BEILINSON-FLACH ELEMENTS AND EULER SYSTEMS II: THE BIRCH AND SWINNERTON-DYER CONJECTURE FOR HASSE-WEIL-ARTIN L-SERIES

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Abstract. Let E be an elliptic curve over $\mathbb{Q}$ and let ϱ be an odd, irreducible two-dimensional Artin representation. This article proves the Birch and Swinnerton-Dyer conjecture in analytic rank zero for the Hasse-Weil-Artin L-series $L(E,\varrho,s)$, namely, the implication

$$L(E,\varrho,1) \neq 0 \quad \Rightarrow \quad (E(H)\otimes\varrho)^{\mathrm{Gal}(H/\mathbb{Q})} = 0,$$

where H is the finite extension of $\mathbb{Q}$ cut out by ϱ. The proof relies on p-adic families of global Galois cohomology classes arising from Beilinson-Flach elements in a tower of products of modular curves.

Contents

Introduction

This paper completes the project undertaken in [BDR12] of proving the Birch and Swinnerton-Dyer conjecture in analytic rank 0 for the ϱ-isotypic components of Mordell-Weil groups of elliptic curves, when ϱ is an odd, irreducible two-dimensional Artin representation.

In the early 90's, Kato unveiled a new approach to the Birch and Swinnerton-Dyer conjecture and to the cyclotomic Iwasawa main conjecture for (modular) elliptic curves, resting on Beilinson elements in the second K-groups of modular curves and on their p-adic deformations. This approach led to a proof of the implication

$$L(E,\chi,1)\neq 0 \quad \Rightarrow \quad \mathrm{Hom}_{G_{\mathbb{Q}}}(\chi, E(\bar{\mathbb{Q}})\otimes\mathbb{C}) = 0,$$

for all elliptic curves E over $\mathbb{Q}$ and all Dirichlet characters χ (viewed as Galois characters via class field theory). The article [BCDDPR] places Kato's strategy in the broader framework of "Euler systems of Garrett-Rankin-Selberg type". Roughly speaking, the global objects exploited by Kato are distinguished elements in the higher Chow group $\mathrm{CH}^2(X_1(Np^s),2)$ attached to a pair of modular units, whose logarithmic derivatives are Eisenstein series of weight two. Replacing either one, or both, of these Eisenstein series by cuspidal eigenforms leads to the study of *Beilinson-Flach* elements in $\mathrm{CH}^2(X_1(Np^s)^2,1)$ and of *Gross-Kudla-Schoen* diagonal cycles in the Chow group $\mathrm{CH}^2(X_1(Np^s)^3)$ of codimension two cycles on the triple product of $X_1(Np^s)$. Chapters 2.2 and 2.3. of [BCDDPR] explain how the images of these elements under p-adic étale regulators and Abel-Jacobi maps, when made to vary in p-adic families and specialised at suitable classical weight one points, might be parlayed into a proof of the implications

$$L(E,\varrho,1)\neq 0 \quad\Rightarrow\quad \mathrm{Hom}_{G_{\mathbb{Q}}}(\varrho, E(\bar{\mathbb{Q}})\otimes\mathbb{C})=0, \tag{1}$$

$$L(E,\varrho_1\otimes\varrho_2,1)\neq 0 \quad\Rightarrow\quad \mathrm{Hom}_{G_{\mathbb{Q}}}(\varrho_1\otimes\varrho_2, E(\bar{\mathbb{Q}})\otimes\mathbb{C})=0, \tag{2}$$

where ϱ, ϱ_1 and ϱ_2 are odd, irreducible two-dimensional (complex) Artin representations for which $\varrho_1\otimes\varrho_2$ is self-dual. The details of this program are carried out in [DR2] to obtain the proof of (2). The goal of the present work is to deliver on the other promise made in [BCDDPR] by proving the following result:

Theorem A. *Let E be an elliptic curve over $\mathbb{Q}$ and let ϱ be an odd, irreducible, two-dimensional Artin representation. Assume that the conductors of E and ϱ are prime to each other. Then* (1) *holds.*

Prior to Theorem A, the only irreducible two-dimensional Artin representations for which (1) had been proved were those induced from *ring class characters* of

(i) imaginary quadratic fields, by a series of works building on the methods of Kolyvagin [Ko89];
(ii) real quadratic fields, under a mild analytic non-vanishing hypothesis, by Corollary A1 of [DR2].

In addition to Artin representations with projective image isomorphic to A_4, S_4 and A_5, Theorem A also applies to a large class of two-dimensional representations induced from general ray class characters of quadratic fields:

Theorem B. *Let E be an elliptic curve over $\mathbb{Q}$ of conductor N, let K be a quadratic field of discriminant D, and let $\psi : \mathrm{Gal}\,(H/K) \longrightarrow \mathbb{C}^\times$ be a ray class character of conductor $\mathfrak{f}\lhd\mathcal{O}_K$. Assume that* $\gcd(N, D\cdot\mathrm{norm}(\mathfrak{f}))=1$*, and that ψ is of mixed signature if K is real quadratic. Then*

$$L(E/K,\psi,1)\neq 0 \quad\Rightarrow\quad E(H)^\psi=0,$$

where

$$E(H)^\psi := \{P\in E(H)\otimes\mathbb{C} \quad \text{s.t.} \quad \sigma P=\psi(\sigma)P, \quad \forall\sigma\in\mathrm{Gal}\,(H/K)\}.$$

Even for imaginary quadratic fields, Theorem B goes well beyond what can be obtained by the methods of Kolyvagin which apply only to ring class characters. For real quadratic K, there is simply no overlap between Theorem B and [DR2, Cor. A1], since ring class characters are either totally even or totally odd.

It is worth pointing out that the representation ϱ arising in Theorem A is typically not self-dual. Because of this, the root number arising in the functional equation of $L(E,\varrho,s)$ does not control the parity of its order of vanishing at $s=1$, and it is expected that $L(E,\varrho,1)$ vanishes only in rare, sporadic instances, so that the non-vanishing hypothesis in Theorem A is almost always satisfied. Any quantitative non-vanishing statement along these lines would find immediate, unconditional applications to the arithmetic of elliptic curves via Theorems A and B. It should also be noted that $L(E,\varrho,1)\neq 0$ if and only if $L(E,\varrho^\vee,1)\neq 0$, where $\varrho^\vee$ is the contragredient representation of ϱ. Likewise, the triviality of the complex vector space $\mathrm{Hom}_{G_{\mathbb{Q}}}(\varrho,E(\bar{\mathbb{Q}})\otimes\mathbb{C}) = (E(\bar{\mathbb{Q}})\otimes\varrho^\vee)^{G_{\mathbb{Q}}}$ is equivalent to that of $\mathrm{Hom}_{G_{\mathbb{Q}}}(\varrho^\vee,E(\bar{\mathbb{Q}})\otimes\mathbb{C})$.

The proof of Theorem A is based on so-called *Beilinson-Flach elements* whose definition is introduced in §2 below, and on their *variation in p-adic families.* For the purpose of the introduction, suffice it to say that Beilinson-Flach elements are distinguished elements in the higher Chow group $\mathrm{CH}^2(X_1(N)^2,1)$, which can also be interpreted as the motivic cohomology group $H^3_{\mathcal{M}}(X_1(N)^2,\mathbb{Q}(2))$, They were first introduced in the 1980's by Beilinson [Bei, Ch. 2, §6], who related their image under his complex regulator to the value at $s=2$ of the Rankin L-series $L(f\otimes g,s)$ attached to the convolution of weight two newforms f and g on $\Gamma_1(N)$.

In the early 1990's, Flach [Fl] exploited (a slight variant of) these elements to prove the finiteness of the Selmer group of the symmetric square of an elliptic curve. Shortly thereafter, these finiteness results found a spectacular application in Wiles' epoch-making proof of the Shimura-Taniyama conjecture and Fermat's Last Theorem. Flach's approach to bounding the Selmer group of the symmetric square featured prominently in Wiles' original strategy, but was later obviated by the simpler and more flexible approach based on "Taylor-Wiles systems" introduced in [TW].

The power and versatility of Taylor-Wiles systems for questions surrounding the deformation theory of Galois representations may explain why, with a few exceptions ([MWe], [We02], ...), Beilinson-Flach elements garnered relatively less attention in the intervening decades. Yet their arithmetic relevance is not confined to the symmetric square motive of a modular form: guided by an analogy between Beilinson-Flach elements and the study of Gross-Kudla-Schoen diagonal cycles undertaken in [DR1] and [DR2], the article [BDR12] established a p-adic analogue of Beilinson's formula for Beilinson-Flach elements, pointing out a number of arithmetic applications of this formula based on the "Euler system philosophy" propounded by Kato and Perrin-Riou, and lying ostensibly outside the scope of the Taylor-Wiles method. (Cf. the introduction of loc.cit., and [BCDDPR, Sec. 2.2.].)

The recent work of Kings, Lei, Loeffler and Zerbes makes significant strides in furthering the program initiated in [BDR12], applying the ideas of the present work to more general settings and introducing key technical improvements along the way. Thus, [LLZ1] constructs a full cyclotomic Euler system of Beilinson-Flach elements and applies it to the study of the Selmer group of the tensor product of two motives attached to cusp forms, while [LLZ2] applies similar techniques to study the Iwasawa theory of elliptic curves over the full two-variable $\mathbb{Z}_p$-extension of an imaginary quadratic field. The recent preprint [KLZ] exploits an interpolation of the Beilinson-Flach elements in all weights at once based on the notion of "Rankin-Iwasawa classes" arising in [Ki13], to establish a general explicit reciprocity law relating these families to Hida's three variable p-adic Rankin L-function, with important applications to the associated three variable Main conjecture. It is worth pointing out that the approach to Theorem A in the present work rests solely on p-adic families of Beilinson-Flach elements of minimal level and on their local behaviour at p; it can therefore be compared to the strategy followed by Coates-Wiles [CoWi] to prove the finiteness of Mordell-Weil groups of CM elliptic curves. By exploiting the full collection of Beilinson-Flach elements and their Euler system properties, notably their "tame deformations" at auxiliary primes $\ell \neq p$, [KLZ] also establishes (just as in Thaine and Rubin's strengthening of the Coates-Wiles method) the finiteness of certain p-parts of the relevant Selmer and Shafarevich-Tate groups in the setting of Theorem A.

Shortly after [BDR12], Dasgupta [Da] had the idea of exploiting Beilinson-Flach elements to factor Hida's p-adic Rankin L-function of the tensor square $f \otimes f$ into the Kubota-Leopoldt p-adic zeta-function and the Coates-Schmidt p-adic L-function attached to the symmetric square of f. The main result of [Da] is a deep manifestation of the Artin formalism for Hida's p-adic L-series and rests on a comparison between circular units and a new construction of units in number fields arising from Beilinson-Flach elements. Dasgupta's approach, which is inspired by a theorem of Gross [Gr] for the Katz p-adic L-function of an imaginary quadratic field and builds on the ideas of the present work, also makes essential use of the improvements described in [KLZ].

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1. Strategy of proof

Not surprisingly, modularity plays a key part in the proof of Theorem A. The elliptic curve E is associated to a weight 2 newform $f = f_E \in S_2(N_f)$ thanks to [BCDT], while the Artin representation ϱ is associated to a weight 1 newform $g = g_\varrho \in S_1(N_g, \chi)$ thanks to [KW09]. The Hasse-Weil-Artin L-series $L(E, \varrho, s)$ can therefore be identified with the convolution L-series $L(f \otimes g, s)$, whose analytic continuation follows from Rankin's method.

Let $L \subset \mathbb{Q}^{\mathrm{ab}}$ denote a field of coefficients over which ϱ can be described, and let H be the finite extension of $\mathbb{Q}$ cut out by ϱ (i.e., the fixed field of the kernel of ϱ). Write

$$E(H)^\varrho_L := \hom_{G_\mathbb{Q}}(V_\varrho, E(H) \otimes L)$$

for the ϱ-isotypic part of the Mordell-Weil group of E. Theorem A asserts the triviality of this vector space when $L(E, \varrho, 1) \neq 0$.

The standard approach to bounding $E(H)^{\varrho}_{L}$ consists in embedding it in an appropriate *Selmer group* attached to the choice of a rational prime p, which throughout the paper is assumed to be prime to $N_f N_g$, and *ordinary* for f, i.e., p does not divide $a_p(f)$. To define this Selmer group, let

$$V_f := V_p(E) := H^1(E_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(1) = (\lim_{\leftarrow, n} E[p^n]) \otimes \mathbb{Q}_p$$

denote Deligne's p-adic representation attached to f, which occurs as a direct factor of the étale cohomology $H^1_{\mathrm{et}}(X_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(1)$, and let $V_g = V_{g,\sigma}$ denote the Artin representation attached to g, viewed as a p-adic representation by fixing some embedding σ of its coefficient field L into $L_p \subset \bar{\mathbb{Q}}_p$. The four dimensional L_p-vector space

$$V_p(E) \otimes V_g = V_f \otimes V_g,$$

where the tensor product is taken over $\mathbb{Q}_p$, is a p-adic representation of $G_{\mathbb{Q}}$. The connecting homomorphism of Kummer theory arising from the p-power descent exact sequence on $E(H)$, composed with the inverse of the restriction map from $\mathbb{Q}$ to H, gives a linear injection of L_p-vector spaces

$$E(H)^{\varrho}_{L_p} := E(H)^{\varrho}_{L} \otimes_L L_p \longrightarrow H^1(H, V_p(E) \otimes V_g^{\vee})^{\mathrm{Gal}\,(H/\mathbb{Q})} \stackrel{\mathrm{res}_H^{-1}}{\longrightarrow} H^1(\mathbb{Q}, V_f \otimes V_g^{\vee}),$$

denoted δ, where $V_g^{\vee} := V_g \otimes \det(V_g)^{-1}$ is the L_p-linear dual of the representation V_g. For each place v of $\mathbb{Q}$ the corresponding map on local points yields an injection

$$\delta_v : E(H_v)^{\varrho}_{L_p} \longrightarrow H^1(\mathbb{Q}_v, V_f \otimes V_g^{\vee}),$$

where $\mathrm{Gal}\,(H/\mathbb{Q})$ acts in the natural way on the local points $E(H_v) := \oplus_{w|v} E(H_w)$. The image of δ_v in $H^1(\mathbb{Q}_v, V_f \otimes V_g^{\vee})$ is called the *finite part* of the local cohomology at v, and is denoted $H^1_{\mathrm{fin}}(\mathbb{Q}_v, V_f \otimes V_g^{\vee})$. Likewise, $H^1_{\mathrm{fin}}(\mathbb{Q}_v, V_f \otimes V_g)$ is defined to be the image of $E(H_v)^{\varrho^{\vee}}$ by the appropriate local coboundary map.

