p-ADIC FAMILIES OF DIAGONAL CYCLES

by

Henri Darmon and Victor Rotger

Abstract. — This note provides the construction of a three-variable family of cohomology classes arising from diagonal cycles on a triple product of towers of modular curves, and proves a reciprocity law relating it to the three variable triple-product *p*-adic *L*-function associated to a triple of Hida families by means of Perrin-Riou's Λ-adic regulator.

To Bernadette Perrin-Riou on her 65-th birthday

Contents

Introduction

The main purpose of this article is to supply a construction of a three-variable family of cycles interpolating the generalized diagonal cycles introduced in [**DR14**], and to prove a reciprocity law relating this family to the three variable triple-product p-adic L-function associated to a triple of Hida families by means of Perrin-Riou's Λ-adic regulator.

In order to give a flavor of our construction, let us describe in more detail the organization and contents of this article.

After reviewing some background in the first section, in section 2 we construct for every $r \geq 1$ a completely explicit family of cycles in the cube X_r^3 of the modular curve $X_r = X_1(Mp^r)$ of $\Gamma_1(Mp^r)$-level structure. This family is parametrized by the space of $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$-orbits of the set

$$\Sigma_r := ((\mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/p^r\mathbf{Z})')^3 \subset ((\mathbf{Z}/p^r\mathbf{Z})^2)^3$$

of triples of primitive row vectors of length 2 with entries in $\mathbf{Z}/p^r\mathbf{Z}$, on which $\mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z})$ acts diagonally by right multiplication. Any triple in Σ_r gives rise to a twisted diagonal embedding of the modular curve $\mathbb{X}(p^r)$ of $\Gamma_1(M) \cup \Gamma(p^r)$-level structure into the three-fold X_r^3 and the associated cycle is defined as the image of this map: we refer to (2.4) for the precise recipe.

The parameter space $\Sigma_r/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$ is closely related to $((\mathbf{Z}/p^r\mathbf{Z})^\times)^3$ and as shown throughout §2, the associated family of global cohomology classes introduced in Definition 2.9 can be packaged into a global Λ-adic cohomology class parametrized by three copies of weight space.

Along §3 and §4 we study the higher weight and cristalline specialisations of this family and we eventually prove in Theorem 4.1 that they interpolate the classes introduced in [**DR14**] as claimed above.

Finally, in §5 we recall Garrett-Hida's triple product p-adic L-function associated to a triple of Hida families $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ and prove in Theorem 5.1 a reciprocity law expressing the latter as the image of our three-variable cohomology classes (as specified in Definition 5.2) under Perrin-Riou's Λ-adic regulator.

It is instructive to compare the construction of our family to the approach taken in [**DR17**], which associated to a triple $(f, \mathbf{g}, \mathbf{h})$ consisting of a *fixed* newform f and a pair $(\mathbf{g}, \mathbf{h})$ of Hida families a *one-variable* family of cohomology classes instead of the two-variable family that one might have felt entitled to a priori. This shortcoming of the earlier approach can be understood by noting that the space of embeddings of $\mathbb{X}(p^r)$ into $X_1(M) \times X_r \times X_r$ as above in which the projection to the first factor is fixed is naturally parametrized by the coset space $M_2(\mathbf{Z}/p^r\mathbf{Z})'/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$, where $M_2(\mathbf{Z}/p^r\mathbf{Z})'$ denotes the set of 2×2 matrices whose rows are not divisible by p. The resulting cycles are therefore parametrized by the coset space $\mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z})/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z}) = (\mathbf{Z}/p^r\mathbf{Z})^\times$, whose inverse limit with r is the one dimensional p-adic space $\mathbf{Z}_p^\times$ rather than a two-dimensional one.

As mentioned already in our previous article in this volume, these cycles are of interest in their own right, and shed a useful complementary perspective on the construction of the Λ-adic cohomology classes for the triple product when compared to

[**BSVa**]. Indeed, their study forms the basis for the ongoing PhD thesis of David Lilienfeldt [**Li**], and has let to interesting open questions as those that are explored by Castella and Hsieh in [**CS20**].

