$\mathcal{L}$-INVARIANTS AND DARMON CYCLES ATTACHED TO MODULAR FORMS

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Abstract. Let f be a modular eigenform of even weight $k \geq 2$ and new at a prime p dividing exactly the level with respect to an indefinite quaternion algebra. The theory of Fontaine-Mazur allows to attach to f a monodromy module $\mathbf{D}_f^{FM}$ and an $\mathcal{L}$-invariant $\mathcal{L}_f^{FM}$. The first goal of this paper is building a suitable p-adic integration theory that allows us to construct a new monodromy module $\mathbf{D}_f$ and $\mathcal{L}$-invariant $\mathcal{L}_f$, in the spirit of Darmon. The two monodromy modules are isomorphic, and in particular the two $\mathcal{L}$-invariants are equal.

Let K be a real quadratic field and assume the sign of the functional equation of the L-series of f over K is -1. The Bloch-Beilinson conjectures suggest that there should be a supply of elements in the Selmer group of the motive attached to f over the tower of narrow ring class fields of K. Generalizing work of Darmon for $k = 2$, we give a construction of local cohomology classes which we expect to arise from global classes and satisfy an explicit reciprocity law, accounting for the above prediction.

Contents

1. Introduction

Let $X/\mathbb{Q}$ denote the canonical model of the smooth projective Shimura curve attached to an Eichler order $\mathcal{R}$ in an indefinite quaternion algebra $\mathcal{B}$ over $\mathbb{Q}$. When $\mathcal{B} \simeq \mathrm{M}_2(\mathbb{Q})$ (respectively

$\mathcal{B}$ is a division algebra), X is the coarse moduli space parametrizing generalized elliptic curves (resp. abelian surfaces with multiplication by a maximal order in $\mathcal{B}$) together with a Γ_0-level structure.

Let $k \geq 2$ be an even integer and let $n := k-2$ and $m := n/2$. As explained in [Ja], [Sc] for $\mathcal{B} \simeq \mathrm{M}_2(\mathbb{Q})$ and in [IS, 10.1] when $\mathcal{B}$ is division, there exists a Chow motive $\mathcal{M}_n$ over $\mathbb{Q}$ attached to the space $S_k(X)$ of cusp forms of weight k on X. Attached to any eigenform $f \in S_k(X)$, there exists a Grothendieck motive $\mathcal{M}_{n,f}$ over $\mathbb{Q}$ with coefficients over the field $L_f := \mathbb{Q}(\{a_\ell(f)\})$ generated by the eigenvalues of f under the action of the Hecke operators T_ℓ for all prime ℓ, which is constructed as the f-isotypical factor of $\mathcal{M}_n$ in the category of Grothendieck motives (cf. [Sc, Thm. 1.2.4]).

Fix a prime p and let $H_p(\mathcal{M}_n)$ denote the p-adic étale realization of $\mathcal{M}_n$ obtained as the $(m+1)$-th Tate twist of the p-adic étale cohomology of a suitable Kuga-Sato variety. It is a finite dimensional continuous representation of $G_{\mathbb{Q}} = \mathrm{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q})$ over $\mathbb{Q}_p$, endowed with a compatible action of a Hecke algebra. Similarly, for any eigenform $f \in S_k(X)$ let $V_p(f)$ denote the p-adic realization of $\mathcal{M}_{n,f}$, a two-dimensional representation over $L_{f,p} := L_f \otimes \mathbb{Q}_p$.

Assume now that p divides exactly the level of $\mathcal{R}$. Let $\mathbb{T}$ denote the maximal quotient of the algebra generated by the Hecke operators acting on $S_k(X)^{p-new}$ and let $V_p := H_p(\mathcal{M}_n)^{p-new}$ denote the p-new quotient of $H_p(\mathcal{M}_n)$.

The restriction of V_p to a decomposition subgroup $D_p \simeq \mathrm{Gal}\,(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is a *semistable* representation and the filtered (ϕ, N)-module $\mathbf{D}^{FM} = D_{\mathrm{st}}(V_p)$ attached by Fontaine and Mazur to V_p is a two-dimensional monodromy $\mathbb{T}\otimes\mathbb{Q}_p$-module over $\mathbb{Q}_p$ in the sense of [IS, Definition 2.2]. An important invariant of its isomorphism class is the $\mathcal{L}$-invariant $\mathcal{L}^{FM} := \mathcal{L}(\mathbf{D}^{FM}) \in \mathbb{T}\otimes\mathbb{Q}_p$ that one may associate to it. We refer the reader to [Ma] and [IS, §2] (and to Proposition 4.6 below) for details. Similarly, let $\mathbf{D}^{FM}_f$ and $\mathcal{L}^{FM}_f \in L_{f,p}$ respectively denote the two-dimensional monodromy module over $L_{f,p}$ and $\mathcal{L}$-invariant associated with f.

An illustrative explicit example arises when $k = 2$, since then $n = 0$ and $\mathcal{M}_0$ can simply be interpreted as the Jacobian J of X. Then $\mathcal{M}_{0,f} = A_f$ is the abelian variety attached to f by Shimura (cf. [Sh1]). As is well-known, if f is an eigenform in $S_k(X)^{p-new}$ then A_f has purely toric reduction at p and Tate-Morikawa's theory allows to attach to it an $\mathcal{L}$-invariant $\mathcal{L}(A_f) \in L_{f,p}$ purely in terms of the p-adic rigid analytic description of this variety. When $E = A_f$ is an elliptic curve, for instance, this $\mathcal{L}$-invariant is simply

$$\mathcal{L}(E) = \frac{\log(q)}{\mathrm{ord}_p(q)},$$

where $q = q(E)$ is the Tate period of E.

Thanks to the work of several authors (Greenberg-Stevens, Kato-Kurihara-Tsuji, Coleman-Iovita, Colmez) we now know that $\mathcal{L}^{FM}_f = \mathcal{L}(A_f)$. The importance of this invariant partly relies on the fact that, when $a_p = 1$, it accounts for the discrepancy between the special values of the classical L-series $L(f,s)$ and the p-adic L-function $L_p(f,s)$ at $s = 1$. This phenomenon was predicted by Mazur, Tate and Teitelbaum as the *exceptional zero conjecture* and was first proved by Greenberg and Stevens.

For higher weights $k \geq 4$ similar phenomena occur, and several a priori different $\mathcal{L}$-invariants attached to a p-new eigenform f were defined by several authors (Teitelbaum, Coleman, Darmon and Orton, Breuil) besides the aforementioned Fontaine-Mazur $\mathcal{L}^{FM}_f$. Let us stress that the definition of all these invariants is not always available in the general setting of this introduction. However, we again know now, thanks to the previously mentioned works together with [Br], [BDI] and [IS], that all these invariants are equal whenever they are defined. See the above references for a detailed account of the theory.

The $\mathcal{L}$-invariant $\mathcal{L}^{D}_f$ introduced by Darmon in the foundational work [Dar] (and generalized by Orton in [Or] and Greenberg in [Gr]) is the one that is most germane to this article (cf.

also é6). Darmon's $\mathcal{L}$-invariant is only available when $\mathcal{B} \simeq \mathrm{M}_2(\mathbb{Q})$ and when $\mathcal{B}$ is an indefinite quaternion algebra but $k = 2$. Note that when $\mathcal{B} \simeq \mathrm{M}_2(\mathbb{Q})$ its construction heavily relies on the theory of modular symbols, which in turn is based on the presence of cuspidal points on the modular curve X. This feature is simply absent when $\mathcal{B}$ is a division algebra.

The first goal of this paper is providing a construction of an $\mathcal{L}$-invariant $\mathcal{L}$ attached to the space of p-new cusp forms for all quaternion algebras $\mathcal{B}$ in the spirit of Darmon, Greenberg and Orton even in the case $k > 2$. This is achieved in 3.2 as a culmination of the results gathered in 2 and 3, which show the existence of a suitable p-adic integration theory and form the technical core of this paper. One of the main results of this first part of the article is Theorem 3.5, which the reader may find of independent interest. It is an avatar of the classical Amice-Velu-Vishik theorem and the comparison theorem of Stevens in [St]. The proof exploits the modular representations of the quaternion algebra $\mathcal{B}$ studied intensively by Teitelbaum and others (cf. 2 below for details).

In view of the above discussion it is natural to expect that our invariant $\mathcal{L}$ equals $\mathcal{L}^{FM}$; this has been proved by the second author in [Se2]: cf. Theorem 4.7 for the precise statement. In 4.2 we construct a monodromy module $\mathbf{D}$ out of the $\mathcal{L}$-invariant $\mathcal{L}$ which is shown to be isomorphic to $\mathbf{D}^{FM}$.

Let us now describe the second goal and main motivation of this article, to which 5 is devoted as an application of the material in 2, 3 and 4.

Let K be a number field, which for simplicity we assume to be unramified over p. As in [BK] and [Ne2], for every place v of K define $H^1_{\mathrm{st}}(K_v, V_p)$ to be the kernel of

$$H^1(K_v, V_p) \to \begin{cases} H^1(K_v^{unr}, V_p) & \text{if } v \nmid p \\ H^1(K_v, \mathbf{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V_p) & \text{if } v \mid p \end{cases} \tag{1}$$

where K_v^{unr} is the maximal unramified extension of K_v and $\mathbf{B}_{\mathrm{st}}$ stands for Fontaine's ring (cf. loc. cit.). Define the (semistable) *Selmer group* of the representation V_p as

$$H^1_{\mathrm{st}}(K, V_p) := \ker\big(H^1(K, V_p) \overset{\prod \mathrm{res}_v}{\longrightarrow} \prod\nolimits_v \frac{H^1(K_v, V_p)}{H^1_{\mathrm{st}}(K_v, V_p)}\big). \tag{2}$$

For any motive $\mathcal{M}$ over a field κ and any integer j, let $\mathrm{CH}^j(\mathcal{M})$ denote the Chow group of cycles on $\mathcal{M}$ of codimension j with rational coefficients and let $\mathrm{CH}^j(\mathcal{M})_0$ denote its subgroup of null-homologous cycles. By the work of Nekovář (cf. [IS, 7] for precise statements in our general quaternionic setting), the classical p-adic étale Abel-Jacobi map induces a commutative diagram:

$$\begin{array}{ccc} \mathrm{CH}^{m+1}(\mathcal{M}_n \otimes K)_0 & \overset{cl^{m+1}_{0,K}}{\to} & H^1_{\mathrm{st}}(K, V_p) \\ \downarrow & & \downarrow \mathrm{res}_v \\ \mathrm{CH}^{m+1}(\mathcal{M}_n \otimes K_v)_0 & \overset{cl^{m+1}_{0,K_v}}{\to} & H^1_{\mathrm{st}}(K_v, V_p) \end{array} \tag{3}$$

for any place v of K over p. Note that in this situation we have $\mathrm{CH}^{m+1}(\mathcal{M}_n)_0 = \mathrm{CH}^{m+1}(\mathcal{M}_n)$, as proved in [IS, Lemma 10.1]. Composing with the natural projection $V_p \to V_p(f)$, we obtain a map $\mathrm{CH}^{m+1}(\mathcal{M}_n \otimes K)_0 \overset{cl^{m+1}_{0,K}(f)}{\to} H^1_{\mathrm{st}}(K, V_p(f))$. As a generalization of the conjecture of Birch and Swinnerton-Dyer, the conjectures of Bloch and Beilinson (cf. [Ne, 4]) predict that

$$cl^{m+1}_{0,K} \otimes \mathbb{Q}_p : \mathrm{CH}^{m+1}(\mathcal{M}_n \otimes K)_0 \otimes \mathbb{Q}_p \overset{\sim ?}{\longrightarrow} H^1_{\mathrm{st}}(K, V_p) \text{ is an isomorphism} \tag{4}$$

and

$$\mathrm{rank}_{L_{f,p}}(cl^{m+1}_{0,K}(f)) \overset{?}{=} \mathrm{ord}_{s=k/2} L(f \otimes K, s). \tag{5}$$

Let $N^- = \mathrm{disc}(\mathcal{B}) \geq 1$ denote the reduced discriminant of $\mathcal{B}$ and let pN^+ denote the level of $\mathcal{R}$. We have $(N^-, pN^+) = 1$ and, as we already mentioned, $p \nmid N^+$.

Assume now that K is quadratic, either real or imaginary, satisfying the following *Heegner hypothesis*:

- The discriminant D_K of K is coprime to $N := pN^+N^-$.
- All prime factors of N^- remain inert in K.
- All prime factors of N^+ split in K.
- p splits (remains inert) in K, if K is imaginary (real, respectively).

Thanks to the first condition, the sign of the functional equation of $L(f \otimes K, s)$ is simply $(\frac{-N}{K})$. The last three conditions imply that this sign is -1. In particular, $L(f \otimes K, k/2) = 0$. Let now $c \geq 1$ be a positive integer and let H_c/K denote the narrow ring class field of conductor c, whose Galois group $G_c := \mathrm{Gal}(H_c/K)$ is canonically isomorphic via Artin's reciprocity map to the narrow Picard group $\mathrm{Pic}(\mathcal{O}_c)$ of the order $\mathcal{O}_c \subset K$ of conductor c. Assuming $(c, N) = 1$, for any character $\chi : G_c \to \mathbb{C}^\times$ the root number of the twisted L-series $L(f \otimes K, \chi, s)$ continues to be -1 and the L-series of $f \otimes H_c$ admits the factorisation

$$L(f \otimes H_c, s) = \prod_{\chi \in G_c^\vee} L(f \otimes K, \chi, s).$$

It follows that

$$\mathrm{ord}_{s=k/2} L(f \otimes H_c, s) \geq h(\mathcal{O}_c) := |G_c|$$

and the Bloch-Beilinson conjecture (5) predicts that $\mathrm{rank}_{L_{f,p}}(cl_{0,K}^{m+1}(f)) \geq h(\mathcal{O}_c)$. In crude terms, there should be a systematic way of producing a collection of nontrivial elements

$$\{s_c \in H^1_{\mathrm{st}}(H_c, V_p(f))\} \tag{6}$$

in the Selmer group of f with coefficients on the tower of class fields H_c/K for $c \geq 1$,$(c, N) = 1$.

When K is imaginary, and $N^- = 1$, Nekovář [Ne] was able to construct these sought-after elements as images by the p-adic étale Abel-Jacobi map $cl_{0,K}^{m+1}(f)$ of certain Heegner cycles on $\mathcal{M}_n$ whose construction exploits, as in the classical case $k = 2$, the theory of complex multiplication on elliptic curves. This construction was later extended to arbitrary discriminants $N^- \geq 1$ by Besser (cf. [IS, §8] for a review).

Assume for the rest of the article that K is a real quadratic field. The aim of 5 is exploiting the p-adic integration theory established in 3 in order to propose a conjectural construction of suitable analogues of Heegner cycles for real quadratic fields.

Namely, our construction yields local cohomology classes $s_c \in H^1_{\mathrm{st}}(K_p, V_p)$ that we expect to arise from global cohomology classes in $H^1_{\mathrm{st}}(H_c, V_p)$. Notice that this makes sense, as H_c naturally embeds in K_p because p is inert in K.

More precisely, we produce local cohomology classes $s_\Psi \in H^1_{\mathrm{st}}(K_p, V_p)$ for every oriented optimal embedding $\Psi : \mathcal{O}_c \hookrightarrow \mathcal{R}$. We expect them to be global over H_c and we conjecture that they satisfy a reciprocity law that describes the Galois action of G_c on them. In addition, one further expects these classes to be related, via a Gross-Zagier formula, to the first derivative of $L(f \otimes K, s)$ at $s = k/2$. See §5 for precise statements. This provides a higher weight generalization of the theory of points due to Darmon [Dar] and continued in [Das], [Gr], [DG], [LRV] and [LRV2].

A fundamental difference of this construction when compared with Nekovář's approach is that these cohomology classes are not defined (at least not a priori) as the image of any cycles on $\mathrm{CH}^{m+1}(\mathcal{M}_n \otimes K_p)_0$.

Instead, letting $\mathbf{P}_n$ denote the space of polynomials of degree $\leq n$ in K_p, the role of the Chow group in our setting is played by the module $H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p)(K_p) \otimes \mathbf{P}_n)$. The choice of this module is motivated by the fact that one can naturally attach a 1-cycle y_Ψ to each optimal embedding Ψ, in a manner that is reminiscent of the p-adic construction of Heegner points for imaginary quadratic fields, and is a straightforward generalization of the points defined by M. Greenberg in [Gr] (cf. also [LRV2]). For this reason, the cycles y_Ψ may be called Stark-Heegner cycles (following loc. cit.) or also, as we suggest here, *Darmon cycles.*

Here, $\Gamma \subseteq (\mathcal{B} \otimes \mathbb{Q}_p)^\times$ is a group whose definition is recalled in 2 and already makes its appearance in classical works of Ihara and in [Dar]. The module $\mathrm{Div}(\mathcal{H}_p)(K_p)$ is the subgroup of divisors with coefficients on $\mathcal{H}_p(\bar{K}_p) := \bar{K}_p \setminus \mathbb{Q}_p$ that are invariant under the action of the Galois group $\mathrm{Gal}\,(\bar{K}_p/K_p)$.

We define s_Ψ as the image of y_Ψ by a composition of morphisms

$$H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p)(K_p) \otimes \mathbf{P}_n) \stackrel{\Phi^{\mathrm{AJ}}}{\longrightarrow} \frac{D \otimes K_p}{F^m(\mathbf{D} \otimes K_p)} \simeq \frac{D^{FM} \otimes K_p}{F^m(\mathbf{D}^{FM} \otimes K_p)} \simeq H^1_{\mathrm{st}}(K_p, V_p) \tag{7}$$

where the first map is introduced in (56) and should be regarded as an analogue of the p-adic Abel-Jacobi map; the second map is the isomorphism given by Theorem 4.7; the last map is the isomorphism provided by Bloch-Kato's exponential. Cf. 5.2 for more details.

The last section of this manuscript is devoted to the particular cases $k = 2$ in 6.1 and $N^- = 1$ in 6.2. For $k = 2$ we quickly review the work of [Gr] and [LRV], comparing it to our constructions. For $N^- = 1$ we rephrase the theory in the convenient language of modular symbols. This formulation is employed in [Se], where Conjecture 5.7 (iii) and Conjecture 5.8 are proved for suitable genus characters of K.

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2. Modular representations of quaternion algebras

2.1. **Quaternion algebras and Hecke modules.** Let $\mathcal{B}$ be a quaternion algebra over $\mathbb{Q}$ and let $N^- \geq 1$ denote its reduced discriminant. Let $b \mapsto \bar{b}$ denote the canonical anti-involution of $\mathcal{B}$ and write $\mathrm{Tr}(b) = b + \bar{b}$, $\mathrm{n}(b) = b\bar{b}$ for the reduced trace and norm of elements of $\mathcal{B}$.

Assume $\mathcal{B}$ is indefinite, that is, N^- is the square-free product of an even number of primes. Equivalently, there is an isomorphism $\iota_\infty : \mathcal{B} \otimes \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$, that we fix for the rest of the article.

For any primes ℓ, write $\mathcal{B}_\ell = \mathcal{B} \otimes \mathbb{Q}_\ell$ and fix isomorphisms $\iota_\ell : \mathcal{B}_\ell \simeq \mathrm{M}_2(\mathbb{Q}_\ell)$ for $\ell \nmid N^-$ and $\mathcal{B}_\ell \simeq \mathbb{H}_\ell$ for $\ell \mid N^-$. Here, $\mathbb{H}_\ell$ stands for a fixed choice of a division quaternion algebra over $\mathbb{Q}_\ell$, which is unique up to isomorphism. Throughout, for each place $l \leq \infty$ of $\mathbb{Q}$ we shall regard $\mathcal{B}$ as embedded in $\mathrm{M}_2(\mathbb{Q}_\ell)$ or $\mathbb{H}_\ell$ via the above fixed isomorphisms.

Let $N^+ \geq 1$ be a positive integer coprime to N^- and fix a prime $p \nmid N^+N^-$. Write $N = pN^+N^-$ and let $\mathcal{R}_0(pN^+) \subset \mathcal{R}_0(N^+)$ be Eichler orders in $\mathcal{B}$ of level pN^+ and N^+. Let $\Gamma_0(pN^+)$ (resp. $\Gamma_0(N^+)$) denote the subgroup of $\mathcal{R}_0(pN^+)^\times$ (resp. $\mathcal{R}_0(N^+)^\times$) of elements of reduced norm 1. Choose an element $\omega_p \in \mathcal{R}_0(pN^+)$ of reduced norm p normalizing $\Gamma_0(pN^+)$ and set $\hat{\Gamma}_0(N^+) := \omega_p \Gamma_0(N^+) \omega_p^{-1}$. In order to lighten the notation, there is no reference to the discriminant N^- in the symbols chosen to denote these orders and groups; this should cause no confusion, as the quaternion algebra $\mathcal{B}$ will always be fixed in our discussion.

Both $\Gamma_0(pN^+)$ and $\Gamma_0(N^+)$ are naturally embedded in $\mathrm{SL}_2(\mathbb{R})$ and act discretely and discontinuously on Poincaré's upper half-plane $\mathcal{H}$ through Möbius transformations, with compact quotient if and only if $N^- > 1$. Let $X_0^{N^-}(pN^+)$, resp. $X_0^{N^-}(N^+)$, denote Shimura's canonical model over $\mathbb{Q}$ of (the cuspidal compactification of, if $N^- = 1$) these quotients (cf. [Sh1, 9.2]).

For reasons that will become clear later, it will also be convenient to consider the Eichler $\mathbb{Z}[1/p]$-order $\mathcal{R} := \mathcal{R}_0(N^+)[1/p]$. Similarly as above, let Γ denote the subgroup of elements of reduced norm 1 of $\mathcal{R}^\times$. This group was first studied by Ihara and also makes an appearance in the works [Dar], [Das], [Gr] and [LRV].

If A is a module endowed with an action of $\mathcal{B}^\times$ and G is either $\Gamma_0(pN^+)$, $\Gamma_0(N^+)$, $\hat{\Gamma}_0(N^+)$ or Γ, the homology and cohomology groups $H_i(G, A)$ and $H^i(G, A)$ are naturally modules over a Hecke algebra

$$\mathcal{H}(G) := \mathbb{Z}[T_\ell : \ell \nmid N_G; U_\ell : \ell \mid N_G^+, W_\ell^- : \ell \mid N^-, W_p, W_\infty],$$

where $N_G^+ = pN^+$ for $G = \Gamma_0(pN^+), \Gamma$ and $N_G^+ = N^+$ otherwise, and $N_G = N_G^+ N^-$. If $A_1 \to A_2$ is a morphism of $\mathcal{B}^\times$-modules, the corresponding maps

$$H^i(G, A_1) \to H^i(G, A_2) \tag{8}$$

are then morphisms of $\mathcal{H}(G)$-modules. Cf. e.g. [AS, §1], [Gr, §3] and [LRV, §2] for details.