When $v = p$, the finite part of the local cohomology admits an alternate description in terms of the general local conditions attached by Bloch and Kato [BK, §3] to any p-adic representation V of $G_{\mathbb{Q}_p}$:

$$\begin{aligned}
H^1_{\mathrm{exp}}(\mathbb{Q}_p, V) &:= \ker\left(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes \mathbf{B}_{\mathrm{cris}}^{\phi=1})\right), \\
H^1_{\mathrm{fin}}(\mathbb{Q}_p, V) &:= \ker\left(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes \mathbf{B}_{\mathrm{cris}})\right) = \mathrm{Ext}^1_{\mathrm{cris}}(\mathbb{Q}_p, V), \qquad (3) \\
H^1_{\mathrm{geom}}(\mathbb{Q}_p, V) &:= \ker\left(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes \mathbf{B}_{\mathrm{dR}})\right) = \mathrm{Ext}^1_{\mathrm{dR}}(\mathbb{Q}_p, V),
\end{aligned}$$

where $\mathbf{B}_{\mathrm{cris}} \subset \mathbf{B}_{\mathrm{dR}}$ are Fontaine's rings of periods for cristalline and de Rham representations, respectively, and ϕ denotes the cristalline frobenius. The subscripts exp, fin, and geom stand for "exponential", "finite", and "geometric" respectively, and one has the obvious inclusions

$$H^1_{\mathrm{exp}}(\mathbb{Q}_p, V) \subset H^1_{\mathrm{fin}}(\mathbb{Q}_p, V) \subset H^1_{\mathrm{geom}}(\mathbb{Q}_p, V).$$

When $V = V_f \otimes V_g$ or $V_f \otimes V_g^{\vee}$, the condition $p \nmid N_f N_g$ shows, as in the proof of equation (18), that the Dieudonné module of V has no ϕ-invariant vectors (for reasons of weight), and thus

$$\delta_p(E(H_p)^{\varrho}_{L_p}) = H^1_{\mathrm{exp}}(\mathbb{Q}_p, V_f \otimes V_g^{\vee}) = H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_f \otimes V_g^{\vee}) = H^1_{\mathrm{geom}}(\mathbb{Q}_p, V_f \otimes V_g^{\vee}).$$

The cup product in Galois cohomology combined with the Weil pairing $V_f \times V_f \to \mathbb{Q}_p(1)$ gives rise to the perfect local Tate pairing

$$\langle\ ,\ \rangle_v : H^1(\mathbb{Q}_v, V_f \otimes V_g) \times H^1(\mathbb{Q}_v, V_f \otimes V_g^{\vee}) \longrightarrow H^2(\mathbb{Q}_v, \mathbb{Q}_p(1)) = \mathbb{Q}_p.$$

For all v, the finite subspaces $H^1_{\mathrm{fin}}(\mathbb{Q}_v, V_f \otimes V_g)$ and $H^1_{\mathrm{fin}}(\mathbb{Q}_v, V_f \otimes V_g^{\vee})$ are orthogonal complements of each other under the local Tate pairing. Let

$$H^1_{\mathrm{sing}}(\mathbb{Q}_v, V_f \otimes V_g) := \frac{H^1(\mathbb{Q}_v, V_f \otimes V_g)}{H^1_{\mathrm{fin}}(\mathbb{Q}_v, V_f \otimes V_g)}$$

denote the *singular quotient* of the local cohomology at v, and write

$$\partial_v : H^1(\mathbb{Q}, V_f \otimes V_g) \longrightarrow H^1_{\mathrm{sing}}(\mathbb{Q}_v, V_f \otimes V_g)$$

for the natural projection, which is referred to as the *residue map* at v. Both the finite part and the singular quotient of $H^1(\mathbb{Q}_p, V_f \otimes V_g)$ are two-dimensional over L_p. By local Tate duality, similar statements hold for the finite part and the singular quotient of $H^1(\mathbb{Q}_p, V_f \otimes V_g^{\vee})$.

The Selmer group attached to $(E, V_g^\vee)$ is the subgroup of $H^1(\mathbb{Q}, V_f \otimes V_g^\vee)$, denoted $H^1_{\rm fin}(\mathbb{Q}, V_f \otimes V_g^\vee)$, which fits into the cartesian square

$$\begin{tikzcd} H^1_{\rm fin}(\mathbb{Q}, V_f \otimes V_g^\vee) \arrow[r] \arrow[d] & H^1(\mathbb{Q}, V_f \otimes V_g^\vee) \arrow[d] \\ \prod_v H^1_{\rm fin}(\mathbb{Q}_v, V_f \otimes V_g^\vee) \arrow[r] & \prod_v H^1(\mathbb{Q}_v, V_f \otimes V_g^\vee). \end{tikzcd}$$

The Selmer group attached to (E, V_g) is defined likewise, by replacing $V_g^\vee$ by V_g.

The standard approach for bounding $E(H)_L^\varrho$ rests on the following statement whose proof is explained in Prop. 6.3 of [DR2].

Lemma 1.1. *Assume that the residue map ∂_p attached to the p-adic representation $V_f \otimes V_{g,\sigma}$ is a surjective map of L_p-vector spaces, for all embeddings $\sigma : L \longrightarrow L_p$. Then $E(H)_L^\varrho = 0$.*

The proof of theorem A has thus been reduced to the problem of constructing two independent classes in the global cohomology $H^1(\mathbb{Q}, V_f \otimes V_g)$ whose images generate the singular quotient $H^1_{\rm sing}(\mathbb{Q}_p, V_f \otimes V_g)$. Let α_g and β_g be the two (not necessarily distinct) roots of the Hecke poynomial

$$x^2 - a_p(g)x + \chi(p) = (x - \alpha_g)(x - \beta_g)$$

attached to g at p, and let g_α and g_β denote the associated p-stabilisations of g satisfying

$$U_p g_\alpha = \alpha_g g_\alpha, \qquad U_p g_\beta = \beta_g g_\beta.$$

When $\alpha_g \neq \beta_g$, these stabilisations are distinct, and the unramified representation V_g decomposes as a direct sum

$$V_g = V_g^\alpha \oplus V_g^\beta$$

of one-dimensional eigenspaces for the arithmetic frobenius element at p attached to these two eigenvalues. When $\alpha_g = \beta_g$, set $V_g^\alpha := V_g$. Theorem A will now follow from the following result.

Theorem 1.2. *There exists a global cohomology class*

$$\kappa(f, g_\alpha) \in H^1(\mathbb{Q}, V_f \otimes V_g)$$

attached to to the p-stabilised form g_α, satisfying:

(1) *The residue $\partial_p(\kappa(f, g_\alpha))$ belongs to $H^1_{\rm sing}(\mathbb{Q}_p, V_f \otimes V_g^\beta)$;*
(2) *$\partial_p(\kappa(f, g_\alpha)) \neq 0$ if and only if $L(f \otimes g, 1) \neq 0$.*

To see why Theorem 1.2 implies Theorem A, choose the descent prime p in such a way that the frobenius element σ_p acts on V_g with distinct eigenvalues α_g and β_g. Such a choice is always possible, in view of the Chebotarev density theorem and the fact that the two-dimensional V_g is irreducible and hence the image of ϱ contains a non-scalar matrix. When $L(E, \varrho, 1) \neq 0$, Theorem 1.2 implies that the residues of the global classes $\kappa(f, g_\alpha)$ and $\kappa(f, g_\beta)$ generate the two-dimensional L_p-vector space $H^1_{\rm sing}(\mathbb{Q}_p, V_f \otimes V_g)$. Since this reasoning applies to the representation $V_{g,\sigma}$ for *any* embedding σ of L into L_p, Theorem A follows from Lemma 1.1.

The "singular" class $\kappa(f, g_\alpha)$ of Theorem 1.2 is not expected to arise from the images of algebraic cycles or elements in K-theory under an appropriate étale Abel-Jacobi or regulator map, since such images are typically cristalline, at least in scenarios of good reduction of the ambient variety. It is constructed instead as a p-adic limit of classes $\kappa(f, g_x)$ attached to (sufficiently many) weight two specialisations g_x of a Hida family specialising to g_α in weight one. These "weight two classes" are obtained via a geometric construction, as the images of Beilinson-Flach elements under the p-adic étale regulator map to Galois cohomology.

Section 2 describes the construction of a geometric class $\kappa(f, g)$ attached to a classical eigenform g of *weight two* and p-power nebentypus character, and summarises its key properties, including the relation between its p-adic logarithm and the values of Hida's p-adic Rankin L-function attached to f and g. This relation extends the main result of [BDR12] to settings where the modular curve has bad reduction at the prime p, and relies crucially on the work of Nekovar and Niziol [NeNi] on syntomic cohomology and p-adic regulators for arbitrary semistable varieties, which in turn allowed Besser, Loeffler and Zerbes [BLZ] to generalize Besser's finite polynomial cohomology to this setting.

Section 3 explains how the weight two classes $\kappa(f,g)$, as g ranges over all the classical specialisations of weight two and p-power nebentypus character of a Hida family $\underline{g}$, can be interpolated into a Λ-adic cohomology class $\kappa(f,\underline{g})$. Since these ordinary weight two points are dense in the rigid analytic topology on the relevant components of the Coleman-Mazur eigencurve, it follows that $\kappa(f,\underline{g})$ is uniquely determined by its weight two specialisations.

The class $\kappa(f,g_\alpha)$ of Theorem 1.2 is obtained by letting $\underline{g}$ be a Hida family specialising to g_α in weight one, and specialising the resulting $\kappa(f,\underline{g})$ to g_α. The p-adic Beilinson formula of Section 2 is then parlayed into the explicit reciprocity law relating the Bloch-Kato dual exponential of $\kappa(f,g_\alpha)$ to the special value of the same p-adic L-function at the point attached to g_α. Since this latter point lies in the range of classical interpolation defining the p-adic L-function, it is directly related to a classical L-value, and this is what leads to the proof of Theorem 1.2.

2. A p-adic Beilinson formula

2.1. **Siegel units and Eisenstein series.** Fix a modulus $M = Np^s \geq 3$ with $p \nmid N$ and $s \geq 0$. Let

$$Y_s := Y_1(Np^s) \quad \subset \quad X_s := X_1(Np^s)$$

be canonical models over $\mathbb{Q}$ of the (affine and projective, respectively) modular curves classifying triples (A, i_N, i_p) where A is a (generalised) elliptic curve and $i_N : \mu_N \to A$ and $i_p : \mu_{p^s} \to A$ are embeddings of finite group schemes. The standard notations $Y_0(Np)$ and $X_0(Np)/\mathbb{Q}$ are adopted for the affine and projective modular curves classifying pairs (A, C_{Np}) where $C_{Np} \subset A$ is a cyclic subgroup scheme of A of order Np. We will denote by $w_M : X_1(M) \longrightarrow X_1(M)$ the standard Atkin-Lehner involution, and will simply denote it as w when the level M is clear from the context.

For $a \in (\mathbb{Z}/M\mathbb{Z})^\times$, let $\mathfrak{g}_{a;M}$ be the *Siegel unit* denoted $g_{0,a/M}$ in [Ka98], whose q-expansion is given by

$$\mathfrak{g}_{a;M} := (1-\zeta_M^a) q^{1/12} \prod_{n=1}^{\infty} (1-\zeta_M^a q^n)(1-\zeta_M^{-a} q^n), \tag{4}$$

where ζ_M is a fixed primitive M-th root of unity. The unit $\mathfrak{g}_{a;M}$ can naturally be viewed as belonging to $\mathcal{O}_{Y_1(M)}^\times \otimes \mathbb{Q}$, and its q-expansion is defined over $\mathbb{Q}(\mu_M)$, while the unit

$$\mathfrak{g}_{a;M}^w := w \mathfrak{g}_{a;M}$$

is defined over $\mathbb{Q}$. If χ is a primitive even Dirichlet character of conductor M, taking values in a field denoted $\mathbb{Q}(\chi)$, let

$$\mathcal{G}(\chi) = \sum_{a=1}^{M} \chi(a) \zeta_M^a$$

denote the Gauss sum attached to χ and the choice of ζ_M, and set

$$\mathfrak{g}_\chi = \frac{1}{\varphi(M)} \cdot \sum_{a=1}^{M} \chi^{-1}(a) \otimes \mathfrak{g}_{a;M} \in \mathcal{O}_{Y_1(M)}^\times \otimes \mathbb{Q}(\chi), \qquad \mathfrak{g}_\chi^w := w \mathfrak{g}_\chi. \tag{5}$$

Given a pair (χ_1, χ_2) of Dirichlet characters and an integer $k \geq 2$ such that $\chi_1\chi_2(-1) = (-1)^k$, define also the Eisenstein series

$$E_k(\chi_1, \chi_2) = \left\{ \begin{array}{cc} \frac{1}{2} \cdot L(\chi_2, 1-k) & \text{if } \chi_1 = 1 \\ 0 & \text{otherwise} \end{array} \right\} + \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi_1(n/d) \chi_2(d) d^{k-1} \right) q^n. \tag{6}$$

A direct calculation using (4), (5) and (6) shows that the logarithmic derivative of $\mathfrak{g}_\chi$ is

$$d \log \mathfrak{g}_\chi = \frac{2\mathcal{G}(\chi^{-1})}{\varphi(M)} \cdot E_2(\chi, 1) \frac{dq}{q} \in \Omega^1_{Y_1(M)}. \tag{7}$$

2.2. **Beilinson-Flach elements.** The Beilinson-Flach elements of this section are elements in the higher Chow group over $\mathbb{Q}$ of the surface

$$S_s := X_0(Np) \times X_s.$$

For any field F, let $C^2(S_s, 1)_F$ denote the group of finite formal linear combinations $\mathrm{D} = \sum_i a_i(Z_i, u_i)$ of pairs (Z_i, u_i) with coefficients $a_i \in F$, where Z_i are irreducible curves embedded in S_s and u_i are rational functions on Z_i. Write $Z^2(S_s, 1)_F \subset C^2(S_s, 1)_F$ for the subspace of elements D satisfying

$$\mathrm{div}(\mathrm{D}) := \sum a_i \mathrm{div}(u_i) = 0.$$

Any pair $\{\varphi, \psi\}$ of non-zero rational functions on S_s gives rise to the element $\sum_i(Z_i, u_i)$ in $Z^2(S_s, 1)_F$, where $\{Z_i\}$ is the set of irreducible components in S_s of the divisors of φ or ψ, and, after setting $a_i = \mathrm{ord}_{Z_i}(\varphi)$ and $b_i = \mathrm{ord}_{Z_i}(\psi)$, the rational function u_i on Z_i is given by the tame symbol $u_i = \prod(-1)^{a_i b_i}(\frac{\varphi^{b_i}}{\psi^{a_i}})_{|Z_i}$. Writing $B^2(S_s, 1)_F$ for the subspace of $Z^2(S_s, 1)_F$ formed by such elements, Bloch's higher Chow group of the surface S_s with coefficients in F is defined as

$$\mathrm{CH}^2(S_s, 1)_F = Z^2(S_s, 1)_F / B^2(S_s, 1)_F.$$

Letting

$$\tilde{\pi}_s : X_s \longrightarrow X_0(Np) \tag{8}$$

denote the natural forgetful projection of modular curves which is compatible with the U_p correspondence acting on both curves, define

$$\iota_s = (\tilde{\pi}_s, \mathrm{Id}) : X_s \hookrightarrow S_s, \tag{9}$$

to be the closed embedding given by $\iota_s(x) = (\tilde{\pi}_s(x), x)$. Let

$$\Delta_s := \iota_s(X_s) \subset S_s$$

denote the resulting embedded curve in S_s.