1. Background

1.1. Basic notations. — Fix an algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$. All the number fields that arise will be viewed as embedded in this algebraic closure. For each such K, let $G_K := \mathrm{Gal}\,(\bar{\mathbf{Q}}/K)$ denote its absolute Galois group. Fix an odd prime p and an embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$; let ord_p denote the resulting p-adic valuation on $\bar{\mathbf{Q}}^\times$, normalized in such a way that $\mathrm{ord}_p(p) = 1$.

For a variety V defined over $K \subset \bar{\mathbf{Q}}$, let $\bar{V}$ denote the base change of V to $\bar{\mathbf{Q}}$. If $\mathcal{F}$ is an étale sheaf on V, write $H^i_{\text{ét}}(\bar{V}, \mathcal{F})$ for the ith étale cohomology group of $\bar{V}$ with values in $\mathcal{F}$, equipped with its continuous action by G_K.

Given a prime p, let $\mathbf{Q}(\mu_{p^\infty}) = \cup_{r \geq 1}\mathbf{Q}(\zeta_r)$ be the cyclotomic extension of $\mathbf{Q}$ obtained by adjoining to $\mathbf{Q}$ a primitive p^r-th root of unity ζ_r. Let

$$\varepsilon_{\mathrm{cyc}} : G_{\mathbf{Q}} \longrightarrow \mathrm{Gal}\,(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \xrightarrow{\simeq} \mathbf{Z}_p^\times$$

denote the p-adic cyclotomic character. It can be factored as $\varepsilon_{\mathrm{cyc}} = \omega\langle\varepsilon_{\mathrm{cyc}}\rangle$, where

$$\omega : G_{\mathbf{Q}} \longrightarrow \mu_{p-1} \qquad \langle\varepsilon_{\mathrm{cyc}}\rangle : G_{\mathbf{Q}} \longrightarrow 1 + p\mathbf{Z}_p$$

are obtained by composing $\varepsilon_{\mathrm{cyc}}$ with the projection onto the first and second factors in the canonical decomposition $\mathbf{Z}_p^\times \simeq \mu_{p-1} \times (1 + p\mathbf{Z}_p)$. If $\mathcal{M}$ is a $\mathbf{Z}_p[G_{\mathbf{Q}}]$-module and j is an integer, write $\mathcal{M}(j) = \mathcal{M} \otimes \varepsilon_{\mathrm{cyc}}^j$ for the j-th Tate twist of $\mathcal{M}$.

Let

$$\mathring{\Lambda}_r := \mathbf{Z}_p[(\mathbf{Z}/p^r\mathbf{Z})^\times], \qquad \mathring{\Lambda} := \mathbf{Z}_p[[\mathbf{Z}_p^\times]] := \varprojlim_r \mathring{\Lambda}_r$$

denote the group ring and completed group ring attached to the profinite group $\mathbf{Z}_p^\times$. The ring $\mathring{\Lambda}$ is equipped with $p - 1$ distinct algebra homomorphisms $\omega^i : \mathring{\Lambda} \to \Lambda$ (for $0 \leq i \leq p - 2$) to the local ring

$$\Lambda = \mathbf{Z}_p[[1 + p\mathbf{Z}_p]] = \varprojlim \mathbf{Z}_p[1 + p\mathbf{Z}/p^r\mathbf{Z}] \simeq \mathbf{Z}_p[[T]],$$

where ω^i sends a group-like element $a \in \mathbf{Z}_p^\times$ to $\omega^i(a)\langle a\rangle \in \Lambda$. These homomorphisms identify $\mathring{\Lambda}$ with the direct sum

$$\mathring{\Lambda} = \bigoplus_{i=0}^{p-2} \Lambda.$$

The local ring Λ is called the one variable Iwasawa algebra. More generally, for any integer $t \geq 1$, let