Choose an element $\omega_p \in \mathcal{R}_0(pN^+)$ (resp. ω_∞) of reduced norm p (resp. -1) that normalizes $\mathcal{R}_0(pN^+)$; such elements exist and are unique up to multiplication by elements of $\Gamma_0(pN^+)$. The operators W_p and W_∞ mentioned above are the (Atkin-Lehner) involutions defined as the double-coset operators attached to ω_p and ω_∞, respectively. For any $\mathbb{Z}[W_\infty]$-module A and sign $\epsilon \in \{\pm 1\}$ we set $A^\epsilon := A/(W_\infty - \epsilon)$. Up to 2-torsion, $A \simeq A^+ \oplus A^-$.

For any element γ in $\mathrm{GL}_2(\mathbb{Q}_p)$ or $\mathrm{GL}_2(\mathbb{R})$, write $\hat{\gamma} := \omega_p \gamma \omega_p^{-1}$. For any subgroup G of $\mathrm{GL}_2(\mathbb{Q}_p)$ or $\mathrm{GL}_2(\mathbb{R})$, write $\hat{G} = \{\hat{g}, g \in G\}$. Note that $\hat{\Gamma}_0(pN^+) = \Gamma_0(pN^+)$, $\hat{\Gamma} = \Gamma$, whereas $\hat{\Gamma}_0(N^+) \neq \Gamma_0(N^+)$. In fact,

$$\Gamma = \Gamma_0(N^+) \star_{\Gamma_0(pN^+)} \hat{\Gamma}_0(N^+) \tag{9}$$

is the amalgamated product of $\Gamma_0(N^+)$ with $\hat{\Gamma}_0(N^+)$ over $\Gamma_0(pN^+) = \Gamma_0(N^+) \cap \hat{\Gamma}_0(N^+)$.

2.2. **The Bruhat-Tits tree.** Let $\mathcal{T}$ denote Bruhat-Tits' tree attached to $\mathrm{PGL}_2(\mathbb{Q}_p)$, whose set $\mathcal{V}$ of vertices is the set of homothety classes of rank two $\mathbb{Z}_p$-submodules of $\mathbb{Q}_p^2$. Write $\mathcal{E}$ for the set of oriented edges of the tree. Given $e \in \mathcal{E}$, write $s(e)$ and $t(e)$ for the source and target of the edge, and $\bar{e}$ for the edge in $\mathcal{E}$ such that $s(\bar{e}) = t(e)$ and $t(\bar{e}) = s(e)$. Cf. e.g. [DT, §1.3.1] for more details.

Write v_*, $\hat{v}_*$ for the vertices associated with the standard lattice $L_* := \mathbb{Z}_p \times \mathbb{Z}_p$ and the lattice $\hat{L}_* := \mathbb{Z}_p \times p\mathbb{Z}_p$, respectively. Note that ω_p acts on $\mathcal{T}$, mapping v_* to $\hat{v}_*$. In general, for any vertex $v \in \mathcal{V}$, write $\hat{v} := \omega_p(v)$.

Let e_* be the edge with source $s(e_*) = v_*$ and $t(e_*) = \hat{v}_*$. Let $\mathcal{V}^+$ (resp. $\mathcal{V}^-$) denote the subset of vertices $v \in \mathcal{V}$ which lie at *even* (resp. *odd*) distance from v_*. Similarly, write $\mathcal{E}^+$ (resp. $\mathcal{E}^-$) for the subset of edges e in $\mathcal{E}$ such that $s(e) \in \mathcal{V}^+$ (resp. $\mathcal{V}^-$).

Let G be a subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ (as the ones already introduced in the previous section) and let A be any left G-module. For any set $\mathcal{S}$, e.g. $\mathcal{S} = \mathcal{V}$ or $\mathcal{E}$, write $C(\mathcal{S}, A)$ for the group of functions on $\mathcal{S}$ with values in A.

Let also $C_0(\mathcal{E}, A)$ be the subgroup of functions c in $C(\mathcal{E}, A)$ such that $c(\bar{e}) = -c(e)$ for all $e \in \mathcal{E}$, and

$$C_{har}(A) = \{c \in C_0(\mathcal{E}, A) : \sum_{s(e)=v} c(e) = 0 \quad \forall v \in \mathcal{V}\}$$

be the subgroup of A-valued *harmonic cocycles*. These groups are naturally endowed with a left action of G by the rule $({}^\gamma c)(e) := \gamma(\, c(\gamma^{-1}e)\,)$ and it is easy to see that they sit in the exact sequences (cf. [Gr, Lemma 24] for the first one):

$$\begin{array}{ccccccc} 0 & \to & C_{har}(A) & \to & C_0(\mathcal{E}, A) & \overset{\varphi}{\to} & C(\mathcal{V}, A) & \to & 0 \\ & & & & \varphi(c)(v) & := & \sum_{s(e)=v} c(e), & & \end{array} \tag{10}$$

$$\begin{array}{ccccccc} 0 & \to & A & \to & C(\mathcal{V}, A) & \overset{\partial^*}{\to} & C_0(\mathcal{E}, A) & \to & 0 \\ & & & & (\partial^* c)(e) & := & c(s(e)) - c(t(e)). & & \end{array} \tag{11}$$

2.3. **Rational representations.** In this section we recall a construction of a rational representation $\mathbb{V}_n$ of $\mathcal{B}^\times$ for each even integer $n \geq 0$ that already appears in [BDIS, §1.2] and [IS, §5]. Its relevance will be apparent in the next section, as according to the Eichler-Shimura isomorphism (cf. (15) below) the cohomology groups of $\mathbb{V}_n$ provide a natural rational structure for the spaces of holomorphic modular forms with respect to the arithmetic subgroups of $\mathcal{B}^\times$.

Let $\mathcal{B}_0 = \{b \in \mathcal{B}, \mathrm{Tr}(b) = 0\} \subset \mathcal{B}$, endowed with a right action of $\mathcal{B}^\times$ by the rule $b \cdot \beta := \beta^{-1} b \beta$ for $\beta \in \mathcal{B}^\times$ and $b \in \mathcal{B}_0$. The pairing

$$\langle b_1, b_2 \rangle := \frac{1}{2}\mathrm{Tr}(b_1 \cdot \bar{b}_2) \tag{12}$$

is non-degenerate and symmetric on $\mathcal{B}_0$, and allows to identify $\mathcal{B}_0$ with its own dual, whence to regard $\mathcal{B}_0$ as a left $\mathcal{B}^\times$-module.

For any $r \geq 0$, the r-th symmetric power $\mathrm{Sym}^r(\mathcal{B}_0)$ of $\mathcal{B}_0$ is naturally a left $\mathcal{B}^\times$-module endowed with the pairing induced by (12), that we continue to denote $\langle -, - \rangle$. For $r \geq 2$, the Laplace operator

$$\Delta_r : \ \mathrm{Sym}^r(\mathcal{B}_0) \ \to \ \mathrm{Sym}^{r-2}(\mathcal{B}_0)$$

attached to $\langle -, - \rangle$ is defined by the rule

$$\Delta_r(b_1 \cdot \ldots \cdot b_r) := \textstyle\sum_{1 \leq i < j = r} \langle b_i, b_j \rangle \, b_1 \cdot \ldots \cdot \widehat{b_i} \cdot \ldots \cdot \widehat{b_j} \cdot \ldots \cdot b_r.$$

The Laplace operator Δ_r is a morphism of $\mathcal{B}^\times$ -modules because, as one checks, $\langle b_1 \cdot \beta, b_2 \cdot \beta \rangle = \langle b_1, b_2 \rangle$ for all $\beta \in \mathcal{B}^\times$, $b_1, b_2 \in \mathcal{B}_0$.

Definition 2.1. Let $\mathbb{V}_0 = \mathbb{Q}$, $\mathbb{V}_2 = \mathcal{B}_0$ and, for any *even* integer $n \geq 4$, let $m := n/2$ and

$$\mathbb{V}_n := \ker \Delta_m.$$

If R is a commutative $\mathbb{Q}$-algebra, write $\mathbb{V}_n(R) := \mathbb{V}_n \otimes R$. For $n = 0$, $\mathbb{V}_0 = \mathbb{Q}$ is endowed simply with the trivial action of $\mathcal{B}^\times$. For arbitrary n, we may regard the spaces $\mathbb{V}_n$ both as right and left $\mathcal{B}^\times$-modules, the pairing $\langle -, - \rangle$ identifying one with another (cf. [BDIS, 1.2]). As such, the general theory reviewed in 2.1 and 2.2 applies in particular to these modules.

Over a base field $K/\mathbb{Q}$ that splits $\mathcal{B}$, the modules $\mathbb{V}_n(K)$ admit a much simpler and classical description, that we now review. For any even integer $n \geq 0$ let $\mathbf{P}_n$ denote the $\mathbb{Q}$-vector space of polynomials of degree at most n with rational coefficients, and write $\mathbf{P}_n(R) := \mathbf{P}_n \otimes R$ for any algebra R as above. It can be endowed with a right action of $\mathrm{GL}_2(R)$ by the rule

$$P(x) \cdot \gamma := \frac{(cx+d)^n}{\det(\gamma)^{n/2}} \cdot P\big(\frac{ax+b}{cx+d}\big), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \, P \in \mathbf{P}_n(R).$$

This way, $\mathbf{V}_n(R) = \mathbf{P}_n^\vee(R) := \mathrm{Hom}_R(\mathbf{P}_n(R), R)$, the dual of $\mathbf{P}_n(R)$, inherits a left $\mathrm{GL}_2(R)$-action, that actually descends to $\mathrm{PGL}_2(R)$.

Let K be a field of characteristic 0 such that $\mathcal{B} \otimes_{\mathbb{Q}} K \simeq \mathrm{M}_2(K)$, and identify these two algebras by fixing an isomorphism between them. The function

$$\begin{array}{ccc} \mathcal{B}_0 \otimes_{\mathbb{Q}} K & \longrightarrow & \mathbf{P_2}(K) \\ b & \mapsto & \mathrm{tr}(b \cdot \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix}) \end{array}$$

is an isomorphism of right $\mathcal{B}^\times$-modules. Identifying $\mathcal{B}_0$ with its own dual via (12), it induces an isomorphism of left $\mathcal{B}^\times$-modules (we omit the details; cf. [BDIS, 1.2], where the definitions of the pairings and actions are the same as the ones taken here, and [IS, 5], [JL, 2]):

$$\mathbb{V}_n(K) \simeq \mathbf{V}_n(K). \tag{13}$$

Notice that we already fixed in 2.1 isomorphisms $\iota_\ell : \mathcal{B} \otimes \mathbb{Q}_\ell \simeq \mathrm{M}_2(\mathbb{Q}_\ell)$ for places $l \leq \infty$, $l \nmid N^-$. Accordingly, in the sequel we shall freely identify $\mathbb{V}_n(\mathbb{Q}_\ell)$ with $\mathbf{V}_n(\mathbb{Q}_\ell)$.

2.4. **Modular forms and the Eichler-Shimura isomorphism.** For any even integer $n \geq 0$ set $k = n + 2 = 2(m+1)$. Let G denote $\Gamma_0(pN^+)$, $\Gamma_0(N^+)$, $\hat{\Gamma}_0(N^+)$ or Γ.

Definition 2.2. Let M be a $\mathcal{H}(G)$-module and fix a prime $\ell \nmid N_G$. We say that M admits an *ℓ-Eisenstein/Cuspidal decomposition* (of weight k) whenever there exists a decomposition of $\mathcal{H}(G)$-modules $M = M^{\mathrm{Eis}} \oplus M^c$ such that $t_\ell := T_\ell - \ell^{k-1} - 1$ vanishes on M^{Eis} and is invertible on M^c.

Remark 2.3. If such decomposition exists, it is easy to check that it is unique.

Furthermore, let M_i, $i = 1, 2$, be $\mathcal{H}(G_i)$-modules, where G_1, G_2 is any choice of groups in either the set $\{\Gamma_0(pN^+), \Gamma\}$ or the set $\{\Gamma_0(N^+), \hat{\Gamma}_0(N^+)\}$. Let $f : M_1 \to M_2$ be a morphism that is equivariant for the action of the good Hecke operators of $\mathcal{H}(G_1)$ and $\mathcal{H}(G_2)$. If both M_1 and M_2 admit an ℓ-Eisenstein/Cuspidal decomposition for some $\ell \nmid N_{G_1}N_{G_2}$, then f decomposes accordingly as $f = f^{\mathrm{Eis}} \oplus f^c$. In particular, $\ker(f)$ and $\mathrm{coker}(f)$ also admit an ℓ-Eisenstein/Cuspidal decomposition.

Finally, if

$$0 \to M_1 \to M_2 \to M_3 \to M_4 \to M_5 \to 0$$

is a Hecke equivariant exact sequence of Hecke modules such that M_1, M_2, M_4 and M_5 admit an ℓ-Eisenstein/Cuspidal decomposition for a given prime ℓ, then so does M_3.

In all the instances of $\mathcal{H}(G)$-modules M we shall be considering, the ℓ-Eisenstein/Cuspidal decomposition is in fact independent of the choice of the prime $\ell \nmid N_G$, and we shall simply refer to it as the Eisenstein/Cuspidal decomposition of M, dropping the prime ℓ from the notations.

Let now G denote either $\Gamma_0(pN^+)$ or $\Gamma_0(N^+)$. Let $M_k(G)$ denote the $\mathbb{C}$-vector space of weight k holomorphic modular forms with respect to G and $S_k(G)$ denote its cuspidal subspace. Let $\mathbb{T}_G$ (resp. $\tilde{\mathbb{T}}_G$) be the maximal quotient of the Hecke algebra $\mathcal{H}(G) \otimes \mathbb{Q}$ that acts faithfully on $S_k(G)$ (resp. on $M_k(G)$).

As a basic example, $M = M_k(G)$ admits an Eisenstein/Cuspidal decomposition with $M^c = S_k(G)$ and $M^{\mathrm{Eis}} = E_k(G)$, the space of modular forms generated by the Eisenstein series. These series are only defined for $N^- = 1$; in order to have uniform notations, we set this space to be $\{0\}$ when $N^- > 1$.

By [Sh1, Theorem 3.51] and the Jacquet-Langlands correspondence,

$$\dim_{\mathbb{C}} S_k(G) = \dim_{\mathbb{Q}} \mathbb{T}_G \tag{14}$$

and in fact $S_k(G)$ is a free module of rank one over $\mathbb{T}_G \otimes \mathbb{C}$.

The Eichler-Shimura isomorphism yields an identification of exact sequences (see [Hi, Ch. 6] and [Fr, Ch. III])

$$\begin{array}{ccccccccc} 0 & \to & S_k(G) \otimes_{\mathbb{R}} \mathbb{C} & \to & (S_k(G) \otimes_{\mathbb{R}} \mathbb{C}) \oplus E_k(G) & \to & E_k(G) & \to & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \to & H^1_{\mathrm{par}}(G, \mathbf{V}_n(\mathbb{C})) & \to & H^1(G, \mathbf{V}_n(\mathbb{C})) & \stackrel{\mathrm{res}}{\to} & H^1_{\mathrm{Eis}}(G, \mathbf{V}_n(\mathbb{C})) & \to & 0, \end{array} \tag{15}$$

where $H^1_{\mathrm{Eis}}(G, \mathbf{V}_n(\mathbb{C}))$ is the image of the restriction map

$$H^1(G, \mathbf{V}_n(\mathbb{C})) \longrightarrow \bigoplus_{i=1}^{t} H^1(G_{s_i}, \mathbf{V}_n(\mathbb{C})). \tag{16}$$

Here, $C_G = \{s_1, ..., s_t\}$ denotes a set of representatives for the cusps of G and for any $s \in C_G$, G_s denotes the stabilizer of s in G.

Thanks to (13) and to the theorem of Universal Coefficients, there is an isomorphism of Hecke modules

$$H^1(G, \mathbf{V}_n(\mathbb{C})) \simeq H^1(G, \mathbb{V}_n) \otimes \mathbb{C}. \tag{17}$$

Note that the map (16) can in fact be viewed as the base change to $\mathbb{C}$ of the natural restriction map

$$H^1(G, \mathbb{V}_n) \longrightarrow \bigoplus_{i=1}^{t} H^1(G_{s_i}, \mathbb{V}_n) \tag{18}$$

As a point of caution, the reader may notice that when $N^- > 1$ the cohomology groups appearing in (18) make no sense if we replace the module of coefficients $\mathbb{V}_n$ by $\mathbf{V}_n$. The kernel and image of (18) can thus be taken as the definition of $H^1_{\mathrm{par}}(G, \mathbb{V}_n)$ and $H^1_{\mathrm{Eis}}(G, \mathbb{V}_n)$, respectively. Their direct sum yields an Eisenstein/Cuspidal decomposition of $H^1(G, \mathbb{V}_n)$.

Remark 2.4. It thus follows from (14), (15) and (17) that $H^1(G, \mathbb{V}_n)^c$ is a free module of rank two over $\mathbb{T}_G$. More precisely, since $\mathrm{n}(\omega_\infty) = -1$, it follows from the work [Sh3] that the Atkin-Lehner involution W_∞ acts on $H^1(G, \mathbb{V}_n(\mathbb{C}))^c$ as complex conjugation. Hence, for each choice of sign $\epsilon \in \{\pm 1\}$, $H^1(G, \mathbb{V}_n)^{c,\epsilon}$ is a module of rank one over $\mathbb{T}_G$.

Remark 2.5. If $f \in S_k(G)$ is a primitive normalized eigenform for the action of $\mathbb{T}_G$, it corresponds via the Eichler-Shimura isomorphism to an element $c_f \in H^1(G, \mathbb{V}_n(L_f))^c$, where L_f is the number field generated over $\mathbb{Q}$ by the eigenvalues of f. This is a consequence of multiplicity one and the fact that $H^1(G, \mathbb{V}_n)^c$ is a rational structure for $S_k(G)$ that is preserved by the action of the Hecke algebra. Hence, if K is a field that contains all the eigenvalues for the action of the Hecke operators, then $H^1(G, \mathbb{V}_n(K))^c$ admits a basis of eigenvectors for this action.

Remark 2.6. For all $i \geq 0$, the spaces $H^i(G, \mathbb{V}_n)$ also admit an Eisenstein/Cuspidal decomposition. For $i = 1$ this is the content of the above discussion. For $i > 2$, these groups vanish because the cohomological dimension of G is 2.

For $i = 0$: if $n > 0$, $H^0(G, \mathbb{V}_0) = \{0\}$ by [Hi, p. 162, Prop. 1; p. 165, Lemma 2] and there is nothing to prove; if $n = 0$, the action of the Hecke operators T_ℓ for $\ell \nmid N_G$ is given by multiplication by $\ell + 1$ and therefore

$$H^0(G, \mathbb{V}_0)^c = \{0\}. \tag{19}$$

For $i = 2$: $H^2(G, \mathbb{V}_n) \simeq H^0_c(G, \mathbb{V}_n)^\vee$ by Poincaré duality and the paragraph above applies. Here, the latter group stands for the cohomology group with compact support of G with coefficients on $\mathbb{V}_n$. See e.g.[Fr, Ch. III] and [MS] for more details.

Definition 2.7. Let

$$\begin{array}{rlcl} \mathrm{cor}: & H^1(\Gamma_0(pN^+), \mathbb{V}_n) & \to & H^1(\Gamma_0(N^+), \mathbb{V}_n) \\ \hat{\mathrm{cor}}: & H^1(\Gamma_0(pN^+), \mathbb{V}_n) & \to & H^1(\hat{\Gamma}_0(N^+), \mathbb{V}_n) \end{array}$$

denote the corestriction maps induced by the inclusions $\Gamma_0(pN^+) \subset \Gamma_0(N^+), \hat{\Gamma}_0(N^+)$ and let

$$H^1(\Gamma_0(pN^+), \mathbb{V}_n)^{p-new} := \mathrm{Ker}(\mathrm{cor} \oplus \hat{\mathrm{cor}}).$$

Similarly, we may define $H^1(\Gamma_0(pN^+), \mathbb{V}_n)^{p-old,c} := \mathrm{Im}(\mathrm{res} + \hat{\mathrm{res}})^c$, where

$$H^1(\Gamma_0(N^+), \mathbb{V}_n) \oplus H^1(\Gamma_0(N^+), \mathbb{V}_n) \overset{\mathrm{res} + \hat{\mathrm{res}}}{\longrightarrow} H^1(\Gamma_0(pN^+), \mathbb{V}_n)$$

is the sum of the natural restriction maps.

Obviously, over $\mathbb{C}$ the above corestriction maps admit a parallel description purely in terms of modular forms and degeneracy maps, via (15). Via the above identifications, the Petersson inner product induces on $H^1(\Gamma_0(pN^+), \mathbf{V}_n(\mathbb{C}))^c$ a perfect pairing with respect to which $H^1(\Gamma_0(pN^+), \mathbf{V}_n(\mathbb{C}))^{p-old,c}$ is the orthogonal complement of $H^1(\Gamma_0(pN^+), \mathbf{V}_n(\mathbb{C}))^{p-new,c}$.

2.5. **The cohomology of Γ.** Besides the relationship between $H^1(\Gamma_0(pN^+),\mathbb{V}_n)$ and modular forms provided by the Eichler-Shimura isomorphism, these groups can also be related to the cohomology of the group Γ introduced in 2.1 with values in the modules of functions on Bruhat-Tits's tree $\mathcal{T}_p$, as we now review.

The long exact sequence in cohomology arising from (10) with $A=\mathbb{V}_n$ gives rise to an exact sequence of $\mathcal{H}(\Gamma)$-modules (cf. 2.1)

$$\to H^0(\Gamma,C(\mathcal{V},\mathbb{V}_n))\to H^1(\Gamma,C_{har}(\mathbb{V}_n))\stackrel{s}{\to} H^1(\Gamma,C_0(\mathcal{E},\mathbb{V}_n))\ \to\ H^1(\Gamma,C(\mathcal{V},\mathbb{V}_n)). \tag{20}$$

By Shapiro's lemma, for all $i\geq 0$ there are isomorphisms

$$\begin{array}{rlcl} \mathscr{S}_{\mathcal{V}}: & H^i(\Gamma,C(\mathcal{V},\mathbb{V}_n)) & \simeq & H^i(\Gamma_0(N^+),\mathbb{V}_n)^2, \\ \mathscr{S}_{\mathcal{E}}: & H^i(\Gamma,C_0(\mathcal{E},\mathbb{V}_n)) & \simeq & H^i(\Gamma_0(pN^+),\mathbb{V}_n), \end{array} \tag{21}$$

where throughout, by a slight abuse of notation, by $H^i(\Gamma_0(N^+),\mathbb{V}_n)^2$ we actually mean $H^i(\Gamma_0(N^+),\mathbb{V}_n)\oplus H^i(\hat{\Gamma}_0(N^+),\mathbb{V}_n)$. Note that conjugation by ω_p induces a canonical isomorphism

$$H^i(\Gamma_0(N^+),\mathbb{V}_n)\simeq H^i(\hat{\Gamma}_0(N^+),\mathbb{V}_n) \tag{22}$$

that we will sometimes use in order to identify these two spaces without further comment.