Lemma 2.1. *For any $u \in \mathcal{O}_{Y_s}^\times$, there exists a negligible element $\theta_s \in C^2(S_s, 1)_F$ satisfying*

$$\mathrm{div}\, \theta_s = \mathrm{div}(\Delta_s, u). \tag{10}$$

Proof. The proof is given in Lemma 3.1. of [BDR12], and relies on the fact the the group of degree zero divisors supported at the cusps is torsion in the jacobian of X_s by the Manin-Drinfeld theorem. □

Thanks to Lemma 2.1, the class $(\Delta_s, u) - \theta_s$ lies in $Z^2(S_s, 1)_\mathbb{Q}$, and its image

$$\mathbf{BF}(u) := [(\Delta_s, u) - \theta_s] \quad \text{in} \quad \mathrm{CH}^2_{\mathrm{neg}}(S_s, 1)_\mathbb{Q}(\mathbb{Q})$$

is independent of the particular choice of θ_s satisfying (10). It is called the *Beilinson-Flach element* attached to the modular unit $u \in \mathcal{O}_{Y_s}^\times$.

Definition 2.2. The elements

$$\mathbf{BF}(a; Np^s) := \mathbf{BF}(\mathfrak{g}^w_{a;Np^s}), \qquad \mathbf{BF}_s := \mathbf{BF}(1; Np^s) \quad \text{in} \quad \mathrm{CH}^2_{\mathrm{neg}}(S_s, 1)_\mathbb{Q}(\mathbb{Q})$$

are called the *Beilinson-Flach elements of level Np^s*.

2.3. **Etale and syntomic regulators.** Let $\varphi = \sum a_n(\varphi) q^n \in S_k(N_\varphi, \chi)$ be a normalized newform of weight k, level N_φ and character χ, and let $\omega_\varphi := \varphi(q)\frac{dq}{q}$ be the associated regular differential form on $X_1(N_\varphi)$. Let K_φ be the finite extension of $\mathbb{Q}_p$ generated by the fourier coefficients of φ, let $\mathcal{O}_\varphi$ be its valuation ring, and let V_φ denote the two-dimensional Galois representation over $\mathcal{O}_\varphi$ associated by Deligne to φ. This Galois representation is pure of weight -1 and is characterized up to isomorphism by the property that for any prime $\ell \nmid pN_\varphi$, the characteristic polynomial of an arithmetic Frobenius element Fr_ℓ is equal to the Hecke polynomial $x^2 - a_\ell(\varphi)x + \chi(\ell)\ell^{k-1}$. Write α_φ and β_φ be the roots of this Hecke polynomial at $\ell = p$, ordered so that $\mathrm{ord}_p(\alpha_\varphi) \leq \mathrm{ord}_p(\beta_\varphi)$.

If N is a multiple of N_φ and $\mathbb{T}_N$ is the Hecke algebra generated by the good Hecke operators T_ℓ with $\ell \nmid N$, the φ-isotypic component of a $\mathbb{T}_N$-module S is defined to be

$$S[\varphi] \quad := \quad \{v \in S \otimes \mathcal{O}_\varphi \ : \ T_\ell(v) = a_\ell(\varphi)v \text{ for all } \ell \nmid N\}. \tag{11}$$

Generalising the setting of §1, and modifying slightly the notations therein, let $f \in S_2(Np)[f_0]$ now denote a normalised eigenform with trivial Nebentypus character arising from a newform f_0 on $\Gamma_0(N_f)$

for some $N_f \mid N$. In the context of §1, f_0 is the eigenform associated to the elliptic curve E. More specifically, we take

$$f(q) = \sum_{d \mid \frac{Np}{N_f}} \lambda_d f_0(q^d) \in S_2(Np)[f_0] \tag{12}$$

to be an eigenvector for all Hecke operators U_ℓ, $\ell \mid Np$, and assume f is ordinary, i.e., that $\alpha_f = \alpha_{f_0}$ is a p-adic unit.

In addition, for the rest of this section let $g \in S_2(Np^s, \chi \cdot \chi_p)$ be an eigenform of weight 2, level Np^s and nebentype character $\chi\chi_p$, that we have factored in such a way that

$$\mathrm{cond}(\chi) \mid N, \qquad \mathrm{cond}(\chi_p) \mid p^s.$$

Assume g arises from a newform g_0 of level $N_g p^s$ for some $N_g \mid N$ and that the conductor of χ_p is equal to p^s. In the terminology introduced by Mazur-Wiles [MWi84], g is said to be *primitive* at p. Assume also that g is ordinary at p, i.e., that $\alpha_g = a_p(g)$ is a p-adic unit.

For any variety X over $\mathbb{Q}$, the notation $\bar{X} := X \times \bar{\mathbb{Q}}$ will be used to designate the base change of X to $\bar{\mathbb{Q}}$. Let

$$V_0(Np) = H^1_{\mathrm{et}}(\bar{X}_0(Np), \mathbb{Z}_p)(1) \qquad \text{and} \qquad V_s = H^1_{\mathrm{et}}(\bar{X}_s, \mathbb{Z}_p)(1) \ \text{ for } \ s \geq 1 \tag{13}$$

denote the Galois representation afforded by the Tate module of $X_0(Np)$ and X_s, respectively.

Let

$$\mathrm{pr}_{1,1} : H^2_{\mathrm{et}}(\bar{S}_s, \mathbb{Z}_p)(2) \longrightarrow V_0(Np) \otimes V_s$$

denote the natural projection of $G_{\mathbb{Q}}$-modules induced by the Künneth decomposition, and let

$$\mathrm{reg}_{\mathrm{et}} : \mathrm{CH}^2(S_s, 1)(\mathbb{Q}) \longrightarrow H^1(\mathbb{Q}, H^2_{\mathrm{et}}(\bar{S}_s, \mathbb{Z}_p)(2)) \xrightarrow{\mathrm{pr}_{1,1}} H^1(\mathbb{Q}, V_0(Np) \otimes V_s)$$

denote the composition of the p-adic étale regulator map (as defined for instance in chapter 2 of [Fl]) with the map induced by $\mathrm{pr}_{1,1}$ in Galois cohomology.

The image under $\mathrm{reg}_{\mathrm{et}}$ of the subgroup generated by negligible element vanishes, for the cohomology class of such elements take values in the Künneth component

$$H^2_{\mathrm{et}}(\bar{X}_0(Np), \mathbb{Z}_p) \otimes H^0_{\mathrm{et}}(\bar{X}_s, \mathbb{Z}_p)(2) \ \oplus \ H^0_{\mathrm{et}}(\bar{X}_0(Np), \mathbb{Z}_p) \otimes H^2_{\mathrm{et}}(\bar{X}_s, \mathbb{Z}_p)(2).$$

Hence the étale regulator descends to a map which we continue to denote with the same symbol

$$\mathrm{reg}_{\mathrm{et}} : \mathrm{CH}^2_{\mathrm{neg}}(S_s, 1)(\mathbb{Q}) \longrightarrow H^1(\mathbb{Q}, V_0(Np) \otimes V_s). \tag{14}$$

Definition 2.3. The Beilinson-Flach cohomology classes of level Np^s are defined to be

$$\begin{aligned} \kappa(a; Np^s) \quad &:= \quad \mathrm{reg}_{\mathrm{et}}(\mathbf{BF}(a; Np^s)) \in H^1(\mathbb{Q}, V_0(Np) \otimes V_s), \\ \kappa_s \quad &:= \quad \kappa(1; Np^s). \end{aligned}$$

We now turn to providing an analytic description of the restriction to $G_{\mathbb{Q}_p}$ of the classes κ_s, which will pave the way for establishing its relation to the values of the Hida-Rankin p-adic L-function associated to Hida families passing through f and g. A first step towards this task is the description of $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_{fg})$ in terms of the de Rham cohomology of the relevant modular curves.

Recall Fontaine's ring $\mathbf{B}_{\mathrm{cris}}$ of cristalline periods, and let

$$D_f := (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_f)^{G_{\mathbb{Q}_p}}, \quad D_g := (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_g)^{G_{\mathbb{Q}_p(\mu_{p^s})}}, \quad D_{fg} := (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_{fg})^{G_{\mathbb{Q}_p(\mu_{p^s})}} \tag{15}$$

be the (potentially crystalline) Dieudonné modules associated to V_f, V_g and V_{fg}, respectively. Note that $D_{fg} \simeq D_f \otimes D_g$ in the category of filtered frobenius vector spaces over the field K. Let

$$g^* = \bar{g} = \sum \bar{a}_n(g) q^n \in S_2(Np^s, \chi^{-1}\chi_p^{-1}), \qquad g^w := wg \tag{16}$$

be the modular forms which both belong to the automorphic representation attached to the contragredient of g; these modular forms are eigenvectors for the good Hecke operators, and

$$V_{g^*} = V_g \otimes \chi^{-1}\chi_p^{-1} = V_g^\vee(1).$$

The Dieudonné module of the Kummer dual $V_{fg}^\vee(1) = V_{fg^*}(-1)$ of V_{fg}, denoted $D_{fg^*}(-1)$, is canonically identified with D_{fg^*} but with a shift in the Hodge filtration. The dimensions of the relevant pieces of the Hodge filtration on D_{fg} and $D_{fg^*}(-1)$ are summarised in the table below:

	Fil^{-2}	Fil^{-1}	Fil^0	Fil^1	Fil^2
D_{fg}	4	3	1	0	0
$D_{fg^*}(-1)$	4	4	3	1	0.

The Bloch-Kato logarithm $\log_p$ of [BK] associated to the Galois representation V_{fg} gives an isomorphism

$$\log_p : H^1_{\exp}(\mathbb{Q}_p(\mu_{p^s}), V_{fg}) \longrightarrow \frac{D_{fg}}{\mathrm{Fil}^0 D_{fg} + D_{fg}^{\phi=1}},$$

where $H^1_{\exp}(\mathbb{Q}_p(\mu_{p^s}), V_{fg})$ is as defined in (3), with $\mathbb{Q}_p$ replaced by $\mathbb{Q}_p(\mu_{p^s})$. The cristalline frobenius ϕ acts on D_{fg} with eigenvalues

$$\alpha_f\alpha_g, \qquad \alpha_f\beta_g, \qquad \beta_f\alpha_g, \qquad \beta_f\beta_g. \tag{17}$$

These eigenvalues are algebraic numbers of absolute value p (relative to any complex embedding of $\bar{\mathbb{Q}}$) and therefore $D_{fg}^{\phi=1} = 0$. It follows from [BK, Cor. 3.8.4] that

$$H^1_{\exp}(\mathbb{Q}_p(\mu_{p^s}), V_{fg}) = H^1_{\mathrm{fin}}(\mathbb{Q}_p(\mu_{p^s}), V_{fg}). \tag{18}$$

The Bloch-Kato logarithm can therefore be recast in the current context as a map

$$\log_p : H^1_{\mathrm{fin}}(\mathbb{Q}_p(\mu_{p^s}), V_{fg}) \longrightarrow \frac{D_{fg}}{\mathrm{Fil}^0(D_{fg})} = \mathrm{Fil}^0(D_{fg^*}(-1))^\vee, \tag{19}$$

where the last identification arises from the natural duality between D_{fg} and $D_{fg^*}(-1)$, relative to which $\mathrm{Fil}^0(D_{fg})$ and $\mathrm{Fil}^0(D_{fg^*}(-1))$ are orthogonal complements of each other.

Assume furthermore that none of the eigenvalues in (17) are equal to p. This assumption is satisfied with at most finitely many exceptions when f is fixed and g runs over the weight two specialisations of a Hida family, hence will suffice for the applications in this paper. It implies, by another application of [BK, cor. 3.8.4], that $H^1_{\exp}(\mathbb{Q}_p(\mu_{p^s}), V_{fg^*}(-1)) = H^1_{\mathrm{fin}}(\mathbb{Q}_p(\mu_{p^s}), V_{fg^*}(-1))$, and hence, by duality, that

$$H^1_{\mathrm{fin}}(\mathbb{Q}_p(\mu_{p^s}), V_{fg}) = H^1_{\mathrm{geom}}(\mathbb{Q}_p(\mu_{p^s}), V_{fg}). \tag{20}$$

The argument used in the proof of Lemma 2.8 of [Fl] shows that the image of the étale regulator $\mathrm{reg}_{\mathrm{et}}^{fg}$ consists of extensions of Galois representations occurring in the étale cohomology of varieties with potentially semistable reduction at p, which hence are de Rham at p. This image is therefore contained in $H^1_{\mathrm{geom}}(\mathbb{Q}_p(\mu_{p^s}), V_{fg})$, and thus by (18) in the finite subspace of the local cohomology.

Let

$$D_0(Np) := (\mathbf{B}_{\mathrm{dR}} \otimes V_0(Np))^{G_{\mathbb{Q}_p}}, \qquad D_s := (\mathbf{B}_{\mathrm{dR}} \otimes V_s)^{G_{\mathbb{Q}_p(\mu_{p^s})}}, \tag{21}$$

be the de Rham Dieudonné filtered modules attached to the Galois representations $V_0(Np)$ and V_s. Note that these p-adic Galois representations are $\mathbf{B}_{\mathrm{dR}}$ but not $\mathbf{B}_{\mathrm{cris}}$-admissible, but that their f_0 and g_0-isotypic parts, denoted $V_0(Np)[f] \simeq V_f^i$ and $V_s[g] \simeq V_g^j$, are cristalline over $\mathbb{Q}_p$ and $\mathbb{Q}_p(\mu_{p^s})$ respectively.

