

KOLYVAGIN CLASSES VERSUS NON-CRISTALLINE DIAGONAL CLASSES

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ABSTRACT. Let E/\mathbb{Q} be an elliptic curve having multiplicative reduction at a prime p . Let (g, h) be a pair of eigenforms of weight 1 arising as the theta series of an imaginary quadratic field K , and assume that the triple-product L -function $L(f, g, h, s)$ is self-dual and does not vanish at the central critical point $s = 1$. The main result of this article is a formula expressing the p -adic iterated integrals introduced in [DLR15] to the Kolyvagin classes associated by Bertolini and Darmon to a system of Heegner points on E .

1. INTRODUCTION

Let E/\mathbb{Q} be an elliptic curve with square-free conductor N_f and let $f \in S_2(N_f)$ be the normalized newform attached to it by modularity. Let K be an imaginary quadratic field of discriminant $-D_K$ relatively prime to N_f . Let

$$\psi_g, \psi_h : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$$

be two finite order Hecke characters of K of conductors $\mathfrak{c}_g, \mathfrak{c}_h$ respectively and let $g := \theta(\psi_g)$ and $h := \theta(\psi_h)$ denote the theta series associated to them. Assume that the central characters of g and h are mutually inverse. Then

$$g \in M_1(N_g, \chi), \quad h \in M_1(N_h, \chi^{-1})$$

are weight one modular forms of level $N_g = D_K \cdot N_{K/\mathbb{Q}}(\mathfrak{c}_g)$, $N_h = D_K \cdot N_{K/\mathbb{Q}}(\mathfrak{c}_h)$ respectively. For simplicity we assume throughout that

$$(1.1) \quad \gcd(N_f, N_g N_h) = 1$$

and we set $N := \text{lcm}(N_f, N_g, N_h)$. Let p be an odd prime number such that

$$(1.2) \quad p \parallel N_f \quad \text{and} \quad p \text{ is inert in } K.$$

Hence E has multiplicative reduction at p and the completion K_p of K at p is the unramified quadratic extension of \mathbb{Q}_p . Let

$$\varphi_{\text{Tate}} : \bar{\mathbb{Q}}_p^\times / q_E^\mathbb{Z} \xrightarrow{\cong} E(\bar{\mathbb{Q}}_p)$$

be Tate's uniformization, with $q_E \in p\mathbb{Z}_p$, and assume throughout that

$$(1.3) \quad p \nmid \text{ord}_p(q_E).$$

Set

$$a := a_p(E) = a_p(f) \in \{\pm 1\}.$$

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In other words, $a = 1$ (resp. $a = -1$) according to whether E has split (non-split) multiplicative reduction at p .

Let α_g, β_g (resp. α_h, β_h) denote the roots of the p -th Hecke polynomial of g (resp. of h). Note that (1.2) implies that

$$\alpha_g = -\beta_g \quad \text{and} \quad \alpha_h = -\beta_h.$$

Since the nebentype character of h is the inverse of that of g , one either has

$$(\alpha_h, \beta_h) = (1/\alpha_g, -1/\alpha_g) \quad \text{or} \quad (\alpha_h, \beta_h) = (-1/\alpha_g, 1/\alpha_g).$$

Throughout this article we fix the ordering of the pair (α_h, β_h) in such a way that

$$(1.4) \quad \alpha_g \cdot \alpha_h = -a.$$

Fix throughout algebraic closures $\bar{\mathbb{Q}}$ and $\bar{\mathbb{Q}}_p$ of \mathbb{Q} and \mathbb{Q}_p respectively and an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Let $L \subset \bar{\mathbb{Q}}$ denote the number field generated by the traces of ψ_g and ψ_h , together with the roots $\alpha_g, \beta_g, \alpha_h, \beta_h$. and let L_p denote the completion of L within $\bar{\mathbb{Q}}_p$.

Set $g_\alpha(z) := g(z) - \beta_g g(pz)$ and define h_α analogously. Let $\mathbf{f}, \mathbf{g}_\alpha, \mathbf{h}_\alpha$ be Hida families passing through f, g_α, h_α respectively. As explained in [DR14] and [DR17], associated to any choice of Λ -adic test vectors $(\check{\mathbf{f}}, \check{\mathbf{g}}_\alpha, \check{\mathbf{h}}_\alpha)$ of tame level N there is a three-variable p -adic L -function

$$\mathcal{L}_p^g(\check{\mathbf{f}}, \check{\mathbf{g}}_\alpha, \check{\mathbf{h}}_\alpha).$$

This p -adic L -function interpolates the square-roots of the central values of the classical L -function $L(\check{f}_k \otimes \check{g}_\ell \otimes \check{h}_m, s)$ attached to the specializations of the choice of test vectors at classical points of weights k, ℓ, m with $k, \ell, m \geq 2$ and $\ell \geq k + m$.

Note that the point $(2, 1, 1)$ corresponding to our triple of modular forms (f, g, h) lies outside the region of classical interpolation for $\mathcal{L}_p^g(\check{\mathbf{f}}, \check{\mathbf{g}}_\alpha, \check{\mathbf{h}}_\alpha)$. As in [DLR15] and [GGMR19] we are interested in studying the p -adic L -value

$$I_p(f, g_\alpha, h_\alpha) := \mathcal{L}_p^g(\check{\mathbf{f}}, \check{\mathbf{g}}_\alpha, \check{\mathbf{h}}_\alpha)(2, 1, 1)$$

regarded as an element of $\bar{\mathbb{Q}}_p^\times / L^\times$. As explained in [DR17] and [DLR15], different choices of Λ -adic test vectors yield the same p -adic value $I_p(f, g_\alpha, h_\alpha)$ up to algebraic scalars in L^\times . This entitles us to denote it simply $I_p(f, g_\alpha, h_\alpha)$, dropping from the notation this choice.

The triple-product L -function $L(f, g, h, s) = L(E, \rho, s)$ may be recast as the Hasse-Weil L -series of the twist of E by the tensor product $\rho = \rho_g \otimes \rho_h$ of the two Artin representations associated to g and h . In light of (1.1), the order of vanishing of $L(f, g, h, s)$ at $s = 1$ is always even and one thus expects the central critical value $L(E, \rho, 1)$ to be generically nonzero. We assume throughout that we fall indeed in this generic case, that is to say, we have

Assumption 1.1. $L(E, \rho, 1) \neq 0$.

The main purpose of this article is proving a formula relating the p -adic iterated integral $I_p(f, g_\alpha, h_\alpha)$ to the Kolyvagin classes associated by Bertolini and Darmon in [BD97].

In order to recall briefly the latter, for every $m \geq 1$ let F_m/H be the layer of degree p^m within the anticyclotomic \mathbb{Z}_p -extension of H . In [BD97] Bertolini and Darmon associated to a canonical collection of *Heegner points* $\{\alpha_m \in E(F_m)\}$ a global cohomology class $\mathbf{K} \in H^1(H, V_f)$ that is referred to in loc. cit. as the *Kolyvagin class* associated to E/H .

Fix a prime \mathfrak{p} of H above p and denote $\Phi_{m, \mathfrak{p}}$ the p -primary part of the group of connected components of the Néron model of E over the completion of F_m at the unique prime ideal above \mathfrak{p} . Denote $\Phi_{\infty, \mathfrak{p}} := \varprojlim_m \Phi_{m, \mathfrak{p}}$ the projective limit with respect to the natural projection maps $\Phi_{m, \mathfrak{p}} \rightarrow \Phi_{m-1, \mathfrak{p}}$.

As explained in Lemma 2.1 there is a canonical isomorphism $\varphi : \Phi_{\infty, \mathfrak{p}} \xrightarrow{\cong} \mathbb{Z}_p$ induced by Tate's uniformization. Denote by $\bar{\alpha}_m$ the image of α_m in $\Phi_{m, \mathfrak{p}}$ and set $\bar{\alpha} := (\bar{\alpha}_m)_m \in \Phi_{\infty, \mathfrak{p}}$. Define the period

$$\Pi_p := \varphi(\bar{\alpha}) \in \mathbb{Z}_p.$$

Note that $H^1(K_p, V_f)$ is naturally a $\text{Gal}(K_p/\mathbb{Q}_p)$ -module and we let $H^1(K_p, V_f)^\pm$ denote the \mathbb{Q}_p -subspace on which Fr_p acts with eigenvalue ± 1 .

The Kolyvagin class \mathbf{K} is not in general crystalline at p , that is to say, the local class $\text{res}_p(\mathbf{K}) \in H^1(K_p, V_f)$ is not expected to lie in Bloch-Kato's finite subspace $H_f^1(K_p, V_f)$. Nevertheless, one can show (cf. Proposition 5.11) that $\text{res}_p(\mathbf{K})^a := \text{res}_p(\mathbf{K}) + a \text{res}_p(\mathbf{K}^{\text{Fr}_p})$ does lie in $H_f^1(K_p, V_f)^a$ and therefore there exists a local point

$$Q_p^a \in (E(K_p) \otimes \mathbb{Q}_p)^a \quad \text{such that} \quad \delta_p(Q_p^a) = \text{res}_p(\mathbf{K})^a,$$

where $\delta_p : E(K_p) \otimes \mathbb{Q}_p \xrightarrow{\sim} H_f^1(K_p, V_f)$ stands for Kummer's isomorphism. In spite of the notation we have chosen, beware that Q_p^a is *not* expected to be the a -component of any local point $Q_p \in E(K_p) \otimes \mathbb{Q}_p$, precisely because $\text{res}_p(\mathbf{K})$ does not lie in $H_f^1(K_p, V_f)$. In other words, while $\text{res}_p(\mathbf{K})^a$ is crystalline, the local class $\text{res}_p(\mathbf{K})^{-a}$ is not.

Let $c = \prod_{v \nmid N\infty} c_v(\check{f}, \check{g}_\alpha, \check{h}_\alpha) \in L$ denote the product of local automorphic factors appearing in [DLR15, Prop. 2.1 (a) (iii)] associated to the choice of test vectors at $(2, 1, 1)$, and define the algebraic L -value

$$L^{\text{alg}}(E \otimes \rho, 1) := \frac{L(E \otimes \rho, 1)}{\pi^4 \langle f, f \rangle^2}.$$

It follows from the work of Harris and Kudla [HK91] that the above ratio lies in L and is in fact non-zero by (1.1). The following is the main theorem of this note.

Theorem A. For any choice of test vectors, we have

$$I_p(f, g_\alpha, h_\alpha) = \frac{\sqrt{c} \cdot \sqrt{L^{\text{alg}}(E \otimes \rho, 1)}}{\Pi_p \cdot \mathcal{L}_{g_\alpha}} \times \log_p(Q_p^a) \quad (\text{mod } L^\times)$$

where

- $\mathcal{L}_{g_\alpha} \in K_p$ is a period on which Fr_p acts as multiplication by -1 and only depends on g_α ,
- $\log_p : E(K_p) \rightarrow K_p$ denotes the p -adic logarithm.

Remark 1.2. Period \mathcal{L}_{g_α} was first introduced in [DR16] and is well-defined only up to scalars in L^\times ; cf. (3.7) for more details. Let H_g be the number field cut out by the Artin representation attached to the adjoint of g and fix a completion $H_{g,p}$ of H_g at a prime above p . In [DR16] it is conjectured that $\mathcal{L}_{g_\alpha} = \log_p(u_{g_\alpha}) \pmod{L^\times}$, where $\log_p : \mathcal{O}_{H_{g,p}}^\times \otimes L \rightarrow H_{g,p} \otimes L$ is the usual p -adic logarithm and $u_{g_\alpha} \in \mathcal{O}_{H_g}[1/p]^\times \otimes L$ is the so-called Gross-Stark unit attached to g_α [DLR15, §1.2]. See (3.7) for the explicit definition of \mathcal{L}_{g_α} .

The body of the paper is devoted to the proof of this result. In §2 we relate Bloch-Kato's dual exponential map on E to the group of connected components. This is an exercise in p -adic Hodge theory which is surely well-known to experts, but we included because we did not find precise references in the literature. We are most grateful to A. Iovita for his assistance in this section. In §3 we review the theory of Hida families and Selmer groups, and culminates with the description of an explicit basis of the relaxed Selmer group associated to the triple (f, g, h) . In §4 we exploit the fine work [BSVb] of Bertolini, Seveso and Venerucci in order to prove a formula relating the iterated integral $I_p(f, g_\alpha, h_\alpha)$ in terms of the basis described in the previous section. Finally, in the

last section we introduce Kolyvagin classes and prove our main result, which is found in Corollary 5.13.

Theorem A is vaguely similar to the main theorem of [DLR15, §3], although both the statements and specially the proofs are ostensibly different. While in loc. cit. the motivic elements appearing in the statement (both the diagonal classes and Heegner points) are cristalline at p , in our article they are not and this makes the calculations substantially different. Moreover, in [DLR15, §3] one makes crucial use of a factorization of p -adic L -series (which follows rather trivially from a comparison of critical L -values), while in our setting the analogous p -adic L -functions associated to K are not available (because in our scenario p remains inert in K). It would be of great interest to investigate whether the recent p -adic L -functions of Andreatta and Iovita [AI19] could be exploited in order to provide an alternative proof of our formula, although this does not appear to be a straight-forward project.

As we have already pointed out, one of the key ingredients of our proof are the results of Bertolini, Seveso and Venerucci in [BSVb]. In loc. cit. the authors study the exceptional zero phenomena appearing in the scenario of diagonal cycles at weights $(2, 1, 1)$ when $\alpha_g \alpha_h = \pm a$. An arithmetic application of these results is obtained in [BSVa] and [DRb] to the theory of Stark-Heegner points on elliptic curves, in the setting where $\alpha_g \alpha_h = a$ and an *improved* p -adic L -function plays a relevant role. The present work focus instead in the opposite setting where $\alpha_g \alpha_h = -a$ and it is rather an *improved* diagonal cohomology class (cf. [BSVb, §8.3] and §4.2 of this note) which plays a prominent role.

2. p -ADIC HODGE THEORY FOR ELLIPTIC CURVES WITH MULTIPLICATIVE REDUCTION

Let E/\mathbb{Q}_p be an elliptic curve of multiplicative reduction at a prime p . Let

$$\chi_E : G_{\mathbb{Q}_p} \longrightarrow \{\pm 1\}$$

be the trivial character if E has split multiplicative reduction over \mathbb{Q}_p , and the quadratic unramified character if E has non-split multiplicative reduction. For any $G_{\mathbb{Q}_p}$ -module M , let $M(\chi_E)$ denote the twist of M by χ_E .

Tate's uniformisation provides a $G_{\mathbb{Q}_p}$ -equivariant isomorphism

$$(2.1) \quad \varphi_{\text{Tate}} : \bar{\mathbb{Q}}_p^\times(\chi_E)/q_E^{\mathbb{Z}} \xrightarrow{\cong} E(\bar{\mathbb{Q}}_p)$$

for some $q_E \in p\mathbb{Z}_p$. As in (1.3), assume throughout that $p \nmid n := \text{ord}_p(q_E)$.

Let $T = T_E := \varprojlim E[p^m]$ denote the Tate module associated to E and set $V := T \otimes \mathbb{Q}_p$. Let $E_0(\bar{\mathbb{Q}}_p)$ denote the set of points in $E(\bar{\mathbb{Q}}_p)$ that stay nonsingular in the special fiber of E at p . The module

$$T^+ := \varprojlim E_0[p^m]$$

fits in an exact sequence of $\mathbb{Z}_p[G_{\mathbb{Q}_p}]$ -modules

$$(2.2) \quad 0 \longrightarrow T^+ \xrightarrow{\iota} T \xrightarrow{\pi} T^- \longrightarrow 0,$$

where $T^- := T/T^+$. The modules T^+ and T^- are free over \mathbb{Z}_p of rank one, and Tate's uniformization (2.1) induces $G_{\mathbb{Q}_p}$ -equivariant isomorphisms

$$(2.3) \quad \mathbb{Z}_p(\chi_E)(1) \xrightarrow{\cong} T^+, \quad \mathbb{Z}_p(\chi_E) = \varprojlim_m (\bar{\mathbb{Q}}_p^\times(\chi_E)/q_E^{\mathbb{Z}})[p^m]/\mu_{p^m}(\chi_E) \xrightarrow{\cong} T^-.$$

More precisely, if we fix compatible systems

$$\tilde{\varepsilon} := (\tilde{\varepsilon}^{(m)})_m \in \mathbb{Z}_p(\chi_E)(1), \quad \tilde{q} := (q_E^{1/p^m})_m \in \varprojlim (\bar{\mathbb{Q}}_p^\times/q_E^{\mathbb{Z}})[p^m]$$

then

$$\varepsilon := \varphi_{\text{Tate}}(\tilde{\varepsilon}), \quad q := \varphi_{\text{Tate}}(\tilde{q})$$

form a \mathbb{Z}_p -basis of T . Moreover, if $\chi_{\text{cyc}} : G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_p^{\times}$ denotes the cyclotomic character, then

- (1) ε is a basis of T^+ on which $G_{\mathbb{Q}_p}$ acts as the character $\chi_E \chi_{\text{cyc}}$;
- (2) $\bar{q} := \pi(q)$ is a basis of T^- on which $G_{\mathbb{Q}_p}$ acts via χ_E .

Let K be an imaginary quadratic field in which p is inert and let K_p denote the completion of K at p . Notice that $\chi_E|_{G_{K_p}} = 1$ and thus (2.2) gives an exact sequence of $\mathbb{Q}_p[G_{K_p}]$ -modules

$$(2.4) \quad 0 \longrightarrow V^+ \xrightarrow{\iota} V \xrightarrow{\pi} V^- \longrightarrow 0$$

such that $\dim_{\mathbb{Q}_p} V^+ = \dim_{\mathbb{Q}_p} V^- = 1$, G_{K_p} acts on V^+ via χ_{cyc} , and G_{K_p} acts trivially on V^- .

2.1. The group of connected components. Fix a ring class field H of K of conductor c prime to p and for every integer $m \geq 1$, let $H(p^m)$ be the ring class field of K of conductor $c \cdot p^m$. The Galois group $\text{Gal}(H(p^m)/H)$ is cyclic of order $e_m := (p+1)p^{m-1}$. Let F_m be the intermediate field $H \subseteq F_m \subseteq H(p^{m+1})$ such that $\text{Gal}(F_m/H)$ is cyclic of order p^m . Since p is inert in K , the prime ideal $p\mathcal{O}_K$ splits completely in H . Fix once and for all a prime \mathfrak{p} of H above p ; it ramifies totally in $H(p^m)$ as $\mathfrak{p}\mathcal{O}_{F_m} = \mathfrak{p}_m^{p^m}$. Let us denote $F_{m,\mathfrak{p}}$ the completion of F_m at \mathfrak{p}_m , $\mathcal{O}_{m,\mathfrak{p}}$ its ring of integers and $\mathbb{F}_{m,\mathfrak{p}}$ its residue field.

Let \mathcal{E} be the Néron model of E over \mathbb{Z}_p , and let $\tilde{\mathcal{E}} := \mathcal{E} \times_{\mathbb{Z}_p} \mathbb{F}_p$ denote its special fiber. For all $m \geq 1$, let $\Phi_{m,\mathfrak{p}}$ denote the p -Sylow subgroup of the group

$$(2.5) \quad \tilde{\mathcal{E}}(\mathbb{F}_{m,\mathfrak{p}})/\tilde{\mathcal{E}}_0(\mathbb{F}_{m,\mathfrak{p}}) \cong E(F_{m,\mathfrak{p}})/E_0(F_{m,\mathfrak{p}})$$

of connected components of the base change of \mathcal{E} to $\mathcal{O}_{m,\mathfrak{p}}$.

