ON ABELIAN AUTOMORPHISM GROUPS OF MUMFORD CURVES

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ABSTRACT. We use rigid analytic uniformization by Schottky groups to give a bound for the order of the abelian subgroups of the automorphism group of a Mumford curve in terms of its genus.

INTRODUCTION

Let X be a smooth irreducible projective algebraic curve of genus $g \geq 2$ over a field k. The automorphism group $\mathrm{Aut}(X)$ is always finite and it is an interesting problem to determine its size with respect to the genus. When the ground field k has characteristic 0, it is known that the Hurwitz bound holds:

$$|\mathrm{Aut}(X)| \leq 84(g-1). \tag{1}$$

Moreover, this bound is best possible in the sense that there exist curves of genus g that admit $84(g-1)$ automorphisms for infinitely many different values of g.

When $\mathrm{char}(k) = p > 0$, $|\mathrm{Aut}(X)|$ is bounded by a polynomial of degree four in g. In fact, it holds that

$$|\mathrm{Aut}(X)| \leq 16g^4,$$

[14], provided X is not any of the Fermat curves $x^{q+1} + y^{q+1} = 1$, $q = p^n$, $n \geq 1$, which have even larger automorphism group [9].

For Mumford curves X over an algebraic extension of the p-adic field $\mathbb{Q}_p$, F. Herrlich [5] was able to improve Hurwitz's bound (1) by showing that actually

$$|\mathrm{Aut}(X)| \leq 12(g-1),$$

provided $p \geq 7$.

Moreover, the first author in a joint work with G. Cornelissen and F. Kato [2] proved that a bound of the form

$$|\mathrm{Aut}(X)| \leq \max\{12(g-1), 2\sqrt{g}(\sqrt{g}+1)^2\}$$

holds for Mumford curves defined over non-archimedean valued fields of characteristic $p > 0$.

For ordinary curves X over an algebraically closed field of characteristic $p > 0$, Guralnik and Zieve [4] announced that there exists a sharp bound of the order of $g^{8/5}$ for $|\mathrm{Aut}(X)|$.

In [11], S. Nakajima employs the Hasse-Arf theorem to prove that

$$|\mathrm{Aut}(X)| \leq 4g + 4$$

for any algebraic curve X whose group of automorphisms is abelian.

The results of Herrlich compared to those of Hurwitz and those of [2] compared to Guralnik-Zieve's indicate that if we restrict ourselves to Mumford curves with

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abelian automorphism group a stronger bound than the one of Nakajima should be expected.

The aim of this note is studying the size of the abelian subgroups of the automorphism group $\mathrm{Aut}(X)$ of a Mumford curve over a complete field k with respect to a non-archimedean valuation. These curves are rigid analytically uniformized by a Schottky group $\Gamma \subset \mathrm{PGL}_2(k)$ and their automorphism group is determined by the normalizer N of Γ in $\mathrm{PGL}_2(k)$.

Our results are based on the *Gauss-Bonet* formula of Karass-Pietrowski-Solitar, which relates the rank of the free group Γ to the index $[N : \Gamma]$ and on the characterization of the possible abelian stabilizers $N_v \subset N$ of the vertices $v \in \mathcal{T}_k$ on the Bruhat-Tits tree of k acted upon by the group N.

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1. Abelian automorphism groups of Mumford Curves

Let k be a complete field with respect to a non-archimedean valuation. Let $\bar{k}$ denote the residue field of k and write $p = \mathrm{char}(\bar{k})$ for its characteristic. Choose a separable closure K of k.

Let $\Gamma \subset \mathrm{PGL}_2(k)$ be a Schottky group, that is, a discrete finitely generated subgroup consisting entirely of hyperbolic elements acting on $\mathbb{P}^1_k$ with limit set $\mathcal{L}_\Gamma$ (cf. [3]). By a theorem of Ihara, Γ is a free group. The rigid analytic curve

$$\Gamma \backslash (\mathbb{P}^1_k - \mathcal{L}_\Gamma)$$

turns out to be the analytic counterpart of a smooth algebraic curve of genus $g = \mathrm{rank}(\Gamma) \geq 1$ over k which we shall denote X_Γ/k. In a fundamental work, D. Mumford [10] showed that the stable reduction of X_Γ is a $\bar{k}$-split degenerate curve: all its connected components are rational over $\bar{k}$ and they meet at ordinary double points rational over $\bar{k}$. Conversely, he showed that all such curves admit a rigid analytic uniformization by a Schottky subgroup of $\mathrm{PGL}_2(k)$.