The étale regulator map (14) on $\mathrm{CH}^2(S_s, 1)_{\mathrm{neg}}$ and the Bloch-Kato logarithm associated to the Galois representation $V_0(Np) \otimes V_s$ give rise to a diagram of maps

$$\begin{tikzcd} \mathrm{CH}^2(S_s,1)_{\mathrm{neg}}(\mathbb{Q}_p(\mu_{p^s})) \arrow[r, "\mathrm{reg}_{\mathrm{et}}"] & H^1_{\mathrm{geom}}(\mathbb{Q}_p(\mu_{p^s}), V_0(Np) \otimes V_s) & \\ & H^1_{\exp}(\mathbb{Q}_p(\mu_{p^s}), V_0(Np) \otimes V_s) \arrow[u, hook] \arrow[r, "\log_p"] & \mathrm{Fil}^0(D_0(Np) \otimes D_s(-1))^\vee. \end{tikzcd} \tag{22}$$

Let

$$\mathrm{reg}_p^{fg} : \mathrm{CH}^2(S_s, 1)(\mathbb{Q}_p(\mu_{p^s}))[fg] \longrightarrow \mathrm{Fil}^0(D_0(Np)[f] \otimes D_s(-1)[g^*])^\vee \tag{23}$$

be the map obtained by applying the (f, g)-isotypic projection to (22), invoking (18) and (20) to show that the vertical inclusion becomes an isomorphism after restricting to these (f, g)-isotypic subspaces.

The main goal of the next sections is proving an analytic formula for the value of $\mathrm{reg}_p^{fg}(\mathbf{BF}_s)$ at a suitable vector in $\mathrm{Fil}^0(D_0(Np)[f]\otimes D_s(-1)[g^*])$, which we now proceed to describe.

By the comparison theorem between étale and de Rham cohomology, there are isomorphisms of two-dimensional filtered frobenius modules

$$D_f \simeq H^1_{\mathrm{dR}}(X_0(N_f)_{/\mathbb{Q}_p})(1)[f_0], \quad D_g \simeq H^1_{\mathrm{dR}}(X_1(N_gp^s)_{/\mathbb{Q}_p(\mu_{p^s})})(1)[g_0] \tag{24}$$

such that

$$\mathrm{Fil}^0 D_f \simeq \Omega^1(X_0(N_f))[f_0] = \langle \omega_{f_0}\rangle \quad \text{and} \quad \mathrm{Fil}^0 D_g \simeq \Omega^1(X_1(N_gp^s))[g_0] = \langle \omega_{g_0}\rangle.$$

For any projective curve X/F, write

$$\langle \ , \ \rangle_X : H^1_{\mathrm{dR}}(X_{/F}) \times H^1_{\mathrm{dR}}(X_{/F}) \longrightarrow F$$

for the canonical Poincaré pairing on X. Let η_{f_0} be an element in the one-dimensional unit-root subspace of D_f, normalised so that

$$\langle \omega_{f_0}, \eta_{f_0}\rangle_{X_0(N_f)} = 1.$$

Fix a finite extension $K/\mathbb{Q}_p$ containing K_f and K_g, and let $\mathcal{O}$ denote its valuation ring. For each positive divisor d of Np/N_f, write $\pi_d : X_1(Np) \longrightarrow X_1(N_f)$ for the standard degeneracy map induced by multiplication by d on the upper half plane. Let

$$\pi_{d*} : V_0(Np)[f_0] \longrightarrow V_f, \qquad V_f := V_0(N_f)[f_0] = H^1_{\mathrm{et}}(\bar{X}_0(N_f), \mathcal{O})(1)[f_0]$$

be the map in étale cohomology induced by π_d by covariant functoriality. Our choice of f in $S_2(Np)[f_0]$ stated in equation (12) determines the projection

$$\varpi_f := \sum \lambda_d \pi_{d*} : V_0(Np)\otimes \mathcal{O} \longrightarrow V_f. \tag{25}$$

Committing a slight abuse of notation, denote by ϖ_f the map induced by (25) by functoriality on Dieudonné modules and the first isomorphism in (24), and write ϖ_f^* for the dual map arising from Poincaré duality:

$$\varpi_f : D_0(Np) \longrightarrow D_f, \qquad \varpi_f^* : D_f \longrightarrow D_0(Np). \tag{26}$$

The maps $\varpi_g : D_s \longrightarrow D_g$ and $\varpi_g^* : D_g \longrightarrow D_s$ are defined similarly, and we let

$$\varpi_f^* \otimes \varpi_g^* : D_{fg} \longrightarrow D_0(Np)\otimes D_s$$

denote the map induced by ϖ_f^* and ϖ_g^* and the Künneth decomposition.

Define

$$\eta_f := \varpi_f^*(\eta_{f_0}) \in D_0(Np)[f], \qquad \omega_g := \varpi_g^*(\omega_{g_0}) \in D_s[g], \qquad \omega_{g^w} := w\omega_g \in D_s[g^*].$$

Note that η_f is ordinary, by the compatibility of ϖ_f^* with the U_p-operator. Since η_f belongs to $\mathrm{Fil}^{-1}(D_f) = D_f$ and ω_{g^w} belongs to $\mathrm{Fil}^0(D_{g^*}) \subset \Omega^1(X_s)$, the tensor product $\eta_f \otimes \omega_{g^w}$ belongs to $\mathrm{Fil}^{-1}(D_0(Np)[f]\otimes D_s[g^*])$. Because of the assumption on g leading to (20), the expression

$$\mathrm{reg}_p^{fg}(\mathbf{BF}_s)(\eta_f \otimes \omega_{g^w})$$

is defined. The main result of §2.5 is a formula for its value. As a preparation for that, we first recall in §2.4 a few needed facts on the rigid geometry of the modular curve $X_1(Np^s)$.

2.4. **The rigid cohomology of modular curves.** This section briefly summarises the description of the de Rham cohomology of X_s viewed as a curve over a p-adic field, following the treatment in [DR2, §3.1], with suitable modifications aimed at giving a similar description of the cohomology of the affine curve Y_s. We adopt the same notations as in loc.cit., which the reader is encouraged to consult for a more detailed description of the objects that are used here.

More specifically, let F be any extension of $\mathbb{Q}_p(\mu_{p^s})$ over which X_s admits a semistable model, and over which the closed points of $\Sigma := X_s - Y_s$ are defined.

The étale cohomology groups of X_s and Y_s are related by the Gysin exact sequence of semistable representations of $G_F = \mathrm{Gal}(\bar{F}/F)$:

$$0 \longrightarrow H^1_{\mathrm{et}}(\bar{X}_s, \mathbb{Q}_p) \longrightarrow H^1_{\mathrm{et}}(\bar{Y}_s, \mathbb{Q}_p) \longrightarrow \mathbb{Q}_p(-1)^t \longrightarrow 0, \tag{27}$$

where $t = |\Sigma| - 1$.

Let D_{st} be Fontaine's comparison functor attached to the ring of semistable periods $\mathbf{B}_{\mathrm{st}}$. By applying D_{st} to the sequence (27), and invoking the comparison theorem between étale and de Rham cohomology, one obtains a corresponding sequence of filtered Frobenius monodromy (FFM) modules

$$0 \longrightarrow H^1_{\mathrm{dR}}(X_{s/F}) \xrightarrow{\iota} H^1_{\mathrm{dR}}(Y_{s/F}) \longrightarrow F(-1)^t \longrightarrow 0. \tag{28}$$

(See for example [CI10], Proposition 7.6 for details.) Note that the above FFM modules are isomorphic as filtered Frobenius modules to those obtained by applying the construction of the potentially crystalline Dieudonné module of Section 2.3, in which $\mathbf{B}_{\mathrm{st}}$ is replaced by $\mathbf{B}_{\mathrm{cris}}$. The use of $\mathbf{B}_{\mathrm{st}}$ equips these modules with the additional structure of a monodromy operator.

In particular, it follows from (28) that the Frobenius operator Φ associated to Y_s acts on the subspace of $H^1_{\mathrm{dR}}(Y_{s/F})$ generated by classes of Eisenstein series with eigenvalues of absolute value p.

In order to make (28) more explicit, we recall, following [CI10], the Maier-Vietoris exact sequences describing the de Rham cohomology groups of X_s and Y_s. Fix a proper semistable model $\mathcal{X}_s$ of X_s over the ring of integers $\mathcal{O}_F$ of F, whose special fiber C is a union of smooth geometrically connected components $C_1, \ldots, C_n$ intersecting at ordinary double points defined over the residue field $\mathbb{F}$ of F. Since $s \geq 1$, there are at least two components, i.e., $n \geq 2$. Write $\mathcal{G}$ for the dual graph of C, and $\mathcal{V}$, resp. $\mathcal{E}$ for its set of vertices, resp. oriented edges. Recall that

$$\mathcal{V} = \{C_1, \ldots, C_n\}, \qquad \mathcal{E} = \{(x, C_i, C_j) : x \in C_i \cap C_j\}.$$

Given $e = (x, C_i, C_j)$ in $\mathcal{E}$, we say that $s(e) := C_i$ is the source of e and $t(e) := C_j$ is the target of e. Write

$$\mathrm{red}: \ X_s(\mathbb{C}_p) \longrightarrow C(\bar{\mathbb{F}})$$

for the reduction map, and identify $X_s(\mathbb{C}_p)$ with the rigid-analytic space over F attached to X_s. The wide-open space associated to $v \in \mathcal{V}$ is defined as $\mathcal{W}_v := \mathrm{red}^{-1}(v)$. Its underlying connected affinoid space $\mathcal{A}_v$ is obtained as the inverse image by reduction of the component v with the singular points removed. The wide-open annulus associated to $e = (x, v_i, v_j) \in \mathcal{E}$ is defined as $\mathcal{W}_e := \mathrm{red}^{-1}(x)$ (so that $\mathcal{W}_e = \mathcal{W}_{v_i} \cap \mathcal{W}_{v_j}$). The collection $\{\mathcal{W}_v\}_{v \in \mathcal{V}}$ forms an *admissible covering* of $X_s(\mathbb{C}_p)$ by wide open spaces. Likewise, an admissible covering of $Y_s(\mathbb{C}_p)$ is given by $\{\mathcal{W}'_v\}_{v \in \mathcal{V}}$, where $\mathcal{W}'_v := \mathcal{W}_v - \Sigma$. Note that

$$\mathcal{W}'_{v_i} \cap \mathcal{W}'_{v_j} = \mathcal{W}_{v_i} \cap \mathcal{W}_{v_j} = \mathcal{W}_e,$$

with $e = (x, v_i, v_j) \in \mathcal{E}$. Since $H^1_{\mathrm{dR}}(X_s/F)$ and $H^1_{\mathrm{dR}}(Y_{s/F})$ are identified with the Cech hypercohomology groups of their associated admissible coverings, they arise as the middle terms in the following commutative diagram originating from the Maier-Vietoris sequences attached to these coverings:

(29)

$$\begin{tikzcd}
0 \arrow[r] & \frac{\oplus_{e\in\mathcal{E}} H^0_{\mathrm{dR}}(\mathcal{W}_e)}{\delta(\oplus_{v\in\mathcal{V}} H^0_{\mathrm{dR}}(\mathcal{W}_v))} \arrow[r, "i"] \arrow[d, "="'] & H^1_{\mathrm{dR}}(X_s/F) \arrow[r, "r"] \arrow[d, "\iota"'] & \ker\left(\oplus_{v\in\mathcal{V}} H^1_{\mathrm{dR}}(\mathcal{W}_v) \xrightarrow{\delta} \oplus_{e\in\mathcal{E}} H^1_{\mathrm{dR}}(\mathcal{W}_e)\right) \arrow[r] \arrow[d, "\bar{\iota}"'] & 0 \\
0 \arrow[r] & \frac{\oplus_{e\in\mathcal{E}} H^0_{\mathrm{dR}}(\mathcal{W}_e)}{\delta(\oplus_{v\in\mathcal{V}} H^0_{\mathrm{dR}}(\mathcal{W}_v))} \arrow[r, "i'"] & H^1_{\mathrm{dR}}(Y_{s/F}) \arrow[r, "r'"] & \ker\left(\oplus_{v\in\mathcal{V}} H^1_{\mathrm{dR}}(\mathcal{W}'_v) \xrightarrow{\delta} \oplus_{e\in\mathcal{E}} H^1_{\mathrm{dR}}(\mathcal{W}_e)\right) \arrow[r] & 0
\end{tikzcd}$$

In (29), the various coboundary maps δ are defined by the rule $\delta\{\alpha_v\}(e) := \alpha_{t(e)}|_{\mathcal{W}_e} - \alpha_{s(e)}|_{\mathcal{W}_e}$, and we have used the fact that $H^0_{\mathrm{dR}}(\mathcal{W}_v) = H^0_{\mathrm{dR}}(\mathcal{W}'_v)$, since Σ is finite. Furthermore, the map i sends the class of a collection $\{\lambda_e\}$ of scalars with $\lambda_e \in \mathbb{C}_p$ to the class of the hypercocycle $(\{0_v\}, \{\lambda_e\})$, while the map r sends the class of the hypercocycle $(\{\omega_v\}, \{f_e\})$ to the class of $\{\omega_v\}$. Likewise for i' and r'.

2.5. **Analytic description of the syntomic regulator.** We now give an analytic description of the image under the p-adic syntomic regulator of the Beilinson-Flach element $\mathbf{BF}_s$ attached to the Siegel unit $\mathfrak{g}_{1,Np^s}$. Recall the modular forms $f \in S_2(Np)$ and the primitive form $g \in S_2(Np^s, \chi\chi_p)$ considered in §2.3, arising from newforms f_0 and g_0 of level N_f and $N_g p^s$, respectively.

Lemma 2.4. *We have the equality*

$$\mathrm{reg}_p^{fg}(\mathbf{BF}_s)(\eta_f \otimes \omega_{g^w}) = \mathrm{reg}_p^{fg}(\mathbf{BF}(\mathfrak{g}^w_{\chi^{-1}\chi_p^{-1}}))(\eta_f \otimes \omega_{g^w}).$$

Proof. This follows by noting that
$$\mathbf{BF}_s = \sum \mathbf{BF}(\mathfrak{g}_\psi^w),$$
where the sum is being taken over the even characters ψ of modulus Np^s. It follows from the equivariance of reg_p under the group of diamond operators acting functorially on both source and target that $\mathrm{reg}_p^{fg}(\mathbf{BF}(\mathfrak{g}_\psi^w))(\eta_f \otimes \omega_{g^w})$ is zero for all $\psi \neq \chi^{-1}\chi_p^{-1}$. □

In light of this lemma, let
$$\omega_{\mathcal{E}} := \mathcal{E}(q)\frac{dq}{q} = \frac{\varphi(Np^s)}{2}\cdot \mathrm{dlog}\mathfrak{g}_{\chi^{-1}\chi_p^{-1}} = \mathcal{G}(\chi\chi_p)E_2(\chi^{-1}\chi_p^{-1},1)\frac{dq}{q}, \qquad \mathcal{E}\in M_2(Np^s,\chi^{-1}\chi_p^{-1}) \tag{30}$$
be the differential of the third kind on $X_1(Np^s)$ and the associated Eisenstein series introduced in equation (7) of §2.1.