These isomorphisms are Hecke-equivariant in the following sense: for every prime $\ell\nmid pN^+N^-$, $T_\ell\circ\mathscr{S}_{\mathcal{V}}=\mathscr{S}_{\mathcal{V}}\circ T_\ell$ and $T_\ell\circ\mathscr{S}_{\mathcal{E}}=\mathscr{S}_{\mathcal{E}}\circ T_\ell$. Although we are using the same symbol for the Hecke operator at ℓ acting on the two cohomology groups, note that they lie in the two different Hecke algebras $\mathcal{H}(\Gamma)$ and $\mathcal{H}(\Gamma_0(N^+))$ (resp., $\mathcal{H}(\Gamma_0(pN^+))$). The compatibility with the isomorphism $\mathscr{S}$ follows from the key fact that T_ℓ can be defined in $\mathcal{H}(G)$ for $G=\Gamma_0(pN^+),\Gamma_0(N^+),\Gamma$ as a double-coset operator by means of the *same* choices of local representatives. Cf. [Das, Prop. A.1], [LRV, §2.3] for more details.

Remark 2.3 and the isomorphisms of (21) can be used to define an Eisenstein/Cuspidal decomposition on $H^1(\Gamma,C_0(\mathcal{E},\mathbb{V}_n))$, $H^1(\Gamma,C(\mathcal{V},\mathbb{V}_n))$ and $H^1(\Gamma,C_{har}(\mathbb{V}_n))$ by transporting it from $H^1(G,\mathbb{V}_n)$, where $G=\Gamma_0(pN^+)$, $\Gamma_0(N^+)$ or $\hat{\Gamma}_0(N^+)$.

Lemma 2.8. *There is a Hecke equivariant isomorphism*

$$H^1(\Gamma,C_{har}(\mathbb{V}_n))^c\stackrel{\simeq}{\to} H^1(\Gamma_0(pN^+),\mathbb{V}_n)^{p-new,c}.$$

Proof. Composing the map s in (20) with Shapiro's isomorphism $\mathscr{S}_{\mathcal{E}}$ in (21), we obtain a map

$$H^1(\Gamma,C_{har}(\mathbb{V}_n))\stackrel{\mathscr{S}_{\mathcal{E}}\circ s}{\longrightarrow} H^1(\Gamma_0(pN^+),\mathbb{V}_n), \tag{23}$$

that we already argued to be Hecke equivariant. By Definition 2.7 and (21), $\mathscr{S}_{\mathcal{E}}\circ s$ maps surjectively onto $H^1(\Gamma_0(pN^+),\mathbb{V}_n)^{p-new}$. By [Hi, p. 165], the $\Gamma_0(pN^+)$-module $\mathbf{V}_n(\mathbb{C})$ is irreducible for $n>0$. Hence $\mathbb{V}_n^{\Gamma_0(pN^+)}=0$ by (13); since $\Gamma_0(pN^+)\subset\Gamma_0(N^+),\hat{\Gamma}_0(N^+)$, the proposition now follows for $n>0$ from the exactness of (20), and (21).

When $n=0$, the action of $\mathcal{B}^\times$ on $\mathbb{V}_n$ is trivial, whence $\mathbb{V}_n^{\Gamma_0(pN^+)}=\mathbb{V}_n$. Since $H^0(G,\mathbb{V}_n)^c=\{0\}$ both for $G=\Gamma_0(N^+)$ and $\hat{\Gamma}_0(N^+)$ by (19), the proposition follows as before. □

The long exact sequence in cohomology arising from (11) with $A=\mathbf{V}_n(K_p)$ is

$$\begin{array}{c} \to H^1(\Gamma_0(N^+),\mathbf{V}_n(K_p))^2\to H^1(\Gamma_0(pN^+),\mathbf{V}_n(K_p))\stackrel{\delta}{\to} \\ \stackrel{\delta}{\to} H^2(\Gamma,\mathbf{V}_n(K_p))\stackrel{\varepsilon}{\to} H^2(\Gamma_0(N^+),\mathbf{V}_n(K_p))^2\to H^2(\Gamma_0(pN^+),\mathbf{V}_n(K_p)), \end{array} \tag{24}$$

once we apply the isomorphisms of (21). Exactly as in (20), all maps in (24) are Hecke equivariant and admit an Eisentein/Cuspidal decomposition.

Lemma 2.9. *The boundary map δ^c restricts to an isomorphism*

$$\delta^c: H^1(\Gamma_0(pN^+), \mathbf{V}_n(K_p))^{p-new,c} \stackrel{\simeq}{\to} H^2(\Gamma, \mathbf{V}_n(K_p))^c.$$

Proof. We have $H^2(\Gamma_0(N^+), \mathbf{V}_n(K_p))^c = 0$ by the remarks following (19). Remark 2.3 implies that *taking cuspidal parts* in an exact functor. It thus follows from (24) that there is an exact sequence

$$(H^1(\Gamma_0(N^+), \mathbf{V}_n(K_p))^c)^2 \to H^1(\Gamma_0(pN^+), \mathbf{V}_n(K_p))^c \stackrel{\delta^c}{\to} H^2(\Gamma, \mathbf{V}_n(K_p))^c \to 0. \tag{25}$$

The lemma now follows from the canonical decomposition

$$H^1(\Gamma_0(pN^+), \mathbf{V}_n(K_p))^c = H^1(\Gamma_0(pN^+), \mathbf{V}_n(K_p))^{p-old,c} \oplus H^1(\Gamma_0(pN^+), \mathbf{V}_n(K_p))^{p-new,c}.$$

□

2.6. **Morita-Teitelbaum's integral representations.** Recall the PGL_2-module $\mathbf{P}_n$ and note that it admits as a natural $\mathbb{Z}$-structure the free $\mathbb{Z}$-module $\mathbf{P}_{n,\mathbb{Z}}$ of polynomials of degree at most n with integer coefficients. For any $\mathbb{Z}$-algebra R, $\mathbf{P}_n(R) := \mathbf{P}_{n,\mathbb{Z}} \otimes R$ is endowed with a right $\mathrm{PGL}_2(R)$-action by the same formula. This way, $\mathbf{V}_n(R) = \mathbf{P}_n^\vee(R) := \mathrm{Hom}_R(\mathbf{P}_n(R), R)$, the dual of $\mathbf{P}_n(R)$, inherits a left $\mathrm{PGL}_2(R)$-action.

For any vertex $v \in \mathcal{V}$, choose any element $\gamma_v \in \mathrm{GL}_2(\mathbb{Q}_p)$ such that $\gamma_v(v) = v_*$ and set

$$\mathbf{V}_{n,v}(\mathbb{Z}_p) := \gamma_v^{-1} \cdot \mathbf{V}_n(\mathbb{Z}_p) \subset \mathbf{V}_n(\mathbb{Q}_p).$$

Notice that this definition does not depend on the choice of γ_v, because the stabilizer of v_* in $\mathrm{PGL}_2(\mathbb{Q}_p)$ is $\mathrm{PGL}_2(\mathbb{Z}_p)$, which leaves $\mathbf{V}_n(\mathbb{Z}_p)$ invariant. Define

$$C^{int}(\mathcal{V}, \mathbf{V}_n(\mathbb{Q}_p)) := \{c \in C(\mathcal{V}, \mathbf{V}_n(\mathbb{Q}_p)) : c(v) \in \mathbf{V}_{n,v}\, \forall v \in \mathcal{V}\}.$$

Similarly, for any oriented edge $e = (v, v') \in \mathcal{E}$, set $\mathbf{V}_{n,e}(\mathbb{Z}_p) := \mathbf{V}_{n,v}(\mathbb{Z}_p) \cap \mathbf{V}_{n,v'}(\mathbb{Z}_p) \subset \mathbf{V}_n(\mathbb{Q}_p)$ and define

$$C^{int}(\mathcal{E}, \mathbf{V}_n(\mathbb{Q}_p)) := \{c \in C(\mathcal{E}, \mathbf{V}_n(\mathbb{Q}_p)) : c(e) \in \mathbf{V}_{n,e}(\mathbb{Z}_p)\, \forall e \in \mathcal{E}\},$$

that is naturally a $\mathbb{Z}_p$-module. Introduce also the $\mathbb{Z}_p$-modules

$$C_0^{int}(\mathcal{E}, \mathbf{V}_n(\mathbb{Q}_p)) := C^{int}(\mathcal{E}, \mathbf{V}_n(\mathbb{Q}_p)) \cap C_0(\mathcal{E}, \mathbf{V}_n(\mathbb{Q}_p))$$

and

$$C_{har}^{int}(\mathcal{E}, \mathbf{V}_n(\mathbb{Q}_p)) := C^{int}(\mathcal{E}, \mathbf{V}_n(\mathbb{Q}_p)) \cap C_{har}(\mathbf{V}_n(\mathbb{Q}_p)).$$

The next result of Teitelbaum should be regarded as a refinement of (10).

Proposition 2.10. [Te2, p. 564-566] *For every* even *integer $n \geq 0$ the natural sequence*

$$0 \to C_{har}^{int}(\mathbf{V}_n(\mathbb{Q}_p)) \to C_0^{int}(\mathcal{E}, \mathbf{V}_n(\mathbb{Q}_p)) \to C^{int}(\mathcal{V}, \mathbf{V}_n(\mathbb{Q}_p)) \to 0 \tag{26}$$

is an exact sequence of $\mathrm{PGL}_2(\mathbb{Q}_p)$*-modules.*

In particular we may regard the above sequence as an exact sequence of Γ-modules by means of the identification $\iota_p : \mathcal{B}_p \simeq \mathrm{M}_2(\mathbb{Q}_p)$. As a piece of notation, by *extended norm* on a space A we mean a function $\|\cdot\| : A \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ satisfying the usual properties of a norm, extended in a natural way to the semigroup of values $\mathbb{R}_{\geq 0} \cup \{+\infty\}$.

Let $K_p/\mathbb{Q}_p$ be a finite field extension, with ring of integers R_p, that we fix for the remainder of this article.

Let $|\cdot|$ denote the absolute value of K_p. Let $|\cdot|_{L_*}$ and $|\cdot|_{\widehat{L}_*}$ be two norms on $\mathbf{P}_n(K_p)$. We require the first one to be $\mathrm{GL}(L_*) = \mathrm{GL}_2(\mathbb{Z}_p)$-invariant and the second one to be $\mathrm{GL}(\widehat{L}_*)$-invariant (cf. 2.2 for notations). Choose also a $\mathrm{GL}(L_*) \cap \mathrm{GL}(\widehat{L}_*)$-invariant norm $|\cdot|_{L_*,\widehat{L}_*}$ on $\mathbf{P}_n(K_p)$. We may choose for example $|\cdot|_{L_*,\widehat{L}_*} = |\cdot|_{L_*} := |\cdot|$ to be the supremum of the absolute values of the coefficients of a polynomial and set $|\cdot|_{\widehat{L}_*} := \left|\omega_p^{-1}\cdot\right|_{L_*}$. By duality we can consider the corresponding norms on $\mathbf{V}_n(K_p)$.

Define extended norms on $C(\mathcal{V}^+, \mathbf{V}_n(K_p))$, $C(\mathcal{V}^-, \mathbf{V}_n(K_p))$ and $C_0(\mathcal{E}, \mathbf{V}_n(K_p))$ by the rules

$$\|c\|_+ := \sup_{v\in\mathcal{V}^+} |\gamma_v \cdot c(v)|_{L_*},$$

$$\|c\|_- := \sup_{v\in\mathcal{V}^-} |\gamma_v \cdot c(v)|_{L_*}.$$

$$\|c\| := \sup_{e\in\mathcal{E}^+} |\gamma_e \cdot c(e)|.$$

Here, γ_v (resp. γ_e) is any element in $\mathrm{GL}_2(\mathbb{Q}_p)$ such that $\gamma_v(v) = v_*$ (resp. $\gamma_e(e) = e_*$). The invariance properties of the above norms imply that the above definitions do not depend on the choice of the sets $\{\gamma_v\}$ and $\{\gamma_e\}$. Use $\|\cdot\|_+$ and $\|\cdot\|_-$ to define a norm on $C(\mathcal{V}, \mathbf{V}_n(K_p))$ as the max. of the two.

Let $C(K_p)$ denote either $C(\mathcal{V}, \mathbf{V}_n(K_p)) := C(\mathcal{V}^+, \mathbf{V}_n(K_p)) \oplus C(\mathcal{V}^-, \mathbf{V}_n(K_p))$, $C_0(\mathcal{E}, \mathbf{V}_n(K_p))$ or $C_{har}(\mathbf{V}_n(K_p))$ and write $C^b(K_p) := \{c \in C(K_p) : \|c\| < \infty\}$. The restriction of $\|\cdot\|$ to $C^b(K_p)$ is a norm with respect to which $C^b(K_p)$ is a Banach space over K_p.

Lemma 2.11. (i) $C^b(K_p) = C^b(\mathbb{Q}_p)\hat{\otimes} K_p$.
(ii) $C^b(\mathbb{Q}_p) = C^{int}(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Proof. We sketch a proof only for $\mathcal{V}$, as the remaining cases work similarly. For (i) we first invoke the following general fact: suppose that L/F is a finite Galois extension and that V is a L-module on which $G_{L/F}$ acts semilineraly, i.e., $\sigma(\lambda v) = \sigma(\lambda)\sigma(v)$ for all $\sigma \in G_{L/F}$, $\lambda \in L$, $v \in V$. Then for every subgroup $H \subset G_{L/F}$ the map

$$L^H \otimes_F \left(V^{G_{L/F}}\right) \to V^H, \quad l \otimes v \mapsto lv \tag{27}$$

is an isomorphism. For the trivial subgroup $H = \{1\}$ this is [Mi, Prop. 16.14], and for arbitrary H it follows from that statement by taking H-invariants.

To derive (i) from this, let $L_p \supset K_p$ be a field extension such that $L_p/\mathbb{Q}_p$ is Galois and let $H = G_{L_p/K_p} \subset G_{L_p/\mathbb{Q}_p}$. Endow $C(L_p)$ with the action of $G_{L_p/\mathbb{Q}_p}$ given by the rule $(\sigma c)(s) := \sigma(c(s))$. This action is easily checked to be well-defined and semilinear; moreover $C(L_p)^{G_{L_p/\mathbb{Q}_p}} = C(\mathbb{Q}_p)$, $C(L_p)^H = C(K_p)$. Note also that, since $\sigma(\gamma_s) = \gamma_s \in \mathbf{GL}_2(\mathbb{Q}_p)$,

$$\|\sigma c\| = \sup_s |\gamma_s \sigma(c(s))| = \sup_s |\sigma(\gamma_s c(s))| = \sup_s |\gamma_s c(s)| = \|c\|.$$

It follows that $G_{L_p/\mathbb{Q}_p}$ acts semilinearly on $C^b(L_p)$ and that $C^b(L_p)^{G_{L_p/\mathbb{Q}_p}} = C^b(\mathbb{Q}_p)$, $C^b(L_p)^H = C^b(K_p)$. We can apply now (27) to $V = C^b(L_p)$, which shows (i).

As for (ii), the inclusion $C^{int}(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \subset C^b(\mathbb{Q}_p)$ is obvious: given $c \in C^{int}(\mathbb{Q}_p)$, $\|c\| = \sup_v |\gamma_v \cdot c(v)|$ is bounded because $\gamma_v \cdot c(v) \in \mathbf{V}_n(\mathbb{Z}_p)$. As for the opposite inclusion, let $c \in C(\mathbb{Q}_p)$ be such that $\|c\| = B < \infty$. Then c can be replaced by a scalar multiple of it such in a way that $\|c\| = \sup_v |\gamma_v \cdot c(v)| \le 1$. This implies that $c \in C^{int}(\mathbb{Q}_p)$. □

The next corollary now follows from Proposition 2.10 and Lemma 2.11.

Corollary 2.12. *For every* even *integer $n \ge 0$ the natural sequence*

$$0 \to C^b_{har}(\mathbf{V}_n(K_p)) \to C^b_0(\mathcal{E}, \mathbf{V}_n(K_p)) \to C^b(\mathcal{V}, \mathbf{V}_n(K_p)) \to 0 \tag{28}$$

is an exact sequence of $\mathrm{PGL}_2(\mathbb{Q}_p)$*-modules.*

Again we may regard the above sequence as an exact sequence of Γ-modules by means of the identification $\iota_p : \mathcal{B}_p \simeq \mathrm{M}_2(\mathbb{Q}_p)$.

3. p-ADIC INTEGRATION AND AN $\mathcal{L}$-INVARIANT

3.1. **The cohomology of distributions and harmonic cocyles.** Let $\mathcal{H}_p$ denote the p-adic upper half-plane over $\mathbb{Q}_p$. It is a rigid analytic variety over $\mathbb{Q}_p$ such that $\mathcal{H}_p(K_p) = K_p \setminus \mathbb{Q}_p$. Let $\mathcal{O}_{\mathcal{H}_p}$ denote the ring of entire functions on $\mathcal{H}_p$, that is a Fréchet space over $\mathbb{Q}_p$ (cf. [DT, Prop. 1.2.6]). For any even integer $k \geq 2$, write $\mathcal{O}_{\mathcal{H}_p}(k)$ for the ring $\mathcal{O}_{\mathcal{H}_p}$ equipped with a right action of $\mathrm{GL}_2(\mathbb{Q}_p)$ given by

$$f|\gamma = \frac{\det(\gamma)^{k/2}}{(cz+d)^k} \cdot f(\gamma z), \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, f \in \mathcal{O}_{\mathcal{H}_p}.$$

Let $\mathcal{H}_p^{int}$ denote the formal scheme over $\mathbb{Z}_p$ introduced by Mumford in [Mu] (cf. also [Te2, p. 567]). The rigid analytic space associated with its generic fiber is $\mathcal{H}_p$. The dual graph of its special fiber is the tree $\mathcal{T}_p$ (cf. [DT] and [Te2] for a detailed discussion).

Let $\omega_{\mathbb{Z}_p}$ denote the sheaf on $\mathcal{H}_p^{int}$ introduced in [Te2, Def. 10], such that $\omega_{\mathbb{Z}_p} \otimes \mathbb{Q}_p$ is the sheaf ω of rigid analytic differential forms on $\mathcal{H}_p$. The map $f(z) \mapsto f(z)dz^{k/2}$ induces an isomorphism of $\mathrm{PGL}_2(\mathbb{Q}_p)$-modules between $\mathcal{O}_{\mathcal{H}_p}(k)$ and $H^0(\mathcal{H}_p, \omega^{k/2})$. Set

$$\mathcal{O}^b_{\mathcal{H}_p}(k) := H^0(\mathcal{H}_p^{int}, \omega_{\mathbb{Z}_p}^{k/2}) \cdot dz^{-k/2} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \subset \mathcal{O}_{\mathcal{H}_p}(k).$$

Recall that $n := k-2 \geq 0$. As follows from e.g. [DT, §2.2.4] or [Te2, Theorem 15], the residue map on $\mathcal{O}_{\mathcal{H}_p}(k)$ yields an epimorphism of $\mathrm{PGL}_2(\mathbb{Q}_p)$-modules

$$\mathrm{Res} : \mathcal{O}_{\mathcal{H}_p}(k) \twoheadrightarrow C_{har}(\mathbf{V}_n(\mathbb{Q}_p)). \tag{29}$$

The following deep result is proved in [Te2, p. 569-574], and will be crucial for our purposes.

Proposition 3.1. *The map* Res *restricts to an isomorphism of* $\mathrm{PGL}_2(\mathbb{Q}_p)$*-modules*

$$\mathrm{Res} : \mathcal{O}^b_{\mathcal{H}_p}(k) \xrightarrow{\sim} C^b_{har}(\mathbf{V}_n(\mathbb{Q}_p)). \tag{30}$$

Definition 3.2. Let $\mathcal{A}_n(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ be the space of K_p-valued locally analytic functions on $\mathbb{P}^1(\mathbb{Q}_p)$ with a pole of order at most n at ∞. More precisely, an element $f \in \mathcal{A}_n$ is a locally analytic function $f : \mathbb{Q}_p \to K_p$ for which there exists an integer N such that f is locally analytic on $\{z \in \mathbb{Q}_p : \mathrm{ord}_p(x) \geq N\}$ and admits a convergent expansion

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0 + \sum_{r \geq 1} a_{-r} z^{-r}$$

on $\{z \in \mathbb{Q}_p : \mathrm{ord}_p(z) < N\}$.

The space $\mathcal{A}_n(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ carries a right action of $\mathrm{GL}_2(\mathbb{Q}_p)$ defined by the rule

$$(f \cdot \gamma)(x) = \frac{(cx+d)^n}{\det(\gamma)^{n/2}} \cdot f(\frac{ax+b}{cx+d})$$

for any $f \in \mathcal{A}_n(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$. Note that $\mathbf{P}_n(K_p)$ is a natural $\mathrm{GL}_2(\mathbb{Q}_p)$-submodule of it.

Definition 3.3. Write $\mathcal{D}_n(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ and $\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ for the strong continuous dual of $\mathcal{A}_n(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ and of its quotient by $\mathbf{P}_n(K_p)$, respectively.

These modules of distributions inherit from $\mathcal{A}_n(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ a left action of $\mathrm{GL}_2(\mathbb{Q}_p)$. As explained in [DT, §2.1.1], $\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ is a Fréchet space over K_p and $\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p), K_p) = \mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Q}_p) \hat{\otimes} K_p$.