Let $\mathcal{T}_k$ denote the Bruhat-Tits tree of k. The set of ends of $\mathcal{T}_k$ is in one-to-one correspondence with the projective line $\mathbb{P}^1(k)$; we thus identify $\mathbb{P}^1(k)$ with the boundary of $\mathcal{T}_k$.

Let N be a finitely generated discrete subgroup of $\mathrm{PGL}_2(k)$ that contains Γ as a normal subgroup of finite index. The group N naturally acts on $\mathcal{T}_k$. By taking an appropriate extension of k, we may assume that all fixed points of N in the boundary are rational. In turn, this implies that N acts on $\mathcal{T}_k$ without inversion.

Theorem 1.1. [3, p. 216] [2] *The group $G = N/\Gamma$ is a subgroup of the automorphism group of the Mumford curve X_Γ. If N is the normalizer of Γ in $\mathrm{PGL}_2(k)$ then $G = \mathrm{Aut}(X_\Gamma)$.*

For every vertex v on $\mathcal{T}_k$ let N_v be the stabilizer of v in N, that is,

$$N_v = \{g \in N : g(v) = v\}.$$

Let $\mathrm{star}(v)$ denote the set of edges emanating from the vertex v. It is known that $\mathrm{star}(v)$ is in one to one correspondence with elements in $\mathbb{P}^1(\bar{k})$. Since N_v fixes v, it acts on $\mathrm{star}(v)$ and describes a natural map

$$\rho : N_v \to \mathrm{PGL}_2(\bar{k}). \tag{2}$$

See [2, Lemma 2.7]) for details. The kernel of ρ is trivial unless N_v is isomorphic to the semidirect product of a cyclic group with an elementary abelian group. In this case, $ker\rho$ is an elementary abelian p-group [2, Lemma 2.10].

Assume that $g \geq 2$. This implies that Γ has finite index in N. Since Γ has finite index in N, both groups N and Γ share the same set of limit points $\mathcal{L}$. We shall denote by $\mathcal{T}_N$ the subtree of $\mathcal{T}_k$ whose end points are the limit points of $\mathcal{L}$.

The tree $\mathcal{T}_N$ is acted on by N and we can consider the quotient graph $T_N := N\backslash\mathcal{T}_N$. The graph $N\backslash\mathcal{T}_N$ is the dual graph of the intersection graph of the special fibre of the quotient curve

$$X_N = G\backslash X_\Gamma = N\backslash(\mathbb{P}^1_k - \mathcal{L}).$$

Notice that T_N is a tree whenever X_N has genus 0.

The quotient graph T_N can be regarded as a graph of groups as follows: For every vertex v (resp. edge e) of T_N, consider a lift v' (resp. e') in $\mathcal{T}_N$ and the corresponding stabilizer $N_{v'}$ (resp. $N_{e'}$). We decorate the vertex v (resp. edge e) with the stabilizer $N_{v'}$ (resp. $N_{e'}$).

Let T be a maximal tree of T_N and let $T' \subset \mathcal{T}_N$ be a tree of representatives of $\mathcal{T}_N$ mod N, *i.e.*, a lift of T in $\mathcal{T}_N$. Consider the set Y of lifts of the remaining edges $T_N - T$ in $\mathcal{T}_N$ such that, for every $E \in Y$, the origin $o(E)$ lies in T'.

The set $Y = \{E_1, \dots, E_r\}$ is finite. There exist elements $g_i \in N$ such that $g_i(t(E_i)) \in T'$, where $t(E_i)$ denotes the terminal vertex of the edge E_i of Y. Moreover, the elements g_i can be taken from the free group Γ.

The elements g_i act by conjugation on the groups $N_{t(E_i)}$ and impose the relations $g_i N_{t(E_i)} g_i^{-1} = N_{g_i(t(E_i))}$. Denote by $M_i := N_{t(E_i)}$ and $N_i := N_{g_i(t(E_i))}$.