Lemma 2.1. *There are isomorphisms of $G_{\mathbb{Q}_p}$ -modules*

$$T^- \cong \mathbb{Z}_p(\chi_E) \cong \Phi_{\infty,\mathfrak{p}}.$$

Proof. The first isomorphism is (2.3). Tate's uniformisation provides a description of the group (2.5) as

$$(2.6) \quad \bar{\varphi}_{\text{Tate}} : F_{m,\mathfrak{p}}^{\times}/q_E^{\mathbb{Z}}\mathcal{O}_{m,\mathfrak{p}}^{\times}(\chi_E) \cong (\mathbb{Z}/\text{ord}_{m,\mathfrak{p}}(q_E)\mathbb{Z})(\chi_E) \xrightarrow{\cong} E(F_{m,\mathfrak{p}})/E_0(F_{m,\mathfrak{p}}),$$

where $\text{ord}_{m,\mathfrak{p}}$ is the discrete valuation on $\mathcal{O}_{m,\mathfrak{p}}$. We have $p\mathcal{O}_{F_m} = \prod_{\mathfrak{p}_m|p} \mathfrak{p}_m^{p^m}$, where $\mathfrak{p}_m = (\pi_m)$ is the maximal ideal of $\mathcal{O}_{m,\mathfrak{p}}$. Hence $q_E = p^n \alpha = \pi_m^{np^m} \alpha'$ where $\alpha, \alpha' \in \mathcal{O}_{F_{\mathfrak{p},m}}^{\times}$, i.e.

$$\text{ord}_{m,\mathfrak{p}}(q_E) = n \cdot p^m.$$

Under condition (1.3), then (2.6) gives the isomorphism

$$\bar{\varphi}_{\text{Tate}} : (\mathbb{Z}/np^m\mathbb{Z})(\chi_E) \cong (\mathbb{Z}/n\mathbb{Z})(\chi_E) \oplus (\mathbb{Z}/p^m\mathbb{Z})(\chi_E) \xrightarrow{\cong} E(F_{m,\mathfrak{p}})/E_0(F_{m,\mathfrak{p}}).$$

The lemma follows after taking p -primary parts and passing to the inverse limit. □

2.2. The G_{K_p} -cohomology of E . Let $B_{dR} = \text{Frac}(B_{dR}^+)$ denote Fontaine's algebra of de Rham periods. For a de Rham G_{K_p} -representation W , denote

$$D_{dR}(W) := (W \otimes_{\mathbb{Q}_p} B_{dR})^{G_{K_p}}.$$

This Dieudonné module comes naturally equipped with a filtration, inherited from the filtration on B_{dR} , i.e.

$$\text{Fil}^j D_{dR}(W) := (W \otimes_{\mathbb{Q}_p} \text{Fil}^j B_{dR})^{G_{K_p}},$$

and in particular we have $\text{Fil}^0 D_{dR}(W) = (W \otimes B_{dR}^+)^{G_{K_p}}$. The inclusion $\alpha : W \rightarrow W \otimes_{\mathbb{Q}_p} B_{dR}$ induces in cohomology the map

$$\alpha_* : H^1(K_p, W) \rightarrow H^1(K_p, W \otimes B_{dR}).$$

Define

$$H_g^1(K_p, W) := \ker(\alpha_*) \subseteq H^1(K_p, W), \quad H_s^1(K_p, W) := H^1(K_p, W) / H_g^1(K_p, W).$$

Similarly, if B_{cris} denotes the ring of crystalline periods, define

$$H_f^1(K_p, W) := \ker(H^1(K_p, W) \rightarrow H^1(K_p, W \otimes B_{\text{cris}})).$$

Let $\log_p : 1 + p\mathbb{Z}_p \rightarrow p\mathbb{Z}_p$ be the usual p -adic logarithm, \exp_p denote its inverse and set $u := \exp_p(p) \in 1 + p\mathbb{Z}_p$. Let

$$(2.7) \quad \text{Art} : K_p^\times \rightarrow G_{K_p}^{\text{ab}}$$

denote the Artin map of local class field theory, where $G_{K_p}^{\text{ab}}$ denotes the Galois group of the maximal abelian extension K_p^{ab} of K_p over K_p . Recall that (2.7) gives an isomorphism once extended to the profinite completion \widehat{K}_p^\times of K_p^\times .

Decomposition

$$K_p^\times = p^\mathbb{Z} \oplus (1 + p\mathcal{O}_{K_p}) \oplus (\mathcal{O}_{K_p}/p\mathcal{O}_{K_p})^\times,$$

corresponds via (2.7) to

$$G_{K_p}^{\text{ab}} \cong \text{Gal}(K_p^{\text{nr}}/K_p) \times \text{Gal}(K_\infty/K_p) \times \text{Gal}(K'_p/K_p)$$

where K_p^{nr} is the maximal unramified extension of K_p , K_∞/K_p is a \mathbb{Z}_p^2 -extension and K'_p/K_p is finite. More precisely, via the Artin map, $p^\mathbb{Z} \cong \text{Gal}(K_p^{\text{nr}}/K_p)$. Fix a prime \mathfrak{p} of H above p , and denote

$$K_p(\mu_{p^\infty}) := \varinjlim K_p(\mu_{p^m}); \quad H(p^\infty)_\mathfrak{p} := \varinjlim H(p^m)_{\mathfrak{p}_m},$$

where \mathfrak{p}_m is the unique prime of $H(p^m)$ lying above \mathfrak{p} . Let

$$K_p \subseteq K_{\text{cyc},p} \subseteq K_p(\mu_{p^\infty}), \quad K_p \subseteq K_{\text{ant},p} \subseteq H(p^\infty)_\mathfrak{p}$$

be maximal sub-extensions such that $\text{Gal}(K_{\text{cyc},p}/K_p) \simeq \text{Gal}(K_{\text{ant},p}/K_p) \simeq \mathbb{Z}_p$. Then

$$K_\infty = K_{\text{cyc},p} \cdot K_{\text{ant},p}.$$

Fix an element $u_\star \in 1 + p\mathcal{O}_{K_p}$ such that $\text{Fr}_p u_\star = -u_\star$ and $\{u, u_\star\}$ is a \mathbb{Z}_p -basis of $1 + p\mathcal{O}_{K_p}$. Then

$$\text{Gal}(K_{\infty,\mathfrak{p}}/K_p) \cong \Gamma_{\text{cyc}} \times \Gamma_{\text{ant}},$$

where

- (1) $\Gamma_{\text{cyc}} := \text{Gal}(K_{\text{cyc},p}/K_p)$ is generated topologically by $\sigma_{\text{cyc}} := \text{Art}(u)$;
- (2) $\Gamma_{\text{ant}} := \text{Gal}(K_{\text{ant},p}/K_p)$ is generated topologically by $\sigma_{\text{ant}} := \text{Art}(u_\star)$.

Set also $\Gamma_{\text{nr}} := \text{Gal}(K_p^{\text{nr}}/K_p)$. Recall the cyclotomic character $\chi_{\text{cyc}} : G_{\mathbb{Q}} \rightarrow \mathbb{Z}_p^\times$; by an abuse of notation, we continue to denote $\chi_{\text{cyc}} : \Gamma_{\text{cyc}} \rightarrow 1 + p\mathbb{Z}_p$ its restriction to Γ_{cyc} . Let also χ_{ant} and χ_{nr} denote the characters of Γ_{ant} and Γ_{nr} such that

$$\chi_{\text{ant}}(\sigma_{\text{ant}}) = u, \quad \chi_{\text{nr}}(\text{Fr}_p) = u$$

respectively.

Note that

$$\{\xi_{\text{nr}} := \log_p(\chi_{\text{nr}}), \xi_{\text{cyc}} := \log_p(\chi_{\text{cyc}}), \xi_{\text{ant}} := \log_p(\chi_{\text{ant}})\}$$

is naturally a \mathbb{Q}_p -basis of $\text{Hom}_{\text{cont}}(G_{K_p}^{\text{ab}}, \mathbb{Q}_p)$. After composing with the isomorphism $\mathbb{Q}_p \xrightarrow{\bar{\varphi}_{\text{Tate}}} V_{f|G_{K_p}}^-$ provided by Tate's uniformization this further yields a \mathbb{Q}_p -basis of $H^1(K_p, V^-)$, which in turn may be regarded as a basis of $H^1(K_p, \Phi_{\infty, p} \otimes \mathbb{Q}_p)$ by means of Lemma 2.1.

For a class $\xi \in H^1(K_p, V^-)$, we denote $\bar{\xi}$ its image in the singular quotient $H_s^1(K_p, V^-)$.

Lemma 2.2. *$\{\bar{\xi}_{\text{cyc}}, \bar{\xi}_{\text{ant}}\}$ is a \mathbb{Q}_p -basis for $H_s^1(K_p, V^-)$.*

Proof. By definition, the submodule $H_g^1(K_p, \mathbb{Q}_p) = H_f^1(K_p, \mathbb{Q}_p)$ of $H^1(K_p, \mathbb{Q}_p) = \text{Hom}_{\text{cont}}(G_{K_p}, \mathbb{Q}_p)$ is given by $H_f^1(K_p, \mathbb{Q}_p(\chi_E)) \cong \text{Hom}_{\text{cont}}(\Gamma_{\text{nr}}, \mathbb{Q}_p)$ and is generated by ξ_{nr} . The lemma follows because $H_s^1(K_p, \mathbb{Q}_p) = H_s^1(K_p, V^-) = H^1(K_p, V^-)/H_f^1(K_p, V^-)$. \square

Write the uniformizer of the elliptic curve E/\mathbb{Q}_p as

$$(2.8) \quad q_E = p^n u^s x \in p\mathbb{Z}_p, \quad n \geq 1, s \in \mathbb{Z}_p, x \in \mu_{p-1},$$

so that $n = \text{ord}_p(q_E)$ and $ps = \log_p(q_E)$. Let

$$\pi_* : H^1(K_p, V) \rightarrow H^1(K_p, V^-)$$

be the morphism induced in G_{K_p} -cohomology by the projection $\pi : V \rightarrow V^-$.

For a $\text{Gal}(K_p/\mathbb{Q}_p)$ -module M , let us write M^\pm for the eigenspace on which Fr_p acts as ± 1 . Set $a := a_p(E)$ as in the introduction.

Proposition 2.3. *Let $x_{\text{cyc}}, x_{\text{ant}} \in H^1(K_p, V)$ be elements such that*

$$(2.9) \quad \pi_*(x_{\text{cyc}}) = n\xi_{\text{cyc}} - s\xi_{\text{nr}}, \quad \pi_*(x_{\text{ant}}) = \xi_{\text{ant}} \text{ in } H^1(K_p, V^-).$$

Then $\{\bar{x}_{\text{cyc}}, \bar{x}_{\text{ant}}\}$ is a \mathbb{Q}_p -basis for $H_s^1(K_p, V)$ and π_ descends to an isomorphism*

$$\bar{\pi}_* : H_s^1(K_p, V) \xrightarrow{\cong} H_s^1(K_p, V^-), \quad \bar{x}_{\text{cyc}} \mapsto n \cdot \bar{\xi}_{\text{cyc}}, \quad \bar{x}_{\text{ant}} \mapsto \bar{\xi}_{\text{ant}}.$$

Moreover, the Frobenius eigenspaces $H_s^1(K_p, V)^a$ and $H_s^1(K_p, V)^{-a}$ are generated respectively by \bar{x}_{cyc} and \bar{x}_{ant} .

Proof. Consider the long exact sequence

$$(2.10) \quad 0 \rightarrow V^- \xrightarrow{\partial^0} H^1(K_p, V^+) \xrightarrow{\iota_*} H^1(K_p, V) \xrightarrow{\pi_*} H^1(K_p, V^-) \xrightarrow{\partial^1} H^2(K_p, V^+) \rightarrow 0$$

induced in G_{K_p} -cohomology by (2.4). The connecting homomorphisms ∂^0, ∂^1 can be expressed in terms of q_E as follows. Cup product and the trace map give a pairing

$$\langle \cdot, \cdot \rangle : H^1(K_p, V^+) \times H^1(K_p, V^-) \rightarrow H^2(K_p, V^+) \cong \mathbb{Q}_p$$

satisfying

$$\langle \delta_p(q_E), \xi \rangle = \xi(\text{Art}(q_E))$$

for all $\xi \in H^1(K_p, V^-)$. If we still call ∂^1 its composition with $H^2(K_p, V^+) \cong \mathbb{Q}_p$, then it coincides with the map $\langle \delta_p(q_E), \cdot \rangle$. Using the notation of (2.8), we have $\text{Art}(q_E) = \text{Fr}_p^n \sigma_{\text{cyc}}^s$ and thus

- (1) $\partial^1(\xi_{\text{cyc}}) = \xi_{\text{cyc}}(\text{Art}(q_E)) = \xi_{\text{cyc}}(\sigma_{\text{cyc}}^s) = \log_p(\chi_{\text{cyc}}(\sigma_{\text{cyc}}^s)) = s \log_p(u) = sp$;
- (2) $\partial^1(\xi_{\text{ant}}) = \xi_{\text{ant}}(\text{Art}(q_E)) = \log_p(\chi_{\text{ant}}(\sigma_{\text{ant}}^0)) = 0$;
- (3) $\partial^1(\xi_{\text{nr}}) = \xi_{\text{nr}}(\text{Art}(q_E)) = \xi_{\text{nr}}(\text{Fr}_p^n) = \log_p(\chi_{\text{nr}}(\sigma_{\text{cyc}}^n)) = n \log_p(u) = np$.

Combining these computations with the exactness of (2.10) we deduce that the image of π_* is generated by $\{n \cdot \xi_{\text{cyc}} - s \cdot \xi_{\text{nr}}, \xi_{\text{ant}}\}$, and this gives the existence of $x_{\text{cyc}}, x_{\text{ant}}$ as in the statement.

From (2.9) and Lemma 2.2 we deduce that $\bar{\pi}_*(\bar{x}_{\text{cyc}}) = n \cdot \bar{\xi}_{\text{cyc}}, \bar{\pi}_*(\bar{x}_{\text{ant}}) = \bar{\xi}_{\text{ant}}$ and that $\bar{\pi}_*$ is surjective. The map ι_* of (2.10) restricts to an isomorphism $\iota_* : H_f^1(K_p, V^+) \cong H_f^1(K_p, V)$, hence $\bar{\pi}_*$ is also injective.

Finally, in order to understand the action of Frobenius, note that $H_s^1(K_p, V) \simeq H_s^1(K_p, V^-)$ is naturally a $\text{Gal}(K_p/\mathbb{Q}_p)$ -module. As explained in the proof of Lemma 2.2, $\{\xi_{\text{cyc}}, \xi_{\text{ant}}\}$ is a basis of

$$(2.11) \quad H_s^1(K_p, V^-) \cong \text{Hom}(\Gamma_{\text{cyc}}, \mathbb{Q}_p(\chi_E)) \oplus \text{Hom}(\Gamma_{\text{ant}}, \mathbb{Q}_p(\chi_E)).$$

We have

$$\xi_{\text{ant}}^{\text{Fr}_p}(\sigma_{\text{ant}}) = \chi_E(\text{Fr}_p) \cdot \xi_{\text{ant}}(\text{Fr}_p \sigma_{\text{ant}} \text{Fr}_p^{-1}) = a \cdot \xi_{\text{ant}}(\sigma_{\text{ant}}^{-1}) = -a \cdot \xi_{\text{ant}}(\sigma_{\text{ant}})$$

and via (2.11) it follows that

$$H_s^1(K_p, V^-)^{-a} \cong \text{Hom}(\Gamma_{\text{ant}}, \mathbb{Q}_p(\chi_E))$$

is generated by $\bar{\xi}_{\text{ant}}$. Analogously, $\bar{\xi}_{\text{cyc}}$ generates $H_s^1(K_p, V^-)^a \cong \text{Hom}(\Gamma_{\text{cyc}}, \mathbb{Q}_p(\chi_E))$. □

2.3. Bloch–Kato logarithm and dual exponential maps. We wish to describe explicitly the Bloch–Kato logarithm and the dual exponential maps relative to the representation $V = V_f$ regarded as a G_{K_p} -module. First we need to study $D_{\text{dR}}(V_{K_p})$, $D_{\text{dR}}(V_{K_p}^\pm)$ and their filtration, where $V_{K_p} := V|_{G_{K_p}}$. Let \mathbb{C}_p be the completion of a fixed algebraic closure of \mathbb{Q}_p , and let

$$\mathcal{R} := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}.$$

Let $W(\mathcal{R})$ be the ring of Witt vectors of \mathcal{R} and denote $[\cdot] : \mathcal{R} \rightarrow W(\mathcal{R})$ the Teichmüller lift. There is an isomorphism

$$\varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p} \xrightarrow{\cong} \mathcal{R}, \quad (x^{(n)})_n \mapsto (x_n := x^{(n)} \pmod{p})_n.$$

Recall the basis $\tilde{\varepsilon} := (\tilde{\varepsilon}^{(m)})_m$ of $\mathbb{Z}_p(1)$ we introduced in §2.1. It can be regarded as an element of $\varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p} \cong \mathcal{R}$, and if \log_{q_E} denotes the branch of the p -adic logarithm such that $\log_{q_E}(q_E) = 0$, then

$$t := \log_{q_E}([\tilde{\varepsilon}])$$

generates the maximal ideal $\text{Fil}^1 B_{\text{dR}}$ of $B_{\text{dR}}^+ = \text{Fil}^0 B_{\text{dR}}$. Analogously, we can regard \tilde{q} as an element of \mathcal{R} and we define

$$q_E^\# := \log_{q_E}([\tilde{q}]/q_E) \in \text{Fil}^1 B_{\text{dR}}$$

Recall that the images ε, q of $\tilde{\varepsilon}, \tilde{q}$ under Tate's uniformization form a basis of V . By elementary calculations one obtains the following lemma.

Lemma 2.4. *Let*

$$e_1 := q \otimes 1 - \varepsilon \otimes \frac{q_E^\#}{t} \in V \otimes \text{Fil}^0 B_{\text{dR}}, \quad e_2 := \varepsilon \otimes \frac{1}{t} \in V \otimes \text{Fil}^{-1} B_{\text{dR}}.$$

These elements are invariant with respect to the action of G_{K_p} , they form a K_p -basis of $D_{\text{dR}}(V_{K_p})$ and e_1 is a K_p -basis of $\text{Fil}^0 D_{\text{dR}}(V_{K_p})$. Moreover, there are isomorphisms

$$\iota_* : D_{\text{dR}}(V_{K_p}^+) / \text{Fil}^0 D_{\text{dR}}(V_{K_p}^+) \xrightarrow{\cong} D_{\text{dR}}(V_{K_p}) / \text{Fil}^0 D_{\text{dR}}(V_{K_p}), \quad \varepsilon \otimes t^{-1} \mapsto e_2;$$

$$\pi_* : \text{Fil}^0 D_{\text{dR}}(V_{K_p}) \xrightarrow{\cong} \text{Fil}^0 D_{\text{dR}}(V_{K_p}^-), \quad e_1 \mapsto \bar{q},$$

where, with a slight abuse of notation, $\bar{q} := \bar{q} \otimes 1$ in $(V^- \otimes B_{\text{dR}}^+)^{G_{K_p}} = \text{Fil}^0 D_{\text{dR}}(V^-)$.