Morita's (or sometimes also called Schneider-Teitelbaum) duality, yields an isomorphism $\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Q}_p) \xrightarrow{\sim} \mathcal{O}_{\mathcal{H}_p}(k)$ that induces an isomorphism

$$I : \mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p), K_p) \xrightarrow{\sim} \mathcal{O}_{\mathcal{H}_p}(k) \hat{\otimes} K_p. \tag{31}$$

We refer the reader to [DT, §2.2] for more details; in the sequel, we shall freely identify these two spaces. Define

$$\mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b := I^{-1}(\mathcal{O}^b_{\mathcal{H}_p}(k)\hat{\otimes}K_p) \subset \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p).$$

Remark 3.4. As a consequence of a variant of the theorem of Amice-Velu-Vishik (cf. e.g. [DT, Theorem 2.3.2]), the space $\mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b$ can alternatively be described as the subspace of distributions $\mu \in \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ for which there is a constant A such that, for $i \geq 0$, $j \geq 0$, and $a \in \mathbb{Z}_p$,

$$|\mu((x-a)^i_{|a+p^j\mathbb{Z}_p})| \leq p^{A-j(i-1-k/2)}.$$

A distribution μ satisfying the above condition is then completely determined.

The composition of (29) with (31) and the natural identification $C_{har}(\mathbf{V}_n(K_p)) \simeq C_{har}(\mathbf{V}_n(\mathbb{Q}_p))\hat{\otimes}K_p$ yields an epimorphism of $\mathrm{GL}_2(\mathbb{Q}_p)$-modules

$$r : \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p) \twoheadrightarrow C_{har}(\mathbf{V}_n(K_p))$$

that can be described purely in terms of distributions by the rule

$$r(\mu)(e)(P) = \int_{U_e} P(t)d\mu(t) := \mu(P \cdot \chi_{U_e}).$$

Here $U_e \subset \mathbb{P}^1(\mathbb{Q}_p)$ is the open compact subset of $\mathbb{P}^1(\mathbb{Q}_p)$ corresponding to the ends leaving from the oriented edge e, and χ_{U_e} stands for its characteristic function.

By Proposition 3.1 and Lemma 2.11, the map r restricts to an isomorphism

$$r : \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b \xrightarrow{\sim} C^b_{har}(\mathbf{V}_n(K_p)), \tag{32}$$

which by abuse of notation we denote with the same symbol r. The same abuse will be made for the several maps that r induces in cohomology below.

The following theorem is the basic piece that shall allow us to introduce a p-adic integration theory on indefinite quaternion algebras.

Theorem 3.5. *There is a commutative diagram of morphisms of Hecke-modules*

$$\begin{array}{ccc} H^1(\Gamma, \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b) & \longrightarrow & H^1(\Gamma, \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)) \\ \downarrow & & \downarrow \\ H^1(\Gamma, C^b_{har}(\mathbf{V}_n(K_p))) & \longrightarrow & H^1(\Gamma, C_{har}(\mathbf{V}_n(K_p))) \end{array}$$

such that the composition $r : H^1(\Gamma, \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b) \xrightarrow{\sim} H^1(\Gamma, C_{har}(\mathbf{V}_n(K_p)))$ *is an isomorphism.*

In the statement, by Hecke-modules we mean modules over the Hecke algebra $\mathcal{H}(\Gamma)$ introduced in 2.1. Since r and the natural inclusions

$$\mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p))^b \hookrightarrow \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p)), \quad C^b_{har}(\mathbf{V}_n(K_p)) \hookrightarrow C_{har}(\mathbf{V}_n(K_p))$$

are morphisms of $\mathrm{GL}_2(\mathbb{Q}_p)$-modules, it follows from the discussion around (8) that there indeed exists a commutative diagram as above, where the maps are morphisms of $\mathcal{H}(\Gamma)$-modules. Notice that, by (32), it suffices to show that the inclusion $C^b_{har}(\mathbf{V}_n(K_p)) \subset C_{har}(\mathbf{V}_n(K_p))$ induces an isomorphism

$$H^1(\Gamma, C^b_{har}(\mathbf{V}_n(K_p))) \simeq H^1(\Gamma, C_{har}(\mathbf{V}_n(K_p))). \tag{33}$$

We devote the rest of the section to prove this statement. In order to prove (33) we need a further preliminary discussion. Quite generally, let $\mathcal{S}$ be a set on which Γ acts transitively. Fix an element $s_* \in \mathcal{S}$ and let $\Gamma_0 \subset \Gamma$ denote its stabilizer in Γ, that we assume to be finitely generated. Let $\{\gamma_s\}_{s\in\mathcal{S}}$ be a set of representatives for the coset space $\Gamma_0\backslash\Gamma$ such that $\gamma_{s_*} = 1$ and $\gamma_s s = s_*$ for all $s \in \mathcal{S}$. Let A be Γ -module endowed with a Γ_0-invariant non-archimedean norm $|\cdot|$ with values in K_p. On the group of functions $C(\mathcal{S}, A)$, define an extended norm by the rule

$$\|c\| := \sup_{s\in\mathcal{S}} |\gamma_s(c(s))| = \sup_{s\in\mathcal{S}} |({}^{\gamma_s}c)(s_*)|\,.$$

As before, let $C^b(\mathcal{S}, A) = \{c \in C(\mathcal{S}, A) : \|c\| < \infty\}$. It is a Γ-submodule of $C(\mathcal{S}, A)$ and the restriction of norm $\|\cdot\|$ to $C^b(\mathcal{S}, A)$ turns out to be Γ-invariant, as can be easily checked. Note that the subspace $C^b(\mathcal{S}, A)$ does not depend on the choice of the set $\{\gamma_s\}$.

Lemma 3.6. *Let G be a finitely generated group and let M be a G-module endowed with a G-invariant non-archimedean norm $[\cdot]$ with values in K_p. Then*

$$[[c]] := \sup_{g\in G}[c(g)]$$

defines a norm on $Z^1(G, M)$.

Proof. Let $\{g_i : i \in I\}$ be a set of generators of G with $\#I < \infty$. Given an element $c \in Z^1(G, B)$ define $K_c := \sup_i\{[c(g_i)], [c(g_i^{-1})] \in \mathbb{R}_{\geq 0}$. Let $g \in G$ be any element. Then $g = g_{i_1}^{\varepsilon_1}...g_{i_k}^{\varepsilon_k}$ for some $i_j \in I$ and $\varepsilon_j \in \{\pm 1\}$. Let us show by induction on k that $[c(g)] \leq K_c$. When $k = 1$ this is clear. When $k > 1$, the cocyle relation $c(g) = c(g_{i_1}^{\varepsilon_1}) + g_{i_1}^{\varepsilon_1} c(g_{i_2}^{\varepsilon_2}...g_{i_k}^{\varepsilon_k})$, together with the G-invariance of $[\cdot]$, imply that

$$\begin{aligned}[c(g)] &\leq \max\left\{[c(g_{i_1}^{\varepsilon_1})], [g_{i_1}^{\varepsilon_1} c(g_{i_2}^{\varepsilon_2}...g_{i_k}^{\varepsilon_k})]\right\} = \\ &= \max\left\{[c(g_{i_1}^{\varepsilon_1})], [c(g_{i_2}^{\varepsilon_2}...g_{i_k}^{\varepsilon_k})]\right\} \leq K_c.\end{aligned}$$

□

Proposition 3.7. *For $i = 0, 1$ the inclusion $\iota : C^b(\mathcal{S}, A) \subset C(\mathcal{S}, A)$ induces an isomorphism*

$$\iota : H^i(\Gamma, C^b(\mathcal{S}, A)) \overset{\simeq}{\to} H^i(\Gamma, C(\mathcal{S}, A)).$$

Proof. Let $i = 0$. We wish to show that every Γ-invariant element of $C(\mathcal{S}, A)$ has bounded norm. By Shapiro's lemma there is an isomorphism $\mathscr{S} : C(\mathcal{S}, A)^\Gamma \overset{\simeq}{\longrightarrow} A^{\Gamma_0}$, whose inverse is given explicitly by the map $a \mapsto c_a$, where $c_a(s) = \gamma_s^{-1}a$. One checks from this description that $\mathscr{S}$ is an isometry. Since $|\cdot|$ is a norm on $A^{\Gamma_0} \subset A$, the proposition follows.

Assume now $i = 1$. Again by Shapiro's lemma, the natural map

$$\pi : Z^1(\Gamma, C(\mathcal{S}, A)) \to Z^1(\Gamma_0, A)$$

induces an isomorphism $[\pi] : H^1(\Gamma, C(\mathcal{S}, A)) \simeq H^1(\Gamma_0, A)$. Let us first construct an explicit section

$$\tau : Z^1(\Gamma_0, A) \to Z^1(\Gamma, C(\mathcal{S}, A))$$

of π with values in the submodule $Z^1(\Gamma, C^b(\mathcal{S}, A))$ of $Z^1(\Gamma, C(\mathcal{S}, A))$.

Given a cocyle $c \in Z^1(\Gamma_0, A)$ define a chain $\tau_c(\gamma, s) := \gamma_s^{-1}c(g_{\gamma,s})$, where $\gamma_s\gamma = g_{\gamma,s}\gamma_{s'}$, with $g_{\gamma,s} \in \Gamma_0$ and $s' \in \mathcal{S}$. An elementary verification shows that $\tau_c \in Z^1(\Gamma, C(\mathcal{S}, A))$ is well-defined and that $\pi\tau = \mathrm{Id}$. Moreover, the morphism $[\tau] : H^1(\Gamma_0, A) \to H^1(\Gamma, C(\mathcal{S}, A))$ that τ induces in cohomology is an explicit inverse of the isomorphism $[\pi]$.

Let us prove that $\tau_c \in Z^1(\Gamma, C^b(\mathcal{S}, A))$. Since Γ_0 is finitely generated and the norm $|\cdot|$ is Γ_0-invariant, it follows from Lemma 3.6 applied to $(G, M, [\cdot]) = (\Gamma_0, A, |\cdot|)$ that there exists a constant $K_c \geq 0$ such that

$$|c(g)| \leq K_c \text{ for all } g \in \Gamma_0.$$

It then follows that for all $\gamma \in \Gamma$:

$$\begin{aligned}\|\tau_c(\gamma,\cdot)\| &= \sup_{s\in\mathcal{S}} |\gamma_s\cdot\tau_c(\gamma,s)| = \sup_{\substack{s\in\mathcal{S}\\ \gamma_s\gamma=g_{\gamma,s}\gamma_{s'}}} \left|\gamma_s\gamma_s^{-1}c(g_{\gamma,s})\right| = \\ &= \sup_{\substack{s\in\mathcal{S}\\ \gamma_s\gamma=g_{\gamma,s}\gamma_{s'}}} |c(g_{\gamma,s})| \le \sup_{g\in\Gamma_0}|c(g)|\le K_c.\end{aligned}$$

We can now easily prove that ι is surjective. Indeed, let $[\tilde{c}]$ denote the class of a cocycle $\tilde{c}\in Z^1(\Gamma, C(\mathcal{S},A))$. Set $c:=\pi(\tilde{c})\in Z^1(\Gamma_0,A)$. By the above discussion, $\tau(c)\in Z^1(\Gamma, C^b(\mathcal{S},A))$ and $[\tilde{c}]=[\iota(\tau(c))]$.

In order to prove that ι is injective, let $[\tilde{c}]\in H^1(\Gamma, C^b(\mathcal{S},A))$. Note that Lemma 3.6 applied to $(G,M,[\cdot])=(\Gamma, C^b(\mathcal{S},A),\|\cdot\|)$ yields the existence of a constant $K_{\tilde{c}}\ge 0$ such that

$$\sup_{\gamma\in\Gamma}\|\tilde{c}(\gamma)\| = \sup_{\gamma\in\Gamma, s\in\mathcal{S}} |\gamma_s\tilde{c}(\gamma,s)|\le K_{\tilde{c}}. \tag{34}$$

Suppose that the class of $\tilde{c}$ vanishes in $H^1(\Gamma, C(\mathcal{S},A))$, that is, there exists $C\in C(\mathcal{S},A)$ such that $\tilde{c}(\gamma)=C-\gamma C$ for all $\gamma\in\Gamma$. Equivalently, for all $s\in\mathcal{S}$ we have

$$\tilde{c}(\gamma,s)=C(s)-\gamma C(\gamma^{-1}s).$$

If C were not bounded, there would exist a sequence $\{s_n\}\subset\mathcal{S}$ such that $|\gamma_{s_n}C(s_n)|\to\infty$. Thus for $n\gg 0$ we would have $|C(s_*)|<|\gamma_{s_n}C(s_n)|$ and by the non-archimedean property of $|\cdot|$ we would conclude that

$$|\gamma_{s_n}\tilde{c}(\gamma_{s_n}^{-1},s_n)| = |\gamma_{s_n}C(s_n)-C(s_*)| = |\gamma_{s_n}C(s_n)|\to\infty.$$

Now (34) yields a contradiction. □

We are now ready to prove Theorem 3.5, which we already reduced to proving (33).

Proof of Theorem 3.5. By Corollary 2.12 we can consider the following commutative diagram, with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & C^b_{har}(\mathbf{V}_n(K_p)) & \to & C^b_0(\mathcal{E},\mathbf{V}_n(K_p)) & \to & C^b(\mathcal{V},\mathbf{V}_n(K_p)) & \to & 0\\ & & \cap & & \cap & & \cap & & \\ 0 & \to & C_{har}(\mathbf{V}_n(K_p)) & \to & C_0(\mathcal{E},\mathbf{V}_n(K_p)) & \to & C(\mathcal{V},\mathbf{V}_n(K_p)) & \to & 0\end{array}$$

The respective long exact sequences in cohomology induce the following commutative diagram, with exact rows.

$$\begin{array}{ccccccc} ...\to & C^b(\mathcal{V},\mathbf{V}_n(K_p))^\Gamma & \to & H^1(\Gamma, C^b_{har}(\mathbf{V}_n(K_p))) & \to & H^1(\Gamma, C^b_0(\mathcal{E},\mathbf{V}_n(K_p))) & \to ...\\ & \downarrow & & \downarrow & & \downarrow & \\ ...\to & C(\mathcal{V},\mathbf{V}_n(K_p))^\Gamma & \to & H^1(\Gamma, C_{har}(\mathbf{V}_n(K_p))) & \to & H^1(\Gamma, C_0(\mathcal{E},\mathbf{V}_n(K_p))) & \end{array}$$

Proposition 3.7, applied to $\mathcal{S}=\mathcal{V}$ and $\mathcal{E}^+$, shows that the first and third vertical arrows are isomorphisms. The same applies to the two vertical maps arising just before and after in the long exact sequence, that we do not draw. By the five lemma the middle vertical arrow is an isomorphism too, which is what we needed to prove. □

3.2. **Higher p-adic Abel-Jacobi maps.** The object of this section is introducing certain integration maps which will lead us to the definition of a map that will play the role of the p-adic Abel-Jacobi map in our context.

Since the choice of the finite field extension $K_p/\mathbb{Q}_p$ is fixed throughout, we shall drop it from the notation and simply write $\mathbf{P}_n=\mathbf{P}_n(K_p)$, $\mathbf{V}_n=\mathbf{V}_n(K_p)$, $\mathcal{A}_n(\mathbb{Q}_p)=\mathcal{A}_n(\mathbb{Q}_p,K_p)$, $\mathcal{D}_n(\mathbb{P}^1(\mathbb{Q}_p))=\mathcal{D}_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)$ and $\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p))=\mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p),K_p)$. Let $k_p/\mathbb{Q}_p$ denote the maximal unramified sub-extension of $K_p/\mathbb{Q}_p$.

Definition 3.8. Define integration maps

$$\begin{array}{lccc} \int\omega^{\log}: & \mathrm{Div}^0(\mathcal{H}_p)(k_p)\otimes\mathbf{P}_n\otimes\mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p)) & \to & K_p \\ & (\tau_2-\tau_1)\otimes P\otimes\mu & \mapsto & \int_{\tau_1}^{\tau_2}P\omega_\mu^{\log} \end{array}$$

$$\begin{array}{lccc} \int\omega^{\mathrm{ord}}: & \mathrm{Div}^0(\mathcal{H}_p)(k_p)\otimes\mathbf{P}_n\otimes\mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p)) & \to & K_p \\ & (\tau_2-\tau_1)\otimes P\otimes\mu & \mapsto & \int_{\tau_1}^{\tau_2}P\omega_\mu^{\mathrm{ord}} \end{array}$$

where for any τ_1, $\tau_2\in\mathcal{H}_p$, $P\in\mathbf{P}_n$ and $\mu\in\mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p))$:

$$\int_{\tau_1}^{\tau_2}P\omega_\mu^{\log}:=\int_{\mathbb{P}^1(\mathbb{Q}_p)}\log_p\left(\frac{t-\tau_2}{t-\tau_1}\right)P(t)d\mu(t)$$

and

$$\int_{\tau_1}^{\tau_2}P\omega_\mu^{\mathrm{ord}}:=\int_{\mathbb{P}^1(\mathbb{Q}_p)}\mathrm{ord}_p\left(\frac{t-\tau_2}{t-\tau_1}\right)P(t)d\mu(t)=\sum_{e:red(\tau_1)\to red(\tau_2)}\int_{U_e}P(t)d\mu(t),$$

where the last equality follows from [BDG, Lemma 2.5], as explained e.g. in the proof of [Se3, Prop. 5.2].

Several comments are in order concerning the definitions above. Recall that $\mathrm{Div}^0(\mathcal{H}_p)(k_p)$ stands for the module of degree zero divisors of $\mathcal{H}_p(\mathbb{Q}_p^{ur})=\mathbb{Q}_p^{ur}\backslash\mathbb{Q}_p$ that are fixed by the action of the Galois group $\mathrm{Gal}(\mathbb{Q}_p^{ur}/k_p)$. We shall regard $\mathrm{Div}^0(\mathcal{H}_p)(k_p)\otimes\mathbf{P}_n$ as a right $\mathrm{GL}_2(\mathbb{Q}_p)$-module by the rule

$$((\tau_2-\tau_1)\otimes P)\cdot\gamma:=(\gamma^{-1}\tau_2-\gamma^{-1}\tau_1)\otimes(P\cdot\gamma)\,.$$

Note that the definition of the first integration map depends on the choice of a branch of a p-adic logarithm $\log_p:K_p^\times\to K_p$; we do not specify a priori any such choice.

Finally, note that in the definition of the second integration map, the fact that $k_p/\mathbb{Q}_p$ is unramified implies that the reduction of any $\tau\in k_p\setminus\mathbb{Q}_p$ is a *vertex* (and not an edge) of the tree $\mathcal{T}$. The sum is taken over the edges of the path joining the two vertices $\mathrm{red}(\tau_1)$ and $\mathrm{red}(\tau_2)$.

Lemma 3.9. *The map*

$$\begin{array}{rcl} \mathrm{Div}^0(\mathcal{H}_p)(k_p)\otimes\mathbf{P}_n & \to & \mathcal{A}_n(\mathbb{Q}_p)/\mathbf{P}_n \\ (\tau_2-\tau_1)\otimes P & \mapsto & \log_p\left(\dfrac{t-\tau_2}{t-\tau_1}\right)P(t) \end{array}$$

is $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant.*

Proof. Write $\theta_{\tau_2-\tau_1}(t):=\frac{t-\tau_2}{t-\tau_1}$. A direct computation shows that $(t-\tau)\cdot\gamma=\det(\gamma)^{-1/2}(a-c\tau)\cdot(t-\bar{\gamma}\tau)$, for all $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Q}_p)$. Here we write $\bar{\gamma}:=\det(\gamma)\gamma^{-1}$. It follows that $\theta_{\tau_2-\tau_1}(t)\cdot\gamma=\frac{a-c\tau_2}{a-c\tau_1}\cdot\theta_{\bar{\gamma}\tau_2-\bar{\gamma}\tau_1}(t)$ and hence

$$[\log_p\theta_{\tau_2-\tau_1}(t)\cdot P(t)]\cdot\gamma=\log_p(\frac{a-c\tau_2}{a-c\tau_1})\cdot(P\gamma)(t)+\log_p\theta_{\bar{\gamma}\tau_2-\bar{\gamma}\tau_1}(t)\cdot(P\gamma)(t).$$

The claim follows, as $\gamma^{-1}\tau=\bar{\gamma}\tau$ for every $\gamma\in\mathrm{GL}_2(\mathbb{Q}_p)$ and $\log_p(\frac{a-c\tau_2}{a-c\tau_1})(P\gamma)(t)\in\mathbf{P}_n$. □

From now on, thanks to Theorem 3.5, we shall make the identification

(35) $$\mathbf{H}(K_p):=H^1(\Gamma,\mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p))^b)=H^1(\Gamma,C_{har}(\mathbf{V}_n)).$$

The natural inclusion $\mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p))^b\subseteq\mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p))$ induces a map

$$H^1(\Gamma,\mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p))^b)\hookrightarrow H^1(\Gamma,\mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p)))$$

that is a monomorphism thanks to Theorem 3.5. Together with Lemma 3.9 and the cap product, the above pairings induce maps

$$\Psi^{\log}, \Psi^{\mathrm{ord}}: \quad H_1(\Gamma, \mathrm{Div}^0(\mathcal{H}_p)(k_p) \otimes \mathbf{P}_n) \quad \longrightarrow \quad \mathbf{H}(K_p)^\vee. \tag{36}$$

Lemma 3.10. *$H^1(\Gamma, \mathbf{V}_n) = 0$.*

Proof. The exact sequence (11) and the identifications (21) provided by Shapiro's lemma induce the long exact sequence

$$0 \to H^0(\Gamma, \mathbf{V}_n) \to H^0(\Gamma_0(N^+), \mathbf{V}_n)^2 \to H^0(\Gamma_0(pN^+), \mathbf{V}_n) \stackrel{\delta}{\to} \tag{37}$$

$$\stackrel{\delta}{\to} H^1(\Gamma, \mathbf{V}_n) \stackrel{\varepsilon}{\to} H^1(\Gamma_0(N^+), \mathbf{V}_n)^2 \stackrel{\partial^*}{\to} H^1(\Gamma_0(pN^+), \mathbf{V}_n) \to$$

Notice first that ε is a monomorphism. Indeed, if $n = 0$, $\mathbf{V}_n = K_p$ is trivial as Γ-module. Thus $H^0(\Gamma, \mathbf{V}_n) = H^0(\Gamma_0(N^+), \mathbf{V}_n) = H^0(\Gamma_0(pN^+), \mathbf{V}_n) = K_p$. The exactness of (37) implies our claim. If $n > 0$, $H^0(\Gamma_0(pN^+), \mathbf{V}_n) = 0$ by [Hi, pag. 165 Lemma 2] and we again deduce that $\ker(\varepsilon) = 0$.

It thus remains left to show that $\mathrm{Ker}(\partial^*) = 0$. Let us first show that $\mathrm{Ker}(\partial^*)^c = 0$. The map ∂^* can be composed with the degeneracy map $\partial_* : H^1(\Gamma_0(pN^+), \mathbf{V}_n) \to H^1(\Gamma_0(N^+), \mathbf{V}_n)^2$ given by $\partial_* = (\mathrm{cor}_{\Gamma_0(pN^+)}^{\Gamma_0(N^+)}, \mathrm{cor}_{\Gamma_0(pN^+)}^{\hat{\Gamma}_0(N^+)})$. The reader may wish to recall the natural identifications already made in (22).