Choose a polynomial $P(x) \in F[x]$ such that the following conditions hold:

(1) $P(\Phi\times\Phi)$ annihilates the class of $\omega_{g^w}\otimes\omega_{\mathcal{E}^w}$ in $H^2_{\mathrm{dR}}(Y^2_{s\,/F})$,

(2) $P(\Phi)$ acts invertibly on $H^1_{\mathrm{dR}}(Y_{s/F})$.

The polynomial P exists because $\Phi\times\Phi$ acts with eigenvalues of complex absolute value $p^{3/2}$ on the subspace of $H^2_{\mathrm{dR}}(Y^2_{s\,/F})$ generated by the frobenius translates of classes of the form $\omega_{g^w}\otimes\omega_{\mathcal{E}^w}$, while Φ acts on $H^1_{\mathrm{dR}}(Y_{s/F})$ with eigenvalues of absolute value 1, $\sqrt{p}$ and p (as is deduced for example by analysing the Mayer-Vietoris sequence (29)).

Just as in [DR2, §3.2], condition (1) in the choice of P means that $P(\Phi\times\Phi)\,(\omega_{g^w}\otimes\omega_{\mathcal{E}^w})$ is exact and hence there exists a system of rigid one-forms
$$\rho_P(g^w,\mathcal{E}^w) = \{\rho_P^{v_1,v_2}(g^w,\mathcal{E}^w)\in\Omega^1_{\mathrm{rig}}(\mathcal{W}'_{v_1}\times\mathcal{W}'_{v_2})\}$$
indexed by pairs $(v_1,v_2)\in\mathcal{V}\times\mathcal{V}$, satisfying
$$d\rho_P^{v_1,v_2}(g^w,\mathcal{E}^w) = P(\Phi\times\Phi)\left(\omega_{g^w}\otimes\omega_{\mathcal{E}^w}\,|_{\mathcal{W}'_{v_1}\times\mathcal{W}'_{v_2}}\right). \tag{31}$$

Note that the choice of the system of one-forms $\rho_P^{v_1,v_2}(g^w,\mathcal{E}^w)$ is not unique, as it is well-defined only up to systems of closed rigid one-forms on $\mathcal{W}'_{v_1}\times\mathcal{W}'_{v_2}$. Nevertheless, we shall use this system to construct a canonical class in $H^1_{\mathrm{dR}}(X_s/F)$ associated to g^w and $\mathcal{E}^w$, independent of all the choices made.

For any $a\in(\mathbb{Z}/Np^s\mathbb{Z})^\times$, let $\langle a\rangle\in\mathrm{Aut}(Y_s)$ denote the diamond operator associated to a on Y_s, and for any pair (a,b) of such elements let $\langle a,b\rangle$ denote the automorphism of $Y_s\times Y_s$ given by the product of $\langle a\rangle$ and $\langle b\rangle$ acting on the first and second variable, respectively.

Fix an $a\in(\mathbb{Z}/Np^s\mathbb{Z})^\times$ such that $\chi\chi_p(a)\neq 1$ for the rest of the section, and define the linear homomorphism
$$\delta_a^*:\oplus_{v_1,v_2}\Omega^1_{\mathrm{rig}}(\mathcal{W}'_{v_1}\times\mathcal{W}'_{v_2})\longrightarrow\oplus_v\Omega^1_{\mathrm{rig}}(\mathcal{W}'_v),\quad \delta_a^* = \mathrm{diag}^*\circ(\mathrm{Id}-\langle a,1\rangle^*)\circ(\mathrm{Id}-\langle 1,a\rangle^*) \tag{32}$$
where diag^* denotes the map induced by pull-back under the diagonal embedding $\mathcal{W}'_v\longrightarrow\mathcal{W}'_v\times\mathcal{W}'_v$.

Lemma 2.5. *The map δ_a^* sends collections of closed 1-forms to collections of exact 1-forms.*

Proof. By the Künneth formula, there is a decomposition
$$H^1_{\mathrm{dR}}(\mathcal{W}'_v\times\mathcal{W}'_v)\simeq(H^0\otimes H^1)\,\oplus\,(H^1\otimes H^0),$$
where we use the shorthands $H^0 = H^0_{\mathrm{dR}}(\mathcal{W}'_v)$ and $H^1 = H^1_{\mathrm{dR}}(\mathcal{W}'_v)$. Since the endomorphism $\mathrm{Id}-\langle a,1\rangle^*$ vanishes on the first factor, and $\mathrm{Id}-\langle 1,a\rangle^*$ vanishes on the second, the claim follows. □

Corollary 2.6. *The image $\Xi_{P,a}(g^w,\mathcal{E}^w)$ of the system of closed rigid 1-forms $\delta_a^*(\rho_P(g^w,\mathcal{E}^w))$ in $\oplus_v H^1_{\mathrm{dR}}(\mathcal{W}'_v)$ does not depend on the choice of solution of the differential equation (31).*

As the notation indicates, the class $\Xi_{P,a}(g^w,\mathcal{E}^w)$ does depend on both our choices of P and a. In order to eliminate this ambiguity, define the class
$$\Xi_P(g^w,\mathcal{E}^w) := (1-\chi\chi_p(a))^{-1}(1-\chi^{-1}\chi_p^{-1}(a))^{-1}\Xi_{P,a}(g^w,\mathcal{E}^w)\in\oplus_{v\in\mathcal{V}}H^1_{\mathrm{dR}}(\mathcal{W}'_v). \tag{33}$$
This class still depends on P, but is independent of the choice of a as above. Indeed, this follows from the very definition of $\Xi_P(g^w,\mathcal{E}^w)$ and the fact that $\langle a\rangle$ acts on the g^w-isotypic (resp. $\mathcal{E}^w$-isotypic) component of $H^1_{\mathrm{dR}}(Y_{s/F})$ as multiplication by $\chi^{-1}\chi_p^{-1}(a)$ (resp. $\chi\chi_p(a)$).

Following [DR2, Def. 3.1], define the subspace $H^1_{\mathrm{dR}}(X_{s/F})^{(1)}$ of *pure classes of weight one* of $H^1_{\mathrm{dR}}(X_{s/F})$ by the condition that Φ acts via eigenvalues of complex absolute value $\sqrt{p}$ on its elements. Likewise, define the subspace $\oplus_v H^1_{\mathrm{dR}}(\mathcal{W}'_v)^{(1)}$ of $\oplus_v H^1_{\mathrm{dR}}(\mathcal{W}'_v)$ by the same condition. Weight considerations show that $\oplus_v H^1_{\mathrm{dR}}(\mathcal{W}'_v)^{(1)}$ is contained in the kernel of the map δ appearing in the commutative diagram (29), and that the map $r' \circ \iota$ defined in the same diagram induces an isomorphism between the above spaces of pure weight one classes. Denote by

$$\mathrm{spl}_X : \oplus_v H^1_{\mathrm{dR}}(\mathcal{W}'_v)^{(1)} \longrightarrow H^1_{\mathrm{dR}}(X_{s/F})^{(1)}$$

the (Frobenius-equivariant) inverse isomorphism. In light of condition (2) in our choice of P, define the class

$$\Xi(g^w, \mathcal{E}^w) := P(\Phi)^{-1}\mathrm{spl}_X\left(\Xi_P(g^w, \mathcal{E}^w)\right) \in H^1_{\mathrm{dR}}(X_s/F)^{(1)}. \tag{34}$$

It is immediate to check that $\Xi(g^w, \mathcal{E}^w)$ not depend on our choice of P.

Let π_s and ϖ_s be the natural degeneracy maps arising in the diagram

$$\tilde{\pi}_s : X_s \xrightarrow{\pi_s} X_0(Np^s) \xrightarrow{\varpi_s} X_0(Np),$$

where $\tilde{\pi}_s$ is the morphism appearing in equation (8). The degeneracy map ϖ_s that we choose is the unique one which is compatible with the U_p operators in level Np^s and Np. Let

$$\eta_{f,s} := \varpi_s^* \eta_f, \qquad \tilde{\eta}_{f,s} := \pi_s^* \eta_{f,s}.$$

The class $\eta_{f,s}$ is ordinary, by the ordinariness of η_f combined with the compatibility of ϖ_s with U_p.

Proposition 2.7. *The equality*

$$\mathrm{reg}_p^{fg}(\mathbf{BF}_s)(\eta_f \otimes \omega_{g^w}) = \frac{2}{\varphi(Np^s)} \langle \tilde{\eta}_{f,s}, \Xi(g^w, \mathcal{E}^w)\rangle_{X_s} \tag{35}$$

holds.

Proof. This formula follows by applying the same argument as in the proof of [BDR12, Proposition 3.3], with the following difference. While in loc. cit. we work with the modular curve $X_1(N)$ which admits a smooth model over $\mathbb{Z}_p$, here the modular curve X_s has bad reduction at p and we are bound to work with a semistable model of this curve. Besser's theory of finite polynomial cohomology has been extended recently to the setting of semistable varieties by Besser, Loeffler and Zerbes in [BLZ], building on previous work of Nekovar and Niziol [NeNi]. The proof of [BDR12, Proposition 3.3] applies verbatim after replacing Besser's [Bes, Prop. 6.3] with [BLZ, Proposition 3.12], taking into account that

$$\omega_{\mathcal{E}^w} = \frac{\varphi(Np^s)}{2} \cdot d\log(\mathfrak{g}^w_{\chi^{-1}\chi_p^{-1}}).$$

□

The fact that the nebentype characters of g^w and $\mathcal{E}^w$ are inverse one of another implies that the class $\Xi(g^w, \mathcal{E}^w)$ arises as the pull-back of a class – denoted with the same symbol by abuse of notation – in the cohomology of $X_0(Np^s)$ under π_s.

Proposition 2.8. *For all $s \geq 1$,*

$$\mathrm{reg}_p^{fg}(\mathbf{BF}_s)(\eta_f \otimes \omega_{g^w}) = \langle \eta_{f,s}, \Xi(g^w, \mathcal{E}^w)\rangle_{X_0(Np^s)}. \tag{36}$$

Proof. Because the degree of π_s is $\frac{1}{2}\varphi(Np^s)$, and the pullback and pushforward maps are adjoint to each other relative to the Poincaré pairing,

$$\langle \tilde{\eta}_{f,s}, \Xi(g^w, \mathcal{E}^w)\rangle_{X_s} = \frac{1}{2}\varphi(Np^s)\langle \eta_{f,s}, \Xi(g^w, \mathcal{E}^w)\rangle_{X_0(Np^s)}. \tag{37}$$

The result follows. □

It will be convenient to consider the anti-ordinary counterparts of the classes η_f and $\eta_{f,s}$:

$$\eta_f^w := w\eta_f, \qquad \eta_{f,s}^w := w\eta_{f,s},$$

where we recall that w refers in the above equations to the Atkin-Lehner involutions w_{Np} and w_{Np^s} of levels Np and Np^s respectively.

Let $\{\mathcal{W}_\infty, \mathcal{W}_0\}$ denote the standard admissible covering of $X_0(Np)$ by wide-open neighbourhoods of the Hasse domain of $X_0(N)$, as described by Coleman in [Co94]. By construction, arguing as in [DR2, Lemma 4.6], the ordinary unit root class $\eta_f \in H^1_{\mathrm{dR}}(X_0(Np))$ is supported on $\mathcal{W}_0$, and has vanishing residues at the supersingular annuli, while the anti-ordinary unit root class η_f^w is supported on $\mathcal{W}_\infty$.

Let $U_p^w = wU_pw$ be the adjoint operator of U_p with respect to the Poincaré pairing on $X_0(Np^s)$, and let

$$e_{\mathrm{ord}} = \lim U_p^{n!} \quad \text{and} \quad e_{\mathrm{ord}}^w = \lim (U_p^w)^{n!} = w e_{\mathrm{ord}} w$$

denote Hida's ordinary and anti-ordinary projectors, respectively. Pull-back under ϖ_s gives rise to an isomorphism between the ordinary subspaces of $H^1_{\mathrm{dR}}(X_0(Np^s))$ and $H^1_{\mathrm{dR}}(X_0(Np))$. Hence $e_{\mathrm{ord}}\Xi(g,\mathcal{E})$ arises as the pull-back under ϖ_s of some ordinary class in $H^1_{\mathrm{dR}}(X_0(Np))$, that we again denote with the same symbol. In the context of the next theorem, $e_{\mathrm{ord}}\Xi(g,\mathcal{E})$ is to be understood in the latter sense. Since $e_{\mathrm{ord}}\Xi(g,\mathcal{E})$ belongs to the subspace of the cohomology which is pure of weight one, its restriction to the wide open $\mathcal{W}_\infty$ has vanishing residues along the supersingular annuli. Therefore, it also makes sense to pair it with the (restriction of) the anti-ordinary class η_f^w.

Theorem 2.9. *For all $s \geq 1$,*

$$\mathrm{reg}_p^{fg}(\mathbf{BF}_s)(\eta_f \otimes \omega_{g^w}) = \alpha_f^{s-1} \cdot \langle \eta_f^w, e_{\mathrm{ord}}\Xi(g,\mathcal{E})\rangle_{\mathcal{W}_\infty}.$$

Proof. Since the class $\eta_{f,s}$ is ordinary, it is fixed by e_{ord} and thus

$$\begin{aligned} \langle \eta_{f,s}, \Xi(g^w,\mathcal{E}^w)\rangle_{X_0(Np^s)} &= \langle e_{\mathrm{ord}}\eta_{f,s}, \Xi(g^w,\mathcal{E}^w)\rangle_{X_0(Np^s)} \\ &= \langle \eta_{f,s}, e_{\mathrm{ord}}^w\Xi(g^w,\mathcal{E}^w)\rangle_{X_0(Np^s)} \\ &= \langle \eta_{f,s}^w, e_{\mathrm{ord}}\,\Xi(g,\mathcal{E})\rangle_{X_0(Np^s)}, \end{aligned} \tag{38}$$

where the self-adjointness of the involution w has been exploited to derive the last equation.