According to [13, Lemma 4, p. 34], the group N can be recovered as the group generated by

$$N := \langle N_v, g_i \rangle = \langle g_1, \dots, g_r, K |\, \mathrm{rel}\, K, g_1 M_1 g_1^{-1} = N_1, \dots, g_r M_r g_r^{-1} = N_r \rangle,$$

where K is the tree product of T'.

Assume that the tree T' of representatives has κ edges and $\kappa + 1$ vertices. Let v_i denote the order of the stabilizer of the i-th vertex and e_i the order of the stabilizer of the i-th edge. If $f_i = |M_i|$, we define the *volume* of the fundamental domain as

$$\mu(T_N) := \left(\sum_{i=1}^{r} \frac{1}{f_i} + \sum_{i=1}^{\kappa} \frac{1}{e_i} - \sum_{i=1}^{\kappa+1} \frac{1}{v_i} \right).$$

Notice that when $r = 0$, *i.e.* the quotient graph T_N is a tree, this definition coincides with the one given in [2].

Karrass, Pietrowski and Solitar proved in [8] the following *discrete Gauss-Bonnet* theorem:

Proposition 1.2. *Let N, T_N, g be as above. The following equality holds:*

$$|N/\Gamma| \cdot \mu(T_N) = g - 1.$$

In order to obtain an upper bound for the group of automorphisms with respect to the genus, we aim for a lower bound for $\mu(T_N)$. Observe that if we restrict the above sum to the maximal tree T of T_N, we deduce the following inequality

$$\mu(T) := \sum_{i=1}^{\kappa} \frac{1}{e_i} - \sum_{i=1}^{\kappa+1} \frac{1}{v_i} \leq \mu(T_N),$$

where equality is achieved if and only if T_N is a tree, *i.e.*, the genus of X_N is 0.

In what follows we pursue lower bounds for $\mu(T)$, where T is a maximal tree. These should be lower bounds for $\mu(T_N)$ as well.

Lemma 1.3. *Let G be a finite abelian subgroup of $\mathrm{PGL}_2(\mathbb{F}_{p^n})$ acting on $\mathbb{P}^1(\mathbb{F}_{p^n})$. Let S be the subset of $\mathbb{P}^1(\bar{\mathbb{F}}_p)$ of ramified points of the cover*

$$\mathbb{P}^1 \to G\backslash\mathbb{P}^1.$$

Then, either

(1) $G \simeq \mathbb{Z}/n\mathbb{Z}$, *where* $(n,p)=1$, $S=\{P_1,P_2\}$ *and the ramification indices are* $e(P_1)=e(P_2)=n$, *or*
(2) $G \simeq D_2 = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, $p\neq 2$ *and* $S=\{P_1,P_2,P_3\}$ *with ramification indices* $e(P_1)=e(P_2)=e(P_3)=2$, *or*
(3) $G \simeq E(r) = \mathbb{Z}/p\mathbb{Z}\times \overset{(r)}{\cdots} \times\mathbb{Z}/p\mathbb{Z}$ *for some* $r\geq 0$ *and* $S=\{P\}$, *with ramification index* $e(P)=p^r$.

Proof. The finite subgroups of $\mathrm{PGL}_2(\mathbb{F}_{p^n})$ were classified by L. E. Dickson (cf. [6, II.8.27], [16], [2, Theorem 2.9]). The list of abelian groups follows by selecting the abelian groups among the possible finite subgroups of $\mathrm{PGL}_2(\mathbb{F}_{p^n})$. Notice that the case $E(r)\rtimes\mathbb{Z}/n\mathbb{Z}$, where $(n,p)=1$ and $n\mid p-1$, is never abelian. Indeed, this is due to the fact that $\mathbb{Z}/n\mathbb{Z}$ acts on $E(r)$ by means of a primitive n-th root of unity [12, cor 1. p.67]. The description of the ramification locus S in each case is given in [16, th 1.]. □

Lemma 1.4. *Let* v *be a vertex of* T_N. *If the finite group* N/Γ *is abelian, then* N_v *is abelian. Moreover, the map* $\rho: N_v \to \mathrm{PGL}_2(\bar{k})$ *is injective unless* $N_v = E(r_1)$. *In this case,* $\ker(\rho)\simeq E(r_2)$ *for some* $r_2\leq r_1$.