$$\mathring{\Lambda}^{\otimes t} := \mathring{\Lambda}\hat{\otimes}_{\mathbf{Z}_p} \overset{t}{\ldots} \hat{\otimes}_{\mathbf{Z}_p}\mathring{\Lambda}, \qquad \Lambda^{\otimes t} = \Lambda\hat{\otimes}_{\mathbf{Z}_p} \overset{t}{\ldots} \hat{\otimes}_{\mathbf{Z}_p}\Lambda \simeq \mathbf{Z}_p[[T_1, \ldots T_t]].$$

The latter ring is called the Iwasawa algebra in t variables, and is isomorphic to the power series ring in t variables over $\mathbf{Z}_p$, while

$$\mathring{\Lambda}^{\otimes t} = \bigoplus_\alpha \Lambda^{\otimes t},$$

the sum running over the $(p-1)^t$ distinct $\mathbf{Z}_p^\times$ valued characters of $(\mathbf{Z}/p\mathbf{Z})^{\times t}$.

1.2. Modular forms and Galois representations. — Let

$$\phi = q + \sum_{n \geq 2} a_n(\phi)q^n \in S_k(M, \chi)$$

be a cuspidal modular form of weight $k \geq 1$, level M and character $\chi : (\mathbf{Z}/M\mathbf{Z})^\times \to \mathbf{C}^\times$, and assume that ϕ is an eigenform with respect to all good Hecke operators T_ℓ, $\ell \nmid M$.

Fix an odd prime number p (that in this section may or may not divide M). Let $\mathcal{O}_\phi$ denote the valuation ring of the finite extension of $\mathbf{Q}_p$ generated by the fourier coefficients of ϕ, and let $\mathbb{T}$ denote the Hecke algebra generated over $\mathbf{Z}_p$ by the good Hecke operators T_ℓ with $\ell \nmid M$ and by the diamond operators acting on $S_k(M, \chi)$. The eigenform ϕ gives rise to an algebra homomorphism

$$\xi_\phi : \mathbb{T} \longrightarrow \mathcal{O}_\phi$$

sending T_ℓ to $a_\ell(\phi)$ and the diamond operator $\langle \ell \rangle$ to $\chi(\ell)$.

A fundamental construction of Shimura, Deligne, and Serre-Deligne attaches to ϕ an irreducible Galois representation

$$\varrho_\phi : G_\mathbf{Q} \longrightarrow \mathrm{Aut}(V_\phi) \simeq \mathrm{GL}_2(\mathcal{O}_\phi)$$

of rank 2, unramified at all primes $\ell \nmid Mp$, and for which

$$(1.1) \qquad \det(1 - \varrho_\phi(\mathrm{Fr}_\ell)x) = 1 - a_\ell(\phi)x + \chi(\ell)\ell^{k-1}x^2,$$

where Fr_ℓ denotes the arithmetic Frobenius element at ℓ. This property characterizes the semi-simplification of ϱ_ϕ up to isomorphism.

When $k := k_\circ + 2 \geq 2$, the representation V_ϕ can be realised in the p-adic étale cohomology of an appropriate Kuga-Sato variety. Since this realisation is important for the construction of generalised Kato classes, we now briefly recall its salient features. Let $Y = Y_1(M)$ and $X = X_1(M)$ denote the open and closed modular curve representing the fine moduli functor of isomorphism classes of pairs (A, P) formed by a (generalised) elliptic curve A together with a torsion point P on A of exact order M. Let

$$(1.2) \qquad\qquad\qquad \pi : \mathcal{A}_\circ \longrightarrow Y$$

denote the universal elliptic curve over Y.