But since $e_{\mathrm{ord}}\Xi(g,\mathcal{E})$ arises as a pullback under ϖ_s, we can write

$$\begin{aligned} \langle \eta_{f,s}^w, e_{\mathrm{ord}}\,\Xi(g,\mathcal{E})\rangle_{X_0(Np^s)} &= \langle \varpi_s(\eta_{f,s}^w), e_{\mathrm{ord}}\,\Xi(g,\mathcal{E})\rangle_{X_0(Np)} \\ &= \langle \varpi_s w \varpi_s^*(\eta_f), e_{\mathrm{ord}}\,\Xi(g,\mathcal{E})\rangle_{X_0(Np)} \\ &= \langle w \circ U_p^{s-1}(\eta_f), e_{\mathrm{ord}}\,\Xi(g,\mathcal{E})\rangle_{X_0(Np)} \\ &= \alpha_f^{s-1}\langle \eta_f^w, e_{\mathrm{ord}}\,\Xi(g,\mathcal{E})\rangle_{\mathcal{W}_\infty}, \end{aligned} \tag{39}$$

where the last identity follows from the fact that the anti-ordinary unit root class η_f^w is supported on $\mathcal{W}_\infty$. The theorem now follows from Proposition 2.8 combined with (38) and (39). ☐

We are now in a position to state the main result of this section, namely, the evaluation of the p-adic syntomic regulator attached to $\mathbf{BF}_s$ in terms of the q-expansions of concrete (overconvergent) p-adic modular forms. For a p-adic modular form $\varphi = \sum a_n(\varphi)q^n$, let

$$\varphi^{[p]} = \sum_{p\nmid n} a_n(\varphi)q^n$$

denote its "p-deprived" counterpart.

The Atkin-Serre operator $d = q\frac{d}{dq}$ sends p-adic overconvergent modular forms of weight 0 to overconvergent modular forms of weight two. The image of the p-depleted classical weight two Eisenstein series $E_2^{[p]}(\chi^{-1}\chi_p^{-1},1)$ under d^{-1} is the p-adic overconvergent Eisenstein series of weight 0:

$$d^{-1}E_2^{[p]}(\chi^{-1}\chi_p^{-1},1) = E_0^{[p]}(1,\chi^{-1}\chi_p^{-1}).$$

The product $g^{[p]} \times E_0^{[p]}(1,\chi^{-1}\chi_p^{-1})$ is therefore an overconvergent cusp form of weight two with trivial nebentypus character, and hence its image under the ordinary projection e_{ord} corresponds to a classical weight two cusp form on $\Gamma_0(Np)$, which can be viewed as an element of the first de Rham cohomology of $X_0(Np)$. The inner product $\langle\ ,\ \rangle_{X_0(N)}$ in the theorem below denotes the usual Poincaré duality on this de Rham cohomology group.

Theorem 2.10. *For all $s \geq 1$,*

$$\mathrm{reg}_p^{fg}(\mathbf{BF}_s)(\eta_f \otimes \omega_{g^w}) = \mathcal{G}(\chi\chi_p) \cdot \frac{\alpha_f^{s-1}}{1-\chi^{-1}(p)\alpha_f^{-1}\alpha_g} \cdot \langle \eta_f^w, e_{\mathrm{ord}}(g^{[p]} \cdot E_0^{[p]}(1,\chi^{-1}\chi_p^{-1}))\rangle_{X_0(Np)}.$$

Proof. The restriction to $\mathcal{W}_\infty$ of $e_{\rm ord}\Xi(g,\mathcal{E})$ is the cohomology class of a specific overconvergent weight two p-adic modular form representing a rigid analytic differential form on $\mathcal{W}_\infty$ with vanishing annular residues. The class $e_{\rm ord}\Xi(g,\mathcal{E})$ is defined by exactly the same analytic recipe as the class denoted $e_{\rm ord}\xi(\omega_{\breve{g}},\omega_{\breve{h}})$ in equation (85) of [DR2], after replacing $\breve{g}$ by g and setting

$$\breve{h} = w_{p^s}\mathcal{E} \quad (= \mathcal{G}(\chi)E_2(\chi^{-1},\chi_p)).$$

In [DR2], the form $\breve{h}$ was assumed to be cuspidal while $w_{p^s}\mathcal{E}$ is an Eisenstein series, but the calculations of loc.cit. remain valid in the latter setting. In particular, the expression

$$w_{p^s}\breve{h} = \mathcal{G}(\chi_p^{-1})\alpha_h^{-s}\omega_{\breve{h}^\iota}$$

that appears in the statement of Prop. 4.13 of [DR2] can be replaced by $\omega_{\mathcal{E}}$, yielding

$$e_{\rm ord}\Xi_P(g,\mathcal{E}) = e_{\rm ord}\xi_P(\omega_g,\omega_{w_{p^s}\mathcal{E}}) = -2e_{\rm ord}(g\times d^{-1}\mathcal{E}^{[p]}) \tag{40}$$

$$= -2\mathcal{G}(\chi\chi_p)e_{\rm ord}(g\times E_0^{[p]}(1,\chi^{-1}\chi_p^{-1})), \tag{41}$$

where $P(x)$ is the linear polynomial $2(1-p^{-1}\chi^{-1}(p)\alpha_g x)$, and $\Xi_P(g,\mathcal{E})$ is defined as in (33). Combining equation (40) with Theorem 2.9, we obtain

$$\mathrm{reg}_p^{fg}(\mathbf{BF}_s)(\eta_f\otimes\omega_{g^w}) = -2\mathcal{G}(\chi\chi_p)\alpha_f^{s-1}\cdot\langle\eta_f^w, P(\Phi)^{-1}e_{\rm ord}(g\cdot E_0^{[p]}(1,\chi^{-1}\chi_p^{-1}))\rangle_{\mathcal{W}_\infty}. \tag{42}$$

Note that g can be replaced by $g^{[p]}$ in equation (42), as a direct calculation shows that

$$(g^{[p]}-g)\cdot E_0^{[p]}(1,\chi^{-1}\chi_p^{-1})$$

belongs to the kernel of U_p, and a fortiori of $e_{\rm ord}$. (See for instance Lemma 2.17 of [DR1].) Finally, for any class Ξ in $H^1_{\rm dR}(\mathcal{W}_\infty)$, we have

$$\langle\eta_f^w,\Phi\Xi\rangle_{\mathcal{W}_\infty} = \alpha_f^{-1}\langle\Phi\eta_f^w,\Phi\Xi\rangle_{\mathcal{W}_\infty} = p\alpha_f^{-1}\langle\eta_f^w,\Xi\rangle_{\mathcal{W}_\infty} = \beta_f\langle\eta_f^w,\Xi\rangle_{\mathcal{W}_\infty},$$

which in turn implies

$$\langle\eta_f^w, P(\Phi)^{-1}\Xi\rangle_{\mathcal{W}_\infty} = P(\beta_f)^{-1}\langle\eta_f^w,\Xi\rangle_{\mathcal{W}_\infty}. \tag{43}$$

In light of the explicit description of $P(x)$ given above, combined with equations (42) and (43),

$$\mathrm{reg}_p^{fg}(\mathbf{BF}_s)(\eta_f\otimes\omega_{g^w}) = \mathcal{G}(\chi\chi_p)\cdot\frac{\alpha_f^{s-1}}{1-\chi^{-1}(p)\alpha_f^{-1}\alpha_g}\cdot\langle\eta_f^w, e_{\rm ord}(g^{[p]}\cdot E_0^{[p]}(1,\chi^{-1}\chi_p^{-1}))\rangle_{\mathcal{W}_\infty}.$$

Proceeding just as in the proof of Theorem 2.9, Theorem 2.10 follows from the compatibility between the de Rham pairings on $\mathcal{W}_\infty\subset X_0(Np)$ and on $X_0(Np)$, using the fact that η_f^w is anti-ordinary and supported on $\mathcal{W}_\infty$. □

2.6. **Hida's p-adic L-function.** A point of the rigid analytic space

$$\Omega := \mathrm{Hom}_{\rm cts}(\mathbb{Z}_p^\times,\mathbb{C}_p^\times)$$

is said to be *arithmetic* if it is of the form

$$\nu_{k,\epsilon}(x) := x^{k-1}\epsilon(x), \qquad (\text{for all } x\in\mathbb{Z}_p^\times),$$

where $k\geq 2$ is an integer, and ϵ is a finite order character factoring through $(\mathbb{Z}/p^s\mathbb{Z})^\times$. The space Ω is referred to as the *weight space* and plays a basic role in the theory of p-adic familes of modular forms.

A *Hida family* of tame character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}_p^\times$ is a pair $(\Omega_\varphi,\underline{\varphi})$, where

(1) Ω_φ is a rigid analytic space equipped with a finite morphism

$$\mathrm{wt} : \Omega_\varphi \longrightarrow \Omega,$$

(2) $\underline{\varphi} = \sum_{n=1}^\infty a_n(\underline{\varphi})q^n$ is a formal q-series with coefficients in the ring $\Lambda_\varphi := \mathcal{A}(\Omega_\varphi)$ of rigid analytic functions on Ω_φ,

such that, for all points $x \in \Omega_\varphi$ of arithmetic weight $\mathrm{wt}(x) = \nu_{k,\epsilon}$, the specialisation

$$\varphi_x := \sum_{n=1}^{\infty} a_n(\underline{\varphi})(x) q^n$$

is the q-series of a normalised eigenform of weight k, level Np^s and character $\chi_{\varphi_x} = \chi\epsilon\omega^{1-k}$, where ω is the Teichmüller character of conductor p.

Let g be the weight one modular form considered in Section 1, and let g_α be one of its ordinary p-stabilisations. A theorem of Hida ensures the existence of a Hida family, denoted $\underline{g}$, of tame character χ which specialises to g_α at a suitable point $y_0 \in \Omega_{\underline{g}}$ lying above $\nu_{1,1}$. For any arithmetic point $y \in \Omega_{\underline{g}}$ above $\nu_{\ell,\epsilon}$, for some $\ell \geq 1$ and some character ϵ of p-power conductor, denote by $g_y \in S_\ell(Np^s, \chi\epsilon\omega^{1-\ell})$ the specialisation of $\underline{g}$ at y.

Write $\underline{E}(1, \chi^{-1})$ for the ordinary Hida family of Eisenstein series, whose specialisation at the point $\nu_{\ell,\epsilon}$ in weight space is equal to the classical Eisenstein series

$$\underline{E}(1, \chi^{-1})_{\nu_{\ell,\epsilon}} = E_\ell(1, \chi^{-1}\epsilon\omega^{1-\ell})$$

of weight ℓ and character $\chi^{-1}\epsilon\omega^{1-\ell}$. The Iwasawa algebra admits an involution inducing the map $\nu_{\ell,\epsilon} \mapsto \nu_{2-\ell,\epsilon^{-1}}$ on weight space. Applying the change of scalars to $\underline{E}(1, \chi^{-1})$ via this involution yields a Λ-adic modular form , denoted $\underline{E}^*(1, \chi^{-1})$, whose specialisation at the arithmetic point $\nu_{\ell,\epsilon}$ is the overconvergent modular form

$$\underline{E}^*(1, \chi^{-1})_{\nu_{\ell,\epsilon}} = E_{2-\ell}(1, \chi^{-1}\epsilon^{-1}\omega^{\ell-1}).$$

The expression $g_y^{[p]} \cdot E_0^{[p]}(1, \chi^{-1}\chi_p^{-1})$ appearing in Theorem 2.10 is the specialisation at the point y of weight $\nu_{2,\chi_p\omega}$ of the Λ-adic modular form

$$\underline{g}\underline{E}^{*[p]}(1, \chi^{-1}),$$

whose specialisation at a point y of weight $\nu_{\ell,\epsilon}$ is given by

$$\left\{\underline{g}^{[p]} \cdot \underline{E}^{*[p]}(1, \chi^{-1})\right\}_x = g_y^{[p]} \cdot E_{2-\ell}^{[p]}(1, \chi^{-1}\epsilon^{-1}\omega^{\ell-1}). \tag{44}$$

Note that for all arithmetic y, this specialisation is an overconvergent modular form of weight 2 and trivial character, whose ordinary projection is therefore classical of level Np; hence,

$$e_{\mathrm{ord}}(\underline{g}^{[p]} \cdot \underline{E}^{*[p]}(1, \chi^{-1}) \text{ belongs to } S_2(\Gamma_0(Np), \mathbb{Q}_p) \otimes \Lambda_{\underline{g}}.$$

The reader is referred to Section 4.5 of [DR2] for details on dual families of modular forms, which motivate the definition of the Λ-adic modular form of equation (44).

Let $L_p(\underline{f}, \underline{g})(x, y, j)$ be the 3-variable Rankin p-adic L-function on $\Omega_{\underline{f}} \times \Omega_{\underline{g}} \times \Omega$ introduced in [BDR12, §2.2], where $\underline{f}$ is the Hida family passing through f. It interpolates the algebraic part of the critical values $L(f_x \otimes g_y, j)$, where f_x and g_y are classical specialisations of $\underline{f}$ and $\underline{g}$ whose weights k and ℓ satisfy

$$k > \ell, \qquad \frac{k+\ell-1}{2} \leq j \leq k-1. \tag{45}$$

Consider the restriction of $L_p(\underline{f}, \underline{g})(x, y, j)$ to the set of points $(x_0, y, \nu_{\ell,\epsilon})$, where x_0 is such that $\underline{f}_{x_0} = f$, and $y \in \Omega_{\underline{g}}$ is a point lying above $\nu_{\ell,\epsilon}$. In the notations of [BDR12, §2.2.1], this amounts to considering only the values of $L_p(\underline{f}, \underline{g})(k, \ell, j)$ when $k = 2$, $j = \ell$, $t = 1 - \ell$ and $m = \ell$. In light of equations (27) and (24) in loc.cit., combined with the equality

$$d^{1-\ell} E_\ell^{[p]}(\chi^{-1}\epsilon^{-1}\omega^{\ell-1}, 1) = E_{2-\ell}^{[p]}(1, \chi^{-1}\epsilon^{-1}\omega^{\ell-1}),$$

this restriction agrees, up to a non-zero fudge factor which depends only on the fixed form f, with the (p-adic analytic) function described for all such y by

$$L_p(f, \underline{g})(y) := \langle \eta_f^w, e_{\mathrm{ord}}(g_y^{[p]} \cdot E_{2-\ell}^{[p]}(1, \chi^{-1}\epsilon^{-1}\omega^{\ell-1})) \rangle_{X_0(Np)}. \tag{46}$$

Note that the points of the form $(x_0, y, \nu_{\ell,\epsilon})$ lie outside the range of classical interpolation for $L_p(\underline{f}, \underline{g})$, unless g_y is classical and of weight one. The next result relates the values of $L_p(f, \underline{g})$ at arithmetic points y of weight $\nu_{2,\epsilon}$ to the p-adic regulators of Beilinson-Flach elements:

Theorem 2.11. *Let ϵ be a character of conductor p^s, let $y \in \Omega_{\underline{g}}$ be a point over $\nu_{2,\epsilon}$ and let $g = g_y \in S_2(Np^s, \chi\epsilon\omega^{-1})$ denote the specialization of $\underline{g}$ at y. Then*

$$\mathrm{reg}_p^{fg}(\mathbf{BF}_s)(\eta_f \otimes \omega_{g^w}) = \mathcal{G}(\chi\epsilon\omega^{-1}) \cdot \frac{\alpha_f^{s-1}}{1 - \chi(p)^{-1}\alpha_f^{-1}\alpha_g} \cdot L_p(f, \underline{g})(y).$$

Proof. This follows directly by combining equation (46) with Theorem 2.10. □

Another important property of $L_p(f, \underline{g})$ pertains to its specialisations at classical weight one points attached to the Hida family $\underline{g}$.