Proof. The composition
$$N_v \subset N \to N/\Gamma,$$
is injective, since it is not possible for an element of finite order to be cancelled out by factoring out the group Γ. Hence N_v is a abelian. The possible kernels of ϱ are collected in [2, Lemma 2.10]. □

Let v be a vertex of T_N decorated by the group N_v and assume that there exist $s\geq 1$ edges in its star, decorated by groups $N^v_{e_\nu}\subset N_v$, $\nu=1,...,s$. We define the *curvature* $c(v)$ of v as
$$c(v) := \frac{1}{2}\sum_{i=1}^{s}\frac{1}{|N^v_{e_\nu}|} - \frac{1}{|N_v|}.$$

It is obvious that the following formula holds:
$$\mu(T) = \sum_{v\in\mathrm{Vert}(T)} c(v).$$

In what follows, we shall provide lower bounds for the curvature of each vertex.

We shall call a tree of groups *reduced* if $|N_v| > |N_{e_\nu}|$ for all vertices v and edges $e_\nu\in\mathrm{star}(v)$. Notice that, if $N_v = N_e$ for a vertex v and an edge $e\in\mathrm{star}(v)$, then the opposite vertex v' of e is decorated by a group $N_{v'}\supseteq N_e$. The contribution of e to the tree product is the amalgam $N_v *_{N_e} N_{v'} = N_{v'}$. This means that e can be contracted without altering the tree product. From now on we shall assume that *the tree* T *is reduced.*

For an element $\gamma\in N$, define the *mirror* of γ to be the smallest subtree $M(\gamma)$ of $\mathcal{T}_k$ generated by the point-wise fixed vertices of $\mathcal{T}$ by γ.

Let $\gamma\in N$ be an elliptic element (*i.e.*, an element of N of finite order with two distinct eigenvalues of the same valuation). Then γ has two fixed points in $\mathbb{P}^1(k)$ and $M(\gamma)$ is the geodesic connecting them.

If $\gamma\in N$ is a parabolic element (*i.e.*, an element in N having a single eigenvalue), then it has a unique fixed point z on the boundary $\mathbb{P}^1(k)$.

Lemma 1.5. *Let* P_1,P_2,Q_1,Q_2 *be four distinct points on the boundary of* $\mathcal{T}_k$. *Let* $g(P_1,P_2)$, $g(Q_1,Q_2)$ *be the corresponding geodesic on* $\mathcal{T}_k$ *connecting* P_1,P_2 *and* Q_1,Q_2 *respectively. For the intersection of the geodesics* $g(P_1,P_2)$ *and* $g(Q_1,Q_2)$ *there are the following possibilities:*

(1) *$g(P_1, P_2)$, $g(Q_1, Q_2)$ have empty intersection.*
(2) *$g(P_1, P_2)$, $g(Q_1, Q_2)$ intersect at only one vertex of $\mathcal{T}_k$.*
(3) *$g(P_1, P_2)$, $g(Q_1, Q_2)$ have a common interval as intersection.*

Proof. It immediately follows from the fact that $\mathcal{T}_k$ is simply-connected. □

We refer to [7, prop. 3.5.1] for a detailed description on the arrangement of the geodesics with respect to the valuations of the cross-ratio of the points P_1, P_2, Q_1, Q_2.

Lemma 1.6. *Two non-trivial elliptic elements $\gamma, \gamma' \in \mathrm{PGL}_2(k)$ have the same set of fixed points in $\mathbb{P}^1(k)$ if and only if $\langle \gamma, \gamma' \rangle$ is a cyclic group.*

Proof. If γ and γ' generate a cyclic group, there exists an element σ such that $\sigma^i = \gamma$ and $\sigma^{i'} = \gamma'$ for some $i, i' \geq 1$. Since any non-trivial elliptic element has exactly two fixed points, it is immediate that γ, γ' and σ have the same set of fixed points.