A computation now shows that the endomorphism $\partial_* \circ \partial^*$ of $H^1(\Gamma_0(N^+), \mathbf{V}_n)^2$ is

$$\begin{pmatrix} p+1 & p^{-m}T_p \\ p^{-m}T_p & p+1 \end{pmatrix}.$$

Fix an embedding of K_p into the field $\mathbb{C}$ of complex numbers. By Deligne's bound, the complex absolute value of the eigenvalues of the Hecke operator T_p acting on $H^1(\Gamma_0(N^+), \mathbf{V}_n(\mathbb{C}))^c$ are bounded above by $2\sqrt{p^{k-1}} = 2p^m\sqrt{p}$. It thus follows that $\partial_* \cdot \partial^*$ restricts to a linear automorphism of the cuspidal part of $H^1(\Gamma_0(N^+), \mathbf{V}_n(\mathbb{C}))$, and thus $(\partial^*)^c$ is injective.

In order to conclude, let us now show that $\mathrm{Ker}(\partial^*)^{\mathrm{Eis}} = 0$ when $N^- = 1$. Let $C_{\Gamma_0(N^+)} = \{s_1, ..., s_t\}$ be a set of representatives for the cusps of $\Gamma_0(N^+)$. One then can check that $C_{\hat{\Gamma}_0(N^+)} = \{\widehat{s}_i := \omega_p s_i\}$ and $C_{\Gamma_0(pN^+)} = \{s_i, \hat{s}_i\}$ are systems of representatives for the cusps of $\hat{\Gamma}_0(N^+)$ and $\Gamma_0(pN^+)$, respectively.

It follows from (15) and the discussion around it that for our purposes it suffices to show that for each $i = 1, ..., t$ the natural map

$$H^1(\Gamma_0(N^+)_{s_i}, \mathbf{V}_n(\mathbb{C})) \xrightarrow{\partial^*} H^1(\Gamma_0(pN^+)_{s_i}, \mathbf{V}_n(\mathbb{C}))$$

induced by ∂^* by restriction is a monomorphism (and analogously for $\hat{\Gamma}_0(N^+)$ and $\hat{s}_i$). But this is clear because ∂^* is the restriction map $\mathrm{res}_{\Gamma_0(pN^+)_{s_i}}^{\Gamma_0(N^+)_{s_i}}$, which is injective by a similar reason as before: the composition with the corresponding corestriction map is multiplication by the index $[\Gamma_0(N^+)_{s_i} : \Gamma_0(pN^+)_{s_i}]$, which is finite as it divides $[\Gamma_0(N^+) : \Gamma_0(pN^+)] = p + 1$. □

Consider the exact sequence of Γ-modules

$$0 \to \mathrm{Div}^0(\mathcal{H}_p)(k_p) \to \mathrm{Div}(\mathcal{H}_p)(k_p) \to \mathbb{Z} \to 0. \tag{38}$$

Taking the tensor product with $\mathbf{P}_n$ and forming the long exact sequence in homology yields a connecting map

$$H_2(\Gamma, \mathbf{P}_n) \stackrel{\partial_2}{\to} H_1(\Gamma, \mathrm{Div}^0(\mathcal{H}_p)(k_p) \otimes \mathbf{P}_n). \tag{39}$$

Recall from 2.4 and 2.5 that the cuspidal part $\mathbf{H}(K_p)^c$ of $\mathbf{H}(K_p)$ is naturally identified with the p-new space of cuspidal modular forms of level pN^+ on the quaternion algebra $\mathcal{B}$. Let

$\mathrm{pr}_c : \mathbf{H}(K_p)^\vee \longrightarrow (\mathbf{H}(K_p)^c)^\vee$ denote the natural projection. We shall use the same symbol pr_c for the map $\mathrm{pr}_c \oplus \mathrm{pr}_c$.

Theorem 3.11. *For every $n \geq 0$ the morphism*

$$\mathrm{pr}_c \circ \Psi^{\mathrm{ord}} \circ \partial_2 : H_2(\Gamma, \mathbf{P}_n) \to (\mathbf{H}(K_p)^c)^\vee$$

is surjective and induces an isomorphism

$$(\Psi^{\mathrm{ord}} \circ \partial_2)^c : H_2(\Gamma, \mathbf{P}_n)^c \overset{\simeq}{\to} (\mathbf{H}(K_p)^c)^\vee.$$

Proof. Let us rewrite the homomorphism Ψ^{ord} as a composition of several natural maps. First, consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Div}^0(\mathcal{H}_p)(k_p) \otimes \mathbf{P}_n & \to & \mathrm{Div}(\mathcal{H}_p)(k_p) \otimes \mathbf{P}_n & \to & \mathbf{P}_n & \to & 0 \\ & & \mathrm{red} \otimes 1 \downarrow & & \mathrm{red} \otimes 1 \downarrow & & \| & & \\ 0 & \to & \mathrm{Div}^0(\mathcal{V}) \otimes \mathbf{P}_n & \to & \mathrm{Div}(\mathcal{V}) \otimes \mathbf{P}_n & \to & \mathbf{P}_n & \to & 0. \end{array} \quad (40)$$

The long exact sequence in homology yields a commutative diagram

$$\begin{array}{ccc} H_2(\Gamma, \mathbf{P}_n) & \overset{\partial_2}{\to} & H_1(\Gamma, \mathrm{Div}^0(\mathcal{H}_p)(k_p) \otimes \mathbf{P}_n) \\ & \overset{\partial_{\mathcal{V}}}{\searrow} & \downarrow \Psi_1 \\ & & H_1(\Gamma, \mathrm{Div}^0(\mathcal{V}) \otimes \mathbf{P}_n). \end{array} \quad (41)$$

Second, let $\overline{\mathrm{Div}}(\mathcal{E})$ be the quotient of $\mathrm{Div}(\mathcal{E})$ obtained by imposing the relation $\overline{e} + e = 0$ for all $e \in \mathcal{E}$. Note that the morphisms

$$\begin{array}{cccc} \mathrm{path} : & \mathrm{Div}^0(\mathcal{V}) & \to & \overline{\mathrm{Div}}(\mathcal{E}) \\ & v_1 - v_2 & \mapsto & \sum_{e : v_1 \to v_2} e \end{array}$$

$$\begin{array}{cccc} \partial : & \overline{\mathrm{Div}}(\mathcal{E}) & \to & \mathrm{Div}^0(\mathcal{V}) \\ & e & \mapsto & s(e) - t(e) \end{array}$$

are mutually inverse and identify the two Γ-modules. In particular we obtain from path the commutative diagram

$$\begin{array}{ccc} H_2(\Gamma, \mathbf{P}_n) & \overset{\partial_{\mathcal{V}}}{\to} & H_1(\Gamma, \mathrm{Div}^0(\mathcal{V}) \otimes \mathbf{P}_n) \\ & \overset{\partial_{\mathcal{E}}}{\searrow} & \downarrow \Psi_2 \\ & & H_1(\Gamma, \overline{\mathrm{Div}}(\mathcal{E}) \otimes \mathbf{P}_n). \end{array} \quad (42)$$

where the morphism $\partial_{\mathcal{E}}$ is obtained from the second row of (40) and the identification $\mathrm{Div}^0(\mathcal{V}) = \overline{\mathrm{Div}}(\mathcal{E})$.

Third, consider the exact sequence obtained from (11) with $A = \mathbf{V}_n$:

$$0 \to \mathbf{V}_n \to C(\mathcal{V}, \mathbf{V}_n) \overset{\partial^*}{\to} C_0(\mathcal{E}, \mathbf{V}_n) \to 0. \quad (43)$$

The dual exact sequence of (43) is canonically identified with the exact sequence obtained from the second row of (40) and the identification $\mathrm{Div}^0(\mathcal{V}) = \overline{\mathrm{Div}}(\mathcal{E})$:

$$0 \to \overline{\mathrm{Div}}(\mathcal{E}) \otimes \mathbf{P}_n \overset{\partial \otimes \mathrm{Id}}{\longrightarrow} \mathrm{Div}(\mathcal{V}) \otimes \mathbf{P}_n \to \mathbf{P}_n \to 0. \quad (44)$$

More precisely the duality between (43) and (44) is induced by the evaluation pairings:

$$\begin{array}{cccc} \langle -, - \rangle_{\mathcal{V}} : & \mathrm{Div}(\mathcal{V}) \otimes \mathbf{P}_n \otimes C(\mathcal{V}, \mathbf{V}_n) & \to & K_p \\ & v \otimes P \otimes c & \mapsto & c(v, P) \end{array}$$

$$\begin{array}{cccc} \langle -, - \rangle_{\mathcal{E}} : & \overline{\mathrm{Div}}(\mathcal{E}) \otimes \mathbf{P}_n \otimes C_0(\mathcal{E}, \mathbf{V}_n) & \to & K_p \\ & e \otimes P \otimes c & \mapsto & c(e, P). \end{array}$$

By cap product, these pairings yield the following commutative diagram:

$$\begin{array}{ccc} H_2(\Gamma, \mathbf{P}_n) & \stackrel{\partial_{\mathcal{E}}}{\to} & H_1(\Gamma, \overline{\mathrm{Div}}(\mathcal{E}) \otimes \mathbf{P}_n) \\ \downarrow & & \downarrow \Psi_3 \\ H^2(\Gamma, \mathbf{V}_n)^\vee & \stackrel{\delta^\vee}{\to} & H^1(\Gamma, C_0(\mathcal{E}, \mathbf{V}_n))^\vee. \end{array} \tag{45}$$

The Universal Coefficients theorem guarantees that the above vertical arrows are isomorphisms. With these notations the morphism Ψ^{ord} is obtained as follows. Let Ψ_4 be the dual of the morphism

$$\mathbf{H}(K_p) = H^1(\Gamma, C_{har}(\mathbf{V}_n)) \to H^1(\Gamma, C_0(\mathcal{E}, \mathbf{V}_n)).$$

Then we have

$$\Psi^{\mathrm{ord}} = \Psi_4 \circ \Psi_3 \circ \Psi_2 \circ \Psi_1.$$

Hence the morphism $\mathrm{pr}_c \circ \Psi^{\mathrm{ord}}$ is obtained by further composition with the morphism pr_c dual to the inclusion $i^c : \mathbf{H}(K_p)^c \subset \mathbf{H}(K_p)$:

$$\mathrm{pr}_c \circ \Psi^{\mathrm{ord}} \circ \partial_2 = \mathrm{pr}_c \circ \Psi_4 \circ \Psi_3 \circ \Psi_2 \circ \Psi_1.$$

From the commutativity of diagrams (41), (42) and (45), we obtain the commutative diagram

$$\begin{array}{ccccc} H_2(\Gamma, \mathbf{P}_n) & \stackrel{\partial_2}{\to} & H_1(\Gamma, \mathrm{Div}^0(\mathcal{H}_p) \otimes \mathbf{P}_n) & & \\ \downarrow & & \downarrow \Psi_3 \circ \Psi_2 \circ \Psi_1 & & \\ H^2(\Gamma, \mathbf{V}_n)^\vee & \stackrel{\delta^\vee}{\to} & H^1(\Gamma, C_0(\mathcal{E}, \mathbf{V}_n))^\vee & \stackrel{\mathrm{pr}_c \circ \Psi_4}{\to} & (\mathbf{H}(K_p)^c)^\vee. \end{array}$$

As already mentioned, the left vertical arrow is an isomorphism. To prove the second statement of the theorem, it remains to prove that $\mathrm{pr}_c \circ \Psi_4 \circ \delta^\vee$ restricts to an isomorphism on the cuspidal parts; the first statement about surjectivity will then follow from Remark 2.3.

Equivalently, since the composition $\mathrm{pr}_c \circ \Psi_4$ is dual to the morphism

$$\mathbf{H}(K_p)^c \subset \mathbf{H}(K_p) \to H^1(\Gamma, C_0(\mathcal{E}, \mathbf{V}_n)),$$

we need to show that the morphism

$$\mathbf{H}(K_p) = H^1(\Gamma, C_{har}(\mathbf{V}_n)) \to H^1(\Gamma, C_0(\mathcal{E}, \mathbf{V}_n)) \to H^2(\Gamma, \mathbf{V}_n) = H_2(\Gamma, \mathbf{P}_n)^\vee$$

induces an isomorphism when restricted to the cuspidal parts. This is the content of Lemmas 2.8 and 2.9. □

Remark 3.12. When $n > 0$ and $N^- > 1$ we have $\mathrm{pr}_c = \mathrm{Id}$. Furthermore $H_2(\Gamma, \mathbf{P}_n)^c = H_2(\Gamma, \mathbf{P}_n)$, $\mathbf{H}(K_p)^c = \mathbf{H}(K_p)$ and the morphism Φ^{ord} is an isomorphism.

Let $\mathbb{T} := \mathbb{T}^{p-new}_{\Gamma_0(pN^+)}$ denote the maximal quotient of the Hecke algebra $\mathcal{H}(\Gamma_0(pN^+)) \otimes \mathbb{Q}$ acting on $S_k(\Gamma_0(pN^+))^{p-new}$ and put $\mathbb{T}_p = \mathbb{T} \otimes_{\mathbb{Q}} \mathbb{Q}_p$, $\mathbb{T}_{K_p} = \mathbb{T} \otimes_{\mathbb{Q}} K_p$.

Corollary 3.13. *There exists a unique endomorphism* $\mathcal{L} \in \mathrm{End}_{\mathbb{T}_{K_p}}((\mathbf{H}(K_p)^c)^\vee)$ *such that*

$$\mathrm{pr}_c \circ \Psi^{\log} \circ \partial_2 = \mathcal{L} \circ \ \mathrm{pr}_c \circ \Psi^{\mathrm{ord}} \circ \partial_2 : H_2(\Gamma, \mathbf{P}_n) \to (\mathbf{H}(K_p)^c)^\vee. \tag{46}$$

Proof. Let $i : (\mathbf{H}(K_p)^c)^\vee \tilde{\to} H_2(\Gamma, \mathbf{P}_n)^c$ be the inverse of the isomorphism $(\Psi^{\mathrm{ord}} \circ \partial_2)^c$ of Theorem 3.11 and define $\mathcal{L} = (\Psi^{\log} \circ \partial_2)^c \circ i$. Since $i \circ \Psi^{\mathrm{ord}} \circ \partial_2$ is the natural projection $H_2(\Gamma, \mathbf{P}_n) \to H_2(\Gamma, \mathbf{P}_n)^c$, it is clear that (46) holds true with this choice of $\mathcal{L}$. As for the uniqueness, let $\tilde{\mathcal{L}} \in \mathrm{End}_{\mathbb{T}_{\mathbb{Q}_p}}((\mathbf{H}(K_p)^c)^\vee)$ be any endomorphism satisfying (46) and let $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}^{\mathrm{Eis}} \oplus \tilde{\mathcal{L}}^c$ denote its Eisenstein/cuspidal decomposition (cf. Remark 2.3). Since the Eisenstein subspace of $(\mathbf{H}(K_p)^c)^\vee$ is trivial, it follows that $\tilde{\mathcal{L}}^{\mathrm{Eis}} = 0$. Hence $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}^c$ and, by Theorem 3.11, $\tilde{\mathcal{L}}^c = \mathcal{L}$ is necessarily the endomorphism defined above. □

Definition 3.14. The $\mathcal{L}$-invariant of the space $S_k(\Gamma_0(pN^+))^{p-new}$ of p-new modular forms is the endomorphism

$$\mathcal{L} \in \mathrm{End}_{\mathbb{T}_{K_p}}((\mathbf{H}(K_p)^c)^\vee)$$

appearing in the above corollary.

By Remark 2.4, $(\mathbf{H}(K_p)^c)^\vee$ is a free rank one $\mathbb{T}_{K_p}$-module. Hence $\mathcal{L} \in \mathbb{T}_{K_p}$. But we can even claim that $\mathcal{L} \in \mathbb{T}_p$, because our construction of the $\mathcal{L}$-invariant is valid for any finite field extension $K_p/\mathbb{Q}_p$ and it is clear from Corollary 3.13 that it is invariant under base change.

4. Monodromy modules

4.1. **Fontaine-Mazur theory.** As in 3.2, let $k_p/\mathbb{Q}_p$ denote the maximal unramified sub-extension of $K_p/\mathbb{Q}_p$. Write $\sigma \in \mathrm{Aut}(k_p)$ for the absolute Frobenius of k_p. Throughout this section, $k \geq 2$ is a fixed positive even integer.

Let $\mathbb{T}_p$ be a finite dimensional commutative $\mathbb{Q}_p$ -algebra and write $\mathbb{T}_{k_p} = \mathbb{T}_p \otimes k_p$ and $\mathbb{T}_{K_p} = \mathbb{T}_p \otimes K_p$. Set $\sigma_{\mathbb{T}_{k_p}} := \mathrm{Id} \otimes \sigma$ on $\mathbb{T}_{k_p}$.

Definition 4.1. A *two-dimensional monodromy $\mathbb{T}_p$-module over K_p* is a 4-tuple $(D, \varphi, N, F^\cdot)$ where D is a $\mathbb{T}_{k_p}$-module, $\varphi : D \to D$ is σ-linear endomorphism (i.e. $\varphi(ax) = \sigma(a)x$ for all $a \in k_p, x \in D$) and $N : D \to D$ is a $\mathbb{T}_{k_p}$-linear endomorphism such that

(a) $F^\cdot$ is a filtration on the K_p-vector space $D \otimes_{k_p} K_p$ of the form

$$D \otimes K_p = F^0 \supset F^1 = ... = F^{k-1} \supset F^k = 0$$

where F^{k-1} is a free $\mathbb{T}_{K_p}$-module of rank one;

(b) $D \otimes K_p = F^{k-1} \oplus N_{K_p}(D \otimes K_p)$ as a $\mathbb{T}_{K_p}$-module, with $N_{K_p} : F^{k-1} \to N_{K_p}(D \otimes K_p)$ a $\mathbb{T}_{K_p}$-module isomorphism.

(c) $N \circ \varphi = p\varphi \circ N$ and, for any $T \in \mathbb{T}_{k_p}$, $\varphi \circ T = \sigma_{\mathbb{T}_{k_p}}(T) \circ \varphi$.

The integer $k_D := k$ appearing in (a) is called the *weight* of the monodromy $\mathbb{T}_p$-module D. See [CI], [IS, §2] and [Ma, §9, p. 12] for related but slightly different notions, and for proofs of some of the claims below.

Let $D = (D, \varphi, N, F^\cdot)$ be a two-dimensional monodromy $\mathbb{T}_p$ -module over K_p, that by an abuse of notation sometimes will be denoted simply as D. When we forget the $\mathbb{T}_p$-structure, it is customary to call D a filtered Frobenius monodromy module, or simply a (φ, N)-module over K_p. Write $\mathrm{MF}_{K_p}(\varphi, N)$ for the category of such objects, in which a morphism is a homomorphism of k_p-modules preserving the filtrations and commuting with φ and N. As an illustrative example, multiplication by a scalar $a \in k_p$ on D is an endomorphism of vector spaces over k_p that is a morphism in $\mathrm{MF}_{K_p}(\varphi, N)$ if and only if $a \in \mathbb{Q}_p$. The category $\mathrm{MF}_{K_p}(\varphi, N)$ is an additive tensor category admitting kernels and cokernels.

Remark 4.2. If $K_p^+ \supseteq K_p$ is a complete field extension of $\mathbb{Q}_p$ containing K_p, then the maximal unramified sub-extension k_p^+ of $K_p^+/\mathbb{Q}_p$ contains k_p, and there is a natural obvious notion of base change of monodromy modules: $D_{K_p^+} := (D \otimes_{k_p} k_p^+, \varphi_{k_p} \otimes \sigma_{k_p^+/k_p}, N_{k_p} \otimes k_p^+, F^\cdot \otimes K_p^+)$ is a two-dimensional monodromy $\mathbb{T}_p$-module over K_p^+.

In our applications in 4.2, we shall be working with monodromy modules over the quadratic unramified extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$ that in fact can be obtained as the base change of a monodromy module over $\mathbb{Q}_p$. Consider the slope decomposition

$$D = \bigoplus_{\alpha \in \mathbb{Q}} D^\alpha$$

where for $\alpha = r/s$, $r, s \in \mathbb{Z}$, $s > 0$, $(r, s) = 1$, $D^\alpha \subset D$ is the largest k_p-vector subspace of D that has an $\mathcal{O}_{k_p}$-stable lattice D_0 with $\varphi^s(D_0) = p^r D_0$.

Since $N \neq 0$ by (b) and $N(D^{\alpha+1}) \subset D^{\alpha}$ by (c), it is an exercise in commutative algebra to show that there exists $\lambda \in \mathbb{Q}$ such that $D^{\lambda}, D^{\lambda+1} \neq 0$ are free $\mathbb{T}_{k_p}$-modules of rank 1 and the map $N : D^{\lambda+1} \to D^{\lambda}$ is non-zero. It then follows that D is free of rank two over $\mathbb{T}_{k_p}$ and we deduce that such a λ is unique; we call it the *slope* of D. It is easy to check that

$$D = D^{\lambda} \oplus D^{\lambda+1}, \quad D^{\lambda} = \ker N = N(D), \tag{47}$$

$$\text{with } D^{i} \simeq \mathbb{T}_{k_p} \text{ for } i = \lambda, \lambda+1. \tag{48}$$

Definition 4.3. The *$\mathcal{L}$-invariant $\mathcal{L}_D$ of* $D = (D, \varphi, N, F^{\cdot})$ is defined to be the unique element $\mathcal{L}_D \in \mathbb{T}_{K_p}$ such that

$$x - \mathcal{L}_D N_{K_p}(x) \in F^{k-1} \text{ for every } x \in D^{\lambda+1} \otimes K_p.$$

The existence and uniqueness of $\mathcal{L}_D \in \mathbb{T}_{K_p}$ are again easy to check.

Lemma 4.4. $\mathbb{T}_p \simeq \mathrm{End}_{\mathrm{MF}_{K_p}(\varphi,N)}(D)$.