Theorem 2.12. *Let $y \in \Omega_{\underline{g}}$ be a point over $\nu_{1,1}$, corresponding to a classical p-stabilised weight one form $g = g_y \in S_1(Np, \chi)$. Then*

$$L_p(f, \underline{g})(y) \neq 0 \quad \Leftrightarrow \quad L(f \otimes g, 1) \neq 0.$$

Proof. By equation (46),

$$\begin{aligned} L_p(f, \underline{g})(y) &= \langle \eta_f^w, e_{\mathrm{ord}}\left(g_y^{[p]} \cdot E_1^{[p]}(1, \chi^{-1})\right)\rangle_{X_0(Np)} \\ &= \langle \eta_f^w, g_y^{[p]} \cdot E_1^{[p]}(1, \chi^{-1})\rangle_{X_0(Np^2)} \pmod{\bar{\mathbb{Q}}^\times}, \end{aligned} \tag{47}$$

where the fact that $g_y^{[p]} E_1^{[p]}(1, \chi^{-1})$ is a classical modular form of level Np^2 has been used to deduce the second equality. The expression in (47) is an explicit non-zero multiple of the value of the classical complex Rankin-Selberg L-function $L(f \otimes g, s)$ of the convolution of f and g at $s = 1$. (See equation (27) and the one immediately following it in [BDR12, §2.2.2].) The theorem follows. □

3. Λ-adic classes

3.1. **Norm-compatible systems of elements.** For all $s \geq 0$, the natural U_p-compatible projections

$$\pi_{s+1,s} : X_{s+1} \to X_s \tag{48}$$

of modular curves give rise to maps

$$\pi_{s+1,s} : S_{s+1} \longrightarrow S_s \tag{49}$$

on the associated surfaces, that we continue to denote with the same symbol by a slight abuse of notation. Write also $\pi_{s+1,s} : \mathcal{O}_{X_{s+1}}^\times \longrightarrow \mathcal{O}_{X_s}^\times$ for the norm maps on units induced by $\pi_{s+1,s}$.

Proposition 3.1. *For all $s \geq 1$,*

$$\pi_{s+1,s}(\mathfrak{g}_{a;Np^{s+1}}^w) = \mathfrak{g}_{a;Np^s}^w.$$

Proof. Lemma 2.12 of [Ka98] implies that the units $g_{a;Np^{s+1}}$ are compatible relative to the pushforward via the maps $\pi'_{s+1,s} : X_{s+1} \longrightarrow X_s$ that are compatible with the operator U'_p. This compatibility of the units $\mathfrak{g}_{a;Np^s}$ relative to the $\pi'_{s+1,s} := w_{p^s}\pi_{s+1,s}w_{p^{s+1}}$ implies the $\pi_{s+1,s}$-compatiblity of the units $\mathfrak{g}_{a;Np^s}^w$. See also §2.11 and §2.13 of [Ka98], or Thm. 2.24 of [LLZ1], for a convenient summary of these and further calculations in the more general setting treated in [Ka98]. (Recall that the unit $\mathfrak{g}_{a;Np^s}$ is denoted $g_{0,\frac{a}{Np^s}}$ in [Ka98] and [LLZ1].) □

Write

$$\pi_{s+1,s} : \mathrm{CH}^2(S_{s+1}, 1) \to \mathrm{CH}^2(S_s, 1)$$

for the norm maps on higher Chow groups induced by push-forward under the maps $\pi_{s+1,s}$ of (49). Note that $\pi_{s+1,s}$ preserves the subspaces of negligible classes, and hence descends to a well-defined map

$$\pi_{s+1,s} : \mathrm{CH}^2_{\mathrm{neg}}(S_{s+1}, 1) \to \mathrm{CH}^2_{\mathrm{neg}}(S_s, 1). \tag{50}$$

The norm compatibilities of the units $\mathfrak{g}_{a;Np^s}$ are inherited by the associated Beilinson-Flach elements:

Proposition 3.2. *For all $s \geq 1$,*

$$\pi_{s+1,s}(\mathbf{BF}(a; Np^{s+1})) = \mathbf{BF}(a; Np^s), \qquad \pi_{s+1,s}(\mathbf{BF}_{s+1}) = \mathbf{BF}_s \quad \textit{in} \quad \mathrm{CH}^2_{\mathrm{neg}}(S_s, 1)(\mathbb{Q}). \tag{51}$$

Proof. This follows directly from Proposition 3.1 in light of the fact that the map $\pi_{s+1,s}$ of (49) maps the diagonally embedded $X_{s+1} \simeq \Delta_{s+1} \subset S_{s+1}$ to Δ_s, and is identified with the projection $\pi_{s+1,s}$ of (48). $\square$

Corollary 3.3. *The Beilinson-Flach cohomology classes κ_s of Definition 2.3 are compatible under the norm maps, i.e.,*

$$\pi_{s+1,s}(\kappa_{s+1}) = \kappa_s.$$

In particular, the classes κ_s can be packaged together into the inverse limit class

$$\kappa_\infty := (\kappa_s)_{s\geq 1} \in H^1(\mathbb{Q}, V_0(Np) \otimes \mathbb{V}_\infty),$$

where

$$\mathbb{V}_\infty := \lim_{\leftarrow, s} V_s.$$

3.2. **The Λ-adic Beilinson-Flach cohomology class.** The module $\mathbb{V}_\infty$ of the previous section becomes more manageable when replaced by its image under the ordinary projection. Let

$$V_s^{\mathrm{ord}} := e_{\mathrm{ord}} V_s, \qquad \mathbb{V}_\infty^{\mathrm{ord}} := e_{\mathrm{ord}} \mathbb{V}_\infty = \lim_{\leftarrow, s} e_{\mathrm{ord}} V_s.$$

Note that the action of the group D_s of diamond operators on X_s endows the $G_{\mathbb{Q}}$-module $\mathbb{V}_\infty^{\mathrm{ord}}$ with a natural structure of module over the Iwasawa algebra $\tilde{\Lambda} = \mathbb{Z}_p[[D_\infty]] := \varprojlim \mathbb{Z}_p[D_s]$. A theorem of Hida asserts that this module is finitely generated and locally free over this algebra. Its Hecke eigenspaces realise the Λ-adic representations attached to ordinary families of eigenforms.

The Λ-adic Beilinson-Flach cohomology class is defined by setting

$$\begin{aligned} \kappa_s^{\mathrm{ord}} &:= e_{\mathrm{ord}} \kappa_s \in H^1(\mathbb{Q}, V_0(Np) \otimes V_s^{\mathrm{ord}}), \\ \kappa_\infty^{\mathrm{ord}} &:= e_{\mathrm{ord}} \kappa_\infty = (\kappa_s^{\mathrm{ord}})_{s\geq 1} \in H^1(\mathbb{Q}, V_0(Np) \otimes \mathbb{V}_\infty^{\mathrm{ord}}). \end{aligned}$$

Recall the cusp form $f = q + \sum_{n\geq 2} a_n(f) q^n \in S_2(Np)$ introduced in §2.3, on which U_p acts with eigenvalue α_f. Define

$$\psi_f : G_{\mathbb{Q}_p} \longrightarrow \mathcal{O}^\times$$

to be the unramified character of $G_{\mathbb{Q}_p}$ such that $\psi_f(\mathrm{Fr}_p) = \alpha_f$ and let $\epsilon_{\mathrm{cyc}} : G_{\mathbb{Q}_p} \longrightarrow \mathbb{Z}_p^\times$ denote the cyclotomic character. There is an exact sequence of $G_{\mathbb{Q}_p}$-modules

$$0 \to V_f^+ \to V_f \to V_f^- \to 0 \qquad \text{with } V_f^+ \simeq \mathcal{O}(\psi_f^{-1}\epsilon_{\mathrm{cyc}}) \text{ and } V_f^- \simeq \mathcal{O}(\psi_f) \tag{52}$$

as $\mathcal{O}[G_{\mathbb{Q}_p}]$-modules.

Let

$$\underline{\epsilon}_{\mathrm{cyc}} : G_{\mathbb{Q}_p} \longrightarrow \Lambda^\times$$

denote the Λ-adic cyclotomic character satisfying the interpolation property

$$\nu_{\ell,\epsilon} \circ \underline{\epsilon}_{\mathrm{cyc}} = \epsilon \epsilon_{\mathrm{cyc}}^{\ell-1} \omega^{1-\ell}$$

for any $\ell \geq 1$ and for any Dirichlet character ϵ of p-power conductor.

Let $\underline{g} = \sum_{n\geq 1} \mathbf{a}_n(\underline{g}) q^n$ be a Λ-adic cuspidal eigenform of tame level N and tame character χ, arising from a Λ-adic newform of level $N_{\underline{g}}$, and let $\Lambda_{\underline{g}}$ denote the finite flat extension of the Iwasawa algebra $\Lambda := \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$ generated by the coefficients $\mathbf{a}_n(\underline{g})$. Hida and Wiles associated to $\underline{g}$ a two-dimensional Galois representation $\mathbb{V}_{\underline{g}}$ over $\Lambda_{\underline{g}}$, again characterized up to isomorphism by the property that the characteristic polynomial of Fr_ℓ is $T^2 - \mathbf{a}_\ell(\underline{g})T + \underline{\epsilon}_{\mathrm{cyc}}(\ell)$ for primes $\ell \nmid Np$.

Wiles [Wi88] proved a Λ-adic analogue of (52), which asserts that (cf. also [Oh00])

$$0 \to \mathbb{V}_{\underline{g}}^+ \to \mathbb{V}_{\underline{g}} \to \mathbb{V}_{\underline{g}}^- \to 0, \qquad \text{with } \mathbb{V}_{\underline{g}}^+ \simeq \Lambda_{\underline{g}}(\psi_{\underline{g}}^{-1}\chi\underline{\epsilon}_{\mathrm{cyc}}) \text{ and } \mathbb{V}_{\underline{g}}^- \simeq \Lambda_{\underline{g}}(\psi_{\underline{g}}) \tag{53}$$

as $\Lambda_{\underline{g}}[G_{\mathbb{Q}_p}]$-modules. Here

$$\psi_{\underline{g}} : G_{\mathbb{Q}_p} \longrightarrow \Lambda_{\underline{g}}^\times$$

is the unramified character sending Fr_p to $\mathbf{a}_p(\underline{g})$.

Fix once and for all a choice of the isomorphisms in (53). Similarly to the previous sections, the Λ-adic form $\underline{g}$ gives rise to an epimorphism

$$\varpi_{\underline{g}} : \mathbb{V}_\infty^{\mathrm{ord}} \longrightarrow \mathbb{V}_{\underline{g}}$$

of $\tilde{\Lambda}[G_{\mathbb{Q}}]$-modules. Set

$$\mathbb{V}_{f,\underline{g}} := V_f \otimes \mathbb{V}_{\underline{g}}, \qquad \varpi_{f,\underline{g}} := \varpi_f \otimes \varpi_{\underline{g}} : V_0(Np) \otimes \mathbb{V}_\infty^{\mathrm{ord}} \longrightarrow \mathbb{V}_{f,\underline{g}}.$$

Definition 3.4. The Λ-adic Beilinson-Flach cohomology class associated to the pair $(f, \underline{g})$ is

$$\kappa(f, \underline{g}) = \varpi_{f,\underline{g}}(\kappa_\infty^{\mathrm{ord}}) \in H^1(\mathbb{Q}, \mathbb{V}_{f,\underline{g}}).$$

Let $\ell \geq 1$ be an integer and ϵ be a Dirichlet character of conductor p^s for some $s \geq 1$. Let y be a point in $\Omega_{\underline{g}} = \mathrm{Spf}(\Lambda_{\underline{g}})$ such that $\mathrm{wt}(y) = \nu_{\ell,\epsilon}$, and assume that the specialization g_y of $\underline{g}$ at y is a *classical* eigenform (which is always the case if $\ell \geq 2$). Then g_y belongs to $M_\ell(Np^s, \chi\epsilon\omega^{1-\ell})$, and is actually a cuspform if $\ell \geq 2$. Define

$$\kappa(f, g_y) \in H^1(\mathbb{Q}, V_{fg_y}) \tag{54}$$

to be the specialization at y of the Λ-adic class $\kappa(f, \underline{g})$. Let

$$\kappa_p(f, \underline{g}) \in H^1(\mathbb{Q}_p, \mathbb{V}_{f\underline{g}}), \quad \kappa_p(f, g_y) \in H^1(\mathbb{Q}_p, V_{fg_y})$$

denote the restriction to $G_{\mathbb{Q}_p}$ of the Λ-adic class $\kappa(f, \underline{g})$ and of its specialization $\kappa(f, g_y)$.

Since the p-adic syntomic regulator map is the composition of the étale regulator with Bloch-Kato's logarithm map, Theorem 2.11 translates into the following statement:

Theorem 3.5. *Let $y \in \Omega_{\underline{g}}$ be an arithmetic point of weight-character $\nu_{2,\epsilon}$ for some character ϵ of conductor p^s. Then*

$$\log_p \kappa_p(f, g_y)(\eta_f \otimes \omega_{g_y^w}) = \mathcal{G}(\chi\epsilon\omega^{-1}) \cdot \frac{\alpha_f^{s-1}}{1 - \chi(p)^{-1}\alpha_f^{-1}\alpha_{g_y}^{-1}} \cdot L_p(f, \underline{g})(y).$$

3.3. **The explicit reciprocity law.** The $\Lambda_{\underline{g}}$-module $\mathbb{V}_{f\underline{g}}$ admits a natural $G_{\mathbb{Q}_p}$-stable filtration, given by

$$\mathbb{V}_{f\underline{g}}^{++} := V_f^+ \otimes \mathbb{V}_{\underline{g}}^+ \quad \subset \quad \mathbb{V}_{f\underline{g}}^+ := V_f \otimes \mathbb{V}_{\underline{g}}^+ + V_f^+ \otimes \mathbb{V}_{\underline{g}} \quad \subset \quad \mathbb{V}_{f\underline{g}}.$$

Note that $\mathbb{V}_{f\underline{g}}^{++}$ and $\mathbb{V}_{f\underline{g}}^+$ have rank 1 and 3 over $\Lambda_{\underline{g}}$, respectively. We shall use similar notations for the analogous filtration of any classical specialization of $\mathbb{V}_{f\underline{g}}$.