Conversely if γ, γ' have the same set of fixed points, say $0, \infty$, then a simple computation shows that γ, γ' are of the form

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \text{ and } \gamma' = \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix},$$

where a/d and a'/d' are roots of unity. Hence there exists $\sigma \in \mathrm{PGL}_2(k)$ such that $\sigma^i = \gamma$ and $\sigma^{i'} = \gamma'$. □

Lemma 1.7. *Assume that N/Γ is an abelian group and let $\gamma, \gamma' \in N$, $\gamma \neq \gamma'$, be elements of prime-to-p finite order. If $M(\gamma) \cap M(\gamma') \neq \emptyset$, then $\langle \gamma, \gamma' \rangle$ is isomorphic to either D_2 or a cyclic group.*

Proof. By Lemma 1.4, the stabilizers N_v of those vertices v such that $(|N_v|, p) = 1$ are abelian subgroups of $\mathrm{PGL}_2(\bar{k})$. Let F_γ and $F_{\gamma'}$ denote the sets of the fixed points of γ and γ' in $\mathbb{P}^1(k)$, respectively.

If $M(\gamma) = M(\gamma')$ then $F_\gamma = F_{\gamma'}$ and it follows from Lemma 1.6 that $\langle \gamma, \gamma' \rangle$ is a cyclic group.

On the other hand, if $M(\gamma) \neq M(\gamma')$, then $F_\gamma \neq F_{\gamma'}$ and $\langle \gamma, \gamma' \rangle$ cannot be cyclic, again by Lemma 1.6. In this case, any vertex $v \in M(\gamma) \cap M(\gamma')$ is fixed by $\langle \gamma, \gamma' \rangle$, which must be isomorphic to D_2 by Lemma 1.3. □

Lemma 1.8. *Assume that N/Γ is an abelian group. If N_v is a p-group for some vertex v in T, then $N_e = \{1\}$ for all $e \in \mathrm{star}(v)$.*

Proof. Assume that $v \in \mathrm{Vert}(T)$ is lifted to $v' \in \mathrm{Vert}(\mathcal{T}_N)$ and that $N_{v'}$ is an elementary abelian group. Recall the map $\rho : N_{v'} \to \mathrm{PGL}_2(\bar{k})$, which describes the action of $N_{v'}$ on $\mathrm{star}(v')$.

If $\mathrm{Im}(\rho) = \{\mathrm{Id}_{\mathrm{PGL}_2(\bar{k})}\}$, then every edge $e' \in \mathrm{star}(v')$ would be fixed by $N_{v'} = \ker(\rho)$. If one of the edges $e' \in \mathrm{star}(v')$ were reduced in T to an edge e with non trivial stabilizer then it would follow that $N_v = N_e$ and the tree would not be reduced.

Suppose now that $\mathrm{Im}(\rho) \neq \{\mathrm{Id}_{\mathrm{PGL}_2(\bar{k})}\}$. By the classification of Valentini-Madan there exists exactly one edge e' in the star of v' which is fixed by the whole group $\mathrm{Im}(\rho)$ and all other edges emanating from v' are not fixed by $\mathrm{Im}(\rho)$. Therefore the edge e' is fixed by the whole group $N_{v'}$. If the edge e' were reduced in T to an edge e with non trivial stabilizer, then it would follow that $N_v = N_e$ and the tree would not be reduced.

Assume now that there exist two vertices v_1, v_2 on T joint by an edge e such that N_{v_1}, N_{v_2} are elementary abelian groups and N_e is a nontrivial proper subgroup both of N_{v_1} and N_{v_2}. Let us show that this can not happen.

Let σ, τ be two commuting parabolic elements of $\mathrm{PGL}_2(\bar{k})$. They fix a common point in the boundary of $\mathbb{P}^1(k)$. Indeed, every parabolic element fixes a single point in the boundary. If P is the unique fixed point of σ, then

$$\sigma(\tau P) = \tau(\sigma P) = \tau P$$

and $\tau(P)$ is fixed also by σ. Since the fixed point of σ in the boundary is unique, we have $\tau(P) = P$.

Let v_1', v_2' be two lifts of v_1, v_2 on the Bruhat-Tits tree. The apartment $[v_1', v_2']$ is contracted to the edge e and it is fixed by N_e, but not by a larger subgroup.