The $k_\circ$-th open Kuga-Sato variety over Y is the $k_\circ$-fold fiber product

$$(1.3) \qquad\qquad\qquad \mathcal{A}_\circ^{k_\circ} := \mathcal{A}_\circ \times_Y \overset{(k_\circ)}{\cdots} \times_Y \mathcal{A}_\circ$$

of $\mathcal{A}_\circ$ over Y. The variety $\mathcal{A}_\circ^{k_\circ}$ admits a smooth compactification $\mathcal{A}^{k_\circ}$ which is fibered over X and is called the $k_\circ$-th Kuga-Sato variety over X; we refer to Conrad's appendix in [**BDP13**] for more details. The geometric points in $\mathcal{A}^{k_\circ}$ that lie above Y are in bijection with isomorphism classes of tuples $[(A, P), P_1, \ldots, P_{k_\circ}]$, where (A, P) is associated to a point of Y as in the previous paragraph and $P_1, ..., P_{k_\circ}$ are points on A.

The representation V_ϕ is realised (up to a suitable Tate twist) in the middle degree étale cohomology $H^{k_\circ+1}_{\text{ét}}(\bar{\mathcal{A}}^{k_\circ}, \mathbf{Z}_p)$. More precisely, let

$$\mathcal{H}_r := R^1\pi_* \mathbf{Z}/p^r\mathbf{Z}(1), \qquad \mathcal{H} := R^1\pi_* \mathbf{Z}_p(1),$$

and for any $k_\circ \geq 0$, define

$$(1.4) \qquad \mathcal{H}^{k_\circ}_r := \text{TSym}^{k_\circ}(\mathcal{H}_r), \qquad \mathcal{H}^{k_\circ} := \text{TSym}^{k_\circ}(\mathcal{H})$$

to be the sheaves of symmetric $k_\circ$-tensors of $\mathcal{H}_r$ and $\mathcal{H}$, respectively. As defined in e.g. [**BDP13**, (2.1.2)], there is an idempotent $\epsilon_{k_\circ}$ in the ring of rational correspondences of $\mathcal{A}^{k_\circ}$ whose induced projector on the étale cohomology groups of this variety satisfy:

$$(1.5) \qquad \epsilon_{k_\circ}\left(H^{k_\circ+1}_{\text{ét}}(\bar{\mathcal{A}}^{k_\circ}, \mathbf{Z}_p(k_\circ))\right) = H^1_{\text{ét}}(\bar{X}, \mathcal{H}^{k_\circ}).$$

Define the $\mathcal{O}_\phi$-module

$$(1.6) \qquad V_\phi(M) := H^1_{\text{ét}}(\bar{X}, \mathcal{H}^{k_\circ}(1)) \otimes_{\mathbb{T}, \xi_\phi} \mathcal{O}_\phi,$$

and write

$$(1.7) \qquad \varpi_\phi : H^1_{\text{ét}}(\bar{X}, \mathcal{H}^{k_\circ}(1)) \longrightarrow V_\phi(M)$$

for the canonical projection of $\mathbb{T}[G_{\mathbf{Q}}]$-modules arising from (1.6). Deligne's results and the theory of newforms show that the module $V_\phi(M)$ is the direct sum of several copies of a locally free module V_ϕ of rank 2 over $\mathcal{O}_\phi$ that satisfies (1.1).

Let α_ϕ and β_ϕ the two roots of the p-th Hecke polynomial $T^2 - a_p(\phi)T + \chi(p)p^{k-1}$, ordered in such a way that $\text{ord}_p(\alpha_\phi) \leq \text{ord}_p(\beta_\phi)$. (If α_ϕ and β_ϕ have the same p-adic valuation, simply fix an arbitrary ordering of the two roots.) We set $\chi(p) = 0$ whenever p divides the primitive level of ϕ and thus $\alpha_\phi = a_p(\phi)$ and $\beta_\phi = 0$ in this case. The eigenform ϕ is said to be *ordinary* at p when $\text{ord}_p(\alpha_\phi) = 0$. In that case, there is an exact sequence of $G_{\mathbf{Q}_p}$-modules

$$(1.8) \qquad 0 \to V^+_\phi \longrightarrow V_\phi \longrightarrow V^-_\phi \to 0, \qquad V^+_\phi \simeq \mathcal{O}_\phi(\varepsilon^{k-1}_{\text{cyc}}\chi\psi^{-1}_\phi), \quad V^-_\phi \simeq \mathcal{O}_\phi(\psi_\phi),$$

where ψ_ϕ is the unramified character of $G_{\mathbf{Q}_p}$ sending Fr_p to α_ϕ.