Lemma 3.6. (i) *There is an isomorphism $\mathbb{V}_{f\underline{g}}/\mathbb{V}_{f\underline{g}}^+ \simeq \Lambda_g(\psi_f\psi_{\underline{g}})$ of $\Lambda_{\underline{g}}[G_{\mathbb{Q}_p}]$-modules.*

(ii) *The quotient $\mathbb{V}_{f\underline{g}}^+/\mathbb{V}_{f\underline{g}}^{++}$ decomposes as a $\Lambda_{\underline{g}}[G_{\mathbb{Q}_p}]$-module as*

$$\mathbb{V}_{f\underline{g}}^+/\mathbb{V}_{f\underline{g}}^{++} \simeq \mathbb{V}_{f\underline{g}}^f \oplus \mathbb{V}_{f\underline{g}}^g \quad \textit{where } \mathbb{V}_{f\underline{g}}^f = \Lambda_{\underline{g}}(\psi_f\psi_{\underline{g}}^{-1}\chi \cdot \underline{\epsilon}_{\mathrm{cyc}}) \textit{ and } \mathbb{V}_{f\underline{g}}^g = \Lambda_{\underline{g}}(\psi_f^{-1}\psi_{\underline{g}}\epsilon_{\mathrm{cyc}}).$$

Proof. The lemma follows directly from the definitions of $\mathbb{V}_{f\underline{g}}^{++}$ and $\mathbb{V}_{f\underline{g}}^+$, and the formulae described in (52) and (53). ☐

Note that according to our definitions, $\mathbb{V}_{f\underline{g}}^f \simeq V_f^- \otimes \mathbb{V}_{\underline{g}}^+$ and $\mathbb{V}_{f\underline{g}}^g \simeq V_f^+ \otimes \mathbb{V}_{\underline{g}}^-$. The natural inclusion $\mathbb{V}_{f\underline{g}}^+ \hookrightarrow \mathbb{V}_{f\underline{g}}$ induces a homomorphism $H^1(\mathbb{Q}_p, \mathbb{V}_{f\underline{g}}^+) \hookrightarrow H^1(\mathbb{Q}_p, \mathbb{V}_{f\underline{g}})$, which is injective because $H^0(\mathbb{Q}_p, \mathbb{V}_{f\underline{g}}/\mathbb{V}_{f\underline{g}}^+) = 0$. Thus $H^1(\mathbb{Q}_p, \mathbb{V}_{f\underline{g}}^+)$ is identified with a subgroup of $H^1(\mathbb{Q}_p, \mathbb{V}_{f\underline{g}})$.

Define $\xi_{fg} := 1 - \alpha_f a_p(\underline{g}) \in \Lambda_{\underline{g}}$. For the remainder of the article, we replace the ring $\Lambda_{\underline{g}}$ and all modules over it by their localization at the multiplicative set generated by the powers of ξ_{fg}. Since this element vanishes at no classical point of weight $\ell \geq 1$, this modification does not affect any of the subsequent arguments concerning the specialization at these points.

Lemma 3.7. *The local class $\kappa_p(f, \underline{g})$ belongs to $H^1(\mathbb{Q}_p, \mathbb{V}_{f\underline{g}}^+) \subset H^1(\mathbb{Q}_p, \mathbb{V}_{f\underline{g}})$.*

Proof. It follows from known properties of the étale regulator map, as explained in [Ne98] and [Fl, §2], that for all points y of weight $\nu_{2,\epsilon}$, the specializations $\kappa_p(f, g_y)$ belong to the image of $H^1(\mathbb{Q}_p, V_{fg_y}^+)$ in $H^1(\mathbb{Q}_p, V_{fg_y})$. Since these points form a dense subset of $\Omega_{\underline{g}}$ for the rigid analytic topology, we conclude that $\kappa_p(f, \underline{g})$ belongs to the kernel of the map

$$H^1(\mathbb{Q}_p, \mathbb{V}_{f\underline{g}}) \longrightarrow H^1(I_p, \mathbb{V}_{f\underline{g}}/\mathbb{V}_{f\underline{g}}^+).$$

In order to prove the lemma, it thus suffices to show that the kernel of the restriction map

$$H^1(\mathbb{Q}_p, \mathbb{V}_{f\underline{g}}/\mathbb{V}^+_{f\underline{g}}) \xrightarrow{\mathrm{res}} H^1(I_p, \mathbb{V}_{f\underline{g}}/\mathbb{V}^+_{f\underline{g}})$$

is trivial. Note that such kernel is a quotient of $H^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, (\mathbb{V}_{f\underline{g}}/\mathbb{V}^+_{f\underline{g}})^{I_p})$, which by Lemma 3.6 (i) is isomorphic to $H^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, \Lambda_{\underline{g}}(\psi_f\psi_{\underline{g}}))$. This module is ξ_{gh}-torsion. In view of the redefinition of $\Lambda_{\underline{g}}$, this implies the sought-after claim. □

Set $\underline{\Psi} := \psi_f\psi_{\underline{g}}^{-1}\chi$. The two previous lemmas allow us to define

$$\kappa_p^f(f, \underline{g}) \in H^1(\mathbb{Q}_p, \Lambda_g(\underline{\Psi} \cdot \underline{\epsilon}_{\mathrm{cyc}}))$$

as the projection of $\kappa_p(f, \underline{g})$ to the first factor $\mathbb{V}^f_{f\underline{g}}$.

Let $y \in \Omega_{\underline{g}}$ be a classical point of weight $\mathrm{wt}(y) = \nu_{\ell,\epsilon}$ with $\ell \geq 2$ and $\epsilon \neq \omega^{\ell-1}$, and let $K_y/\mathbb{Q}_p$ denote the residue field of y. Write $\Psi_y : G_{\mathbb{Q}_p} \longrightarrow K_y^\times$ for the specialisation at y of the unramified Λ-adic character $\underline{\Psi}$. The classical eigenform $g_y \in S_\ell(Np^s, \chi\epsilon\omega^{1-\ell})$ is new at p^s, and the specialization of $\kappa_p^f(f, \underline{g})$ at y yields a local class

$$\kappa_p^f(f, g_y) \in H^1(\mathbb{Q}_p, K_y(\Psi_y \cdot \epsilon \cdot \epsilon_{\mathrm{cyc}}^{\ell-1} \cdot \omega^{1-\ell})).$$

Lemma 3.8. *For all arithmetic points y of weight $\nu_{\ell,\epsilon}$ with $\ell \geq 2$, the local cohomology group $H^1(\mathbb{Q}_p, K_y(\Psi_y \cdot \epsilon \cdot \epsilon_{\mathrm{cyc}}^{\ell-1} \cdot \omega^{1-\ell}))$ is one-dimensional over K_y and equal to $H^1_{\mathrm{exp}}(\mathbb{Q}_p, K_y(\Psi_y \cdot \epsilon \cdot \epsilon_{\mathrm{cyc}}^{\ell-1} \cdot \omega^{1-\ell}))$.*

Proof. Write simply $V = V^f_{fg_y}$ for the specialization at y of $\mathbb{V}^f_{f\underline{g}}$ and $D = D_{\mathrm{dR}}(V^f_{fg_y})$ for its de Rham Dieudonné module. By [BK, Def. 3.10], [Bel, Ex. 2.19], the Bloch-Kato logarithm map is an isomorphism

$$\log_p : H^1_{\mathrm{exp}}(\mathbb{Q}_p, V) \to D/(\mathrm{Fil}^0 D + D^{\phi=1}). \tag{55}$$

By [Bel, §2.1.3], $\dim_{K_y} H^1(\mathbb{Q}_p, V) = e_0 + e_1 + 1$ where $e_0 = \dim H^0(\mathbb{Q}_p, V)$ and e_1 is the multiplicity of $K_y(1)$ as a quotient of V. Since $\ell \geq 2$, $e_0 = 0$. Since $\epsilon \neq \omega^{\ell-1}$, $e_1 = 0$ and thus $H^1(\mathbb{Q}_p, V)$ is one-dimensional over K_y. Moreover, since the Hodge-Tate weight of V is $1 - \ell \leq -1$, the subspace of D on which frobenius acts as the identity is $D^{\phi=1} = 0$, and $\mathrm{Fil}^{1-\ell}(D) \supsetneq \mathrm{Fil}^{2-\ell}(D)$. Since (55) is an isomorphism and $\dim H^1_{\mathrm{exp}}(\mathbb{Q}_p, V) \leq \dim H^1(\mathbb{Q}_p, V) = 1$, it follows that in fact $\mathrm{Fil}^{1-\ell}(D) = D \supset \mathrm{Fil}^0(D) = 0$, and thus (55) gives rise to an isomorphism

$$\log_p : H^1_{\mathrm{exp}}(\mathbb{Q}_p, V) = H^1(\mathbb{Q}_p, V) \longrightarrow D \tag{56}$$

of one-dimensional K_y-vector spaces. The result follows. □

For each arithmetic point $y \in \Omega_{\underline{g}}$ of weight $\nu_{\ell,\epsilon}$ with $\ell \geq 2$, the choice of a period $\mho_y \in D_{\mathrm{dR}}(V^{g_y^*}_{fg_y^*})$ attached to the dual $V^{g_y^*}_{fg_y^*}$ of $V^f_{fg_y}$ determines an isomorphism

$$\log_{\mho_y} : H^1(\mathbb{Q}_p, K_y(\Psi_y \cdot \epsilon \cdot \epsilon_{\mathrm{cyc}}^{\ell-1} \cdot \omega^{1-\ell})) \longrightarrow K_y, \qquad \log_{\mho_y}(\kappa) := \log_p(\kappa)(\mho_y). \tag{57}$$

Fix a choice of $\mho_y$ by defining, for all arithmetic points y of weight $\nu_{2,\epsilon}$ with $\epsilon \neq 1$ of conductor p^s,

$$\mho_y \quad := \quad \alpha_f^{1-s}\mathcal{G}(\chi\epsilon\omega^{-1})^{-1} \cdot (\eta_f \otimes \omega_{g_y^w}). \tag{58}$$

Ohta's Λ-adic Eichler-Shimura isomorphism can be used to show that these periods agree with those arising in Perrin-Riou's generalised Coleman maps attached to families of cyclotomic twists of a fixed Galois representations and their generalisations to twists by unramified Λ-adic characters, as described in [LZ, Thm. 4.15]. (Cf. the discussions in [DR2, §5.2, §5.3], or in [KLZ, §7.4, §7.6, Thm. 7.6.1].) It follows that, for all $\kappa \in H^1(\mathbb{Q}, \mathbb{V}^f_{f\underline{g}})$, the function $y \mapsto \log_{\mho_y}(\kappa)$ interpolates to an element of the fraction field of $\Lambda_{\underline{g}}$, denoted $\mathcal{L}(\kappa)$.

Recall the point $y \in \Omega_g$ of weight-character $\nu_{1,1}$ satisfying $\underline{g}_y = g_\alpha$, and let $\Lambda_{\underline{g}}{}'$ denote the localisation of $\Lambda_{\underline{g}}$ at the prime ideal $\ker(y)$.

Theorem 3.9. *For the map $\mathcal{L}$ as above,*

$$\mathcal{L}(\kappa_p^f(f, \underline{g})) = L_p(f, \underline{g}) \pmod{(\Lambda_{\underline{g}}{}')^\times}. \tag{59}$$

Proof. Theorem 3.5 implies that the elements

$$\mathcal{L}(\kappa_p^f(f,\underline{g})), \qquad (1-\chi(p)^{-1}\alpha_f^{-1}\alpha_{\underline{g}}^{-1})^{-1}L_p(f,\underline{g})$$

agree on all points $y \in \Omega_g$ of weight $\nu_{2,\epsilon}$, as ϵ ranges over all the characters of p-power conductor. Since both elements belong to the fraction field of Λ_g, it follows that they must be equal. Theorem 3.9 follows after noting that the factor $(1-\chi(p)^{-1}\alpha_f^{-1}\alpha_{\underline{g}}^{-1})$ does not lie in the kernel of the weight one point y attached to g_α (for reasons of weight, since α_f has complex absolute $\sqrt{p}$ while α_{g_y} is a root of unity), and hence belongs to $(\Lambda_{\underline{g}}{}')^\times$. □

At the weight one point y attached to $g = g_\alpha$, we have $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_{fg}^f) = 0$ and thus $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_{fg}^f) = H^1(\mathbb{Q}_p, V_{fg}^f)$; in particular the class $\kappa_p^f(f,g_\alpha)$ can only be cristalline if it is 0. Let

$$\exp_p^*: H^1(\mathbb{Q}_p, V_{fg}^f) \longrightarrow \mathrm{Fil}^0 D(V_{fg}^f)$$

denote the dual exponental map of Bloch and Kato associated to V_{fg}^f.

Theorem 3.10. *Let $y = \in \Omega_{\underline{g}}$ be the point over $\nu_{1,1}$, corresponding to the classical p-stabilised weight one form $g_y = g_\alpha \in S_1(Np,\chi)$. Then*

$$\exp_p^*(\kappa_p^f(f,g_\alpha)) \neq 0 \quad \Leftrightarrow \quad L(f\otimes g,1)\neq 0.$$

Proof. Perrin-Riou's local reciprocity law and its extension in [LZ, Thm. 4.15] asserts that, at the weight one point y attached to g_α,

$$\mathcal{L}(\kappa_p(f,\underline{g}))(y_0) \sim \exp^* \kappa_p^f(f,g_\alpha),$$

where the symbol $\sim$ denotes agreement up to an (explicit) non-zero element of $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_{fg_\alpha})$. (The Euler factor that arises in the p-adic interpolation process is non-vanishing at y, since the associated Galois representation V_{fg} is pure of weight -1.) The result now follows by combining Theorems 3.9 and 2.12. □

Note that it follows from the definitions in Lemma 3.6 that $V_{fg}^f \simeq V_f^- \otimes V_g^+ \simeq V_f^- \otimes V_g^\beta$. In addition, a standard exercise in p-adic Hodge theory (following e.g. [Bel, §2.2]) shows that $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_{fg}) = H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_{fg}^+) = H^1(\mathbb{Q}_p, V_f^+\otimes V_g)$ and hence $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_{fg}^+) = H^1(\mathbb{Q}_p, V_{fg}^+/V_f^+\otimes V_g) = H^1(\mathbb{Q}_p, V_{fg}^f)$.

This shows that the class $\kappa(f,g_y)=\kappa(f,g_\alpha)$ of equation (54) satisfies the two properties listed in Theorem 1.2. Indeed, part (1) follows from Lemma 3.7 and part (2) is a consequence of Theorem 3.10.

This concludes the proof of Theorems A and B of the Introduction.

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