Since N_e is contained in both abelian groups $N_{v_1'}, N_{v_2'}$, all parabolic elements in $N_{v_1'}, N_{v_2'}$ have the same fixed point P in the boundary $\mathbb{P}^1(k)$. Therefore, the apartment $[v_1', P[$ (resp. $[v_2', P[$) is fixed by $N_{v_1'}$ (resp. $N_{v_2'}$). Moreover, the apartments $[v_1', P[, [v_2', P[$ have nonempty intersection. Since the Bruhat-Tits tree is simply connected, the apartment $[v_1', v_2']$ intersects $[v_2', P[\cap [v_1', P[$ at a bifurcation point Q:

$$[v_1', v_2'] \cap \big([v_2', P[\cap [v_1', P[\big) = \{Q\}.$$

The point Q is then fixed by $N_{v_1'}$ and $N_{v_2'}$ and it is on the apartment $[v_1', v_2']$, a contradiction. □

Lemma 1.9. *Let v be a vertex in T. If $c(v) > 0$, then $c(v) \geq \frac{1}{6}$. Let $s = \#\mathrm{star}(v)$ and let $N_{e_\nu}^v$ denote the stabilizers of the edges in the star of v for $\nu = 1, \ldots, s$. It holds that $c(v) = 0$ if and only if:*

1. *$N_v = D_2$, $s = 1$, $N_{e_1}^v = \mathbb{Z}_2$ or*
2. *$N_v = \mathbb{Z}_2$, $s = 1$, $|N_{e_1}^v| = 1$.*

We have

- *$c(v) = \frac{1}{6}$ if and only if $N_v = \mathbb{Z}_3$ and $s = 1$,*
- *$c(v) = \frac{1}{4}$ if and only if $N_v = D_2$ with $s = 2$ and $|N_{e_1}^v| = |N_{e_2}^v| = |2|$, or $N_v = D_2$ with $s = 1$ and $|N_{e_1}^v| = 1$.*

In the remaining cases we have $c(v) \geq \frac{1}{3}$.

Proof. • Assume that $N_v = D_2$. Then $c(v) \geq 0$. Equality $c(v) = 0$ holds only when $s = 1$ and the only edge leaving v is decorated by a group of order 2. If we assume that $c(v) > 0$, then

$$\frac{1}{4} \leq c(v)$$

and equality is achieved if $s = 2$ and $|N_{e_1}| = |N_{e_2}| = 2$, or if $s = 1$ and $|N_{e_1}^v| = 1$.

• Assume that $N_v = \mathbb{Z}_n$. Then

$$c(v) = \sum_{i=\nu}^{s} \frac{1}{2|N_{e_\nu}|} - \frac{1}{n}.$$

By Lemma 1.7, the stabilizer of each edge in the star of v is trivial. Indeed, if there were an edge $e \in \mathrm{star(v)}$ with $N_e > \{1\}$, then e would be fixed by a cyclic group $\mathbb{Z}_m$, where $m \mid n$. Let σ be a generator of $\mathbb{Z}_n$ and let σ^κ be the generator of $\mathbb{Z}_m$. The elements σ, σ^κ have the same fixed points. Hence a lift of the edge e in $\mathcal{T}_N$ would lie on the mirror of σ. But then $N_e = N_v$ and this is not possible by the reducibility assumption. See also [5, Lemma 1].

If $n > 2$, then

$$\frac{1}{6} \leq \frac{n-2}{2n} \leq \frac{sn-2}{2n} = s\frac{1}{2} - \frac{1}{n} \leq c(v),$$

and equality holds only if $s = 1, n = 3$. If $n = 2$ and $c(v) > 0$ then $s \geq 2$, and $c(v) = s\frac{1}{2} - \frac{1}{2} \geq \frac{1}{2}$.

N	N/Γ	g
$D_2 * \mathbb{Z}_4$	$D_2 \times \mathbb{Z}_4$	2
$\mathbb{Z}_2 * \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	2
$D_2 * D_2$	$\mathbb{Z}_2^4$	4
$\mathbb{Z}_2 * D_2$	$\mathbb{Z}_2^3$	2
$D_2 *_{\mathbb{Z}_2} D_2 *_{\mathbb{Z}_2} D_2$	$\mathbb{Z}_2^4$	2

TABLE 1.