Kummer theory provides isomorphisms

$$\delta_p : \mathcal{O}_{K_p}^\times \otimes \mathbb{Q}_p \xrightarrow{\cong} H_f^1(K_p, \mathbb{Q}_p(1)), \quad E(K_p) \otimes \mathbb{Q}_p \xrightarrow{\cong} H_f^1(K_p, V),$$

which combined with (2.3) and Tate's uniformization yield canonical identifications of $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$ -modules

$$(2.12) \quad \mathcal{O}_{K_p}^\times(\chi_E) \otimes \mathbb{Q}_p \cong H_f^1(K_p, V^+) \xrightarrow{\iota_*} H_f^1(K_p, V) = E(K_p) \otimes \mathbb{Q}_p.$$

Besides, the p -adic logarithm induces a morphism of $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$ -modules

$$\log_p : \mathcal{O}_{K_p}^\times(\chi_E) \otimes \mathbb{Q}_p \longrightarrow K_p(\chi_E).$$

Via the identifications (2.12), it gives rise to the map

$$(2.13) \quad \log_E : E(K_p) \longrightarrow K_p(\chi_E),$$

which is the formal group logarithm on E associated with the invariant differential ω_f on E introduced in §3.1. Finally, (2.12) and the isomorphism

$$D_{\text{dR}}(V_{K_p}) / \text{Fil}^0 D_{\text{dR}}(V_{K_p}) \xrightarrow{\cong} K_p(\chi_E), \quad e_2 \mapsto 1,$$

allow us to identify \log_E with the composition of the Bloch–Kato logarithm

$$\log : H_f^1(K_p, V) \longrightarrow D_{\text{dR}}(V_{K_p}) / \text{Fil}^0 D_{\text{dR}}(V_{K_p})$$

with the pairing with ω_f .

For a p -adic de Rham representation W of G_{K_p} , let

$$\gamma : \text{Fil}^0 D_{\text{dR}}(W) = (W \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+)^{G_{K_p}} \longrightarrow H^1(K_p, W \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+)$$

be the isomorphism defined by $x \mapsto \gamma(x)$, where $\gamma(x)$ is the cohomology class of the map

$$\sigma \mapsto x \cdot \log_p(\chi_{\text{cyc}}(\sigma)).$$

Definition 2.5. The *Bloch–Kato dual exponential* of W is the map

$$\exp^* : H^1(K_p, W) \xrightarrow{\alpha_*} H^1(K_p, W \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+) \xrightarrow{\gamma^{-1}} \text{Fil}^0 D_{\text{dR}}(W).$$

The map $H^1(K_p, W \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+) \longrightarrow H^1(K_p, W \otimes_{\mathbb{Q}_p} B_{\text{dR}})$ is injective and thus $\ker(\exp^*) = H_g^1(K_p, W)$. We shall regard the dual exponential associated to W as the map

$$\exp^* : H_s^1(K_p, W) = H^1(K_p, W) / H_g^1(K_p, W) \longrightarrow \text{Fil}^0 D_{\text{dR}}(W).$$

Similarly as in the case of the Bloch–Kato logarithm, the dual exponential for $W = K_p$ together with the identification $H_s^1(K_p, \mathbb{Q}_p)(\chi_E) = H_s^1(K_p, \mathbb{Q}_p(\chi_E))$ gives rise to an isomorphism of $K_p[G_{\mathbb{Q}_p}]$ -modules

$$\exp^* : H_s^1(K_p, \mathbb{Q}_p(\chi_E)) \longrightarrow K_p(\chi_E).$$

It coincides with the dual exponential

$$(2.14) \quad \exp^* : H_s^1(K_p, V) \longrightarrow K_p(\chi_E)$$

via the identifications

$$H_s^1(K_p, \mathbb{Q}_p(\chi_E)) \cong H_s^1(K_p, V^-) \cong H_s^1(K_p, V)$$

explained in the previous section, where

$$\mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V_{K_p}) \xrightarrow{\cong} K_p(\chi_E), \quad e_1 \mapsto 1.$$

It follows directly from the very definitions and the $G_{\mathbb{Q}_p}$ -equivariance of the dual exponential map that

$$\exp^*(\xi_{\mathrm{cyc}}) = 1 \quad \text{and} \quad z := \exp^*(\xi_{\mathrm{ant}}) \in K_p^-.$$

The map (2.14) and its \pm -components can be explicitly described as follows.

Proposition 2.6. *The dual exponential $\exp^* : H_s^1(K_p, V) \longrightarrow K_p(\chi_E)$ is characterized by*

$$\exp^*(\bar{x}_{\mathrm{cyc}}) = n, \quad \exp^*(\bar{x}_{\mathrm{ant}}) = z \in K_p^-.$$

Proof. By (the proof of) Proposition 2.3,

$$H_s^1(K_p, \mathbb{Q}_p(\chi_E))^a \oplus H_s^1(K_p, \mathbb{Q}_p(\chi_E))^{-a} \cong \mathrm{Hom}(\Gamma_{\mathrm{cyc}}, \mathbb{Q}_p(\chi_E)) \oplus \mathrm{Hom}(\Gamma_{\mathrm{ant}}, \mathbb{Q}_p(\chi_E)),$$

whose basis $\{\xi_{\mathrm{cyc}}, \xi_{\mathrm{ant}}\}$ is compatible with the above decomposition.

The claim follows by using Lemma 2.4, Proposition 2.3 and the description of (2.14) discussed before this proposition. \square

Corollary 2.7. *There is a commutative diagram whose arrows are isomorphisms of $G_{\mathbb{Q}_p}$ -modules:*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{cyc}}, \Phi_{\infty, \mathfrak{p}}) \oplus \mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{ant}}, \Phi_{\infty, \mathfrak{p}}) & \xrightarrow{\varphi} & H_s^1(K_p, V) \\ \downarrow \frac{1}{p} \mathrm{ev}_{\mathrm{cyc}} \oplus \frac{1}{p} \mathrm{ev}_{\mathrm{ant}} & & \downarrow \exp^* \\ (\Phi_{\infty, \mathfrak{p}} \otimes \mathbb{Q}_p) \oplus (\Phi_{\infty, \mathfrak{p}} \otimes \mathbb{Q}_p) & \xrightarrow{\varrho} & K_p(\chi_E) \end{array}$$

where

- (1) $\varphi := (\bar{\varphi}_{\mathrm{Tate}} \circ \bar{\pi}_*)^{-1}$;
- (2) $\varrho(\bar{q}, 0) = 1$, $\varrho(0, \bar{q}) = z$, where $z = \exp^*(\xi_{\mathrm{ant}}) \in K_p^-$ is as in Proposition 2.6;
- (3) $\mathrm{ev}_{\mathrm{cyc}}$ denotes the valuation at $\sigma_{\mathrm{cyc}} \in \Gamma_{\mathrm{cyc}}$, and analogously for $\mathrm{ev}_{\mathrm{ant}}$.

Moreover, φ decomposes with respect to the action of Fr_p as

$$(2.15) \quad \mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{cyc}}, \Phi_{\infty, \mathfrak{p}}) \longrightarrow H_s^1(K_p, V)^a, \quad \mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{ant}}, \Phi_{\infty, \mathfrak{p}}) \longrightarrow H_s^1(K_p, V)^{-a}.$$

Proof. By definition we have

$$\xi_{\mathrm{cyc}}(\sigma_{\mathrm{cyc}}) = \bar{\varphi}_{\mathrm{Tate}}(\log_p(\chi_{\mathrm{cyc}}(\sigma_{\mathrm{cyc}}))) = \bar{\varphi}_{\mathrm{Tate}}(\log_p(u)) = \bar{\varphi}_{\mathrm{Tate}}(p) = p \cdot \bar{q}$$

and similarly, $\xi_{\mathrm{ant}}(\sigma_{\mathrm{ant}}) = p \cdot \bar{q}$. The statement follows by combining (the proof of) Proposition 2.6, Proposition 2.3 and Lemma 2.1. \square

3. HIDA FAMILIES AND SELMER GROUPS ASSOCIATED TO (f, g, h)

3.1. Filtrations and differentials. Let

$$\varphi \in S_w(M, \varepsilon)$$

be an ordinary newform of weight w , level M and character ε with coefficients in a finite extension L/\mathbb{Q} . Fix an algebraic closure $\bar{\mathbb{Q}}$ of \mathbb{Q} and an embedding $L \subset \bar{\mathbb{Q}}$. Fix also a prime number p , an algebraic closure $\bar{\mathbb{Q}}_p$ of \mathbb{Q}_p and an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, with respect to which we assume φ is ordinary. Let $L_p \subset \bar{\mathbb{Q}}_p$ denote the p -adic closure of L .

Let V_φ denote the p -adic Galois representation attached to φ with coefficients in L_p and let $\alpha_\varphi, \beta_\varphi$ be the roots of the Hecke polynomial $x^2 + a_p(\varphi)x + \varepsilon(p)$ at p of V_φ . Since φ is ordinary, one of these roots, say α_φ , is a p -adic unit. As a $G_{\mathbb{Q}_p}$ -representation, V_φ is equipped with a decreasing filtration of $L_p[G_{\mathbb{Q}_p}]$ -modules

$$(3.1) \quad \text{Fil}^2(V_\varphi) = 0 \subseteq \text{Fil}^1(V_\varphi) \subseteq V_\varphi = \text{Fil}^0(V_\varphi).$$

Let

$$\psi_\varphi : G_{\mathbb{Q}_p} \longrightarrow L_p^\times$$

be the unramified character such that $\psi_\varphi(\text{Fr}_p) = \alpha_\varphi$. The filtration (3.1) satisfies:

- (i) $V_\varphi^+ := \text{Fil}^1(V_\varphi)$ and $V_\varphi^- := V_\varphi / \text{Fil}^1(V_\varphi)$ have dimension one over L_p ;
- (ii) $G_{\mathbb{Q}_p}$ acts on V_φ^+ via $\chi_{\text{cyc}}^{w-1} \varepsilon \psi_\varphi^{-1}$;
- (iii) $G_{\mathbb{Q}_p}$ acts on V_φ^- via ψ_φ . In particular, it is an unramified $G_{\mathbb{Q}_p}$ -module.

When $w \geq 2$ Faltings' comparison theorem yields an isomorphism

$$(3.2) \quad S_w(M)_{L_p}[\varphi] \xrightarrow{\cong} \text{Fil}^1 \text{D}_{\text{dR}}(V_\varphi) \cong \text{D}_{\text{dR}}(V_\varphi^-),$$

where $S_w(M)_{L_p}[\varphi]$ denotes the φ -isotypical component of $S_w(M) \otimes L_p$. We define

$$\omega_\varphi \in \text{D}_{\text{dR}}(V_\varphi^-)$$

to be the image of φ via (3.2).

Denote $\varphi^* := \varphi \otimes \varepsilon^{-1} \in S_w(M, \bar{\varepsilon})$. Consider the pairing

$$\langle \cdot, \cdot \rangle : V_\varphi \times V_{\varphi^*}(1) \longrightarrow L_p$$

given by Poincaré's duality. This in turn induces perfect pairings

$$\langle \cdot, \cdot \rangle : \text{D}_{\text{dR}}(V_\varphi) \times \text{D}_{\text{dR}}(V_{\varphi^*}(1)) \longrightarrow \text{D}_{\text{dR}}(L_p) = L_p$$

and

$$\langle \cdot, \cdot \rangle : \text{D}_{\text{dR}}(V_\varphi^+) \times S_w(M)_{L_p}[\varphi^*] \longrightarrow L_p.$$

Define

$$\eta_\varphi \in \text{D}_{\text{dR}}(V_\varphi^+) \subseteq \text{D}_{\text{dR}}(V_\varphi)$$

to be the differential characterized by

$$\langle \omega_{\varphi^*}, \eta_\varphi \rangle = 1.$$

If φ has weight $w = 1$, then we will always assume that $p \nmid M$. In this case, both α_φ and β_φ are p -adic units, and throughout this paper we will also assume that

$$\alpha_\varphi \neq \beta_\varphi.$$

In this setting, the filtration (3.1) admits a splitting. More precisely, if we denote $V_\varphi^\alpha, V_\varphi^\beta$ the eigenspaces for the action of Fr_p on V_φ , with eigenvalues $\alpha_\varphi, \beta_\varphi$ respectively, then

$$V_\varphi = V_\varphi^\alpha \oplus V_\varphi^\beta, \quad V_\varphi^+ = V_\varphi^\beta, \quad V_\varphi^- = V_\varphi^\alpha$$

as $G_{\mathbb{Q}_p}$ -modules. In this case, there is a pairing

$$(3.3) \quad \text{D}_{\text{dR}}(V_\varphi) \times \text{D}_{\text{dR}}(V_\varphi) \longrightarrow \text{D}_{\text{dR}}(L_p(\chi_\varphi)),$$

and we let

$$(3.4) \quad \omega_{\varphi_\alpha} \in \text{D}_{\text{dR}}(V_\varphi^\alpha), \quad \eta_{\varphi_\alpha} \in \text{D}_{\text{dR}}(V_\varphi^\beta)$$

be the differentials attached by Ohta to the p -stabilized eigenform φ_α as described in [DR16, §2]. They satisfy

$$(3.5) \quad \langle \omega_{\varphi_\alpha}, \eta_{\varphi_\alpha} \rangle = \mathfrak{g}(\varepsilon) \in L_p,$$

where $\mathfrak{g}(\varepsilon)$ denotes the Gauss sum of the nebentype character ε of φ .

Attached to a weight 1 modular form φ there is an Artin representation

$$\rho_\varphi : G_{\mathbb{Q}} \rightarrow \text{GL}(V_\varphi^\circ)$$

where V_φ° is an L -vector space equipped with a (non-canonical) isomorphism $j_\varphi : V_\varphi^\circ \otimes_L L_p \xrightarrow{\cong} V_\varphi$. The choice of j_φ thus induces an L -structure on V_φ by $V_\varphi^L := j_\varphi(V_\varphi^\circ)$.

Let v_φ^α be an L -basis of $V_\varphi^L \cap V_\varphi^\alpha$ and v_φ^β an L -basis of $V_\varphi^L \cap V_\varphi^\beta$. Let H be the number field cut out by ρ_φ , fix once and for all an embedding $H \subseteq \bar{\mathbb{Q}}_p$ and denote H_p the p -adic completion of H inside $\bar{\mathbb{Q}}_p$. Finally, define periods

$$\Omega_{\varphi_\alpha} \in H_p^{\text{Fr}_p=1/\alpha_\varphi}, \quad \Theta_{\varphi_\alpha} \in H_p^{\text{Fr}_p=1/\beta_\varphi}$$

characterized as

$$(3.6) \quad \Omega_{\varphi_\alpha} \otimes v_\varphi^\alpha = \omega_{\varphi_\alpha} \in \text{D}_{\text{dR}}(V_g^\alpha), \quad \Theta_{\varphi_\alpha} \otimes v_\varphi^\beta = \eta_{\varphi_\alpha} \in \text{D}_{\text{dR}}(V_\varphi^\beta).$$

Although the above periods depend on the choice of j_φ , the ratio

$$(3.7) \quad \mathcal{L}_{g_\alpha} := \frac{\Omega_{g_\alpha}}{\Theta_{g_\alpha}}$$

is readily seen to be independent of this choice; it only depends on the choice of basis $\{v_\varphi^\alpha, v_\varphi^\beta\}$ and is thus well defined up to scalars in L^\times .

Let $\Lambda := \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$ be the Iwasawa algebra and $\mathcal{W} := \text{Spf}(\Lambda)$ be the weight space. It is the rigid analytic space whose A -points for any \mathbb{Z}_p -algebra A are

$$\mathcal{W}(A) = \text{Hom}_{\mathbb{Z}_p\text{-alg}}(\Lambda, A) = \text{Hom}_{\text{cts}}(1 + p\mathbb{Z}_p, A^\times).$$

Recall that \mathbb{Z} can be seen as a dense subset of $\mathcal{W}(\mathbb{C}_p)$ identifying an integer $k \in \mathbb{Z}$ with the character $\nu_k : (x \mapsto x^k) \in \text{Hom}_{\text{cts}}(1 + p\mathbb{Z}_p, \mathbb{C}_p^\times)$. More generally, given a Dirichlet character ϵ of p -power conductor, we denote $\nu_{k,\epsilon}$ the point $(x \mapsto \epsilon(x)x^k) \in \mathcal{W}(\mathbb{C}_p)$. A character $\nu \in \mathcal{W}$ is called *classical* if it is of the form $\nu = \nu_{k,\epsilon}$ with $k \geq 2$ and k is called its *weight*. A classical point is *cristalline* if it is of the form ν_{k,ω^k} , where ω is the Teichmüller character.

Let

$$\varphi = \sum a_n(\varphi)q^n \in \Lambda_\varphi[[q]]$$

be a Hida family of tame level M and tame character χ , where Λ_φ is a finite flat extension of Λ , and let $\kappa : \mathcal{W}_\varphi \rightarrow \mathcal{W}$ be the *weight map* induced by the inclusion $\Lambda \subseteq \Lambda_\varphi$. A point $x \in \mathcal{W}_\varphi$

is called *classical* (resp. *cristalline*) if $\kappa(x)$ is, and we denote $\mathcal{W}_\varphi^{\text{cl}}$ (resp. $\mathcal{W}_\varphi^\circ$) the set of classical (resp. cristalline) points of \mathcal{W}_φ . For each $x \in \mathcal{W}_\varphi^{\text{cl}}$ with $\kappa(x) = \nu_{k,\epsilon}$, the specialization

$$\varphi_x := \sum_{n \geq 1} x(a_n(\varphi)) q^n$$

of φ at x is the q -expansion of a classical p -ordinary eigenform of weight k and character $\chi\epsilon\omega^{-k}$. If $x \in \mathcal{W}$ is a point of weight 1, meaning that $\kappa(x) = \nu_{1,\epsilon}$ for some Dirichlet character ϵ of p -power conductor, the specialization φ_x may be a classical modular form or not. (Later we shall regard the p -stabilizations of g and h as weight 1 specializations of a Hida family, and these are classical by construction.)

Notice that the specializations of φ at cristalline points have all nebentype χ . If $x \in \mathcal{W}_\varphi^\circ$ is a cristalline point of weight $k > 2$, then φ_x is the ordinary p -stabilization of a newform $\varphi_x \in S_k(N, \chi)$. If $k = 2$, then φ_x may be either old or new at p . We denote $\varphi_x := \varphi_x$ if it is new, while if φ_x is old at p , we denote φ_x the newform whose p -stabilization is φ_x . If no confusion is likely to arise, we may often denote these forms by an abuse of notation as φ_k and φ_k .

Let V_φ be the Galois representation of $G_\mathbb{Q}$ attached to φ , a Λ_φ -module of rank 2. As a $G_{\mathbb{Q}_p}$ -representation there is again a filtration

$$\text{Fil}^2(V_\varphi) = 0 \subseteq \text{Fil}^1(V_\varphi) \subseteq V_\varphi = \text{Fil}^0(V_\varphi)$$

where $V_\varphi^+ := \text{Fil}^1(V_\varphi)$ has dimension one, and the quotient $V_\varphi^- := V_\varphi / \text{Fil}^1(V_\varphi)$ is unramified. More precisely, $G_{\mathbb{Q}_p}$ acts on V_φ^- via the unramified character

$$\psi_\varphi : G_{\mathbb{Q}_p} \longrightarrow \Lambda_\varphi^\times$$

such that $\psi_\varphi(\text{Fr}_p) = a_p(\varphi)$.