Proof. It follows from (a)-(c) that there is a natural map $\mathbb{T}_p \xrightarrow{\eta} \mathrm{End}_{\mathrm{MF}^{ad}_{K_p}(\varphi,N)}(D)$. The algebra $\mathbb{T}_{K_p}$ preserves F^{k-1} and it follows from (a) that the above map induces an isomorphism $\mathrm{End}_{K_p}(F^{k-1}) = \mathbb{T}_{K_p}$. In particular η is injective. As for surjectivity, let $f \in \mathrm{End}_{\mathrm{MF}_{K_p}(\varphi,N)}(D)$. Since f commutes with φ, it preserves the slope decomposition (47). For $i \in \{\lambda, \lambda+1\}$, let $t_f^i \in \mathbb{T}_{k_p}$ be such that $f_{|D^i} = t_f^i \in \mathrm{End}_{k_p}(D^i)$.

Since $N : D^{\lambda+1} \to D^{\lambda}$ is an isomorphism of $\mathbb{T}_{k_p}$-modules by (47), we may write $D^{\lambda+1} = \mathbb{T}_{k_p} \cdot e$, $D^{\lambda} = \mathbb{T}_{k_p} N(e)$ for some $e \in D^{\lambda+1}$. Since $t_f^{\lambda+1} N(e) = N t_f^{\lambda+1}(e) = N f(e) = f N(e) = t_f^{\lambda} N(e)$ we deduce that $t := t_f^{\lambda+1} = t_f^{\lambda}$.

Finally, since t must commute with the σ-linear automorphism φ, it follows that $t \in \mathbb{T}_p$. □

Along with $\mathcal{L}_D$, one may also attach to D the following invariant. The notation is as in the previous proof.

Definition 4.5. Let $U = U_D \in \mathbb{T}_{k_p}$ be the element such that $\varphi N(e) = U N(e)$.

Notice that U exists and is well-defined, because D^{λ} is preserved by φ and $D^{\lambda} = \mathbb{T}_{k_p} \cdot N(e)$. The reader may check that U does not depend on the choice of the generator e of $D^{\lambda+1}$.

As a final remark in this short review of monodromy modules, we notice that the invariants U_D, $\mathcal{L}_D$ and k_D of a two-dimensional monodromy $\mathbb{T}_p$-module D over K_p completely determine it up to isomorphism. More precisely, we can prove the following statement:

Proposition 4.6. *For any integer $k \in \mathbb{Z}$ and any pair of elements $U \in \mathbb{T}_{k_p}$ and $\mathcal{L} \in \mathbb{T}_{K_p}$ there exists a two-dimensional monodromy $\mathbb{T}_p$-module $D_{U,\mathcal{L},k}$ over K_p such that $U_{D_{U,\mathcal{L}}} = U$, $\mathcal{L}_{D_{U,\mathcal{L}}} = \mathcal{L}$ and $k_D = k$. Moreover, for any two-dimensional monodromy $\mathbb{T}_p$-module D,*

$$D \simeq D_{U,\mathcal{L}} \text{ if and only if } U_D = U,\ \mathcal{L}_D = \mathcal{L} \text{ and } k_D = k. \tag{49}$$

This will be useful for our purposes in 4.2. As we were not able to find an explicit proof of this fact in the literature, let us sketch the details.

Proof. Fix $k, U, \mathcal{L}$ as in the statement and define

$$D_{U,\mathcal{L},k} := \mathbb{T}_{k_p} \oplus \mathbb{T}_{k_p}$$

endowed with:

- a filtration $D_{U,\mathcal{L},k} \otimes K_p = F^0 \supsetneq F^1 = ... = F^{k-1} \supsetneq F^k = 0$, where for all $1 \leq j \leq k-1$,
$$F^j = \left\{(-\mathcal{L}x, x) : x \in \mathbb{T}_{K_p}\right\};$$

- a Frobenius operator $\varphi_{U,\mathcal{L},k}$ given by the rule

$$\varphi_{U,\mathcal{L},k}(x,y) := (\sigma_{\mathbb{T}_{k_p}}(x)U, p\,\sigma_{\mathbb{T}_{k_p}}(y)U);$$

- a monodromy operator $N_{U,\mathcal{L},k}$ defined by the rule

$$N_{U,\mathcal{L},k}(x,y) = (y,0).$$

One immediately checks that $D_{U,\mathcal{L},k}$ is a two-dimensional monodromy $\mathbb{T}_p$-module over K_p, satisfying conditions (a), (b), (c) as required. It also follows from the definitions that $k_{D_{U,\mathcal{L},k}} = k$, $\mathcal{L}_{D_{U,\mathcal{L},k}} = \mathcal{L}$ and $U_{D_{U,\mathcal{L},k}} = U$.

In order to prove the converse, let now $D = (D, \varphi, N, F^{\cdot})$ be any two-dimensional monodromy $\mathbb{T}_p$-module over K_p, say of slope λ, such that $U_D = U$, $\mathcal{L}_D = \mathcal{L}$ and $k_D = k$. As in the proof of Lemma 4.4, we can write $D = D^{\lambda} \oplus D^{\lambda+1} = \mathbb{T}_{k_p}N(e) \oplus \mathbb{T}_{k_p}e$ and this allows us to fix the isomorphism of $\mathbb{T}_{k_p}$-modules $\mu : D \simeq D_{U,\mathcal{L}} = \mathbb{T}_{k_p} \oplus \mathbb{T}_{k_p}$ given by $\mu(e) = (0,1)$, $\mu(N(e)) = (1,0)$.

Let us show that μ is also an isomorphism of monodromy $\mathbb{T}_p$-modules over K_p. It is obvious from the construction that both have the same filtration and that μ intertwines the action of N. It also follows immediately from Definition 4.3 and the equality $\mathcal{L}_D = \mathcal{L}$ that μ preserves the filtration. Finally, μ commutes with φ thanks to the defining property of U, condition (c) of Definition 4.1 and the fact that $N_{|D^{\lambda+1}} : D^{\lambda+1} \to D^{\lambda}$ is an isomorphism. □

Fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ and choose an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ containing K_p. Choose also a prime ideal $\wp$ of $\bar{\mathbb{Q}}$ over p, that we may use to fix an embedding of $G_{\mathbb{Q}_p} := \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ into $G_{\mathbb{Q}} := \mathrm{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q})$.

For a p-adic representation V of G_{K_p} over $\mathbb{Q}_p$ one defines $D_{\mathrm{st}}(V) := (V \otimes B_{\mathrm{st}})^{G_{K_p}}$, where B_{st} is Fontaine's ring defined in [Fo] and from which $D_{\mathrm{st}}(V)$ inherits the structure of a filtered (φ, N)-module over K_p. A p-adic representation V of G_{K_p} is called *semistable* if the canonical monomorphism $D_{\mathrm{st}}(V) \otimes_{k_p} B_{\mathrm{st}} \to V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}$ is an isomorphism. A filtered (φ, N)-module D over K_p is called *admissible* if $D \simeq D_{\mathrm{st}}(V)$ for some semistable representation V. It can be shown that the modules arising from Proposition 4.6 are admissible if and only if the slope is $(k-2)/2$.

The full subcategory $\mathrm{MF}^{ad}_{K_p}(\varphi, N)$ of $\mathrm{MF}_{K_p}(\varphi, N)$ of admissible two-dimensional monodromy $\mathbb{T}_p$-modules over K_p is an abelian tensor category such that exact sequences remain exact in $\mathrm{MF}_{K_p}(\varphi, N)$. The functor D_{st} establishes an equivalence of categories between that of semistable continuous representations of G_{K_p} over $\mathbb{Q}_p$ and $\mathrm{MF}^{ad}_{K_p}(\varphi, N)$.

Let $\mathbb{T} = \mathbb{T}^{p-new}_{\Gamma_0(pN^+)} \otimes \mathbb{Q}$ and put $\mathbb{T}_p := \mathbb{T} \otimes \mathbb{Q}_p$. As recalled in the introduction, let $V_p := H_p(\mathcal{M}_n)^{p-new}$ denote the p-new quotient of the p-adic étale realization of the motive $\mathcal{M}_n$ attached to the space of p-new cusp forms of weight k with respect to $\Gamma_0(pN^+)$. Let us regard V_p as a representation of $\mathrm{Gal}\,(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, by restricting the action of $G_{\mathbb{Q}}$ to the decomposition subgroup of the fixed prime $\wp$ above. As is well-known to the experts, V_p is semistable (cf. [C] and [CI]). Crucial for this is the fact that the Shimura curve $X_0^{N^-}(pN^+)$ has semistable reduction at p.

The admissible filtered (φ, N)-module

$$\mathbf{D}^{FM} := D_{\mathrm{st}}(V_p) \tag{50}$$

attached by Fontaine and Mazur to V_p is in a natural way a two-dimensional monodromy $\mathbb{T}_p$-module over $\mathbb{Q}_p$ in the sense of Definition 4.1, for which $U_{\mathbf{D}^{FM}} = U_p$ is the usual Hecke operator at p and the slope is m; cf. again [C] and [CI]. Let

$$\mathcal{L}^{FM} := \mathcal{L}_{\mathbf{D}^{FM}} \in \mathbb{T}_p$$

denote the $\mathcal{L}$-invariant of $\mathbf{D}^{FM}$; note that, as follows from the definitions, $\mathcal{L}_{\mathbf{D}^{FM}_{K_p}} = \mathcal{L}^{FM} \in \mathbb{T}_p$ is stable under base change to K_p.

4.2. **A monodromy module arising from p-adic integration.** The aim of this section is to explain how the theory developed above allows us to construct a monodromy module attached to the space of p-new modular forms $S_k(\Gamma_0(pN^+))^{p-new}$ and the invariant $\mathcal{L}$ introduced in Definition 3.14.

As before, let $\mathbb{T} = \mathbb{T}^{p-new}_{\Gamma_0(pN^+)} \otimes \mathbb{Q}$ and put $\mathbb{T}_p := \mathbb{T} \otimes \mathbb{Q}_p$. For any field extension $L/\mathbb{Q}$, set

(51) $$\mathbf{H}(L) := H^1(\Gamma, C_{har}(\mathbb{V}_n(L)));$$

recalling the identification $\mathbb{V}_n(L) = \mathbf{V}_n(L)$ of (13) whenever L splits $\mathcal{B}$, this notation is in consonance with (35).

Fix a choice of a sign $w_\infty \in \{\pm 1\}$. Since $\mathbf{H}(\mathbb{Q}_p)^{c,\vee,w_\infty}$ is a free module of rank 1 over $\mathbb{T}_p$ by Remark 2.4, we can fix a generator and identify $\mathbf{H}(\mathbb{Q}_p)^{c,\vee,w_\infty} \simeq \mathbb{T}_p$. Exactly as in the proof of Proposition 4.6, we can attach now to $U_p, \mathcal{L} \in \mathbb{T}_p$ a monodromy module

(52) $$\mathbf{D} := D_{U_p,\mathcal{L}} = \mathbf{H}(\mathbb{Q}_p)^{c,\vee,w_\infty} \oplus \mathbf{H}(\mathbb{Q}_p)^{c,\vee,w_\infty}$$

endowed with the filtration, Frobenius and monodromy operators described in loc. cit. In parallel to the preparation of this note, the second author has proved the equality of the $\mathcal{L}$-invariants of the two monodromy modules just introduced in (50) and (52):

Theorem 4.7. [Se2] $\mathcal{L}_{\mathbf{D}} = \mathcal{L}_{\mathbf{D}^{FM}}$.

In Theorem 4.7, note that the definition of both monodromy modules depends on the choice of a branch of the p-adic logarithm. We assume that the same choice has been made for both $\mathbf{D}$ and $\mathbf{D}^{FM}$.

In view of Proposition 4.6, Theorem 4.7 is equivalent to saying that there is an isomorphism $\mathbf{D} \simeq \mathbf{D}^{FM}$ of two-dimensional monodromy $\mathbb{T}_p$-modules over $\mathbb{Q}_p$ (as $U_{\mathbf{D}} = U_{\mathbf{D}^{FM}}$). Let $\mathbf{D}_{K_p} = (\mathbf{H}(k_p)^{c,\vee,w_\infty} \oplus \mathbf{H}(k_p)^{c,\vee,w_\infty}, \varphi \otimes \sigma_{k_p/\mathbb{Q}_p}, N \otimes k_p, F^{\cdot} \otimes K_p)$ denote the base change to K_p of $\mathbf{D}$ in the sense of Remark 4.2. Let

(53) $$\Psi := -\Psi^{\log} \oplus \Psi^{\mathrm{ord}} : H_1(\Gamma, \mathrm{Div}^0(\mathcal{H}_p)(k_p) \otimes \mathbf{P}_n(K_p)) \longrightarrow \mathbf{H}(K_p)^\vee \oplus \mathbf{H}(K_p)^\vee$$

where $\Psi^{\log}$ and Ψ^{ord} are the integration maps introduced in (36), and set

(54) $$\Phi := -\Phi^{\log} \oplus \Phi^{\mathrm{ord}} : H_1(\Gamma, \mathrm{Div}^0(\mathcal{H}_p)(k_p) \otimes \mathbf{P}_n(K_p)) \to \mathbf{D} \otimes K_p$$

for the natural composition of the above map(s) onto $\mathbf{H}(K_p)^{c,\vee,w_\infty}$.

By definition of Φ, the free $\mathbb{T}_{K_p}$-submodule of rank one

$$F^1 = ... = F^m = ... = F^{k-1} := \left\{(-\mathcal{L}x, x) : x \in \mathbf{H}(K_p)^{c,\vee,w_\infty}\right\}$$

of $\mathbf{D} \otimes K_p$ is $\mathrm{Im}(\Phi \circ \partial_2)$.

As it will be useful for our purposes later in the construction of Darmon cycles, let us recall at this point that, thanks to Lemma 3.10, there is a natural isomorphism

(55) $$H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n(K_p)) \simeq \frac{H_1(\Gamma, \mathrm{Div}^0(\mathcal{H}_p) \otimes \mathbf{P}_n(K_p))}{\mathrm{Im}\,\partial_2}.$$

Definition 4.8. The *p-adic Abel-Jacobi maps* are the morphisms

$$\Psi^{\mathrm{AJ}} : H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n(K_p)) \longrightarrow \frac{\mathbf{H}(K_p)^\vee \oplus \mathbf{H}(K_p)^\vee}{\mathrm{Im}\,\Psi \circ \partial_2}$$

and

(56) $$\Phi^{\mathrm{AJ}} = \mathrm{pr}_c \circ \Psi^{\mathrm{AJ}} : H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n(K_p)) \longrightarrow \mathbf{D} \otimes K_p/F^m.$$

induced by Ψ and Φ, respectively, together with the isomorphism (55).

4.3. **An Eichler-Shimura construction.** Let $\mathbb{T}$ be a finite dimensional semisimple commutative algebra over $\mathbb{Q}$. For any algebraic extension $L/\mathbb{Q}$, set $\mathcal{X}_{\mathbb{T}}(L) := \mathrm{Hom}_{\mathbb{Q}\text{-}\mathrm{alg}}(\mathbb{T}, L)$. By an L -valued system of eigenvalues we shall mean an element $\lambda \in \mathcal{X}_{\mathbb{T}}(L)$.

Let H be a $\mathbb{Q}$-vector space endowed with a linear action of $\mathbb{T}$. Given $\lambda \in \mathcal{X}_{\mathbb{T}}(L)$, a λ−eigenvector in H is a non-zero element $f \in H \otimes_{\mathbb{Q}} L$ such that $T \cdot f = \lambda(T)f$ for all $T \in \mathbb{T}$; write $H_\lambda(L)$ for the subspace of $H \otimes_{\mathbb{Q}} L$ spanned by such elements. When $H_\lambda(L) \neq 0$, we say that λ *occurs* in $H \otimes_{\mathbb{Q}} L$. If $\mathbb{T} \subset \mathrm{End}_{\mathbb{Q}}(H)$, all $\lambda \in \mathcal{X}_{\mathbb{T}}(L)$ occur.

The Galois group $G_{\mathbb{Q}}$ acts on $\mathcal{X}_{\mathbb{T}}(\overline{\mathbb{Q}})$ by composition. Given $\lambda \in \mathcal{X}_{\mathbb{T}}(\overline{\mathbb{Q}})$, we write $[\lambda]$ for the orbit of λ under this action. Note that $\ker(\lambda_1) = \ker(\lambda_2)$ if and only if $[\lambda_1] = [\lambda_2]$.

Set $L_\lambda := \lambda(\mathbb{T})$ so that $\lambda \in \mathcal{X}_{\mathbb{T}}(L_\lambda)$, and $L_{[\lambda]} = \prod_{\lambda' \in [\lambda]} L_{\lambda'} \subset \overline{\mathbb{Q}}$; $L_{[\lambda]}/\mathbb{Q}$ is a Galois extension. Set $H_{[\lambda]}(L_{[\lambda]}) := \oplus_{\lambda' \in [\lambda]} H_{\lambda'}(L_{[\lambda]})$; an easy descent argument shows that there exists a $\mathbb{T}$-submodule $H_{[\lambda]} \subset H$ over $\mathbb{Q}$ such that $H_{[\lambda]}(L_{[\lambda]}) = H_{[\lambda]} \otimes_{\mathbb{Q}} L_{[\lambda]}$.

Define $I_{[\lambda]}$ by the exact sequence

$$0 \longrightarrow I_{[\lambda]} \longrightarrow \mathbb{T} \stackrel{\lambda}{\longrightarrow} L_\lambda \longrightarrow 0. \tag{57}$$

Given $\lambda \in \mathcal{X}_{\mathbb{T}}(\bar{\mathbb{Q}})$, let $\iota : H_{[\lambda]} \subset H$ be the natural inclusion and let $(H^\vee)^\lambda = H^\vee / I_{[\lambda]} \cdot H^\vee$ denote the maximal quotient of $H^\vee$ on which $\mathbb{T}$ acts through λ. Then there is a canonical commutative diagram of $\mathbb{T}$-modules with exact rows

$$\begin{array}{ccccccccc} 0 & \to & I_{[\lambda]}(H^\vee) & \to & H^\vee & \to & (H^\vee)^\lambda & \to & 0 \\ & & & & \| & & \downarrow \wr & & \\ & & & & H^\vee & \stackrel{\iota^\vee}{\to} & H^\vee_{[\lambda]} & \to & 0. \end{array} \tag{58}$$

Let now $\mathbb{T} = \mathbb{T}^{p-new}_{\Gamma_0(pN^+)} \otimes \mathbb{Q}$ and let $\mathbf{H}^{c,w_\infty} = H^1(\Gamma, C_{har}(\mathbb{V}_n(\mathbb{Q})))^{c,w_\infty}$ be the module introduced in (51); note that $\mathrm{End}(\mathbf{H}^{c,w_\infty}) = \mathbb{T}$ by Remark 2.4. By Remark 2.5, Lemma 2.8, the Jacquet-Langlands correspondence and the q-expansion principle, $\dim_{L_\lambda}(\mathbf{H}^{c,w_\infty}_\lambda(L_\lambda)) = 1$ for all $\lambda \in \mathcal{X}_{\mathbb{T}}(\bar{\mathbb{Q}})$.

Given a non-zero eigenvector f, write λ_f for the corresponding system of eigenvalues and put $L_f := L_{\lambda_f}$, $I_{[f]} = I_{[\lambda_f]}$ and $\mathbf{H}^{c,w_\infty}_{[f]} = \mathbf{H}^{c,w_\infty}_{[\lambda_f]}$.

Since the category of admissible filtered Frobenius modules over $\mathbb{Q}_p$ is an abelian category and the elements of $I_{[f]}$ act on $\mathbf{D}$, we can introduce the module $\mathbf{D}_{[f]} \in \mathrm{MF}_{\mathbb{Q}_p}(\varphi, N)$ as the one sitting in the exact sequence

$$0 \to I_{[f]}\mathbf{D} \to \mathbf{D} \stackrel{\lambda_f}{\to} \mathbf{D}_{[f]} \to 0.$$

Tensoring (57) with $\mathbb{Q}_p$ over $\mathbb{Q}$ yields an exact sequence

$$0 \to I_{[\lambda],p} \to \mathbb{T}_p \stackrel{\lambda_p}{\to} L_{\lambda,p} \to 0.$$

Since $\mathbb{T}_p \subset \mathrm{End}_{\mathrm{MF}^{ad}_{\mathbb{Q}_p}(\varphi,N)}(\mathbf{D})$, we have $\mathbf{D}_{[f]} = \mathbf{D}/I_{[\lambda],p}\mathbf{D}$ and it follows that $\mathbf{D}_{[f]}$ is canonically a two-dimensional monodromy $L_{f,p}$-module over K_p. Its $\mathcal{L}$-invariant is

$$\mathcal{L}_{[f]} := \lambda_{f,p}(\mathcal{L}) \in L_{f,p} \tag{59}$$

and its U-invariant is $\lambda_{f,p}(U_p) = a_p(f) = \pm p^m$. In the notation of Proposition 4.6,

$$\mathbf{D}_{[f]} = \mathbf{D}_{a_p(f),\mathcal{L}_{[f]}}. \tag{60}$$

Explicitly, $\mathbf{D}_{[f]}$ can be described as the filtered Frobenius monodromy module over $\mathbb{Q}_p$ whose underlying vector space is

$$\mathbf{D}_{[f]} = (\mathbf{H}^{c,w_\infty,\vee}(\mathbb{Q}_p))^{\lambda_f} \oplus (\mathbf{H}^{c,w_\infty,\vee}(\mathbb{Q}_p))^{\lambda_f} \simeq \mathbf{H}^{c,w_\infty,\vee}_{[f]}(\mathbb{Q}_p) \oplus \mathbf{H}^{c,w_\infty,\vee}_{[f]}(\mathbb{Q}_p), \tag{61}$$

where the latter isomorphism arises from (58). The filtration $F^{\cdot}_{[f]}$ is given as in Definition 4.1, where

$$F^m_{[f]} = \left\{(-\mathcal{L}_{[f]}x, x) : x \in \mathbf{H}^{c,w_\infty,\vee}_{[f]}(\mathbb{Q}_p)\right\}. \tag{62}$$

Let $\mathbf{D}_{[f],K_p}$ denote the base change to K_p of $\mathbf{D}_{[f]}$ in the sense of Remark 4.2. As in (54) and (56), we can introduce the map

$$\Phi_{[f]} : H_1(\Gamma, \mathrm{Div}^0(\mathcal{H}_p) \otimes \mathbf{P}_n) \xrightarrow{\Phi} \mathbf{D} \otimes K_p \overset{\lambda_f}{\twoheadrightarrow} \mathbf{D}_{[f]} \otimes K_p \tag{63}$$

and the Abel-Jacobi map

$$\Phi^{\mathrm{AJ}}_{[f]} : H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n) \overset{\Phi^{\mathrm{AJ}}}{\to} \mathbf{D} \otimes K_p/F^m \overset{\lambda_f}{\twoheadrightarrow} \mathbf{D}_{[f],K_p}/F^m_{[f]}. \tag{64}$$

Of course the monodromy module $\mathbf{D}_{[f]}$ canonically decomposes according to $L_{f,p} = \bigoplus_{\mathfrak{p}|p} L_{f,\mathfrak{p}}$, where $L_{f,\mathfrak{p}}$ denotes the completion of L_f at the prime $\mathfrak{p}$ above p:

$$\mathbf{D}_{[f]} = \textstyle\bigoplus_{\mathfrak{p}|p} \mathbf{D}_{[f],\mathfrak{p}}.$$

In the notation of Proposition 4.6, $\mathbf{D}_{[f],\mathfrak{p}} = \mathbf{D}_{a_p(f),\mathcal{L}_{[f],\mathfrak{p}}}$, where $\mathcal{L}_{[f],\mathfrak{p}}$ denotes the $\mathfrak{p}$-component of $\mathcal{L}_{[f]}$. We can further consider $\Phi_{[f],\mathfrak{p}}$ as well as $\Phi^{\mathrm{AJ}}_{[f],\mathfrak{p}}$.