- Assume that $N_v = E(r)$. Then $s = 1$ and it follows from Lemma 1.8 that $|N_{e_1}| = 1$. If $p^r = 2$ then $c(v) = 0$. Hence if $c(v) > 0$ then $p^r > 2$ and

$$\frac{1}{6} \leq c(v) = \frac{1}{2} - \frac{1}{p^r} = \frac{p^r - 2}{2p^r},$$

and equality holds only if $p^r = 3$. □

Theorem 1.10. *Assume that N/Γ is abelian. If N is not isomorphic neither to $\mathbb{Z}_2 * \mathbb{Z}_3$ nor $D_2 * \mathbb{Z}_3$, then*

$$|N/\Gamma| \leq 4(g-1).$$

If we exclude the groups of Table 1 then

$$|N/\Gamma| \leq 3(g-1).$$

The case $N = \mathbb{Z}_2 * \mathbb{Z}_3$ gives rise to a curve of genus 2 whose automorphism group is a cyclic group of order 6. The case $N = D_2 * \mathbb{Z}_3$ gives rise to a curve of genus 3 with automorphism group $D_2 \times \mathbb{Z}_3$.

Proof. Since $g \geq 2$ and therefore $\mu(T_N) > 0$, we have by Proposition 1.2 that

$$|\mathrm{Aut}(X)| = \frac{1}{\mu(T_N)}(g-1) \leq \frac{1}{\mu(T)}(g-1) \leq \frac{6}{\lambda}(g-1),$$

where

$$\lambda = \#\{v \in \mathrm{Vert}(T_N) : c(v) > 0\}.$$

If $\lambda \geq 2$ the result follows. Assume that there is only one vertex v such that $c(v) > 0$. Since $g \geq 2$, there exist other vertices v' on the tree T_N but their contribution is $c(v') = 0$.

Case 1: $c(v) = \frac{1}{6}$. Then $N_v = \mathbb{Z}_3$ and there exists a single edge e at the star of v. Let v' denote the terminal vertex of e. Since $c(v') = 0$ if and only if there exists a single edge leaving v', the only possibilities for N are $N = D_2 * \mathbb{Z}_3$ and $N = \mathbb{Z}_2 * \mathbb{Z}_3$. Since N/Γ is abelian, the group $\Gamma_1 := [D_2, \mathbb{Z}_3]$ (resp. $\Gamma_1 := [\mathbb{Z}_2, \mathbb{Z}_3]$) is contained in Γ. According to [2, Lemma 6.6], Γ_1 is a maximal free subgroup of N and thus $\Gamma = \Gamma_1$. The rank of Γ is $(4-1)(2-1) = 3$ in the first case and $(3-1)(2-1) = 2$ in the second. Therefore the first amalgam gives rise to a curve of genus 3 with automorphism group $D_2 \times \mathbb{Z}_3$ and the second gives rise to a curve of genus 2 with automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$.

Case 2: $c(v) = \frac{1}{4}$. This occurs only when $N_v = \mathbb{Z}_4, D_2$ and $s = 1$ or $N_v = D_2$, $s = 2$, $|N_{e_1}| = |N_{e_2}| = \frac{1}{2}$. The possible groups are given in Table:1.

In this case we have the following bound

$$|\mathrm{Aut}(X)| \leq \frac{1}{\mu(T)}(g-1) \leq 4(g-1).$$

Case 3: $c(v) \geq \frac{1}{3}$. Similarly as above we obtain that

$$|\mathrm{Aut}(X)| \leq \frac{1}{\mu(T)}(g-1) \leq \frac{3}{\lambda}(g-1) \leq 3(g-1).$$

□

Example: *Subrao Curves.*

Let $(k, |\cdot|)$ be a complete field of characteristic p with respect to a non-archimedean norm $|\cdot|$. Assume $\mathbb{F}_q \subset k$, for some $q = p^r$, $r \geq 1$. Define the curve:

$$(y^q - y)(x^q - x) = c,$$

with $|c| < 1$. This curve was introduced by Subrao in [15] and it has a large automorphism group compared to the genus. This curve is a Mumford Curve [2, p. 9] and has *chessboard* reduction [2, par. 9]. It is a curve of genus $(q-1)^2$ and admits the group $G := \mathbb{Z}_p^r \times \mathbb{Z}_p^r$ as a subgroup of the automorphism group. The group G consists of the automorphisms $\sigma_{a,b}(x, y) = (x + a, y + b)$ where $(a, b) \in \mathbb{F}_q \times \mathbb{F}_q$. The discrete group N' corresponding to G is given by $\mathbb{Z}_{p^r} * \mathbb{Z}_{p^r}$ and the free subgroup Γ giving the Mumford uniformization is given by the commutator $[\mathbb{Z}_{p^r}, \mathbb{Z}_{p^r}]$, which is of rank $(q-1)^2$ [13].