If x is cristalline and L_p/\mathbb{Q}_p is a finite extension containing the values of x , then the Galois representation $V_\varphi \otimes_{\Lambda_\varphi, x} L_p \cong V_{\varphi_x}$ is cristalline at p .

Fix now three cuspidal Hida families

$$\mathbf{f} \in \Lambda_{\mathbf{f}}[[q]], \quad \mathbf{g} \in \Lambda_{\mathbf{g}}[[q]], \quad \mathbf{h} \in \Lambda_{\mathbf{h}}[[q]].$$

Write $\Lambda_{\mathbf{fgh}} = \Lambda_{\mathbf{f}} \hat{\otimes} \Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{h}}$ and $\mathcal{O}_{\mathbf{fgh}} := \Lambda_{\mathbf{f}}[1/p] \hat{\otimes} \Lambda_{\mathbf{g}}[1/p] \hat{\otimes} \Lambda_{\mathbf{h}}[1/p]$. Let $\mathcal{W}_{\mathbf{fgh}} = \text{Spf}(\Lambda_{\mathbf{fgh}}) = \mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}}$ denote the associated three-dimensional product of weight spaces. Denote $\mathcal{W}_{\mathbf{fgh}}^\circ$ the set of cristalline points $(x, y, z) \in \mathcal{W}_{\mathbf{f}}^\circ \times \mathcal{W}_{\mathbf{g}}^\circ \times \mathcal{W}_{\mathbf{h}}^\circ$ of weights $(k, \ell, m) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ such that $k \equiv 2 \pmod{p-1}, \ell \equiv m \equiv 1 \pmod{2(p-1)}$. As above, by abuse of notation we may identify points (x, y, z) of $\mathcal{W}_{\mathbf{fgh}}^\circ$ with their weights (k, ℓ, m) .

Set

$$\mathbb{V} := V_{\mathbf{f}} \otimes V_{\mathbf{g}} \otimes V_{\mathbf{h}} \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi,$$

where $\Xi : G_{\mathbb{Q}_p} \longrightarrow \mathcal{O}_{\mathbf{fgh}}^\times$ is the character defined in [BSVb, §4.6.2] and is characterized by the property that, if $\underline{x} = (k, \ell, m) \in \mathcal{W}_{\mathbf{fgh}}^\circ$ is a triple of cristalline points and $\rho_{\underline{x}} := \nu_k \otimes \nu_\ell \otimes \nu_m : \mathcal{O}_{\mathbf{fgh}}^\times \longrightarrow L_p$ is the corresponding specialisation map, then $\rho_{\underline{x}} \circ \Xi = \chi_{\text{cyc}}^{(4-k-\ell-m)/2}$. We define a filtration on \mathbb{V} by

$$\text{Fil}^q(\mathbb{V}) := [\oplus_{i+j+t=q} \text{Fil}^i(V_{\mathbf{f}}) \hat{\otimes} \text{Fil}^j(V_{\mathbf{g}}) \hat{\otimes} \text{Fil}^t(V_{\mathbf{h}})] \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi,$$

Since $\mathbb{V} / \text{Fil}^2(\mathbb{V})$ has no $G_{\mathbb{Q}_p}$ -invariants, the natural inclusion

$$\text{Fil}^2(\mathbb{V}) = [V_{\mathbf{f}} \otimes V_{\mathbf{g}}^+ \otimes V_{\mathbf{h}}^+ \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi] \oplus [V_{\mathbf{f}}^+ \otimes V_{\mathbf{g}} \otimes V_{\mathbf{h}}^+ \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi] \oplus [V_{\mathbf{f}}^+ \otimes V_{\mathbf{g}}^+ \otimes V_{\mathbf{h}} \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi] \hookrightarrow \mathbb{V}$$

induces an injective morphism in cohomology

$$H^1(\mathbb{Q}_p, \text{Fil}^2(\mathbb{V})) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}),$$

and henceforth we shall identify $H^1(\mathbb{Q}_p, \text{Fil}^2(\mathbb{V}))$ with its image in $H^1(\mathbb{Q}_p, \mathbb{V})$. Defining

$$(3.8) \quad \mathbb{V}^f := V_{\mathbf{f}}^- \otimes V_{\mathbf{g}}^+ \otimes V_{\mathbf{h}}^+ \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi, \quad \mathbb{V}^g := V_{\mathbf{f}}^+ \otimes V_{\mathbf{g}}^- \otimes V_{\mathbf{h}}^+ \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi, \quad \mathbb{V}^h := V_{\mathbf{f}}^+ \otimes V_{\mathbf{g}}^+ \otimes V_{\mathbf{h}}^- \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi,$$

it follows that $H^1(\mathbb{Q}_p, \text{Fil}^2(\mathbb{V})/\text{Fil}^3(\mathbb{V}))$ decomposes as

$$H^1(\mathbb{Q}_p, \mathbb{V}^f) \oplus H^1(\mathbb{Q}_p, \mathbb{V}^g) \oplus H^1(\mathbb{Q}_p, \mathbb{V}^h).$$

For $\varphi \in \{f, g, h\}$, we denote

$$\pi_{\varphi} : H^1(\mathbb{Q}_p, \text{Fil}^2(\mathbb{V})) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}^{\varphi})$$

the natural projection map.

3.2. Selmer groups. Let W be a $L_p[G_{\mathbb{Q}}]$ -module. Given a prime number ℓ , define as customary

$$H_f^1(\mathbb{Q}_{\ell}, W) := \begin{cases} H_{\text{ur}}^1(\mathbb{Q}_{\ell}, W) := H^1(\mathbb{Q}_{\ell}^{\text{ur}}/\mathbb{Q}_{\ell}, W^{I_{\ell}}) & \ell \neq p \\ \ker \left(H^1(\mathbb{Q}_p, W) \rightarrow H^1(\mathbb{Q}_p, W \otimes_{\mathbb{Q}_p} B_{\text{cris}}) \right) & \ell = p, \end{cases}$$

and

$$H_s^1(\mathbb{Q}_{\ell}, W) := H^1(\mathbb{Q}_{\ell}, W) / H_f^1(\mathbb{Q}_{\ell}, W).$$

The Bloch–Kato Selmer group of W is

$$\text{Sel}_p(W) := \{x \in H^1(\mathbb{Q}, W) \mid \text{res}_{\ell}(x) \in H_f^1(\mathbb{Q}_{\ell}, W) \text{ for all } \ell\},$$

where $\text{res}_{\ell} : H^1(\mathbb{Q}, W) \rightarrow H^1(\mathbb{Q}_{\ell}, W)$ denotes the restriction map in Galois cohomology.

For each prime ℓ , we denote by ∂_{ℓ} the composition

$$\partial_{\ell} : H^1(\mathbb{Q}, W) \xrightarrow{\text{res}_{\ell}} H^1(\mathbb{Q}_{\ell}, W) \longrightarrow H_s^1(\mathbb{Q}_{\ell}, W).$$

The relaxed Selmer group is defined as

$$\text{Sel}_{(p)}(W) := \{x \in H^1(\mathbb{Q}, W) \mid \text{res}_{\ell}(x) \in H_f^1(\mathbb{Q}_{\ell}, W) \text{ for all } \ell \neq p\} \supseteq \text{Sel}_p(W).$$

Let now (f, g, h) be the triple of eigenforms of weights $(2, 1, 1)$ introduced at the beginning of the article. Recall Assumption 1.1 imposed on $V := V_f \otimes V_g \otimes V_h$ in the introduction, which implies

$$\text{Sel}_p(V) = 0.$$

Indeed, as discussed in detail in §5.1, the representation V decomposes as a direct sum

$$V = [V_f \otimes V_{\psi_1}] \oplus [V_f \otimes V_{\psi_2}],$$

where, for $i = 1, 2$, the $G_{\mathbb{Q}}$ -representation V_{ψ_i} is induced by a ring class character ψ_i of the imaginary quadratic field K . This decomposition induces factorisations

$$L(E, \rho, s) = L(E/K, \psi_1, s) \cdot L(E/K, \psi_2, s), \quad \text{Sel}_p(V) = \text{Sel}_p(E \otimes \psi_1) \oplus \text{Sel}_p(E \otimes \psi_2).$$

Combining the factorisation of the complex L -function with Assumption 1.1 we obtain that $L(E/K, \psi_i, 1) \neq 0$ for $i = 1, 2$, which, by [BD97, Theorem B], implies that $\text{Sel}_p(E \otimes \psi_1) = \text{Sel}_p(E \otimes \psi_2) = 0$.

Lemma 3.1. *There is an isomorphism*

$$\partial_p : \text{Sel}_{(p)}(V) \xrightarrow{\cong} H_s^1(\mathbb{Q}_p, V)$$

Proof. This follows from Poitou–Tate duality as in [MR04, Theorem 2.3.4], where a similar statement is proved, under the (irrelevant) assumption that $p \nmid N_f$. \square

The characteristic polynomial for the action of Fr_p on V_g is $x^2 - a_p(g)x + \chi(p) = x^2 - \varepsilon(p)$ and we may thus write the eigenvalues of Fr_p as $\alpha_g = \lambda$, $\beta_g = -\lambda$ for some root of unity λ . The same holds for h and since its nebentype is the inverse of that of g we have

$$(\alpha_h, \beta_h) = (1/\lambda, -1/\lambda) \quad \text{or} \quad (\alpha_h, \beta_h) = (-1/\lambda, 1/\lambda).$$

Using the notation of §3.1, denote $\{v_g^\alpha, v_g^\beta\}$ and $\{v_h^\alpha, v_h^\beta\}$ a pair of L -bases for V_g and V_h consisting of eigenvectors with eigenvalue $\alpha_g, \beta_g, \alpha_h, \beta_h$ respectively. Denote V_g^α, V_g^β , etc. the corresponding eigenspaces. Set $V_{gh}^{\alpha\alpha} := V_g^\alpha \otimes V_h^\alpha$ and $V^{\alpha\alpha} := V_f \otimes V_{gh}^{\alpha\alpha}$, and likewise for the remaining pairs of eigenvalues. There is a decomposition of $L_p[G_{\mathbb{Q}_p}]$ -modules

$$(3.9) \quad V := V_{fgh} = V^{\alpha\alpha} \oplus V^{\alpha\beta} \oplus V^{\beta\alpha} \oplus V^{\beta\beta}.$$

Lemma 3.2. *Let $\triangle, \heartsuit \in \{\alpha, \beta\}$. Then*

i) the Bloch–Kato logarithm gives an isomorphism

$$\log_{\triangle\heartsuit} : H_f^1(\mathbb{Q}_p, V^{\triangle\heartsuit}) = H_f^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit}) \xrightarrow{\cong} L_p;$$

ii) the Bloch–Kato dual exponential gives an isomorphism

$$\exp_{\triangle\heartsuit}^* : H_s^1(\mathbb{Q}_p, V^{\triangle\heartsuit}) = H_s^1(\mathbb{Q}_p, V_f^- \otimes V_{gh}^{\triangle\heartsuit}) \xrightarrow{\cong} L_p;.$$

Proof. Let $W := V_{gh}^{\triangle\heartsuit}$. By [DRb, Lemma 2.4.1]

$$(3.10) \quad H_f^1(\mathbb{Q}_p, V^{\triangle\heartsuit}) = \begin{cases} H_f^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit}) & \text{if } \triangle \cdot \heartsuit \cdot a = +1 \\ H_f^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit}) & \text{if } \triangle \cdot \heartsuit \cdot a = -1. \end{cases}$$

Moreover, in the case in which $\triangle \cdot \heartsuit \cdot a = -1$, we have

$$H^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit}) = H_f^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit})$$

(see [DRa, Example 3.1.4]). Since $V_f^+ \cong \mathbb{Q}_p(1)(\chi_E)$, *i)* follows from the fact that

$$V_f^+ \otimes V_{gh}^{\triangle\heartsuit} = L_p(\zeta\chi_{\text{cyc}})$$

where $\zeta : G_{\mathbb{Q}_p} \rightarrow \mathbb{Q}_p^\times$ is the unramified character given by $\zeta(\text{Fr}_p) = \triangle \cdot \heartsuit \cdot a$ (for more details, see e.g. [DRa, Example 3.1.4]). The statement on the singular quotients is proven similarly using (3.10) and the fact that

$$V_f^- \otimes V_{gh}^{\triangle\heartsuit} = L_p(\zeta).$$

\square

Set $L_p^4 = L_p \oplus L_p \oplus L_p \oplus L_p$. We obtain from the previous lemma isomorphisms

$$\log : H_f^1(\mathbb{Q}_p, V) \rightarrow L_p^4, \quad \exp^* : H_s^1(\mathbb{Q}_p, V) \rightarrow L_p^4.$$

where $\log = \log_{\alpha\alpha} \oplus \log_{\alpha\beta} \oplus \log_{\beta\alpha} \oplus \log_{\beta\beta}$ and $\exp^* = \exp_{\alpha\alpha}^* \oplus \exp_{\alpha\beta}^* \oplus \exp_{\beta\alpha}^* \oplus \exp_{\beta\beta}^*$.

Combining Lemma 3.1 with Lemma 3.2 we obtain the following result on the L_p -structure of the relaxed Selmer group attached to V .

Corollary 3.3. *There are isomorphisms*

$$\text{Sel}_{(p)}(V) \xrightarrow{\partial_p} H_s^1(\mathbb{Q}_p, V) \xrightarrow{\exp^*} L_p^4.$$

We now turn to pin down a particular basis of $\text{Sel}_{(p)}(V)$ which is canonical up to multiplication by elements in L^\times . In order to do this, we need a more precise description of the finite and singular parts of $H^1(\mathbb{Q}_p, V^{\heartsuit\Delta})$ in terms of $H^1(K_p, V_f)$. Note that the latter space is equipped with a natural action of $\text{Gal}(K_p/\mathbb{Q}_p)$.

In what follows, for any $\text{Gal}(K_p/\mathbb{Q}_p)$ -module M we denote M^\pm the subspace of M on which Fr_p acts as multiplication by ± 1 . Note that $\alpha_g\alpha_h, \alpha_g\beta_h, \beta_g\alpha_h, \beta_g\beta_h \in \{\pm 1\}$ and thus it makes sense to consider $M^{\alpha_g\alpha_h}$, etc.

Lemma 3.4. *Let $\heartsuit, \Delta \in \{\alpha, \beta\}$. There are canonical isomorphisms of L_p -vector spaces*

$$H_f^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\heartsuit\Delta}) \cong H_f^1(K_p, V_f)^{\Delta\heartsuit} \otimes V_{gh}^{\heartsuit\Delta}$$

and

$$H_s^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\heartsuit\Delta}) \cong H_s^1(K_p, V_f)^{\Delta\heartsuit} \otimes V_{gh}^{\heartsuit\Delta}.$$

Proof. Let χ be the quadratic character of $G_{\mathbb{Q}_p}$ with $\chi(\text{Fr}_p) = -1$. We describe the isomorphisms in the case in which E has split multiplicative reduction at p and $\Delta \cdot \heartsuit = -1$. In this setting,

$$\begin{aligned} H_f^1(\mathbb{Q}_p, V^{\Delta\heartsuit}) &= H_f^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\Delta\heartsuit}) && \text{by Lemma 3.2} \\ &= H_f^1(\mathbb{Q}_p, \mathbb{Q}_p(1) \otimes L_p(\chi)) && \text{by Lemma 3.1 (ii)} \\ &= H_f^1(K_p, \mathbb{Q}_p(1) \otimes L_p(\chi))^{G_{\mathbb{Q}_p}} && \text{by the inflation-restriction exact sequence} \\ &= (H_f^1(K_p, \mathbb{Q}_p(1)) \otimes L_p(\chi))^{G_{\mathbb{Q}_p}} && \text{because } \chi|_{G_{K_p}} = 1 \\ &= (H_f^1(K_p, V_f) \otimes L_p(\chi))^{G_{\mathbb{Q}_p}} && \text{by (2.12).} \end{aligned}$$

The claim follows, because $V_{gh}^{\Delta\heartsuit} = L_p(\chi)$ as $G_{\mathbb{Q}_p}$ -modules and the subspace of $G_{\mathbb{Q}_p}$ -invariants of $H_f^1(K_p, V_f) \otimes V_{gh}^{\Delta\heartsuit}$ is $H_f^1(K_p, V_f)^{\Delta\heartsuit} \otimes V_{gh}^{\heartsuit\Delta}$. Using similar computations as above, one also verifies that

$$H_s^1(\mathbb{Q}_p, V^{\Delta\heartsuit}) \cong (H^1(K_p, V_f) \otimes V_{gh}^{\Delta\heartsuit})^{G_{\mathbb{Q}_p}} / (H_f^1(K_p, V_f) \otimes V_{gh}^{\Delta\heartsuit})^{G_{\mathbb{Q}_p}} \cong (H_s^1(K_p, V_f) \otimes V_{gh}^{\Delta\heartsuit})^{G_{\mathbb{Q}_p}}.$$

The remaining cases are proven similarly. \square

Remark 3.5. The logarithm maps of Lemma 3.2 are twisted versions of (2.13). More precisely, $\log_{\alpha\beta}$ may be accordingly recast as

$$(3.11) \quad \begin{aligned} H_f^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\alpha\beta}) &= E(K_p)^{\beta\alpha} \otimes V_{gh}^{\alpha\beta} \longrightarrow K_p(\chi_E)^{\beta\alpha} \otimes K_p^{\alpha\beta} \otimes L_p \cong L_p \\ x \otimes v_g^\alpha v_h^\beta &\longmapsto \log_E(x) \cdot \langle v_g^\alpha v_h^\beta, \eta_{g_\alpha} \omega_{h_\alpha} \rangle. \end{aligned}$$

Using Proposition 2.6 we can define the following generators k^+ and k^- of $K_p(\chi_E)^+$ and $K_p(\chi_E)^-$ respectively:

$$k^+ := \begin{cases} 1 & \text{if } a = +1; \\ z & \text{if } a = -1 \end{cases} \quad \text{and} \quad k^- := \begin{cases} z & \text{if } a = +1; \\ 1 & \text{if } a = -1. \end{cases}$$

Using (3.5) and (3.6), the latter pairing is

$$\langle v_g^\alpha v_h^\beta, \eta_{g_\alpha} \omega_{h_\alpha} \rangle = \frac{1}{\Omega_{g_\alpha} \Theta_{h_\alpha}} \langle \omega_{g_\alpha} \eta_{h_\alpha}, \eta_{g_\alpha} \omega_{h_\alpha} \rangle = \frac{1}{\Omega_{g_\alpha} \Theta_{h_\alpha}} \otimes 1 \in K_p^{\alpha\beta} \otimes L_p,$$

and the right-most isomorphism of (3.11) is given by

$$\begin{array}{ccc} K_p(\chi_E)^{\beta\alpha} \otimes K_p^{\alpha\beta} & \xrightarrow{\cong} & K_p(\chi_E)^+ \xrightarrow{\cong} \mathbb{Q}_p \\ x \otimes y & \mapsto & xy \\ & & k^+ \mapsto 1. \end{array}$$

Analogously, the dual exponential maps of Lemma 3.2 are twisted versions of (2.14), and $\exp_{\alpha\beta}^*$ may be recast under the identification provided by Lemma 3.4 as

$$\begin{array}{ccc} \exp_{\alpha\beta}^* : H_s^1(K_p, V_f)^{\beta\alpha} \otimes V_{gh}^{\alpha\beta} & \longrightarrow & L_p \\ x \otimes v_g^\alpha v_h^\beta & \longmapsto & \exp^*(x) \cdot \langle v_g^\alpha v_h^\beta, \eta_{g_\alpha} \omega_{h_\alpha} \rangle. \end{array}$$

We will still denote by

$$(3.12) \quad \exp_\pm^* : H_s^1(K_p, V_f)^\pm \longrightarrow \mathbb{Q}_p$$

the composition of (2.14) with the isomorphism $K_p(\chi_E)^\pm \cong \mathbb{Q}_p$ sending k^\pm to 1.