5. Darmon cycles

5.1. **Construction of Darmon homology classes.** The aim of this section is to introduce what we call *Darmon cycles*, that should be regarded as analogues of the classical Heegner cycles attached to imaginary quadratic fields and weight $k \geq 4$ modular forms by Nekovář (c.f. [Ne], [IS]) and of Stark-Heegner points (also called Darmon points in [LRV2]) attached to real quadratic fields and weight 2 modular forms (cf. [Dar], [Gr], [LRV], [LRV2]).

As in the previous sections, fix an even integer $k \geq 2$ and let $n = k-2$, $m = n/2$. Let p be a prime and let N be a positive integer such that $p \mid N$, $p^2 \nmid N$. Let $K/\mathbb{Q}$ be a real quadratic field in which p remains inert. Assume for simplicity that the discriminant D_K of K is prime to N. This induces a factorization of N as $N = pN^+N^-$, where $(N^+, N^-) = 1$ and all prime factors of N^+ (respectively N^-) split (resp. remain inert) in K.

Crucial for our construction is the following *Heegner hypothesis* (see also our general discussion in the introduction), that we assume for the rest of this section.

Assumption. *N^- is the square-free product of an even number of primes.*

In consonance with the notations introduced in §2, let K_p denote the completion of K at p, a quadratic unramified extension of $\mathbb{Q}_p$. Since this field shall be fixed throughout this section and the maximal unramified subextension of K_p is $k_p = K_p$ itself, we shall simply write $\mathcal{H}_p$, $\mathrm{Div}(\mathcal{H}_p)$ and $\mathbf{P}_n$ instead of $\mathcal{H}_p(K_p)$, $\mathrm{Div}(\mathcal{H}_p)(K_p)$ and $\mathbf{P}_n(K_p)$, respectively.

Let $\mathcal{B}$ be the indefinite quaternion algebra of discriminant N^- over $\mathbb{Q}$, $\mathcal{R}$ be a $\mathbb{Z}[1/p]$-Eichler order of level N^+ in $\mathcal{B}$ and Γ be the subgroup of $\mathcal{R}^\times$ of elements of reduced norm 1.

As in §1, fix an embedding $\mathcal{B}^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q}_p)$, that allows us to regard Γ as a subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. Choose also embeddings

$$\sigma_\infty : K \to \mathbb{R} \text{ and } \sigma_p : K \to K_p$$

that we use to regard K as a subfield both of $\mathbb{R}$ and of K_p. In particular we have $D_K^{-\frac{m}{2}} \in K_p$.

Let us denote by $\mathrm{Emb}(K, \mathcal{B})$ the set of $\mathbb{Q}$-algebra embeddings of K into $\mathcal{B}$. Let $\mathcal{O} \subset K$ be a $\mathbb{Z}[1/p]$-order of conductor $c \geq 1$, $(c, N) = 1$, and let

$$\mathrm{Emb}(\mathcal{O}, \mathcal{R}) := \{\Psi : \mathcal{O} \hookrightarrow \mathcal{R} \text{ such that } \Psi(K) \cap \mathcal{R} = \Psi(\mathcal{O})\}$$

be the set of $\mathbb{Z}[1/p]$-optimal embeddings of $\mathcal{O}$ into $\mathcal{R}$. Attached to an embedding $\Psi \in \mathrm{Emb}(\mathcal{O}, \mathcal{R})$ there is the following data:

- the two fixed points $\tau_\Psi, \overline{\tau}_\Psi \in \mathcal{H}_p \cap K$ for the action of $\Psi(K_p^\times)$ on $\mathcal{H}_p \cap K$, labelled in such a way that the action of $K^\times$ on the tangent space at τ_Ψ is given by the character $z \mapsto z/\overline{z}$;
- the unique vertex $v_\Psi \in \mathcal{V}$ which is fixed for the action of $\Psi(K_p^\times)$ on $\mathcal{V}$; we have $v_\Psi = \mathrm{red}(\tau_\Psi) = \mathrm{red}(\overline{\tau}_\Psi)$;
- the unique polynomial up to sign P_Ψ in $\mathbf{P}_2$ that is fixed by the action of $\Psi\left(K_p^\times\right)$ on $\mathbf{P}_2$ and satisfies $\langle P_\Psi, P_\Psi\rangle_{\mathbf{P}_2} = -D_K/4$. We single out one by

$$P_\Psi := \mathrm{Tr}(\Psi(\sqrt{D_K}/2) \cdot \begin{pmatrix} X & -X^2 \\ 1 & -X \end{pmatrix}) \in \mathbf{P}_2;$$

- the stabilizer Γ_Ψ of Ψ in Γ, that is,

$$\Gamma_\Psi = \Psi(K^\times) \cap \Gamma = \Psi(\mathcal{O}_1^\times)$$

  where $\mathcal{O}_1^\times := \{\gamma \in \mathcal{O}^\times, \mathrm{n}(\gamma) = 1\}$;
- the generator $\gamma_\Psi := \Psi(u)$ of $\Gamma_\Psi/\{\pm 1\} \simeq \mathbb{Z}$, where $u \in \mathcal{O}_1^\times$ is the unique generator of $\mathcal{O}_1^\times/\{\pm 1\}$ such that $\sigma(u) > 1$.

For each $\tau \in \mathcal{H}_p$, we say that τ has positive orientation at p if $\mathrm{red}(\tau) \in \mathcal{V}^+$. We write $\mathcal{H}_p^+$ to denote the set of positive oriented elements in $\mathcal{H}_p$. We say that $\Psi \in \mathrm{Emb}(\mathcal{O}, \mathcal{R})$ has positive orientation whenever $v_\Psi \in \mathcal{V}^+$, i.e. $\tau_\Psi, \overline{\tau}_\Psi \in \mathcal{H}_p^+ \cap K$. Put

$$\mathrm{Emb}(\mathcal{O}, \mathcal{R}) = \mathrm{Emb}_+(\mathcal{O}, \mathcal{R}) \sqcup \mathrm{Emb}_-(\mathcal{O}, \mathcal{R})$$

with the obvious meaning. The group Γ acts on $\mathrm{Emb}(\mathcal{O}, \mathcal{R})$ by conjugation, preserving orientations.

The Γ_Ψ-module $K_p \cdot (\tau_\Psi \otimes D_K^{-m/2}) P_\Psi^m \subset \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n$ is endowed with the trivial Γ_Ψ-action (see the computation (65) below). Hence, the choice of the generator γ_Ψ for the cyclic group Γ_Ψ allow us to fix an identification $K_p = H_1(\Gamma_\Psi, K_p \cdot \tau_\Psi \otimes D_K^{-m/2} P_\Psi^m)$. The inclusion $K_p \cdot \tau_\Psi \otimes D_K^{-m/2} P_\Psi^m \subset \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n$ then induces the *cycle class map*

$$cl_\Psi : K_p = H_1(\Gamma_\Psi, K_p \cdot \tau_\Psi \otimes D_K^{-m/2} P_\Psi^m) \to H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n).$$

The group $H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n)$ should be regarded as a substitute of the local Chow group in our real quadratic setting. See 1 for more on this analogy. With this in mind we make the following definition.

Definition 5.1. The *Darmon cycle* attached to an embedding $\Psi \in \mathrm{Emb}(\mathcal{O}, \mathcal{R})$ is

$$y_\Psi := cl_\Psi(1) \in H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n).$$

Note that the Darmon cycle y_Ψ is represented by $\gamma_\Psi \otimes \tau_\Psi \otimes D_K^{-m/2} P_\Psi^m$.

Lemma 5.2. *The homology class $y_\Psi \in H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n)$ does not depend on the choice of Ψ in its conjugacy class of optimal embeddings for the action of Γ.*

Proof. Let $\gamma \in \Gamma$. The assignation $\Psi \mapsto (\tau_\Psi, P_\Psi, \gamma_\Psi)$ behaves under conjugation by γ as

$$(\tau_{\gamma\Psi\gamma^{-1}}, P_{\gamma\Psi\gamma^{-1}}, \gamma_{\gamma\Psi\gamma^{-1}}) = (\gamma\tau_\Psi, \gamma P_\Psi := P_\Psi\gamma^{-1}, \gamma\gamma_\Psi\gamma^{-1}). \tag{65}$$

from what it follows that

$$cl_\Psi(1) = \tau_\Psi \otimes D_K^{-m/2} P_\Psi^m \otimes [\gamma_\Psi] = \gamma \cdot \tau_\Psi \otimes \gamma \cdot D_K^{-m/2} P_\Psi^m \otimes [\gamma\gamma_\Psi\gamma^{-1}] = cl_{\gamma\Psi\gamma^{-1}}(1).$$

□

As a consequence of Lemma 5.2, there is a well-defined morphism

$$y : \Gamma\backslash\mathrm{Emb}(\mathcal{O}, \mathcal{R}) \to H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n)$$

attaching a Darmon cycle $y_{[\Psi]} := y_\Psi$ to any conjugacy class $[\Psi]$ of optimal embeddings. Invoke now the Abel-Jacobi map

$$H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n) \overset{\Phi^{\mathrm{AJ}}}{\to} \mathbf{D} \otimes K_p/F^m$$

introduced in (56).

Definition 5.3. The *Darmon cohomology class* attached to $[\Psi] \in \Gamma\backslash\mathrm{Emb}(\mathcal{O}, \mathcal{R})$ is $s_{[\Psi]} := \Phi^{\mathrm{AJ}}(y_{[\Psi]}) \in \mathbf{D} \otimes K_p/F^m$.

Remark 5.4. This construction can also be formulated from a different (but equivalent) point of view, which reinforces the analogy with the classical case of imaginary quadratic fields. Namely, let $\mathcal{H}_p^{\mathcal{O}} = \{\tau \in \mathcal{H}_p : \tau = \tau_\Psi \text{ for some } \Psi \in \mathrm{Emb}(\mathcal{O}, \mathcal{R})\}$. Note that there is a well-defined action of Γ on $\mathcal{H}_p^{\mathcal{O}}$. With this notation, the above formalism yields a map

$$(66) \qquad d : \Gamma\backslash\mathcal{H}_p^{\mathcal{O}} \overset{\tau_\Psi \mapsto \Psi}{\longrightarrow} \Gamma\backslash\mathrm{Emb}(\mathcal{O}, \mathcal{R}) \overset{y}{\longrightarrow} H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n) \overset{\Phi^{\mathrm{AJ}}}{\longrightarrow} \mathbf{D} \otimes K_p/F^m.$$

Remark 5.5. For every prime $\ell \mid pN^+N^-$, let $\omega_\ell \in \mathcal{R}_0(N^+)$ be an element of reduced norm ℓ lying in the normalizator of Γ. Conjugation by ω_ℓ induces an involution W_ℓ on $\Gamma\backslash\mathrm{Emb}(\mathcal{O}, \mathcal{R})$ given by $W_\ell(\Psi) = \omega_\ell \Psi \omega_\ell^{-1}$.

Besides, conjugation by ω_ℓ also induces an involution W_ℓ both on $H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n)$ and on $\mathbf{D} \otimes K_p/F^m$, as already mentioned in 2.1. It follows as in the proof of Lemma 5.2 and the Hecke equivariance of Φ^{AJ} that there are commutative diagrams

$$\begin{array}{ccccc} d : \Gamma\backslash\mathrm{Emb}(\mathcal{O}, \mathcal{R}) & \to & H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n) & \overset{\Phi^{\mathrm{AJ}}}{\to} & \mathbf{D} \otimes K_p/F^m \\ \downarrow W_\ell & & \downarrow W_\ell & & \downarrow W_\ell \\ d : \Gamma\backslash\mathrm{Emb}(\mathcal{O}, \mathcal{R}) & \to & H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n) & \overset{\Phi^{\mathrm{AJ}}}{\to} & \mathbf{D} \otimes K_p/F^m. \end{array}$$

Recall that an orientation on the Eichler order $\mathcal{R}$ (resp. on the quadratic order $\mathcal{O}$) is the choice, for each $\ell \mid N^+N^-$, of a ring homomorphism $\mathcal{R} \to k_\ell$ (resp. $\mathcal{O} \to k_\ell$), where $k_\ell = \mathbb{F}_{\ell^2}$ (resp. $k_\ell = \mathbb{F}_\ell$) for $\ell \mid N^-$ (resp. $\ell \mid N^+$).

Fix orientations both on $\mathcal{O}$ and on $\mathcal{R}$. An optimal embedding $\Psi : \mathcal{O} \to \mathcal{R}$ is *oriented* if, for all $\ell \mid N^+N^-$, $\Psi \otimes k_\ell$ commutes with the chosen local orientations on $\mathcal{O} \otimes k_\ell$ and $\mathcal{R} \otimes k_\ell$, respectively. Write $\overrightarrow{\mathrm{Emb}}_+(\mathcal{O}, \mathcal{R}) \subset \mathrm{Emb}_+(\mathcal{O}, \mathcal{R})$ for the set of oriented positive optimal embeddings. The action of Γ on $\mathrm{Emb}_+(\mathcal{O}, \mathcal{R})$ leaves $\overrightarrow{\mathrm{Emb}}_+(\mathcal{O}, \mathcal{R})$ stable and thus induces a well-defined action on it.

By Eichler's theory of optimal embeddings, $\overrightarrow{\mathrm{Emb}}_+(\mathcal{O}, \mathcal{R})$ is not empty and the quotient $\Gamma\backslash\overrightarrow{\mathrm{Emb}}_+(\mathcal{O}, \mathcal{R})$ is endowed with a free transitive action of the narrow class group $\mathrm{Pic}(\mathcal{O})$ of the $\mathbb{Z}[\frac{1}{p}]$-order $\mathcal{O}$ (cf. e.g. [Vi, Ch. III, §5C]). Denote this action by

$$([\mathfrak{a}], [\Psi]) \mapsto [\mathfrak{a} \star \Psi], \qquad \text{for } [\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}), \Psi \in \overrightarrow{\mathrm{Emb}}_+(\mathcal{O}, \mathcal{R}).$$

Artin's reciprocity map of global class field theory provides an isomorphism

$$\mathrm{rec} : \mathrm{Pic}(\mathcal{O}) \overset{\simeq}{\to} \mathrm{Gal}\,(H_{\mathcal{O}}/K),$$

where $H_{\mathcal{O}}$ stands for the narrow ring class field attached to $\mathcal{O}$. In order to state our conjectures it is convenient to introduce the following linear combinations of Darmon cycles.

Definition 5.6. Let $\chi : \mathrm{Gal}\,(H_{\mathcal{O}}/K) \to \mathbb{C}^\times$ be a character. The *Darmon cycle* attached to the character χ is

$$y_\chi := \sum_{\sigma \in \mathrm{Gal}\,(H_{\mathcal{O}}/K)} \chi^{-1}(\sigma) y_{[\mathrm{rec}^{-1}(\sigma) \star \Psi]} \in H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n) \otimes K_p(\chi),$$

where $[\Psi]$ is any choice of a class of optimal embeddings in $\Gamma\backslash\overrightarrow{\mathrm{Emb}}_+(\mathcal{O},\mathcal{R})$ and $K_p(\chi)$ is the field generated by the (algebraic) values of χ over K_p. Write

$$s_\chi := \Phi^{\mathrm{AJ}}(y_\chi) \in \mathbf{D}\otimes K_p(\chi)/F^m \otimes K_p(\chi).$$

5.2. **A conjecture on the global rationality of Darmon cycles.** Keep the notations and hypotheses of 5.1. As in §1 and §4.1, let $V_p := H_p(\mathcal{M}_n)^{p-new}$, that we regard this time as a semistable continuous representation of G_{K_p}, by restricting the action of $G_K \subset G_{\mathbb{Q}}$ to the decomposition subgroup of a prime $\bar{\wp}$ of $\bar{\mathbb{Q}}$ over p.

In this section we show how Theorem 4.7 allows us to attach to each Darmon cycle y_Ψ a class s_Ψ in the group $H^1_{\mathrm{st}}(K_p, V_p)$ of local semistable cohomology classes. Cf. (1), or rather [Ne2], for the definition of this group.

In [BK], Bloch and Kato introduced an exponential map which, in the case that concerns us here, induces an isomorphism

$$\exp : \frac{\mathbf{D}^{FM}\otimes K_p}{\mathrm{Fil}^m(\mathbf{D}^{FM}\otimes K_p)} \overset{\simeq}{\to} H^1_{\mathrm{st}}(K_p, V_p), \tag{67}$$

as follows from [IS, Lemma 2.1].

Keeping the notation of §4, assume Conjecture 4.7 and fix an isomorphism $\mathbf{D}^{FM} \simeq \mathbf{D}$ of two-dimensional monodromy $\mathbb{T}_p$-modules over $\mathbb{Q}_p$. The choice of this isomorphism induces an identification

$$\frac{\mathbf{D}\otimes K_p}{\mathrm{Fil}^m(\mathbf{D}^{FM}\otimes K_p)} = \frac{\mathbf{D}^{FM}\otimes K_p}{\mathrm{Fil}^m(\mathbf{D}\otimes K_p)}. \tag{68}$$

In view of (67) and (68), we may regard the Darmon cohomology classes introduced above as cocycles

$$s_\Psi \in H^1_{\mathrm{st}}(K_p, V_p), \quad s_\chi \in H^1_{\mathrm{st}}(K_p(\chi), V_p), \tag{69}$$

for any optimal embedding $\Psi \in \mathrm{Emb}(\mathcal{O},\mathcal{R})$ and any character $\chi : \mathrm{Gal}\,(H_{\mathcal{O}}/K) \to \mathbb{C}^\times$, respectively.

The prime p splits completely in the narrow ring class field $H_{\mathcal{O}}$. Choose and fix once and for all an embedding $\iota_p : H_{\mathcal{O}} \hookrightarrow K_p$. This choice induces a restriction morphism

$$\mathrm{res}_p : H^1_{\mathrm{st}}(H_{\mathcal{O}}, V_p) \longrightarrow H^1_{\mathrm{st}}(K_p, V_p) \simeq \frac{\mathbf{D}^{FM}\otimes K_p}{\mathrm{Fil}^m(\mathbf{D}^{FM}\otimes K_p)} \simeq \frac{\mathbf{D}\otimes K_p}{\mathrm{Fil}^m(\mathbf{D}\otimes K_p)}$$

as in (3). The image of the global Selmer group $H^1_{\mathrm{st}}(H_{\mathcal{O}}, V_p)$ is a $\mathbb{T}_p$-submodule of $\frac{\mathbf{D}\otimes K_p}{\mathrm{Fil}^m(\mathbf{D}\otimes K_p)}$. By Lemma 4.4 every automorphism of $\mathbf{D}$ acts on $\mathbf{D}/\mathrm{Fil}^m\,\mathbf{D}$ by multiplication by an element in $\mathbb{T}_p$. It follows that the image of $H^1_{\mathrm{st}}(H_{\mathcal{O}}, V_p)$ in $\frac{\mathbf{D}\otimes K_p}{\mathrm{Fil}^m(\mathbf{D}\otimes K_p)}$ does not depend on the choice of the isomorphism $\mathbf{D} \simeq \mathbf{D}^{FM}$.

Conjecture 5.7. (i) *For any optimal embedding $\Psi \in \mathrm{Emb}(\mathcal{O},\mathcal{R})$ there is a global cohomology class $\underline{s}_\Psi \in H^1_{\mathrm{st}}(H_{\mathcal{O}}, V_p)$ such that*

$$s_\Psi = \mathrm{res}_p(\underline{s}_\Psi)$$

(ii) *For any $\Psi \in \overrightarrow{\mathrm{Emb}}_+(\mathcal{O},\mathcal{R})$ and any ideal class $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$,*

$$\mathrm{res}_p({}^{\sigma}\underline{s}_\Psi) = s_{\mathfrak{a}\star\psi},$$

where $\sigma = \mathrm{rec}(\mathfrak{a})^{-1} \in \mathrm{Gal}\,(H_{\mathcal{O}}/K)$.

(iii) *For any character $\chi : \mathrm{Gal}\,(H_{\mathcal{O}}/K) \to \mathbb{C}^\times$, $s_\chi = \mathrm{res}_p(\underline{s}_\chi)$ for some $\underline{s}_\chi \in H^1_{\mathrm{st}}(H_\chi, V_p)^\chi$, where H_χ/K is the abelian sub-extension of $H_{\mathcal{O}}/K$ cut out by χ, and $H^1_{\mathrm{st}}(H_\chi, V_p)^\chi$ stands for its χ-isotypical subspace.*

Notice that (iii) is an immediate consequence of (1) and (2) above.

In light of (4), one can go still further and conjecture that for any optimal embedding $\Psi \in \mathrm{Emb}(\mathcal{O}, \mathcal{R})$ there exists an algebraic cycle $Z_\Psi^? \in \mathrm{CH}^{m+1}(\mathcal{M}_n \otimes H_\mathcal{O})_0 \otimes \mathbb{Q}_p$ such that $cl_0^{m+1}(Z_\Psi^?) = \underline{s}_\Psi$ and hence $\mathrm{res}_p(cl_0^{m+1}(Z_\Psi^?)) = s_\Psi$, satisfying a Galois reciprocity law as in (ii). We leave to the reader the task of rephrasing the two conjectures below in terms of algebraic cycles, in the same spirit as above.