The Karass, Pietrowski, Solitar formula yields

$$(q-1)^2 - 1 = q^2 \left(1 - \frac{2}{q}\right),$$

while the bound is given by

$$q^2 = |G| \leq 2(g-1) = 2(q^2 - 2q).$$

Notice that the group N' is a proper subgroup of the normalizer of Γ in $\mathrm{PGL}_2(k)$, since the full automorphism group of the curve is isomorphic to $\mathbb{Z}_p^{2r} \rtimes D_{p^r-1}$ [2]. □

1.1. **Elementary abelian groups.**

Proposition 1.11. *Let ℓ be a prime number and let X_Γ/k be a Mumford curve over a non-archimedean local field k such that $p = \mathrm{char}(\bar{k}) \neq \ell$. Let $A \subset \mathrm{Aut}(X_\Gamma)$ be a subgroup of the group of automorphisms of X_Γ such that $A \simeq \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z} \times \cdots \times \mathbb{Z}/\ell\mathbb{Z}$.*

If $\ell = 2$ then all stabilizers of vertices and edges of the quotient graph T_N are subgroups of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mu(T_N) = a/4$ for some $a \in \mathbb{Z}$. If $\ell > 2$ then all stabilizers of vertices and edges of T_N are subgroups of $\mathbb{Z}/\ell\mathbb{Z}$ and $\mu(T_N) = a/\ell$.

Proof. Let $A \subset \mathrm{Aut}(X_\Gamma)$. There is a discrete finitely generated subgroup $N' \subset N$ such that $\Gamma \lhd N'$ and $N'/\Gamma = A$. Let N_v be the stabilizer of a vertex in T_N and let $N'_v = N_v \cap N'$. The composition

$$N'_v \subset N_v \subset N \to N/\Gamma,$$

is injective, since it is not possible for an element of finite order to be cancelled out by factoring out the group Γ.

The map $\rho : N'_v \to \mathrm{PGL}_2(\bar{k}) = \mathrm{PGL}_2(\mathbb{F}_{p^m})$ of Lemma 1.4 is injective since $(|N'_v|, p) = 1$ and hence we can regard N'_v as a finite subgroup of $\mathrm{PGL}_2(\bar{k}) = \mathrm{PGL}_2(\mathbb{F}_{p^m})$.

Assume first that $\ell = 2$. Then by Lemma 1.3 the only abelian finite subgroups of $\mathrm{PGL}_2(\bar{k})$ for $p \neq 2$ are $\mathbb{Z}/2\mathbb{Z}$ and the dihedral group of order 4. Hence N'_v is a subgroup of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Since the group N acts without inversions, the stabilizer of a vertex is the intersection of the stabilizers of the limiting edges. It again follows that $N_e \subseteq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Finally, we obtain from its very definition that $\mu(T_N) = a/4$ for some $a \in \mathbb{Z}$.

For the case $\ell > 2$ we observe that $\mathbb{Z}/\ell\mathbb{Z}$ is the only abelian subgroup of $\mathrm{PGL}_2(\mathbb{F}_{p^m})$ and it follows similarly that $\mu(T_N) = a/\ell$ for some $a \in \mathbb{Z}$. □

As an immediate corollary of Proposition 1.11 we obtain the following formula for the ℓ-elementary subgroups of the group of automorphisms of Mumford curves.

Notice that the result below actually holds for arbitrary algebraic curves -as it can be proved by applying the Riemann-Hurwitz formula to the covering $X \longrightarrow X/A$.

Corollary 1.12. *Let $\ell \neq \mathrm{char}(\bar{k})$ be a prime number and let X/k be a Mumford curve of genus $g \geq 2$ over a non-archimedean local field k. Let $A \subseteq \mathrm{Aut}(X)$ be a subgroup of the group of automorphisms of X such that $A \simeq \oplus_{i=1}^{s} \mathbb{Z}/\ell\mathbb{Z}$ for some $s \geq 2$.*

(i) *If $\ell \neq 2$ then $\ell^{s-1} \mid g-1$.*
(ii) *If $\ell = 2$ then $2^{s-2} \mid g-1$.*

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