cdot \Big(1 - \frac{p^{s-1}}{\chi(p)\alpha_{g_y^\circ}}\Big) \cdot \Big(1 - \frac{\chi(p)\beta_{g_y^\circ}}{p^s}\Big) \times L^*(g_y, \chi, s),$$

where $L^*(g_y, \chi, s)$ is the algebraic part of $L(g_y, \chi, s)$, defined in [BD, eq.(22)].

Theorem 5.2. *There exists a rigid analytic function $\mathfrak{f}(y,s)$ such that the following equality holds in $\Lambda_{\mathbf{g}} \otimes \Lambda$*

$$L_p(\mathbf{g}, E_1(\chi_h^+, \chi_h^-))(y,s) = \mathfrak{f}(y,s) \cdot L_p(\mathbf{g}, \chi_h^+)(y,s) \cdot L_p(\mathbf{g}, \chi_h^-)(y,s). \tag{34}$$

Here, $\mathfrak{f}(y,s) = (C_{g_y,\chi_h^+,\chi_h^-} \cdot \mathcal{E}(g_y) \cdot \mathcal{E}^*(g_y) \cdot \lambda^+(\ell) \cdot \lambda^-(\ell))^{-1}$*, being* $C_{g_y,\chi_h^+,\chi_h^-}$ *an explicit non-zero algebraic number and*

$$\mathcal{E}(g_y) = 1 - \beta^2_{g_y^\circ} p^{-\ell}, \qquad \mathcal{E}^*(g_y) := 1 - \beta^2_{g_y^\circ} p^{1-\ell}.$$

Proof. In the range of classical interpolation we have an equality of the corresponding L-values. The result follows after gathering together the different factors appearing in the interpolation process, combined with the observation that $\Omega^+_{\mathbf{g}_y,\mathbb{C}} \cdot \Omega^-_{\mathbf{g}_y,\mathbb{C}} = 4\pi^2 \langle \mathbf{g}_y^\circ, \mathbf{g}_y^\circ \rangle$ □

It may be instructive to compare this result with [BD, Theorem 3.4]: there, a two-variable p-adic L-function attached to a cuspidal Hida family and an Eisenstein family which interpolates central critical points is expressed as the product of two Mazur–Kitagawa p-adic L-functions. Unfortunately, our result is not as useful as one would expect: specialization at weights $(y_0, 1)$ establishes a connection between the Hida–Rankin p-adic L-function and the product of $L_p(g, \chi_h^+, 1)$ with $L_p(g, \chi_h^-, 1)$, but up to multiplication by the quantity $\mathfrak{f}(y_0, 1)$. Recall that a similar question arises at [DR4, Proposition 2.6], since the factor $\mathfrak{f}(y,s)$ is essentially (up to multiplication by some explicit factors) the function $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{g}))(y)$. This multiplier is expected to be related with the logarithm of the Gross-Stark unit u_{g_α}. Further, this also connects with [DR3, Conjecture 2.1], since we hope $\lambda^+(1)$ to be eventually related with Ω_{g_α} and $\lambda^-(1)$ with $\Xi^{-1}_{g_\alpha}$.

As it is proved in [Och], there exists a two-variable Euler system (usually referred to as Beilinson–Kato Euler system), $\kappa(\mathbf{g}, \chi)$, satisfying that $L_p(\mathbf{g}, \chi)$ is the image under a suitable Perrin-Riou map of $\kappa(\mathbf{g}, \chi)$. This allows us to obtain a connection between Beilinson–Flach and Beilinson–Kato elements, and also an expression for the regulator involving special values of the Mazur–Kitagawa p-adic L-function. Observe that this is the counterpart of [DLR1, Sections 6,7], where this same situation was considered in the case of rational points over elliptic curves.

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