Corollary 3.6. *Define $X_\pm \in H_s^1(K_p, V_f)^\pm$ such that $\exp_\pm^*(X_\pm) = 1$. The p -relaxed Selmer group $\text{Sel}_{(p)}(V)$ admits a basis*

$$\{\xi^{\alpha\alpha}, \xi^{\alpha\beta}, \xi^{\beta\alpha}, \xi^{\beta\beta}\}$$

characterized as

$$\begin{aligned} \partial_p \xi^{\alpha\alpha} &= X_{\alpha\alpha} \otimes v_g^\alpha \otimes v_h^\alpha, & \partial_p \xi^{\alpha\beta} &= X_{\alpha\beta} \otimes v_g^\alpha \otimes v_h^\beta, \\ \partial_p \xi^{\beta\alpha} &= X_{\beta\alpha} \otimes v_g^\beta \otimes v_h^\alpha, & \partial_p \xi^{\beta\beta} &= X_{\beta\beta} \otimes v_g^\beta \otimes v_h^\beta. \end{aligned}$$

Proof. By Corollary 3.3 and using the decomposition (3.9), there are isomorphisms

$$(3.13) \quad \text{Sel}_{(p)}(V) \xrightarrow{\partial_p} H_s^1(\mathbb{Q}_p, V) \cong H_s^1(\mathbb{Q}_p, V^{\alpha\alpha}) \oplus H_s^1(\mathbb{Q}_p, V^{\alpha\beta}) \oplus H_s^1(\mathbb{Q}_p, V^{\beta\alpha}) \oplus H_s^1(\mathbb{Q}_p, V^{\beta\beta}).$$

By Lemma 3.4, each of the four components in (3.13) is isomorphic to

$$H_s^1(K_p, V_f)^{\beta\beta} \otimes V_{gh}^{\alpha\alpha}, H_s^1(K_p, V_f)^{\beta\alpha} \otimes V_{gh}^{\alpha\beta}, H_s^1(K_p, V_f)^{\alpha\beta} \otimes V_{gh}^{\beta\alpha}, H_s^1(K_p, V_f)^{\alpha\alpha} \otimes V_{gh}^{\beta\beta},$$

respectively. These are in turn isomorphic to $V_{gh}^{\alpha\alpha} \oplus V_{gh}^{\alpha\beta} \oplus V_{gh}^{\beta\alpha} \oplus V_{gh}^{\beta\beta}$ via $\exp_{\alpha\alpha}^* \oplus \exp_{\alpha\beta}^* \oplus \exp_{\beta\alpha}^* \oplus \exp_{\beta\beta}^*$. \square

4. A SPECIAL VALUE FORMULA FOR THE TRIPLE PRODUCT p -ADIC L -FUNCTION IN RANK 0

The aim of this section is to describe the p -adic L -value $I_p(f, g_\alpha, h_\alpha)$ in terms of the basis $\{\xi^{\alpha\alpha}, \xi^{\alpha\beta}, \xi^{\beta\alpha}, \xi^{\beta\beta}\}$ of $\text{Sel}_{(p)}(V)$ appearing in Corollary 3.6. This section lies within the framework of the *exceptional setting* of [BSVb] and we recall here the notation and the main results from loc. cit. that we shall use.

4.1. The triple product p -adic L -function. If $\varphi = \sum a_n(\varphi)q^n$ is a modular form of weight w , level M and nebentype character χ_φ and $p \nmid M$, the Hecke polynomial at p of φ is

$$x^2 - a_p(\varphi)x + \chi_\varphi(p)p^{w-1} = (x - \alpha_\varphi)(x - \beta_\varphi),$$

where we label the eigenvalues so that $\text{ord}_p(\alpha_\varphi) \leq \text{ord}_p(\beta_\varphi)$. Recall that if φ is ordinary at p , then α_φ is a p -adic unit. If $p \mid M$ we have $\alpha_\varphi = a_p(\varphi)$.

Since g has weight $w = 1$, both α_g and β_g are p -adic units. Recall that we are assuming that $\alpha_g \neq \beta_g$, thus g has two different ordinary p -stabilisations. We denote g_α and g_β the stabilisation satisfying $U_p g_\alpha = \alpha_g g_\alpha$ and $U_p g_\beta = \beta_g g_\beta$ respectively. The same holds for h , and we denote

similarly its p -stabilisations. Since E has multiplicative reduction at p , the level of the newform is divisible by p and it is p -stabilised, meaning that $f_\alpha = f$.

Let $\mathbf{f} = \sum_{n \geq 1} a_n(\mathbf{f}) \in \Lambda_{\mathbf{f}}[[q]]$, $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$, $\mathbf{h} \in \Lambda_{\mathbf{h}}[[q]]$ be the Hida families passing through f, g_α, h_α respectively, where $\Lambda_{\mathbf{f}}, \Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ are finite flat extensions of the Iwasawa algebra Λ . These Hida families have tame level $N_f/p, N_g, N_h$ and tame character $1, \chi, \bar{\chi}$ respectively.

Let $N := \text{lcm}(N_f/p, N_g, N_h)$. A *test vector* for $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ is a triple of Hida families $(\check{\mathbf{f}}, \check{\mathbf{g}}, \check{\mathbf{h}})$ of tame level N such that $\check{\mathbf{f}}$ is of the form $\sum \lambda_d \mathbf{f}(q^d) \in \Lambda_{\mathbf{f}}[[q]]$ with $\lambda_d \in \Lambda_{\mathbf{f}}$, where d runs over the divisors of N/N_f , and similarly for $\check{\mathbf{g}}$ and $\check{\mathbf{h}}$.

Recall from §3.1 the set $\mathcal{W}_{\mathbf{fgh}}^\circ$ of cristalline points. Denote $\mathcal{W}_{\mathbf{fgh}}^f := \{(k, \ell, m) \in \mathcal{W}_{\mathbf{fgh}}^\circ \mid k \geq \ell + m\}$ and define $\mathcal{W}_{\mathbf{fgh}}^g$ and $\mathcal{W}_{\mathbf{fgh}}^h$ analogously. Triplets of points in $\mathcal{W}_{\mathbf{fgh}}^f \cup \mathcal{W}_{\mathbf{fgh}}^g \cup \mathcal{W}_{\mathbf{fgh}}^h$ are called *unbalanced*, and points in the complement $\mathcal{W}_{\mathbf{fgh}}^{\text{bal}} := \mathcal{W}_{\mathbf{fgh}}^\circ \setminus (\mathcal{W}_{\mathbf{fgh}}^f \cup \mathcal{W}_{\mathbf{fgh}}^g \cup \mathcal{W}_{\mathbf{fgh}}^h)$ are called *balanced*.

Let

$$\mathcal{L}_p^f(\check{\mathbf{f}}, \check{\mathbf{g}}, \check{\mathbf{h}}), \mathcal{L}_p^g(\check{\mathbf{f}}, \check{\mathbf{g}}, \check{\mathbf{h}}), \mathcal{L}_p^h(\check{\mathbf{f}}, \check{\mathbf{g}}, \check{\mathbf{h}}) : \mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}} \longrightarrow \mathbb{C}_p$$

be the triple product p -adic L -functions attached to $(\check{\mathbf{f}}, \check{\mathbf{g}}, \check{\mathbf{h}})$ constructed in [DR14].

4.2. Exceptional cases and improved Euler systems. As introduced in (3.1), let

$$\mathbb{V} = V_{\mathbf{f}} \otimes V_{\mathbf{g}} \otimes V_{\mathbf{h}} \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi$$

and let

$$\kappa := \kappa(\mathbf{f}, \mathbf{g}, \mathbf{h}) \in H^1(\mathbb{Q}, \mathbb{V})$$

be the diagonal class constructed in [BSVb] and [DRa]. By [BSVb, Corollary 4.7.1] and [DRa, Proposition 3.5.7], the local class

$$\kappa_p := \text{res}_p(\kappa) \in H^1(\mathbb{Q}_p, \mathbb{V})$$

belongs to $H^1(\mathbb{Q}_p, \text{Fil}^2(\mathbb{V}))$. Moreover, for $\varphi \in \{f, g, h\}$, the p -adic L -function $\mathcal{L}_p^\varphi(\check{\mathbf{f}}, \check{\mathbf{g}}, \check{\mathbf{h}})$ can be recast as the image of κ_p under the idoneous Perrin-Riou's Λ -adic logarithm: more precisely, [BSVb, Theorem A], [DRa, Theorem 2.29] assert that

$$(4.1) \quad \mathcal{L}_p^\varphi(\check{\mathbf{f}}, \check{\mathbf{g}}, \check{\mathbf{h}}) = \mathcal{L}_\varphi(\text{res}_p(\kappa)),$$

where $\mathcal{L}_\varphi : H^1(\mathbb{Q}_p, \text{Fil}^2(\mathbb{V})) \longrightarrow \mathcal{O}_{\mathbf{fgh}}$ is the homomorphism described in [BSVb, Proposition 4.6.2], [DRa, Proposition 3.5.6].

Let \mathcal{S} denote the surface cut out by the equation $k = 2 + \ell - m$ in $\mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}}$ and let $\mathcal{F}_{\mathcal{S}}$ be the fraction field of the ring of Iwasawa functions on \mathcal{S} .

Define the two-variable meromorphic Iwasawa function

$$\mathcal{E}_g := \mathcal{E}_g(\mathbf{f}, \mathbf{g}, \mathbf{h}) := 1 - \frac{a_p(\mathbf{g}_\ell)}{\chi(p)a_p(\mathbf{f}_{\ell-m+2})a_p(\mathbf{h}_m)} \in \mathcal{F}_{\mathcal{S}}.$$

Recall from the introduction that we have $p \parallel N_f$, $p \nmid N_g N_h$, and $\alpha_g \alpha_h = -a$. More explicitly, we have

$$(4.2) \quad \beta_g = -\alpha_g, \quad (\alpha_h, \beta_h) = \begin{cases} (-1/\alpha_g, 1/\alpha_g) & \text{if } a = +1; \\ (1/\alpha_g, -1/\alpha_g) & \text{if } a = -1. \end{cases}$$

Hence

$$\mathcal{E}_g(2, 1, 1) = 1 - \frac{\alpha_g \beta_h}{a} = 0.$$

This fact forces the vanishing of the class $\kappa(2, 1, 1)$, as explained in [BSVb] (cf. also Prop. 4.1 below).

Let $\rho_S : \mathcal{F}_{\mathbf{fgh}} \rightarrow \mathcal{F}_S$ be the map taking a function $F(k, \ell, m)$ to its restriction $F(2 + \ell - m, \ell, 1)$ to the plane \mathcal{S} . Denote $\mathbb{V}_{|S} := \mathbb{V} \otimes_{\mathcal{F}_{\mathbf{fgh}, \rho_S}} \mathcal{F}_S$ and let

$$\kappa_{|S} := \rho_{S, \star}(\kappa) \in H^1(\mathbb{Q}, \mathbb{V}_{|S})$$

denote the restriction of κ to \mathcal{S} .

Proposition 4.1. *There exists a global cohomology class*

$$\kappa_g^* \in H^1(\mathbb{Q}, \mathbb{V}_{|S})$$

satisfying, for each $\ell, m \in \mathbb{Z}_{\geq 1}$:

$$(4.3) \quad \kappa(2 + \ell - m, \ell, m) = \mathcal{E}_g(2 + \ell - m, \ell, m) \cdot \kappa_g^*(2 + \ell - m, \ell, m).$$

Proof. This is shown in [BSVb, §8.3] □

Let $\mathcal{C} \subset \mathcal{S}$ denote the curve given on which the set of points $(\ell + 1, \ell, 1)$ for $\ell \in \mathbb{Z}_{\geq 1}$ is dense.

Proposition 4.2. *There exists an analytic function $\mathcal{L}_p^{f, *}$ on \mathcal{C} satisfying, for each $\ell \in \mathbb{Z}_{\geq 1}$:*

$$\mathcal{L}_p^{f, *}(\ell) = \frac{\langle w_N(f_{\ell+1})^{(p)}, h\mathbf{g}_\ell \rangle}{\langle w_N(f_{\ell+1})^{(p)}, w_N(f_{\ell+1})^{(p)} \rangle}$$

and

$$\mathcal{L}_p^f(\check{\mathbf{f}}, \check{\mathbf{g}}, \check{\mathbf{h}})(\ell + 1, \ell, 1) = \left(1 - \frac{a_p(\mathbf{g}_\ell)\alpha_h}{a_p(\mathbf{f}_{\ell+1})}\right) \mathcal{E}_g(\ell + 1, \ell, 1) \mathcal{L}_p^{f, *}(\ell) \pmod{L^\times}.$$

Moreover,

$$(4.4) \quad \mathcal{L}_p^{f, *}(1) = \frac{1}{2(1 - 1/p)} \exp_{\beta\beta}^*(\pi_{\beta\beta} \partial_p \kappa_g^*(2, 1, 1)) = \frac{1}{2(1 - 1/p)} \exp_{\beta\beta}^*(\partial_p \kappa_g^*(2, 1, 1)).$$

Proof. This is proved in [BSVb, Lemma 8.6 + equation (170)]. In particular, combining Theorem A, part 3 of Proposition 8.2 and Lemma 8.6 of loc. cit. it follows that the factor λ_{w_0} in equation (170) is $1/2(1 - 1/p)$. □

Remark 4.3. By (4.2) the factor $1 - \frac{a_p(\mathbf{g}_\ell)\alpha_h}{a_p(\mathbf{f}_{\ell+1})}$ does not vanish in a neighborhood of $(2, 1, 1)$ in \mathcal{C} .

4.3. $I_p(f, g_\alpha, h_\alpha)$ in terms of the basis. In this section we finally obtain, in Theorem 4.7, the formula for $I_p(f, g_\alpha, h_\alpha)$. Recall from the introduction the Galois representation

$$V = V_f \otimes V_g \otimes V_h.$$

Note that the character Ξ introduced in §3.1 specializes at $(2, 1, 1)$ to the trivial character, and thus the specialization of the improved Λ -adic cohomology class of Prop. 4.1 yields a global cohomology class

$$\kappa_g^*(2, 1, 1) \in \text{Sel}_{(p)}(V).$$

Recall the periods

$$(4.5) \quad \Omega_{g_\alpha} \in K_p^{1/\alpha_g}, \quad \Theta_{g_\alpha} \in K_p^{1/\beta_g}, \quad \mathcal{L}_{g_\alpha} := \frac{\Omega_{g_\alpha}}{\Theta_{g_\alpha}} \in K_p^{\beta_g/\alpha_g}$$

introduced in §3.1.

Proposition 4.4. *We have*

$$\kappa_g^*(2, 1, 1) = \Theta_{g_\alpha} \Theta_{h_\alpha} \frac{2(1 - 1/p) \sqrt{c} \sqrt{L(E \otimes \rho, 1)}}{\pi^2 \langle f, f \rangle} \cdot \xi^{\beta\beta},$$

where $c \in L^\times$ is the product of the local terms appearing in [DLR15, Proposition 2.1 (iii)].

Proof. According to [BSVb, §8.3] one has $\text{res}_p(\kappa_g^*(2, 1, 1)) \in H^1(\mathbb{Q}_p, \text{Fil}^2(V))$. Recall that

$$H^1(\mathbb{Q}_p, \text{Fil}^2(V)/\text{Fil}^3(V)) = H^1(\mathbb{Q}_p, V_f^- \otimes V_{gh}^{\beta\beta}) \oplus H^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\alpha\beta}) \oplus H^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\beta\alpha})$$

and hence $\partial_p \kappa_g^*(2, 1, 1) \in H_s^1(\mathbb{Q}_p, V) = H_s^1(\mathbb{Q}_p, V_f^- \otimes V_{gh})$ lies in $H_s^1(\mathbb{Q}_p, V^{\beta\beta})$. Then, by definition of the basis of $\text{Sel}_{(p)}(V)$ described in Corollary 3.6,

$$\kappa_g^*(2, 1, 1) = \frac{\exp_{\beta\beta}^*(\partial_p \kappa_g^*(2, 1, 1))}{\exp_{\beta\beta}^*(\partial_p \xi^{\beta\beta})} \cdot \xi^{\beta\beta}.$$

We need to compute the numerator and denominator of the previous formula. By (4.4),

$$\begin{aligned} \exp_{\beta\beta}^*(\partial_p \kappa_g^*(2, 1, 1)) &= 2(1 - 1/p) \mathcal{L}_p^{f*}(1) \\ &= \frac{2(1 - 1/p) \langle w_N(f), h_g \rangle}{\langle w_N(f), w_N(f) \rangle} \\ &= \frac{2(1 - 1/p) \sqrt{c} \sqrt{L(E \otimes \rho, 1)}}{\pi^2 \langle f, f \rangle}, \end{aligned}$$

where the last equality is given by [Ich08]. Besides,

$$\begin{aligned} \exp_{\beta\beta}^*(\partial_p \xi^{\beta\beta}) &= \langle \exp_{\beta\beta}^* \partial_p \xi^{\beta\beta}, \eta_f \omega_{g_\alpha} \omega_{h_\alpha} \rangle \\ &= \exp^*(X_{\beta\beta}) \langle v_g^\beta \otimes v_h^\beta, \omega_{g_\alpha} \otimes \omega_{h_\alpha} \rangle \\ &= \frac{1}{\Theta_{g_\alpha} \Theta_{h_\alpha}} \langle \eta_{g_\alpha} \otimes \eta_{h_\alpha}, \omega_{g_\alpha} \otimes \omega_{h_\alpha} \rangle \\ &= \frac{1}{\Theta_{g_\alpha} \Theta_{h_\alpha}}. \end{aligned}$$

The proposition follows. \square

Let $\Lambda_{\text{cyc}} = \mathbb{Z}_p[[j+1]]$ denote the usual Iwasawa algebra regarded as the ring of bounded analytic functions on an open disc centered at $j = -1$, and let $\chi_{\text{cyc}} : G_{\mathbb{Q}} \rightarrow \Lambda_{\text{cyc}}^\times$ denote the Λ -adic cyclotomic character characterized by the property that $\nu_j(\chi_{\text{cyc}}) = \chi_{\text{cyc}}^{-j}$ for all $j \in \mathbb{Z}$.