1.3. Hida families and Λ-adic Galois representations. — Fix a prime $p \geq 3$. The formal spectrum

$$\mathcal{W} := \text{Spf}(\Lambda)$$

of the Iwasawa algebra $\Lambda = \mathbf{Z}_p[[1 + p\mathbf{Z}_p]]$ is called the *weight space* attached to Λ. The A-valued points of $\mathcal{W}$ over a p-adic ring A are given by

$$\mathcal{W}(A) = \text{Hom}_{\text{alg}}(\Lambda, A) = \text{Hom}_{\text{grp}}(1 + p\mathbf{Z}_p, A^\times),$$

where the Hom's in this definition denote continuous homomorphisms of p-adic rings and profinite groups respectively. Weight space is equipped with a distinguished collection of *arithmetic points* $\nu_{k_\circ, \varepsilon}$, indexed by integers $k_\circ \geq 0$ and Dirichlet characters $\varepsilon : (1 + p\mathbf{Z}/p^r\mathbf{Z}) \to \mathbf{Q}_p(\zeta_{r-1})^\times$ of p-power conductor. The point $\nu_{k_\circ, \varepsilon} \in \mathcal{W}(\mathbf{Z}_p[\zeta_r])$ is defined by

$$\nu_{k_\circ, \varepsilon}(n) = \varepsilon(n)n^{k_\circ},$$

and the notational shorthand $\nu_{k_\circ} := \nu_{k_\circ,1}$ is adopted throughout. More generally, if $\tilde{\Lambda}$ is any finite flat Λ-algebra, a point $x \in \tilde{\mathcal{W}} := \mathrm{Spf}(\tilde{\Lambda})$ is said to be arithmetic if its restriction to Λ agrees with $\nu_{k_\circ,\varepsilon}$ for some $k_\circ$ and ϵ. The integer $k = k_\circ + 2$ is called the *weight* of x and denoted $\mathrm{wt}(x)$.

Let

$$(1.9) \qquad \underline{\varepsilon}_{\mathrm{cyc}} : G_{\mathbf{Q}} \longrightarrow \Lambda^\times$$

denote the Λ-adic cyclotomic character which sends a Galois element σ to the group-like element $[\langle \varepsilon_{\mathrm{cyc}}(\sigma) \rangle]$. This character interpolates the powers of the cyclotomic character, in the sense that

$$(1.10) \qquad \nu_{k_\circ,\varepsilon} \circ \underline{\varepsilon}_{\mathrm{cyc}} = \varepsilon \cdot \langle \varepsilon_{\mathrm{cyc}} \rangle^{k_\circ} = \varepsilon \cdot \varepsilon_{\mathrm{cyc}}^{k_\circ} \cdot \omega^{-k_\circ}.$$

Let $M \geq 1$ be an integer not divisible by p.

Definition 1.1. — *A Hida family of tame level M and tame character χ : $(\mathbf{Z}/M\mathbf{Z})^\times \to \bar{\mathbf{Q}}_p^\times$ is a formal q-expansion*

$$\phi = \sum_{n \geq 1} a_n(\phi) q^n \in \Lambda_\phi[[q]]$$

with coefficients in a finite flat Λ-algebra Λ_ϕ, such that for any arithmetic point $x \in \mathcal{W}_\phi := \mathrm{Spf}(\Lambda_\phi)$ above $\nu_{k_\circ,\epsilon}$, where $k_\circ \geq 0$ and ε is a character of conductor p^r, the series

$$\phi_x := \sum_{n \geq 1} x(a_n(\phi)) q^n \in \bar{\mathbf{Q}}_p[[q]]$$

is the q-expansion of a classical p-ordinary eigenform in the space $S_k(Mp^r, \chi\varepsilon\omega^{-k_\circ})$ of cusp forms of weight $k = k_\circ + 2$, level Mp^r and nebentype $\chi\varepsilon\omega^{-k_\circ}$.