Let $f \in S_k(\Gamma_0(pN^+))^{p-new}$ be a p-new eigenform. By means of the p-adic Abel-Jacobi map $\Phi_{[f]}^{\mathrm{AJ}}$ introduced in (64), we can specialize the above constructions to the f -eigencomponent $V_p(f)$ of V_p. Conjecture 5.7 then predicts the existence of global cohomology classes

(70) $$\underline{s}_{\Psi,f} \in^\chi (H_\mathcal{O}, V_p(f)) \text{ and } \underline{s}_f^\chi \in H^1_{\mathrm{st}}(H_\chi, V_p(f))^\chi,$$

such that $\mathrm{res}_p(\underline{s}_{\Psi,f}) = s_{\Psi,f}$, $\mathrm{res}_p(\underline{s}_f^\chi) = s_{\Psi,f}^\chi$ and satisfying an explicit reciprocity law as in Conjecture 5.7 (2). The p-adic representation $V_p(f)$ canonically decomposes according to $L_{f,p} = \bigoplus_{\mathfrak{p}|p} L_{f,\mathfrak{p}}$, where $L_{f,\mathfrak{p}}$ denotes the completion of L_f at the prime $\mathfrak{p}$ above p:

$$V_p(f) = \bigoplus\nolimits_{\mathfrak{p}|p} V_\mathfrak{p}(f).$$

Write $\underline{s}_{f,\mathfrak{p}}^\chi$ for the corresponding conjectural cohomology class and write $s_{f,\mathfrak{p}}^\chi$ for the one obtained from the Darmon cohomology class s_f^χ (that can be directly defined by means of $\Phi_{[f],\mathfrak{p}}^{\mathrm{AJ}}$). In light of the results achieved in [Ko] and [Ne] for classical Heegner points and cycles in the imaginary quadratic setting, it seems reasonable to formulate the following conjecture.

Conjecture 5.8. *Assume* $s_{f,\mathfrak{p}}^\chi \neq 0$. *Then*

$$H^1_{\mathrm{st}}(H_\chi, V_\mathfrak{p}(f))^\chi = L_{f,\mathfrak{p}} \underline{s}_{f,\mathfrak{p}}^\chi.$$

Note that Conjecture 5.8 predicts that, although the map res_p above may not be injective, the global cohomology class $\underline{s}_f^\chi$ is determined by s_f^χ whenever $s_f^\chi \neq 0$. In particular, res_p would induce an isomorphism

$$H^1_{\mathrm{st}}(H_\chi, V_\mathfrak{p}(f))^\chi = L_{f,\mathfrak{p}} \underline{s}_{f,\mathfrak{p}}^\chi \xrightarrow{\simeq} L_{f,\mathfrak{p}} s_{f,\mathfrak{p}}^\chi.$$

It is also possible to formulate Gross-Zagier type conjectures for these cycles, although a proof of them seems to be a long way off, as even their counterparts for classical Heegner cycles remain completely open.

Conjecture 5.9.

$$\underline{s}_{f,\mathfrak{p}}^\chi \neq 0 \iff L'(f/K, \chi, k/2) \neq 0$$

and in particular

$$s_{f,\mathfrak{p}}^\chi \neq 0 \;\Rightarrow\; L'(f/K, \chi, k/2) \neq 0.$$

Note that the second statement in the above conjecture makes sense even when it is not known that there exists a global cohomology class $\underline{s}_f^\chi$ inducing s_f^χ as predicted by Conjecture 5.7. See [LRV2] for a proof of an avatar of this formula for Darmon points, where $k = 2$, s_f^χ is replaced by its image on a suitable group of connected components and $L'(f/K, \chi, 1)$ is replaced by the (comparatively much simpler) special value $L(f_0/K, \chi, 1)$ of the L-function of an eigenform $f_0 \in S_2(\Gamma_0(N^+))$.

6. Particular cases

The circle of ideas in this manuscript specialize, in the particular cases of $k = 2$ or $N^- = 1$, to scenarios that can be tackled by means of finer, simpler methods, as we now describe.

For $k = 2$ and any $N^- \geq 1$, the p-adic integration theory of §3 admits a much finer *multiplicative* version, that allows to introduce p-adic Darmon points on Jacobians of Shimura curves, as envisaged by first time by Darmon in [Dar], and completed later in [Das], [Gr], [DG], [LRV] and [LRV2]. We briefly recall these results in §6.1 below.

For $N^- = 1$ and any $k \geq 2$, the presence of cuspidal points on the classical modular curve $X_0(pN^+)$ allows for a re-interpretation of the whole theory in terms of modular symbols. This was again first visioned by Darmon in [Dar], for $k = 2$.

In §6.2 we develop the theory for $k > 2$ in the language of modular symbols, in a way that shall be employed in the forthcoming work [Se]. From this point of view, the p-adic integration theory may be viewed as a lift of Orton's integration theory [Or]. By means of this approach, the work [Se] of the second author offers a nontrivial partial result towards the conjectures posed in §5.2 for $N^- = 1$.

Below, we treat separately the cases $k = 2$ and $N^- = 1$. For the sake of simplicity in the exposition, we leave aside the overlapping case $N^- = 1$, $k = 2$: this is the setting considered in the original paper [Dar] of Darmon, where both methods converge.

6.1. **The case $N^- > 1$ and $k = 2$.** Assume, only for this section, that $N^- > 1$ and $k = 2$. Thus $m = n = 0$. We proved in Theorem 3.11 that there is a surjective homomorphism

$$\mathrm{pr}_c \circ \Psi^{\mathrm{ord}} \circ \partial_2 : H_2(\Gamma, K_p) \to (\mathbf{H}^c)^\vee$$

that yields an isomorphism when restricted to $H_2(\Gamma, K_p)^c$. Notice that, since $k = 2$, $\mathbf{H}^{\mathrm{Eis}}$ is not trivial (take Eisenstein parts of (20)). However, since $N^- > 1$, it follows from (15) and Lemma 2.8 that $\mathbf{H}^c \simeq H^1(\Gamma_0(pN^+), K_p)^{p-new}$.

The theory developed in [Gr], [DG] and [LRV] shows that there is a multiplicative refinement of the above, as we now briefly recall. Setting

$$T^\star(K_p) := \mathrm{Hom}(H_1(\Gamma_0(pN^+), \mathbb{Z})^{p-new}, K_p^\times),$$

it is shown in [LRV, §5] that there is a Hecke-equivariant multiplicative integration map

$$\Phi^0 : H_1(\Gamma, \mathrm{Div}^0(\mathcal{H}_p)) \to T^\star(K_p)$$

such that $\Phi^{\mathrm{ord}} = \mathrm{ord} \circ \Phi^0$ and $\Phi^{\log} = \log \circ \Phi^0$, up to extending scalars from $\mathbb{Z}$ to K_p. Similarly as in (39), let

$$\partial_2^0 : H_2(\Gamma, \mathbb{Z}) \to H_1(\Gamma, \mathrm{Div}^0(\mathcal{H}_p))$$

denote the boundary morphism such that $\partial_2 = \partial_2^0 \otimes_{\mathbb{Z}} K_p$ and one may define

$$L_0 := \mathrm{Im}(\Phi^0 \circ \partial_2^0) \subset T^\star(K_p)$$

It is shown in [LRV, §6] that L_0 is a lattice in $T^\star(\mathbb{Q}_p)$; hence, one may define the rigid analytic torus $J := \frac{T^\star(\mathbb{Q}_p)}{L_0}$ over $\mathbb{Q}_p$. There is a natural action of the involution W_∞ on J which allows to split the torus $J \sim J^+ \times J^-$ up to an isogeny of 2-power degree. The two factors J^+ and J^- are in fact isogenous and the main results of [DG] and [LRV] show that J^+ admits a Hecke-equivariant isogeny with $\mathrm{Jac}(X_0^{N^-}(pN^+))^{p-new}$ over the quadratic unramified extension K_p of $\mathbb{Q}_p$. This is achieved by proving that the $\mathcal{L}$-invariants of J^+ and of $\mathrm{Jac}(X_0^{N^-}(pN^+))^{p-new}$, in the sense of Tate-Morikawa's uniformization theory, are equal. This is a particular instance of Theorem 4.7 above.

Let now K be a real quadratic field in which p is inert, so that K_p is isomorphic to the completion of K at p. Using the above results, it is possible to attach a Darmon point $y_\Psi \in \mathrm{Jac}(X_0^{N^-}(pN^+))^{p-new}(K_p)$ to each optimal embedding $\Psi : \mathcal{O} \hookrightarrow \mathcal{R}$ as in §5.1; see [Gr, §10], [DG] and [LRV2, §3] for full details and for the precise statement of the conjecture that is the analogue of Conjecture 5.7.

6.2. **The case $N^- = 1$ and $k > 2$.** Let $\Delta := \mathrm{Div}\,\mathbb{P}^1(\mathbb{Q})$ and $\Delta^0 := \mathrm{Div}^0\,\mathbb{P}^1(\mathbb{Q})$ be respectively the space of divisors and degree zero divisors supported to the cusps with coefficients in K_p, so that

$$0 \to \Delta^0 \to \Delta \to K_p \to 0. \tag{71}$$

For any K_p-vector space A endowed with an action by $G \subset \mathrm{GL}_2(\mathbb{Q})$ set $\mathcal{BS}(A) := \mathrm{Hom}(\Delta, A)$ and $\mathcal{MS}(A) := \mathrm{Hom}(\Delta^0, A)$, endowed with the natural induced actions. Then there is a canonical exact sequence

$$0 \to A \to \mathcal{BS}(A) \to \mathcal{MS}(A) \to 0. \tag{72}$$

We also write $\mathcal{BS}_G(A) := \mathcal{BS}(A)^G$ and $\mathcal{MS}_G(A) := \mathcal{MS}(A)^G$ to denote the G-invariants.

When $A = \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b$ and $A = C_{har}(\mathbf{V}_n(K_p))$, the corresponding exact sequences are connected by the morphisms induced by the morphism r introduced in §3.1. Taking the long exact sequences induced in Γ-cohomology we find:

Proposition 6.1. *There is a commutative diagram*

$$\begin{array}{ccc} \mathcal{MS}_\Gamma(\mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b) & \stackrel{\delta}{\to} & H^1(\Gamma, \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b) \\ r \downarrow & & \downarrow r \\ \mathcal{MS}_\Gamma(C_{har}(\mathbf{V}_n(K_p))) & \stackrel{\delta}{\to} & H^1(\Gamma, C_{har}(\mathbf{V}_n(K_p))) \end{array} \tag{73}$$

where both vertical maps r and the cuspidal part δ^c of the lower horizontal map are isomorphisms.

Proof. By Theorem 3.5, the right vertical arrow is an isomorphism. Exploiting the isomorphisms

$$\begin{array}{ccc} \mathcal{MS}_\Gamma(C_{har}(\mathbf{V}_n(K_p))) & \hookrightarrow & \mathcal{MS}_\Gamma(C_0(\mathcal{E}, \mathbf{V}_n(K_p))) \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{MS}_{\Gamma_0(pN^+)}(\mathbf{V}_n(K_p))^{p-new} & \hookrightarrow & \mathcal{MS}_{\Gamma_0(pN^+)}(\mathbf{V}_n(K_p)) \end{array} \tag{74}$$

provided by Shapiro's lemma as in (21), a similar but simpler argument shows that the left vertical arrow is also an isomorphism (see [Se, Proposition 2.8] for details). As for the lower horizontal arrow, the Eichler-Shimura isomorphism factors as the composition

$$ES : S_k(\Gamma_0(pN^+)) \otimes_{\mathbb{R}} \mathbb{C} \stackrel{\simeq}{\to} \mathcal{MS}_{\Gamma_0(pN^+)}(\mathbf{V}_n(\mathbb{C}))^c \stackrel{\delta^c}{\to} H^1(\Gamma_0(pN^+), \mathbf{V}_n)^c. \tag{75}$$

Here the morphism δ appearing in (75) is obtained from (72) with $A = \mathbf{V}_n(\mathbb{C})$ and $G = \Gamma_0(pN^+)$. It follows from this description that the morphism δ^c obtained from δ in (73) is identified with the morphism obtained from δ^c in (75) by taking the p-new parts. Since ES is an isomorphism, δ^c in (75) is an isomorphism and the p-new parts of the source and the target are identified by the Hecke equivariance of δ^c; for this reason the lower δ in (73) induces an isomorphism between the cuspidal parts.

Finally, the commutativity of the diagram follows from a rather tedious but elementary diagram-chasing computation. □

Set $\mathbf{MS}(K_p) := \mathcal{MS}_\Gamma(C_{har}(\mathbf{V}_n(K_p)))$. Since the boundary morphisms δ in Proposition 6.1 are Hecke equivariant, they induce an isomorphism $\delta^c : \mathbf{MS}(K_p)^c \stackrel{\simeq}{\to} \mathbf{H}(K_p)^c$ between the cuspidal parts. There is a commutative diagram

$$\begin{array}{ccccc} H_2(\Gamma, \mathbf{P}_n(K_p)) & \stackrel{\partial_2}{\to} & H_1(\Gamma, \mathrm{Div}^0(\mathcal{H}_p)(k_p) \otimes \mathbf{P}_n(K_p)) & \stackrel{\Psi^{\log}, \Psi^{\mathrm{ord}}}{\longrightarrow} & \mathbf{H}(K_p)^\vee \\ \downarrow & & \downarrow & & \downarrow \delta^\vee \\ H_1(\Gamma, \Delta^0 \otimes \mathbf{P}_n(K_p)) & \stackrel{\partial_1}{\to} & (\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p)(k_p) \otimes \mathbf{P}_n(K_p))_\Gamma & \stackrel{\Psi^{\log}_{\mathcal{MS}}, \Psi^{\mathrm{ord}}_{\mathcal{MS}}}{\longrightarrow} & \mathbf{MS}(K_p)^\vee. \end{array} \tag{76}$$

Here, the morphisms $\Psi^{\log}, \Psi^{\mathrm{ord}}$ in the top row are the ones introduced in Section 3.2. Similarly, the corresponding maps $\Psi^{\log}_{\mathcal{MS}}, \Psi_{\mathcal{MS}}$ in the lower row are obtained by performing the obvious formal modifications in the definition of the pairings in Definition 3.8 and in (36).

The connecting map ∂_1 arises from the long exact sequence in homology associated to (38), tensored with $\Delta^0 \otimes \mathbf{P}_n(K_p)$. Quite similarly, the first (second) vertical arrow is the connecting map arising in the long exact sequence induced by the short exact sequence (71) tensored with $\mathbf{P}_n(K_p)$ (respectively, tensored with $\mathrm{Div}^0(\mathcal{H}_p)(k_p) \otimes \mathbf{P}_n(K_p)$).

By Theorem 3.11, $(\Psi^{\mathrm{ord}} \circ \partial_2)^c : H_2(\Gamma, \mathbf{P}_n(K_p))^c \xrightarrow{\simeq} (\mathbf{H}(K_p)^c)^\vee$ is an isomorphism. The same circle of ideas appearing in the proof of this theorem, paying care to the Eisenstein subspaces, shows that

$$(\Psi^{\mathrm{ord}}_{\mathcal{MS}} \circ \partial_1)^c : H_1(\Gamma, \Delta^0 \otimes \mathbf{P}_n(K_p))^c \xrightarrow{\simeq} (\mathbf{MS}(K_p)^c)^\vee \tag{77}$$

is also an isomorphism.

It then follows from the isomorphism $\delta^c : \mathbf{MS}(K_p)^c \xrightarrow{\simeq} \mathbf{H}(K_p)^c$ that the left vertical arrow also induces an isomorphism $H_2(\Gamma, \mathbf{P}_n(K_p))^c \simeq H_1(\Gamma, \Delta^0 \otimes \mathbf{P}_n(K_p))^c$. This is helpful, because it allows to construct an L-invariant $\mathcal{L}$, as the one already introduced in Definition 3.14, purely in terms of modular symbols, as we now explain.

Let as before $\mathrm{pr}_c : \mathbf{MS}(K_p)^\vee \longrightarrow (\mathbf{MS}(K_p)^c)^\vee$ denote the natural projection and write $\Phi^*_{\mathcal{MS}} = \mathrm{pr}_c \circ \Psi^*_{\mathcal{MS}}$ for either $* = \log$ or ord. In light of (77) and reasoning exactly as in the proof of Corollary 3.13, there exists a unique endomorphism $\mathcal{L}_{\mathcal{MS}} \in \mathrm{End}_{\mathbb{T}_p}((\mathbf{MS}(\mathbb{Q}_p)^c)^\vee)$ such that

$$\Phi^{\log}_{\mathcal{MS}} \circ \partial_1 = \mathcal{L}_{\mathcal{MS}} \circ \Phi^{\mathrm{ord}}_{\mathcal{MS}} \circ \partial_1 : H_1(\Gamma, \Delta^0 \otimes \mathbf{P}_n(K_p)) \to (\mathbf{MS}(K_p)^c)^\vee. \tag{78}$$

The invariants $\mathcal{L}$ and $\mathcal{L}_{\mathcal{MS}}$ are equal, as follows from (76), (77) and the definition of the L-invariants. On the f-isotypic component $\mathcal{L}_{\mathcal{MS}}$ specializes to the Orton $\mathcal{L}$-invariant (see [Or]). Hence they induce isomorphic monodromy modules. Indeed, let $w_\infty \in \{\pm 1\}$ be a choice of a sign and define a monodromy module

$$\mathbf{D}_{\mathcal{MS}} = \mathbf{D}^{w_\infty}_{\mathcal{MS}} := \mathbf{MS}(\mathbb{Q}_p)^{c,\vee,w_\infty} \oplus \mathbf{MS}(\mathbb{Q}_p)^{c,\vee,w_\infty} \tag{79}$$

over $\mathbb{Q}_p$ as in (52), providing it with a structure of filtered Frobenius module by formally replacing $\mathbf{H}$ by $\mathbf{MS}$, and $\mathcal{L}$ by $\mathcal{L}_{\mathcal{MS}}$. It follows from the discussion above and the explicit description of both monodromy modules that the isomorphism $\delta^c : \mathbf{MS}^c \xrightarrow{\simeq} \mathbf{H}^c$ induces an isomorphism

$$\mathbf{D} \xrightarrow{\simeq} \mathbf{D}_{\mathcal{MS}}. \tag{80}$$

Finally, we conclude this section by showing how the Darmon cycles that were introduced in §5.1 can also be recovered by means of the theory of modular symbols when $N^- = 1$; this point of view is of fundamental importance in [Se]. As in (54), set

$$\Phi_{\mathcal{MS}} := -\Phi^{\log}_{\mathcal{MS}} \oplus \Phi^{\mathrm{ord}}_{\mathcal{MS}} : (\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p) \otimes \mathbf{P}_n(K_p))_\Gamma \longrightarrow \mathbf{D}_{\mathcal{MS}}(K_p).$$

As in Definition 4.8 and in (56), we would like to be able to use $\Phi_{\mathcal{MS}}$ to construct a morphism $\Phi^{\mathrm{AJ}}_{\mathcal{MS}} : (\Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p)(k_p) \otimes \mathbf{P}_n(K_p))_\Gamma \longrightarrow \mathbf{D}_{\mathcal{MS}}/F^m$. There is however a slight complication here, as $(\Delta^0 \otimes \mathbf{P}_n(K_p))_\Gamma$ is not trivial. The reader may like to compare this situation with the one encountered in §4.2, where the counterpart of $(\Delta^0 \otimes \mathbf{P}_n(K_p))_\Gamma$ is $H^1(\Gamma, \mathbf{V}_n)$, that is trivial by Lemma 3.10. This motivates the following definition.

Definition 6.2. A p-adic Abel-Jacobi map with respect to $\Phi_{\mathcal{MS}}$ is a morphism

$$\Phi^{\mathrm{AJ}}_{\mathcal{MS}} : (\Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p)(k_p) \otimes \mathbf{P}_n(K_p))_\Gamma \longrightarrow \mathbf{D}_{\mathcal{MS}} \otimes K_p/F^m \tag{81}$$

such that the natural diagram

$$\begin{array}{ccc} (\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p) \otimes \mathbf{P}_n(K_p))_\Gamma & \xrightarrow{\Phi_{\mathcal{MS}}} & \mathbf{D}_{\mathcal{MS}}(K_p) \\ \downarrow & & \downarrow \\ (\Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p)(k_p) \otimes \mathbf{P}_n(K_p))_\Gamma & \xrightarrow{\Phi^{\mathrm{AJ}}_{\mathcal{MS}}} & \mathbf{D}_{\mathcal{MS}}(K_p)/F^m \end{array}$$

is commutative.

Such morphisms exist, but they are not unique. Using now the notation introduced in §5.1, there is a diagram

$$(82) \qquad \begin{array}{ccccc} \Gamma\backslash\mathrm{Emb}(\mathcal{O},\mathcal{R}) & \xrightarrow{y} & H_1(\Gamma, \mathrm{Div}(\mathcal{H}_p)(k_p) \otimes \mathbf{P}_n(K_p)) & \xrightarrow{\Phi^{\mathrm{AJ}}} & \mathbf{D} \otimes K_p/F^m \\ & y_{\mathcal{MS}} \searrow & \downarrow & & \downarrow \\ & & (\Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n(K_p))_\Gamma & \xrightarrow{\Phi^{\mathrm{AJ}}_{\mathcal{MS}}} & \mathbf{D}_{\mathcal{MS}} \otimes K_p/F^m. \end{array}$$

Here $y_{\mathcal{MS}}(\Psi)$ is defined to be the class of $\gamma_\Psi x - x \otimes \tau_\Psi \otimes D_K^{-m/2} P_\Psi^m$, where an arbitrary choice of $x \in \mathbb{P}^1(\mathbb{Q})$ has beeen fixed. The map $y_{\mathcal{MS}}$ is indeed well defined, as easily follows by arguing as in Lemma 5.2. Thus, along with the *Darmon cohomology classes* $s_{[\Psi]}$ attached to $[\Psi] \in \Gamma\backslash\mathrm{Emb}(\mathcal{O},\mathcal{R})$ introduced in Definition 5.3, we can also define $s_{\mathcal{MS}}([\Psi]) := \Phi^{\mathrm{AJ}}_{\mathcal{MS}}(y_{\mathcal{MS}}(\Psi)) \in \mathbf{D}_{\mathcal{MS}} \otimes K_p/F^m$.

Although the triangle in (82) is commutative, we warn the reader that the square in (82) may not be. This is due to the fact that an arbitrary choice of a p-adic Abel-Jacobi map $\Phi^{\mathrm{AJ}}_{\mathcal{MS}}$ has been made. Fortunately, it can be shown that the image of Ψ in $\mathbf{D}_{\mathcal{MS}} \otimes K_p/F^m$ does not depend on the choice of $\Phi^{\mathrm{AJ}}_{\mathcal{MS}}$; see [Se, Proposition 2.22] for more details, where it is proved that although the square in (82) may not be commutative, one still has

$$(83) \qquad \Phi^{\mathrm{AJ}}(y_\Psi) = \Phi^{\mathrm{AJ}}_{\mathcal{MS}}(y_{\mathcal{MS}}(\Psi)).$$

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