Recall the three-variable Iwasawa algebra Λ_{fgh} and set

$$\bar{\Lambda}_{\text{fgh}} = \Lambda_{\text{fgh}} \hat{\otimes} \Lambda_{\text{cyc}}, \quad \bar{\mathcal{W}}_{\text{fgh}} := \text{Spf}(\bar{\Lambda}_{\text{fgh}}) = \mathcal{W}_{\text{fgh}} \times \mathcal{W}, \quad \bar{\mathcal{W}}_{\text{fgh}}^\circ := \mathcal{W}_{\text{fgh}}^\circ \times \mathcal{W}^\circ.$$

Let $\Psi : G_{\mathbb{Q}_p} \rightarrow \Lambda_{\text{fgh}}^\times$ denote the unramified character taking Fr_p to $\frac{a_p(\mathbf{g})}{a_p(\mathbf{f})a_p(\mathbf{h})\chi(p)}$. Define

$$\mathbb{M}^g := \Lambda_{\text{fgh}}(\Psi), \quad \bar{\mathbb{M}}^g := \mathbb{M}^g \otimes \Lambda_{\text{cyc}}(\chi_{\text{cyc}})$$

and note that \mathbb{M}^g is the unramified twist of the local Galois representation \mathbb{V}^g introduced in (3.8).

Let $\theta : \bar{\Lambda}_{\text{fgh}} \rightarrow \Lambda_{\text{fgh}}$ denote the homomorphism taking a function $F(k, \ell, m, j)$ to its restriction $F(k, \ell, m, (\ell - k - m)/2)$. As shown in the proof of [BSVb, Prop. 4.6.2], there is an isomorphism

$$(4.6) \quad \theta : \bar{\mathbb{M}}^g \otimes_{\theta} \mathcal{O}_{\text{fgh}} \cong \mathbb{V}^g.$$

Define also the Λ -adic Dieudonné modules

$$\mathbb{D}^g := (\mathbb{M}^g \hat{\otimes} \hat{Z}_p^{\text{ur}})^{G_{\mathbb{Q}_p}}, \quad \bar{\mathbb{D}}^g := \mathbb{D}^g \otimes \Lambda_{\text{cyc}},$$

where \hat{Z}_p^{ur} is the ring of integers of the p -adic completion $\hat{\mathbb{Q}}_p^{\text{ur}}$ of the maximal unramified extension of \mathbb{Q}_p .

As it directly follows from the above definitions, the specialization of $\bar{\mathbb{M}}^g$ at a point $\underline{x} = (k, \ell, m, j) \in \bar{\mathcal{W}}_{\text{fgh}}^\circ$ is

$$\bar{\mathbb{M}}_{\underline{x}}^g := \bar{\mathbb{M}}^g \otimes_{\underline{x}} L_p = L_p(\Psi_{(k, \ell, m)})(-j) = V_{\mathbf{f}_k}^+ \otimes V_{\mathbf{g}_\ell}^- \otimes V_{\mathbf{h}_m}^+ (2 - k - m - j).$$

Bloch–Kato’s logarithm and dual exponential maps give rise to an isomorphism

$$(4.7) \quad \log_{\underline{x}} : H^1(\mathbb{Q}_p, L_p(\Psi_{(k, \ell, m)})(-j)) \longrightarrow D_{\text{dR}}(L_p(\Psi_{(k, \ell, m)})(-j)) \quad \text{if } j < 0$$

$$(4.8) \quad \exp_{\underline{x}}^* : H^1(\mathbb{Q}_p, L_p(\Psi_{(k, \ell, m)})(-j)) \longrightarrow D_{\text{dR}}(L_p(\Psi_{(k, \ell, m)})(-j)) \quad \text{if } j \geq 0.$$

In particular, at $\underline{x} = (2, 1, 1, -1)$ this isomorphism becomes the map

$$\log_{\alpha\beta} : H_f^1(\mathbb{Q}_p, V^{\alpha\beta}) \longrightarrow L_p,$$

after taking the pairing with the differential $\eta_{g_\alpha} \omega_{h_\alpha}$ (cf. Remark 3.5).

Proposition 4.5. *There exists a single homomorphism of $\bar{\Lambda}_{\text{fgh}}$ -modules*

$$\bar{\mathcal{L}}_g : H^1(\mathbb{Q}_p, \bar{\mathbb{M}}^g) \longrightarrow \bar{\mathbb{D}}^g,$$

such that for all $\mathbf{Z} \in H^1(\mathbb{Q}_p, \bar{\mathbb{M}}^g)$ and $\underline{x} = (k, \ell, m, j) = (w, j) \in \bar{\mathcal{W}}_{\text{fgh}}^\circ$ with

$$1 - \Psi_w(\text{Fr}_p)p^{-j-1} \neq 0,$$

we have

$$\bar{\mathcal{L}}_g(\mathbf{Z})(\underline{x}) = \left(1 - \frac{p^j}{\Psi_w(\text{Fr}_p)}\right) (1 - \Psi_w(\text{Fr}_p)p^{-j-1})^{-1} \cdot \begin{cases} \frac{(-1)^{j+1}}{(-j-1)!} \log_{\underline{x}}(\mathbf{Z}(\underline{x})) & j < 0; \\ j! \exp_{\underline{x}}^*(\mathbf{Z}(\underline{x})) & j \geq 0. \end{cases}$$

Proof. The proposition as stated here is [BSVb, Proposition 4.6.1] for \mathbf{g} , and it is a consequence of [LZ14]. \square

By the proof of [BSVb, Proposition 4.6.2], the relation between $\bar{\mathcal{L}}_g$ and the logarithm \mathcal{L}_g that appears in (4.1) is given, for all $\mathbf{Z} \in H^1(\mathbb{Q}_p, \bar{\mathbb{M}}^g)$ by

$$(4.9) \quad \mathcal{L}_g(\theta_*(\mathbf{Z}))(k, \ell, m) = \langle \bar{\mathcal{L}}_g(\mathbf{Z})(k, \ell, m, (\ell - k - m)/2), \omega_{\mathbf{f}_k} \eta_{\mathbf{g}_\ell} \omega_{\mathbf{h}_m} \rangle.$$

Here, if $\underline{x} = (k, \ell, m, (\ell - k - m)/2)$, then

$$\omega_{\mathbf{f}_k} \eta_{\mathbf{g}_\ell} \omega_{\mathbf{h}_m} \in \bar{\mathbb{D}}^g \otimes_{\underline{x}} L_p \cong D_{\text{dR}}(V_{\mathbf{f}_k}^+ \otimes V_{\mathbf{g}_\ell}^- \otimes V_{\mathbf{h}_m}^+ ((\ell - k - m + 4)/2))$$

are the differentials defined as in [BSVb, (228)] and are natural generalizations of the ones introduced in §3.1.

When j is a fixed integer, note that the Euler-like factors appearing above vary analytically. As it will suffice for our purposes, set $j = -1$ and recall the plane \mathcal{S} parametrized by points of weights $(2 + \ell - m, \ell, m)$ in \mathcal{W}_{fgh} . By a slight abuse of notation we also regard \mathcal{S} as the plane in \mathcal{W}_{fgh} parametrizing points of weights $(2 + \ell - m, \ell, m, -1)$.

Define the homomorphism

$$\bar{\mathcal{L}}_g^* : H^1(\mathbb{Q}_p, \bar{\mathbb{M}}_{|\mathcal{S}}^g) \longrightarrow \bar{\mathbb{D}}_{|\mathcal{S}}^g$$

of \mathcal{O}_S -modules given by

$$(4.10) \quad \bar{\mathcal{L}}_g^* = \mathcal{E}_g \times \bar{\mathcal{L}}_{g|S}.$$

In light of the above proposition, for any $\mathbf{Z} \in H^1(\mathbb{Q}_p, \bar{\mathbb{M}}_{|S}^g)$ and $\ell \geq 1$ one then has

$$(4.11) \quad \bar{\mathcal{L}}_g^*(\mathbf{Z})(\ell) = \left(1 - \frac{a_p(\mathbf{g}_\ell)\beta_h}{a_p(\mathbf{f}_{\ell+1})p}\right) \cdot \langle \log_{(\ell+1, \ell, 1, -1)}(\mathbf{Z}(\ell+1, \ell, 1, -1)), \omega_{\mathbf{f}_{\ell+1}} \eta_{\mathbf{g}_\ell} \omega_{h_\alpha} \rangle.$$

Proposition 4.6. *We have*

$$I_p(f, g_\alpha, h_\alpha) = \left(1 - \frac{1}{p}\right) \log_{\alpha\beta}(\pi_{\alpha\beta} \kappa_g^*(2, 1, 1)) \mod L^\times.$$

Proof. To avoid notational clutter, we continue to denote \mathcal{S} the plane in $\bar{\mathcal{W}}_{\mathbf{fgh}}$ containing the points $(2+\ell-m, \ell, m, -1)$ as a dense subset. Let $\tilde{\kappa}_{g,p}^* \in H^1(\mathbb{Q}_p, \bar{\mathbb{M}}_{|S}^g)$ be any lift of $\text{res}_p(\kappa_g^*)$, whose existence is assured by the fact that the map

$$\theta_* : H^1(\mathbb{Q}_p, \bar{\mathbb{M}}^g) \otimes_{\theta} \mathcal{O}_{\mathbf{fgh}} \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}^g)$$

induced by (4.6) is an isomorphism (cf. the proof of [BSVb, Proposition 4.6.2]).

Combining Proposition 4.5 with (4.1), (4.3), (4.9) we deduce

$$\begin{aligned} \mathcal{L}_p^g(\check{\mathbf{f}}, \check{\mathbf{g}}, \check{\mathbf{h}})|_S &= \mathcal{L}_{g|S}(\text{res}_p(\kappa_{|S})) = \mathcal{E}_{g|S} \cdot \mathcal{L}_{g|S}(\text{res}_p(\kappa_g^*)) \\ &= \mathcal{E}_{g|S} \cdot \langle \bar{\mathcal{L}}_{g|S}(\tilde{\kappa}_{g,p}^*), \omega_{\mathbf{f}_{\ell+1}} \eta_{\mathbf{g}_\ell} \omega_{h_\alpha} \rangle. \end{aligned}$$

By (4.10), (4.11) and (4.7), the latter quantity is equal to

$$\langle \bar{\mathcal{L}}_g^*(\tilde{\kappa}_{g,p}^*), \omega_{\mathbf{f}_{\ell+1}} \eta_{\mathbf{g}_\ell} \omega_{h_\alpha} \rangle = \left(1 - \frac{a_p(\mathbf{g}_\ell)\beta_h}{a_p(\mathbf{f}_{\ell+1})p}\right) \cdot \log_{\alpha\beta}(\pi_{\alpha\beta} \kappa_g^*(\ell+1, \ell, 1)).$$

Since $I_p(f, g_\alpha, h_\alpha) = \mathcal{L}_p^g(\check{\mathbf{f}}, \check{\mathbf{g}}, \check{\mathbf{h}})(2, 1, 1)$ by definition, the proposition follows by using (4.2). \square

Define the local point

$$P_{\alpha\beta} \in E(K_p)_{L_p}^{\alpha\beta}$$

as the one satisfying

$$(4.12) \quad \pi_{\alpha\beta} \text{res}_p(\xi^{\beta\beta}) = \delta_p(P_{\alpha\beta}) \otimes v_g^\alpha \otimes v_h^\beta \in \left(H_f^1(K_p, V_f) \otimes V_{gh}^{\alpha\beta}\right)^{G_{\mathbb{Q}_p}},$$

where $\delta_p : E(K_p) \longrightarrow H_f^1(K_p, V_f)$ is the Kummer map and $E(K_p)_{L_p}^{\alpha\beta} := E(K_p)^{\alpha\beta} \otimes L_p$.

Theorem 4.7. *We have*

$$I_p(f, g_\alpha, h_\alpha) = \frac{2(1-1/p)^2 \sqrt{c}}{\pi^2 \langle f, f \rangle} \times \sqrt{L(E \otimes \rho, 1)} \times \frac{\log_p(P_{\alpha\beta})}{\mathcal{L}_{g_\alpha}} \mod L^\times.$$

Proof. By Proposition 4.4, $\kappa_g^*(2, 1, 1) = \frac{\Theta_{g_\alpha} \Theta_{h_\alpha} 2(1 - 1/p) \sqrt{c} \sqrt{L(E \otimes \rho, 1)}}{\pi^2 \langle f, f \rangle} \xi^{\beta\beta}$. It thus follows from Proposition 4.6 that

$$\begin{aligned} I_p(f, g_\alpha, h_\alpha) &= \left(1 - \frac{1}{p}\right) \log_{\alpha\beta}(\pi_{\alpha\beta} \kappa_g^*(2, 1, 1)) \\ &= \frac{\Theta_{g_\alpha} \Theta_{h_\alpha} \sqrt{c} 2(1 - 1/p)^2 \sqrt{L(E \otimes \rho, 1)}}{\pi^2 \langle f, f \rangle} \log_{\alpha\beta}(\pi_{\alpha\beta} \xi^{\beta\beta}) \\ &= \frac{\Theta_{g_\alpha} \Theta_{h_\alpha} \sqrt{c} 2(1 - 1/p)^2 \sqrt{L(E \otimes \rho, 1)}}{\pi^2 \langle f, f \rangle} \log_p(P_{\alpha\beta}) \langle v_g^\alpha \otimes v_h^\beta, \eta_{g_\alpha} \omega_{h_\alpha} \rangle \\ &= \frac{\Theta_{g_\alpha} \Theta_{h_\alpha} \sqrt{c} 2(1 - 1/p)^2 \sqrt{L(E \otimes \rho, 1)}}{\Omega_{g_\alpha} \Theta_{h_\alpha} \pi^2 \langle f, f \rangle} \log_p(P_{\alpha\beta}). \end{aligned}$$

The theorem follows in light of (4.5). \square

5. A SPECIAL VALUE FORMULA FOR $I_p(f, g_\alpha, h_\alpha)$ IN TERMS OF KOLYVAGIN CLASSES

The aim of this section is to relate Theorem 4.7 to Bertolini-Darmon's Kolyvagin classes constructed in [BD97, §6].

5.1. Decomposition of the representation V and consequences. The underlying reason why the p -adic L -value $\mathcal{L}_p^g(\check{\mathbf{f}}, \check{\mathbf{g}}, \check{\mathbf{h}})(2, 1, 1)$ may be related to Kolyvagin classes relies on the decomposition of the Galois representation $V_g \otimes V_h$ as the direct sum of a pair of 2-dimensional representations induced from characters of an imaginary quadratic field.

More precisely, recall that g and h are theta series of two finite order Hecke characters $\psi_g, \psi_h : G_K \rightarrow L^\times$. As explained in the introduction, there is a decomposition

$$(5.1) \quad V_g \otimes V_h \cong V_{\psi_1} \oplus V_{\psi_2}$$

where $\psi_1 := \psi_g \psi_h$, $\psi_2 := \psi_g \psi'_h$ and $V_{\psi_i} := \text{Ind}_{G_K}^{G_{\mathbb{Q}}}(\psi_i)$.

For $i = 1, 2$, ψ_i is a ring class character, i.e. it factors through the absolute Galois group of a number field H_i which is a ring class field of K of some conductor c_i . Recall we have assumed that p does not divide the conductor of ψ_g and ψ_h , hence that the characters ψ_1 and ψ_2 are unramified at p , and this implies that $p \nmid c_1 c_2$. The Artin representation

$$\rho = \rho_g \otimes \rho_h : G_{\mathbb{Q}} \rightarrow \text{GL}(V_g \otimes_{L_p} V_h)$$

thus factors through $\text{Gal}(H/K)$, where H is the ring class field of K of conductor $c := \text{lcm}(c_1, c_2)$. Moreover, since the prime p is inert in K and $p \nmid c$, the principal ideal $p\mathcal{O}_K$ splits completely in H . Fix once and for all a prime \mathfrak{p} of the field H dividing p and denote $H_{\mathfrak{p}}$ the completion of H at \mathfrak{p} . Then $H_{\mathfrak{p}} = K_p$ and, in particular, the restriction of ψ_i to the decomposition group $G_{K_p} = G_{H_{\mathfrak{p}}}$ is trivial for $i = 1, 2$.

Recall that Fr_p acts on V_g with eigenvalues $\alpha_g = \lambda$, $\beta_g = -\lambda$ and on V_h with eigenvalues

$$(\alpha_h, \beta_h) = \begin{cases} (-1/\lambda, 1/\lambda) & \text{if } a = +1; \\ (1/\lambda, -1/\lambda) & \text{if } a = -1, \end{cases}$$

so we always have $\alpha_g \alpha_h = -a$. As in the previous sections, choose an L -basis of eigenvectors v_g^α, v_g^β of V_g and likewise a basis v_h^α, v_h^β of V_h .

Note also that Fr_p acts on V_{ψ_1} and V_{ψ_2} with eigenvalues ± 1 .