By enlarging Λ_ϕ if necessary, we shall assume throughout that Λ_ϕ contains the M-th roots of unity.

Definition 1.2. — *Let $x \in \mathcal{W}_\phi$ be an arithmetic point lying above the point $\nu_{k_\circ,\epsilon}$ of weight space. The point x is said to be*
  — *tame if the character ϵ is tamely ramified, i.e., factors through $(\mathbf{Z}/p\mathbf{Z})^\times$.*
  — *crystalline if $\epsilon\omega^{-k_\circ} = 1$, i.e., if the weight k specialisation of ϕ at x has trivial nebentypus character at p.*
We let $\mathcal{W}_\phi^\circ$ denote the set of crystalline arithmetic points of $\mathcal{W}_\phi$.

Note that a crystalline point is necessarily tame but of course there are tame points that are not crystalline. The justification for this terminology is that the Galois representation V_{ϕ_x} is crystalline at p when x is crystalline.

If x is a crystalline point, then the classical form ϕ_x is always old at p if $k > 2$. In that case there exists an eigenform ϕ_x° of level M such that ϕ_x is the ordinary p-stabilization of ϕ_x°. If the weight is $k = 1$ or 2, ϕ_x may be either old or new at p; if it is new at p then we set $\phi_x^\circ = \phi_x$ in order to have uniform notations.

We say ϕ is residually irreducible if the mod p Galois representation associated to the Deligne representations associated to ϕ_x° for any crystalline classical point is irreducible.

Finally, the Hida family ϕ is said to be *primitive* of tame level $M_\phi \mid M$ if for all but finitely many arithmetic points $x \in \mathcal{W}_\phi$ of weight $k \geq 2$, the modular form ϕ_x arises from a newform of level M_ϕ.

The following theorem of Hida and Wiles associates a two-dimensional Galois representation to a Hida family ϕ (cf. e.g. [**MT90**, Théorème 7]).

Theorem 1.1. — *Assume ϕ is residually irreducible. Then there is a rank two Λ_ϕ-module $\mathbb{V}_\phi$ equipped with a Galois action*

$$(1.11) \qquad \varrho_\phi : G_{\mathbf{Q}} \longrightarrow \operatorname{Aut}_{\Lambda_\phi}(\mathbb{V}_\phi) \simeq \operatorname{GL}_2(\Lambda_\phi),$$

such that, for all arithmetic points $x : \Lambda_\phi \longrightarrow \bar{\mathbf{Q}}_p$,

$$\mathbb{V}_\phi \otimes_{x,\Lambda_\phi} \bar{\mathbf{Q}}_p \simeq V_{\phi_x} \otimes \bar{\mathbf{Q}}_p.$$

Let

$$\psi_\phi : G_{\mathbf{Q}_p} \longrightarrow \Lambda_\phi^\times$$

denote the unramified character sending a Frobenius element Fr_p to $\mathbf{a}_p(\phi)$. The restriction of $\mathbb{V}_\phi$ to $G_{\mathbf{Q}_p}$ admits a filtration
$$(1.12)$$
$$0 \to \mathbb{V}_\phi^+ \to \mathbb{V}_\phi \to \mathbb{V}_\phi^- \to 0 \quad \text{where} \ \ \mathbb{V}_\phi^+ \simeq \Lambda_\phi(\psi_\phi^{-1}\chi\varepsilon_{\mathrm{cyc}}^{-1}\underline{\varepsilon}_{\mathrm{cyc}}) \ \ \text{and} \ \ \mathbb{V}_\phi^- \simeq \Lambda_\phi(\psi_\phi).$$

The explicit construction of the Galois representation $\mathbb{V}_\phi$ plays an important role in defining the generalised Kato classes, and we now recall its main features.