Lemma 5.1. *There is a $G_{\mathbb{Q}}$ -equivariant isomorphism*

$$\Psi : V_g \otimes V_h \cong V_{\psi_1} \oplus V_{\psi_2}.$$

Moreover, V_{ψ_i} admits an L -basis $\{v_i^+, v_i^-\}$ of eigenvectors with relative eigenvalues $\{+1, -1\}$ satisfying:

- if $a = +1$, then

$$\Psi(v_g^\alpha \otimes v_h^\alpha) = v_1^- - v_2^-, \quad \Psi(v_g^\beta \otimes v_h^\beta) = v_1^- + v_2^-, \quad \Psi(v_g^\alpha \otimes v_h^\beta) = v_1^+ + v_2^+, \quad \Psi(v_g^\beta \otimes v_h^\alpha) = v_1^+ - v_2^+.$$

- If $a = -1$, then

$$\Psi(v_g^\alpha \otimes v_h^\alpha) = v_1^+ + v_2^+, \quad \Psi(v_g^\beta \otimes v_h^\beta) = v_1^+ - v_2^+, \quad \Psi(v_g^\alpha \otimes v_h^\beta) = v_1^- - v_2^-, \quad \Psi(v_g^\beta \otimes v_h^\alpha) = v_1^- + v_2^-.$$

Proof. Let ψ be a finite order Hecke character of K , let $\varphi := \theta(\psi)$ and denote $\text{Fr}_p \in G_{\mathbb{Q}} \setminus G_K$ a Frobenius element at p . As explained in [DLR17, §2.2], we can choose a basis $\{u_\varphi, v_\varphi\}$ of V_φ such that, if

$$\rho_\varphi : G_{\mathbb{Q}} \longrightarrow \text{GL}(V_\varphi)$$

is the ℓ -adic representation attached to φ , then $v_\varphi = \rho_\varphi(\text{Fr}_p)u_\varphi$ and with respect to this basis,

$$\rho_\varphi(\sigma) = \begin{pmatrix} \psi(\sigma) & 0 \\ 0 & \psi'(\sigma) \end{pmatrix} \text{ for } \sigma \in G_K; \quad \rho_\varphi(\tau) = \begin{pmatrix} 0 & \eta(\tau) \\ \eta'(\tau) & 0 \end{pmatrix} \text{ for } \tau \in G_{\mathbb{Q}} \setminus G_K.$$

Here ψ' denotes the character defined by $\psi'(\sigma) := \psi(\text{Fr}_p \sigma \text{Fr}_p^{-1})$, and η is a function on $G_{\mathbb{Q}} \setminus G_K$. Recall that the characteristic polynomial for the action of Fr_p on V_φ is $x^2 - a_p(\varphi)x + \chi_\varphi(p) = x^2 - \epsilon_\psi(p)$, where $\chi_\varphi = \chi_K \epsilon_\psi$ and ϵ_ψ is the central character of ψ . We can always choose $\{u_\varphi, v_\varphi\}$ such that $\rho_\varphi(\text{Fr}_p) = \begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix}$, with $\zeta^2 = \epsilon_\psi(p)$. Then a basis of eigenvectors for the action of Fr_p on V_φ is

$$\{v_\varphi^\zeta = u_\varphi + v_\varphi, \quad v_\varphi^{-\zeta} = u_\varphi - v_\varphi\}$$

with eigenvalue $\zeta, -\zeta$ respectively. Using this notation we obtain the basis

$$(u_g, v_g), \quad (u_h, v_h), \quad (u_1, v_1), \quad (u_2, v_2)$$

of $V_g, V_h, V_{\psi_1}, V_{\psi_2}$ where we denoted $u_i := u_{\theta(\psi_i)}$ and $v_i := v_{\theta(\psi_i)}$. The corresponding matrix $\rho_\varphi(\text{Fr}_p)$ is

$$\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/\lambda \\ 1/\lambda & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consider the basis

$$(u_g \otimes u_h, u_g \otimes v_h, v_g \otimes u_h, v_g \otimes v_h)$$

of $V_g \otimes V_h$, and the basis

$$(u_1, v_1, u_2, v_2)$$

of $V_{\psi_1} \oplus V_{\psi_2}$. Comparing the matrices above, we conclude that the isomorphism Ψ is given by the rule

$$u_g \otimes u_h \mapsto u_1, \quad u_g \otimes v_h \mapsto u_2, \quad v_g \otimes u_h \mapsto v_2, \quad v_g \otimes v_h \mapsto v_1.$$

Moreover, since the eigenvalues of Fr_p acting on V_g and V_h are $\lambda, -\lambda, 1/\lambda, 1/\lambda$, a basis of eigenvectors for $V_g \otimes V_h$ is

$$\begin{aligned} v_g^\lambda \otimes v_h^{1/\lambda} &= (u_g + v_g) \otimes (u_h + v_h), & v_g^\lambda \otimes v_h^{-1/\lambda} &= (u_g + v_g) \otimes (u_h - v_h), \\ v_g^{-\lambda} \otimes v_h^{1/\lambda} &= (u_g - v_g) \otimes (u_h + v_h), & v_g^{-\lambda} \otimes v_h^{-1/\lambda} &= (u_g - v_g) \otimes (u_h - v_h). \end{aligned}$$

with eigenvalues $+1, -1, -1, +1$ respectively. On the other hand, a basis of eigenvectors for $V_{\psi_1} \oplus V_{\psi_2}$ is

$$(v_1^+ := u_1 + v_1, \quad v_1^- := u_1 - v_1, \quad v_2^+ := u_2 + v_2, \quad v_2^- := u_2 - v_2)$$

In conclusion,

$$v_g^\lambda \otimes v_h^{1/\lambda} \mapsto v_1^+ + v_2^+, \quad v_g^\lambda \otimes v_h^{-1/\lambda} \mapsto v_1^- - v_2^-, \quad v_g^{-\lambda} \otimes v_h^{1/\lambda} \mapsto v_1^- + v_2^-, \quad v_g^{-\lambda} \otimes v_h^{-1/\lambda} \mapsto v_1^+ - v_2^+.$$

□

Recall that

$$V_{\psi_i} = \text{Ind}_{G_K}^{G_{\mathbb{Q}}}(\psi_i) = \{v : G_{\mathbb{Q}} \longrightarrow L(\psi) \mid v(\sigma\tau) = \psi_i(\sigma)v(\tau) \ \forall \sigma \in G_K, \tau \in G_{\mathbb{Q}}\} = L \oplus L \cdot \text{Fr}_p.$$

The proof of the previous lemma shows that we may choose the vectors v_i^\pm such that

$$(5.2) \quad v_i^+(1) = v_i^-(1) = 1.$$

Any other choice of basis would be of the form $(a_i v_i^+, b_i v_i^-)$ for some scalars $a_i, b_i \in L^\times$, and would yield the same formula for $I_p(f, g_\alpha, h_\alpha)$ up to a scalar in L^\times . This is fine because that is precisely the ambiguity we are working with in our framework, as explained in the introduction.

As in previous sections, denote

$$V := V_f \otimes V_g \otimes V_h, \quad V_1 := V_f \otimes V_{\psi_1}, \quad V_2 := V_f \otimes V_{\psi_2}$$

The map Ψ of Lemma 5.1 induces the isomorphism

$$(5.3) \quad \Psi : V \xrightarrow{\cong} V_1 \oplus V_2.$$

There is then a decomposition of the Selmer group

$$(5.4) \quad \Psi_* : \text{Sel}_p(V) \xrightarrow{\cong} \text{Sel}_p(V_1) \oplus \text{Sel}_p(V_2),$$

and the analogous decomposition holds for the relaxed and the strict Selmer groups associated to V . By Artin's formalism, decomposition (5.3) yields the factorization

$$L(E, \rho, s) = L(E, \psi_1, s) L(E, \psi_2, s)$$

of classical L -series. As explained in §3.2, Assumption 1.1 on the analytic rank of $E \otimes \rho$ implies that

$$\text{Sel}_p(V_1) = \text{Sel}_p(V_2) = 0.$$

Denote $V_{\psi_i}^\pm$ the eigenspace of V_{ψ_i} on which Fr_p acts as ± 1 , and let

$$V_i^\pm := V_f \otimes V_{\psi_i}^\pm.$$

Lemma 5.2. *There are isomorphisms*

$$(5.5) \quad H_f^1(\mathbb{Q}_p, V_i^\pm) \cong H_f^1(K_p, V_f)^\pm \otimes V_{\psi_i}^\pm \quad \text{and} \quad H_s^1(\mathbb{Q}_p, V_i^\pm) \cong H_s^1(K_p, V_f)^\pm \otimes V_{\psi_i}^\pm.$$

The Bloch–Kato logarithm and dual exponential yield isomorphisms

$$(5.6) \quad H_f^1(\mathbb{Q}_p, V_i^\pm) \xrightarrow{\log_\pm} L_p, \quad H_s^1(\mathbb{Q}_p, V_i^\pm) \xrightarrow{\exp_\pm^*} L_p.$$

Moreover, for $i \in \{1, 2\}$, there are isomorphisms

$$(5.7) \quad \text{Sel}_{(p)}(V_i) \xrightarrow{\partial_p} H_s^1(\mathbb{Q}_p, V_i) \xrightarrow{\exp^*} L_p^2.$$

Proof. The isomorphisms are obtained as in the proofs of Lemma 3.4, Lemma 3.2 and Corollary 3.3, using (1.4). \square

Similarly as in Remark 3.5, the Bloch–Kato maps (5.6) are related to the logarithm and the dual exponential of the representation V_f as follows. Since V_{ψ_i} is the Galois representation attached to the theta series $\theta(\psi_i)$, (3.3) yields a pairing

$$(5.8) \quad \langle \cdot, \cdot \rangle : D_{\text{dR}}(V_{\psi_i}^+) \times D_{\text{dR}}(V_{\psi_i}^-) \longrightarrow D_{\text{dR}}(L_p(\chi_i))$$

where χ_i is the Nebentype character of $\theta(\psi_i)$. Since V_{ψ_i} is unramified, we have

$$D_{\text{dR}}(V_{\psi_i}^\pm) = (V_{\psi_i}^\pm \otimes K_p)^{G_{\mathbb{Q}_p}} = V_{\psi_i}^\pm \otimes K_p^\pm.$$

One can introduce differentials $\omega_i^\pm \in D_{\text{dR}}(V_{\psi_i}^\pm)$ as in (3.4), and elements $\Omega_i^\pm \in K_p^\pm$ characterised by the following relation

$$v_i^\pm \otimes \Omega_i^\pm = \omega_i^\pm.$$

Thus (5.8) in turn induces a pairing

$$\langle \cdot, \cdot \rangle : V_{\psi_i}^+ \times D_{\text{dR}}(V_{\psi_i}^-) \longrightarrow L_p$$

by setting

$$\langle v_i^+, \omega \rangle := \frac{1}{\Omega_i^+} \langle \omega_i^+, \omega \rangle$$

for all $\omega \in D_{\text{dR}}(V_{\psi_i}^-)$. Hence we have

$$(5.9) \quad \exp_\pm^*(x \otimes v) = \exp_{V_f}^*(x) \cdot \langle v, \omega_i^\pm \rangle$$

for any $x \in H_s^1(K_p, V_f)^\pm$, $v \in V_{\psi_i}^\pm$.

Recall from Lemma 5.1 the basis $\{v_i^+, v_i^-\}$ for V_{ψ_i} , and from Corollary 3.6 the local classes $X_\pm \in H_s(K_p, V_f)_{L_p}^\pm$ such that $\exp_\pm^*(X_\pm) = 1$.

Corollary 5.3. *For $i \in \{1, 2\}$, $\text{Sel}_{(p)}(V_i)$ admits a basis $\{\xi_i^+, \xi_i^-\}$ characterized as*

$$\partial_p \xi_i^\pm = X_\pm \otimes v_i^\pm.$$

under the identifications given in (5.5).

Proof. The statement follows after applying Lemma 5.2 and using the same argument as in the proof of Corollary 3.6. \square

Proposition 5.4. *Let*

$$\Psi_* : \text{Sel}_{(p)}(V) \xrightarrow{\cong} \text{Sel}_{(p)}(V_1) \oplus \text{Sel}_{(p)}(V_2)$$

denote the isomorphism induced by (5.4).

• *If $a = +1$ then*

$$\Psi_* \xi^{\alpha\alpha} = \xi_1^- - \xi_2^-, \quad \Psi_* \xi^{\alpha\beta} = \xi_1^+ + \xi_2^+, \quad \Psi_* \xi^{\beta\alpha} = \xi_1^+ - \xi_2^+, \quad \Psi_* \xi^{\beta\beta} = \xi_1^- + \xi_2^-.$$

• *If $a = -1$ then*

$$\Psi_* \xi^{\alpha\alpha} = \xi_1^+ + \xi_2^+, \quad \Psi_* \xi^{\alpha\beta} = \xi_1^- - \xi_2^-, \quad \Psi_* \xi^{\beta\alpha} = \xi_1^- + \xi_2^-, \quad \Psi_* \xi^{\beta\beta} = \xi_1^+ - \xi_2^+.$$

Proof. Combining (5.7) and Lemma 3.1, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Sel}_{(p)}(V) & \xrightarrow{\Psi_*} & \mathrm{Sel}_{(p)}(V_1) \oplus \mathrm{Sel}_{(p)}(V_2) \\ \downarrow \partial_p & & \downarrow \partial_p \oplus \partial_p \\ \mathrm{H}_s^1(\mathbb{Q}_p, V) & \xrightarrow{\Psi_*} & \mathrm{H}_s^1(\mathbb{Q}_p, V_1) \oplus \mathrm{H}_s^1(\mathbb{Q}_p, V_2) \end{array}$$

where each arrow is an isomorphism. Moreover, by Lemma 3.4 and Lemma 5.2, we also have the commutative diagram

$$\begin{array}{ccc} \mathrm{H}_s^1(\mathbb{Q}_p, V) & \xrightarrow{\Psi_*} & \mathrm{H}_s^1(\mathbb{Q}_p, V_1) \oplus \mathrm{H}_s^1(\mathbb{Q}_p, V_2) \\ \downarrow & & \downarrow \\ (\mathrm{H}_s^1(K_p, V_f) \otimes V_{gh})^{G_{\mathbb{Q}_p}} & \longrightarrow & (\mathrm{H}_s^1(K_p, V_f) \otimes V_1)^{G_{\mathbb{Q}_p}} \oplus (\mathrm{H}_s^1(K_p, V_f) \otimes V_2)^{G_{\mathbb{Q}_p}} \end{array}$$

where each arrow is an isomorphism and the bottom horizontal arrow is $1 \otimes \Psi$. By definition,

$$\partial_p \xi^{\alpha\alpha} = X_{\alpha\alpha} \otimes v_g^\alpha \otimes v_h^\alpha \in (\mathrm{H}_s^1(K_p, V_f) \otimes V_{gh}^{\alpha\alpha})^{G_{\mathbb{Q}_p}} \subseteq (\mathrm{H}_s^1(K_p, V_f) \otimes V_{gh})^{G_{\mathbb{Q}_p}}.$$

By (1.4) we have

$$X_{\alpha\alpha} = \begin{cases} X_+ & \text{if } \alpha_g \cdot \alpha_h = +1; \\ X_- & \text{if } \alpha_g \cdot \alpha_h = -1, \end{cases}$$

hence we deduce

$$(1 \otimes \Psi)(\partial_p \xi^{\alpha\alpha}) = \begin{cases} X_{\alpha\alpha} \otimes (v_1^+ + v_2^+) = \partial_p \xi_1^+ + \partial_p \xi_2^+ & \text{if } \alpha_g \cdot \alpha_h = +1 \\ X_{\alpha\alpha} \otimes (v_1^- - v_2^-) = \partial_p \xi_1^- - \partial_p \xi_2^- & \text{if } \alpha_g \cdot \alpha_h = -1. \end{cases}$$

Similar computations prove the remaining cases. \square

Lemma 5.5. Let $\delta_p : E(K_p) \otimes \mathbb{Q}_p \longrightarrow \mathrm{H}_f^1(K_p, V_f)$ denote Kummer's map.

- If $a = +1$, there is a local point $P^+ \in E(K_p)^+ \otimes \mathbb{Q}_p$ such that

$$(5.10) \quad \pi_1^+ \mathrm{res}_p(\xi_1^-) = \delta_p P^+ \otimes v_1^+, \quad \pi_2^+ \mathrm{res}_p(\xi_2^-) = \delta_p P^+ \otimes v_2^+, \quad \pi_{\alpha\beta} \mathrm{res}_p(\xi^{\beta\beta}) = \delta_p P^+ \otimes v_g^\alpha \otimes v_h^\beta,$$

- If $a = -1$, there is a local point $P^- \in E(K_p)^- \otimes \mathbb{Q}_p$ such that

$$(5.11) \quad \pi_1^- \mathrm{res}_p(\xi_1^+) = \delta_p P^- \otimes v_1^-, \quad \pi_2^- \mathrm{res}_p(\xi_2^+) = \delta_p P^- \otimes v_2^-, \quad \pi_{\alpha\beta} \mathrm{res}_p(\xi^{\beta\beta}) = \delta_p P^- \otimes v_g^\alpha \otimes v_h^\beta;$$

Proof. We prove the claim when $a = -1$, as the other case works similarly. By Proposition 5.4,

$$\Psi_* \mathrm{res}_p(\xi^{\beta\beta}) = \mathrm{res}_p(\xi_1^+) - \mathrm{res}_p(\xi_2^+) \in \mathrm{H}^1(\mathbb{Q}_p, V_1) \oplus \mathrm{H}^1(\mathbb{Q}_p, V_2)$$

Write

$$\pi_{\alpha\beta} \mathrm{res}_p(\xi^{\beta\beta}) = Q_- \otimes v_g^\alpha \otimes v_h^\beta \in \mathrm{H}_f^1(K_p, V_f)^- \otimes V_{gh}^{\alpha\beta}.$$

Then, by Proposition 5.4 and (5.5)

$$(5.12) \quad \Psi_* \pi_{\alpha\beta} \mathrm{res}_p(\xi^{\beta\beta}) = Q_- \otimes (v_1^- - v_2^-) = Q_- \otimes v_1^- - Q_- \otimes v_2^-$$

in

$$(\mathrm{H}_f^1(K_p, V_f) \otimes (V_{\psi_1}^- \oplus V_{\psi_2}^-))^{G_{\mathbb{Q}_p}} \cong \mathrm{H}_f^1(K_p, V_1^-) \oplus \mathrm{H}_f^1(K_p, V_2^-).$$

On the other hand, by definition of the basis ξ_i^\pm and using (5.5),

$$\pi_1^- \text{res}_p(\xi_1^+) - \pi_2^- \text{res}_p(\xi_2^+) \in H_f^1(\mathbb{Q}_p, V_1^-) \oplus H_f^1(\mathbb{Q}_p, V_2^-) \cong (H_f^1(K_p, V_f) \otimes (V_{\psi_1}^- \oplus V_{\psi_2}^-))^{G_{\mathbb{Q}_p}}.$$

More precisely, we can write

$$(5.13) \quad \pi_1^- \text{res}_p(\xi_1^+) - \pi_2^- \text{res}_p(\xi_2^+) = P_1^- \otimes v_1^- - P_2^- \otimes v_2^-$$

for points $P_i^- \in E(K_p)^-$. Hence, comparing (5.13) and (5.12) we deduce that $Q_- = P_1^- = P_2^-$. \square

Theorem 5.6. *Let $P^\pm \in E(K_p)^\pm$ be the local points of Lemma 5.5. Then*

$$I_p(f, g_\alpha, h_\alpha) = \frac{\sqrt{c} \cdot 2(1 - 1/p)^2 \sqrt{L(E \otimes \rho, 1)}}{\pi^2 \langle f, f \rangle} \times \frac{1}{\mathcal{L}_{g_\alpha}} \times \log_p(P^a) \pmod{L^\times}.$$

Proof. Combine Theorem 4.7 with (4.12) and Lemma 5.5 and use the relation between the Bloch-Kato logarithms given in (3.11). \square

5.2. Kolyvagin classes. In this section we recall briefly the construction and main properties of the Kolyvagin classes defined in [BD97, §6].

Assume for simplicity that $\mathcal{O}_K^\times = \{\pm 1\}$. Let H be a ring class field of K of conductor prime to p , and recall the fields $H \subseteq F_{m-1} \subseteq H(p^m)$ defined in §2.1. For each m , fix a generator σ_m of $G_m := \text{Gal}(F_m/H)$ such that the image of σ_m via the projection

$$(5.14) \quad G_m \longrightarrow G_m / \text{Gal}(F_m/F_{m-1}) \cong G_{m-1}$$

equals σ_{m-1} . Moreover, the elements $\{\sigma_m\}_m$ can be chosen so that $\sigma_{\text{ant}} = (\sigma_m|_{G_{F_m, \mathfrak{p}}})_m$, where σ_{ant} is the generator of Γ_{ant} we fixed in §2.2. In [BD96], Bertolini and Darmon constructed a collection of points $\{\tilde{\alpha}_m \in E(H(p^m))\}_m$ (in the notation of [BD96], $\tilde{\alpha}_m = \alpha_m(1)$) such that

$$(5.15) \quad N_{H(p^m)/H} \tilde{\alpha}_m = 0.$$

For each $m \geq 1$, let

$$\alpha_m := N_{H(p^{m+1})/F_n} \tilde{\alpha}_{m+1} \in E(F_m).$$

To each point α_m , in [BD97] the authors attached a Kolyvagin class $\mathbf{K}_m \in H^1(H, E[p^m])$ in the following way. The *Kolyvagin derivative operator* is the element $D_m := \sum_{i=1}^{p^m-1} i \sigma_m^i \in \mathbb{Z}[G_m]$.

Lemma 5.7. *The Kolyvagin derivative satisfies the following equality in $\mathbb{Z}[G_m]$*

$$(5.16) \quad (\sigma_m - 1)D_m = p^m - N_{F_m/H}.$$

Proof. Note that $D_n = -\sum_{i=1}^{p^n} \frac{\sigma_n^i - 1}{\sigma_n - 1}$. Then $(\sigma_n - 1)D_n = -\sum_{i=1}^{p^n} (\sigma_n - 1) = -N_{F_n/H} + p^n$. \square

Define $P_m := D_m \alpha_m \in E(F_m)$.

Lemma 5.8. (1) *The point $P_m \in E(F_m)$ only depends on the choice of the generator σ_m of G_m up to the multiplication by an element $a_m \in \{1, \dots, p^m - 1\}$. Moreover, if we choose the elements $\{\sigma_m\}_m$ to be compatible in the sense of (5.14), then $a_{m-1} \equiv a_m \pmod{p^{m-1}}$, i.e. $(a_m)_m \in \varprojlim_m (\mathbb{Z}/p^m \mathbb{Z})^\times = \mathbb{Z}_p^\times$.*

(2) *The class $[P_m]$ of P_m in $E(F_m)/p^m E(F_m)$ is fixed by $G_m = \text{Gal}(F_m/H)$.*

Proof. (1) Let σ'_m any generator of $G_m \cong \mathbb{Z}/p^m \mathbb{Z}$ and define $D'_m := \sum i(\sigma'_m)^i$. Then there exists an element $\overline{a_m} \in (\mathbb{Z}/p^m \mathbb{Z})^\times$ such that $\sigma'_m = \sigma_m^{\overline{a_m}}$, where $a_m \in \{1, \dots, p^m - 1\}$. Then $D'_m = \sum i \sigma_n^{\overline{a_m} i} = a_m^{-1} D_m$.

- (2) It suffices to prove that the class of $D_m \alpha_m$ is fixed by the generator σ_m of G_m , i.e. that $\sigma_m D_m \alpha_m - D_m \alpha_m \in p^n E(F_n)$. Combining (5.16) and (5.15) we obtain $(\sigma_m - 1)D_m \alpha_m = p^m \alpha_m - N_{F_m/H} \alpha_m = p^m \alpha_m$.

□

The inflation-restriction and Kummer exact sequences yield the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & H^1(F_m/H, E(F_m)[p^m]) & & \\
 & & & & \downarrow \text{Inf} & & \\
 0 & \longrightarrow & (E(H)/p^m E(H)) & \xrightarrow{\delta_p} & H^1(H, E[p^m]) & \longrightarrow & H^1(H, E)[p^m] \longrightarrow 0 \\
 & & \downarrow \text{Res} & & \downarrow \text{Res} & & \downarrow \text{Res} \\
 0 & \longrightarrow & (E(F_m)/p^m E(F_m))^{G_m} & \xrightarrow{\delta_p} & H^1(F_m, E[p^m])^{G_m} & \longrightarrow & H^1(F_m, E)[p^m]^{G_m} \\
 & & & & \downarrow & & \\
 & & & & H^2(F_m/H, E(F_m)[p^m]) & &
 \end{array}$$

whose rows and columns are exact. Let C be a constant annihilating $E(F_m)[p^m]$, which can be chosen to be independent of m by [BD97, Lemma 6.3]. By Lemma 5.8, we can consider the element $[P_m] \in (E(F_m)/p^m E(F_m))^{G_m}$, and we define

$$\mathbf{K}_m := C\mathbf{K}_m^\circ \in H^1(H, E[p^m]),$$

where \mathbf{K}_m° is a preimage via Res of $-C\delta_p[P_m]$. For each m , denote

$$\mathcal{G}_m := \text{Gal}(F_m/K) \supseteq G_m := \text{Gal}(F_m/H)$$

and fix a complete set $S = S_m = \{\sigma_1^{(m)}, \dots, \sigma_h^{(m)}\}$ of representatives for the G_m -cosets in \mathcal{G}_m . We require these sets to satisfy the following compatibility condition. Let $\pi_m : \mathcal{G}_m \rightarrow \mathcal{G}_{m-1}$ denote the projection induced by the inclusion $F_{m-1} \subseteq F_m$. Then

$$\pi_m(\sigma_i^{(m)}) = \sigma_i^{(m-1)}$$

for all i and all m . In other words, under this compatibility condition we can consider a complete set $\tilde{S} = \{\tilde{\sigma}_1, \dots, \tilde{\sigma}_m\}$ of representatives of G_H -cosets in G_K such that, under the projection $\tilde{\pi}_m : G_K \rightarrow \text{Gal}(F_m/K)$ we have

$$\tilde{\pi}_m(\tilde{\sigma}_i) = \sigma_i^{(m)}.$$

Lemma 5.9. *The classes \mathbf{K}_m form a compatible system*

$$\mathbf{K} := (\mathbf{K}_m)_m \in \varprojlim_m H^1(H, E[p^m]) = H^1(H, T_p E) \subset H^1(H, V_f).$$

Proof. The norm map $N_{F_{m+1}/F_m} : F_{m+1} \rightarrow F_m$ induces a map $N_{F_{m+1}/F_m} : E(F_{m+1}) \rightarrow E(F_m)$. By [BD96, §2.45], we have

$$(5.17) \quad N_{F_{m+1}/F_m}(\alpha_{m+1}) = \alpha_m.$$

Consider the composition

$$f_{m+1} : E(F_{m+1})/p^{m+1}E(F_{m+1}) \xrightarrow{N_{F_{m+1}/F_m}} E(F_m)/p^{m+1}E(F_m) \rightarrow E(F_m)/p^m E(F_m)$$

where the second morphism is the projection given by the inclusion $p^{m+1}E(F_m) \subseteq p^m E(F_m)$. Hence (5.17) implies that

$$(5.18) \quad f_{m+1}([\alpha_{m+1}]) = [\alpha_m].$$

Consider the following commutative diagram

$$\begin{array}{ccc} E(F_{m+1})/p^{m+1}E(F_{m+1}) & \xrightarrow{\delta_p} & H^1(F_{m+1}, E[p^{m+1}]) \\ \downarrow f_{m+1} & & \downarrow f_{m+1,*} \\ E(F_m)/p^m E(F_m) & \xrightarrow{\delta_p} & H^1(F_m, E[p^m]). \end{array}$$

Here $f_{m+1,*}$ is the map induced in cohomology by f_{m+1} , i.e. it is the composition

$$H^1(F_{m+1}, E[p^{m+1}]) \xrightarrow{N_{F_{m+1}/F_m}} H^1(F_m, E[p^{m+1}]) \xrightarrow{p*} H^1(F_m, E[p^m]),$$

where the last map is the composition with the multiplication by p map $E[p^{m+1}] \xrightarrow{p} E[p^m]$. Then (5.18) implies that

$$f_{m+1,*}\delta_p[D_{m+1}\alpha_{m+1}] = \delta_p[D_m\alpha_m].$$

□

Recall the groups $\Phi_{m,p}$ and $\Phi_{\infty,p}$ defined in §2.1. By Lemma 2.2 and Proposition 2.3, there is an injection

$$\mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{ant}}, \Phi_{\infty,p} \otimes \mathbb{Q}_p) \subseteq H_s^1(K_p, V_f).$$

Recall the choice σ_m of generator of G_m ; if we still denote σ_m its restriction to $\mathrm{Gal}(F_{m,p}/H_p) = \mathrm{Gal}(F_{m,p}/K_p)$, then Γ_{ant} is generated by $\sigma_{\mathrm{ant}} := (\sigma_m)_m$.

Recall the map $\partial_p : H^1(H, V_f) \rightarrow H_s^1(K_p, V_f)$ onto the singular quotient.

Proposition 5.10. *The class \mathbf{K} lies in $\mathrm{Sel}_{(p)}(H, V_f)$ and $\partial_p \mathbf{K} \in \mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{ant}}, \Phi_{\infty,p} \otimes \mathbb{Q}_p)$. Moreover, if $\bar{\alpha}_m$ denotes the image of α_m in $\Phi_{m,p}$ and $\bar{\alpha} := (\bar{\alpha}_m)_m \in \Phi_{\infty,p}$, then*

$$\partial_p \mathbf{K}(\sigma_{\mathrm{ant}}) = \bar{\alpha}.$$

Proof. [BD97, Proposition 6.9, 1,2].

□

5.3. $I_p(f, g_\alpha, h_\alpha)$ and Kolyvagin classes. Recall the ring class characters ψ_1, ψ_2 appearing in the decomposition (5.1). Since they are ring class characters unramified at p , they factor through the Galois group $\mathrm{Gal}(H/K)$, where H is a ring class field of conductor prime to p .

Using the Kolyvagin classes described in §5.2, in this section we define elements \mathbf{K}^{ψ_i} in the relaxed Selmer groups $\mathrm{Sel}_{(p)}(K, V_f \otimes \psi_i)$ for $i = 1, 2$. The aim of this section is to compare the classes $\mathbf{K}^{\psi_1}, \mathbf{K}^{\psi_2}$ with the local points in $E(K_p)^\pm$ appearing in Theorem 5.6, in order to obtain a formula for the value $I_p(f, g_\alpha, h_\alpha)$ in terms of these Kolyvagin classes. Using the notation of the previous sections, consider the dual exponential

$$\exp_\pm^* : H_s^1(K_p, V_f)^\pm \longrightarrow \mathbb{Q}_p$$

of (3.12). As explained in §2, Tate's uniformisation induces the isomorphism $\varphi : \Phi_{\infty,p} \xrightarrow{\cong} \mathbb{Z}_p$ (in the notation of Lemma 2.1, $\varphi := \bar{\varphi}_{\mathrm{Tate}}$ maps \bar{q} to 1). Recall the point $\bar{\alpha} \in \Phi_{\infty,p}$ defined in Proposition 5.10, the sign

$$a := a_p(E) \in \{\pm 1\}$$

and the period

$$\Pi_p := \varphi(\bar{\alpha}) \in \mathbb{Z}_p.$$

Proposition 5.11. *We have*

$$\partial_p \mathbf{K} \in H_s^1(K_p, V_f)^{-a} \quad \text{and} \quad \exp_{-a}^* \partial_p \mathbf{K} = \frac{\text{ord}_p(q_E)}{p} \Pi_p.$$

In particular

$$\text{res}_p(\mathbf{K})^a := \text{res}_p(\mathbf{K}) + a \text{res}_p(\mathbf{K}^{\text{Fr}_p})$$

is cristalline, i.e. lies in $H_f^1(\mathbb{Q}_p, V_f)^a$.

Proof. Recall we can regard $\text{Hom}_{\text{cont}}(\Gamma_{\text{ant}}, \Phi_{\infty, p} \otimes \mathbb{Q}_p)$ as a subspace of $H_s^1(K_p, V_f)$. By (2.15) Fr_p acts on $\text{Hom}_{\text{cont}}(\Gamma_{\text{ant}}, \Phi_{\infty, p} \otimes \mathbb{Q}_p)$ as multiplication by $-a$. Hence $\text{res}_p(\mathbf{K})$ belongs to $H_s^1(K_p, V_f)^{-a}$ by Proposition 5.10, and the formula for $\exp_{-a}^* \partial_p \mathbf{K}$ follows from Corollary 2.7. \square

For $i = 1, 2$ let

$$\text{Tr}_i = \sum_{\sigma \in \tilde{S}} \psi_i(\sigma) \sigma : H^1(H, V_f) \longrightarrow H^1(K, V_f \otimes \psi_i)$$

denote the trace map onto the ψ_i -isotypic component and consider the isomorphism given by Shapiro's Lemma

$$(5.19) \quad \text{Sh} : H^1(K, V_f \otimes \psi_i) \xrightarrow{\cong} H^1(\mathbb{Q}, V_f \otimes \text{Ind}_{G_K}^{G_{\mathbb{Q}}}(\psi_i)) =: H^1(\mathbb{Q}, V_i).$$

Define

$$\mathbf{K}^{\psi_i} := \text{Sh}(\text{Tr}_i(\mathbf{K})) \in H^1(\mathbb{Q}, V_i).$$

It follows from Proposition 5.10 that \mathbf{K}^{ψ_i} lies in $\text{Sel}_{(p)}(V_i)$.

Frobenius element $\text{Fr}_p \in G_{\mathbb{Q}_p}$ acts on $H^1(K_p, V_f)$ as an involution and we may consider the decomposition in \pm -eigenspaces

$$H^1(K_p, V_f \otimes L_p) = H^1(K_p, V_f \otimes L_p)^+ \oplus H^1(K_p, V_f \otimes L_p)^-.$$

As one readily verifies, Shapiro's isomorphism restricts to

$$(5.20) \quad \text{Sh}_p : H^1(K_p, V_f \otimes L_p)^{\pm} \xrightarrow{\cong} H^1(\mathbb{Q}_p, V_f \otimes V_{\psi_i}^{\pm}).$$

Corollary 5.12. *The Kolyvagin class \mathbf{K}^{ψ_i} satisfies*

$$\mathbf{K}^{\psi_i} = \frac{h \text{ord}_p(q_E) \Pi_p}{p} \cdot \xi_i^{-a}.$$

Proof. Since $\psi_i|_{G_{K_p}} = 1$, the restriction of Tr_i to $H^1(H_p, V_f \otimes L_p) = H^1(K_p, V_f \otimes L_p)$ is multiplication by $h = [H : K]$. Hence $\text{res}_p(\mathbf{K}^{\psi_i}) = h \cdot \text{Sh}_p(\text{res}_p(\mathbf{K}))$.

By Proposition 5.11 we have $\partial_p \mathbf{K} \in H_s^1(K_p, V_f)^{-a}$, and by (5.20), it follows that $\partial_p \mathbf{K}^{\psi_i} \in H_s^1(\mathbb{Q}_p, V_{\psi_i}^{-a})$. Recall the choice of basis ξ_i^{\pm} made in Corollary 5.3. We may thus write

$$\mathbf{K}^{\psi_i} = \frac{\exp_{-a}^*(\partial_p \mathbf{K}^{\psi_i})}{\exp_{-a}^*(\partial_p \xi_i^{-a})} \cdot \xi_i^{-a}.$$

By definition of the basis $\{\xi_i^+, \xi_i^-\}$ and by (5.9), the denominator in the above expression is $\exp^*(X_{-a}) \langle v_i^{-a}, \omega_i^{-a} \rangle = \langle v_i^{-a}, \omega_i^{-a} \rangle$. Let

$$R \otimes v_i^{-a} \in H_s^1(K_p, V_f) \otimes V_{\psi_i}^{-a}$$

denote the image of $\partial_p \mathbf{K}^{\psi_i}$ via the isomorphism (5.5). Then

$$\mathbf{K}^{\psi_i} = \frac{\exp^*(R) \langle v_i^{-a}, \omega_i^{-a} \rangle}{\langle v_i^{-a}, \omega_i^{-a} \rangle} \cdot \xi_i^{-a} = \exp^*(R) \cdot \xi_i^{-a}.$$

In order to compute the dual exponential of R , we need the following explicit expression for (5.22) for Sh_p . Recall that, as a $L_p[G_{\mathbb{Q}_p}]$ -module,

$$V_{\psi_i} = \text{Ind}_{G_{K_p}}^{G_{\mathbb{Q}_p}}(1) = \{v : G_{\mathbb{Q}_p} \longrightarrow L_p \mid v(\sigma\tau) = v(\tau) \ \forall \sigma \in G_{K_p}, \tau \in G_{\mathbb{Q}_p}\}.$$

Consider the map

$$ev : V_{\psi_i} \longrightarrow L_p, \quad v \mapsto v(1).$$

It is an equivariant G_{K_p} -morphism which is compatible with the inclusion $G_{K_p} \hookrightarrow G_{\mathbb{Q}_p}$, so it induces a morphism

$$(5.21) \quad \text{Sh}_p^{-1} : H^1(\mathbb{Q}_p, V_f \otimes V_{\psi_i}) \longrightarrow H^1(K_p, V_f \otimes L_p) = H^1(K_p, V_f \otimes \psi_i),$$

which is the inverse of Shapiro's isomorphism (5.19) restricted to G_{K_p} . More explicitly, if $\xi : G_{\mathbb{Q}_p} \longrightarrow V_f \otimes V_{\psi_i}$ represents a class in $H^1(\mathbb{Q}_p, V_f \otimes V_{\psi_i})$, then

$$(5.22) \quad \text{Sh}_p^{-1}(\xi) := (\text{id} \otimes ev) \circ \xi|_{G_{K_p}}$$

represents its image via (5.21). Recall that $R \otimes v_i^{-a}$ is the image of $h \cdot \partial_p \mathbf{K}$ via the composition

$$H_s^1(K_p, V_f \otimes L_p)^{-a} \xrightarrow{\text{Sh}_R} H_s^1(\mathbb{Q}_p, V_f \otimes V_{\psi_i}^{-a}) \xrightarrow{\text{Res}} H_s^1(K_p, V_f \otimes V_{\psi_i}^{-a})^{G_{\mathbb{Q}_p}} \cong H_s^1(K_p, V_f)^{-a} \otimes V_{\psi_i}^{-a}.$$

We conclude that the element $R \otimes v_i^{-a}$ is represented by the cocycle

$$h \cdot \text{Res}(\text{Sh}_p(\partial_p \mathbf{K})) : G_{K_p} \longrightarrow V_f \otimes V_{\psi_i}^{-a}$$

satisfying

$$R \cdot (v_i^{-a}(1)) = h \cdot \partial_p \mathbf{K}.$$

In other words, by (5.2), we have

$$\exp^*(R) = h \cdot \exp^*(\partial_p \mathbf{K}).$$

The statement follows by applying Proposition 5.11. □

Corollary 5.13. *We have*

$$I_p(f, g_\alpha, h_\alpha) = \frac{\sqrt{c} \cdot 2p(1 - 1/p)^2}{\text{ord}_p(q_E)} \cdot \frac{\sqrt{L(E \otimes \rho, 1)}}{\pi^2 \langle f, f \rangle \Pi_p} \cdot \frac{1}{\mathcal{L}_{g_\alpha}} \times \log_p(Q_p^a) \pmod{L^\times},$$

where $Q_p^a \in E(K_p)^a$ is characterized by $\delta_p(Q_p^a) = \text{res}_p(\mathbf{K})^a \in H_f^1(K_p, V_f)^a$.

Proof. By (5.20), if $\pi_a : H^1(\mathbb{Q}_p, V_i) \longrightarrow H^1(\mathbb{Q}_p, V_i^a)$ denotes the natural projection, then

$$(5.23) \quad \pi_a \text{res}_p(\mathbf{K}^{\psi_i}) = h \cdot \text{Sh}_p(\text{res}_p(\mathbf{K})^a) \in H_f^1(\mathbb{Q}_p, V_i^a).$$

Arguing as in the proof of Corollary 5.12, write

$$A \otimes v_i^a \in H_f^1(K_p, V_f)^a \otimes V_{\psi_i}^a$$

for the image of (5.23) via the isomorphism (5.5). Then

$$A = \frac{h \cdot \text{res}_p(\mathbf{K})^a}{v_i^a(1)} = h \cdot \text{res}_p(\mathbf{K})^a.$$

Combining this with Corollary 5.12 and Lemma 5.5, we obtain

$$h \cdot \text{res}_p(\mathbf{K})^a \otimes v_i^a = \pi_a \text{res}_p(\mathbf{K}^{\psi_i}) = \frac{h \text{ord}_p(q_E) \Pi_p}{p} \delta_p(P^a) \otimes v_i^a.$$

Then

$$\delta_p(P^a) = \frac{p}{\text{ord}_p(q_E) \Pi_p} \cdot \text{res}_p(\mathbf{K})^a,$$

and the thesis follows by applying Theorem 5.6. □

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