For all $0 \leq r < s$, let

$$X_r := X_1(Mp^r), \qquad X_{r,s} := X_1(Mp^r) \times_{X_0(Mp^r)} X_0(Mp^s),$$

where the fiber product is taken relative to the natural projection maps. In particular,
— the curve $X := X_0 := X_1(M)$ represents the functor of elliptic curves A with $\Gamma_1(M)$-level structure, i.e., with a marked point of order M;
— the curve X_r represents the functor classifying pairs (A, P) consisting of a generalized elliptic curve A with $\Gamma_1(M)$-level structure and a point P of order p^r on A;
— the curve $X_{0,s} = X_1(M) \times_{X_0(M)} X_0(Mp^s)$ classifies pairs (A, C) consisting of a generalized elliptic curve A with $\Gamma_1(M)$ structure and a cyclic subgroup scheme C of order p^s on A;
— the curve $X_{r,s}$ classifies pairs (A, P, C) consisting of a generalized elliptic curve A with $\Gamma_1(M)$ structure, a point P of order r on A and and a cyclic subgroup scheme C of order p^s on A containing P.

The curves X_r and $X_{0,r}$ are smooth geometrically connected curves over $\mathbf{Q}$. The natural covering map $X_r \longrightarrow X_{0,r}$ is Galois with Galois group $(\mathbf{Z}/p^r\mathbf{Z})^\times$ acting on the left via the diamond operators defined by

$$(1.13) \qquad \langle a \rangle(A, P) = (A, aP).$$

Let

$$(1.14) \qquad \varpi_1 : X_{r+1} \longrightarrow X_r$$

denote the natural projection from level $r+1$ to level r which corresponds to the map $(A, P) \mapsto (A, pP)$, and to the map $\tau \mapsto \tau$ on upper half planes. Let

$$\varpi_2 : X_{r+1} \longrightarrow X_r$$

denote the other projection, corresponding to the map $(A, P) \mapsto (A/\langle p^r P \rangle, P + \langle p^r P \rangle)$, which on the upper half plane sends τ to $p\tau$. These maps can be factored as

(1.15)

$$
\begin{array}{ccc}
X_{r+1} & & X_{r+1} \\
\mu \downarrow \quad \searrow^{\varpi_1} & & \mu \downarrow \quad \searrow^{\varpi_2} \\
X_{r,r+1} \xrightarrow{\pi_1} X_r, & & X_{r,r+1} \xrightarrow{\pi_2} X_r.
\end{array}
$$

For all $r \geq 1$, the vertical map μ is a cyclic Galois covering of degree p, while the horizontal maps π_1 and π_2 are non-Galois coverings of degree p. When $r = 0$, the map μ is a cyclic Galois covering of degree $p - 1$ and π_2 are non-Galois coverings of degree $p + 1$.

The Λ-adic representation $\mathbb{V}_\phi$ shall be realised (up to twists) in quotients of the inverse limit of étale cohomology groups arising from the tower

$$X_\infty^* \quad : \quad \cdots \xrightarrow{\varpi_1} X_{r+1} \xrightarrow{\varpi_1} X_r \xrightarrow{\varpi_1} \cdots \xrightarrow{\varpi_1} X_1 \xrightarrow{\varpi_1} X_0$$

of modular curves. Define the inverse limit

(1.16)
$$H^1_{\text{ét}}(\bar{X}_\infty^*, \mathbf{Z}_p) := \varprojlim_{\varpi_{1*}} H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)$$

where the transition maps arise from the pushforward induced by the morphism ϖ_1. This inverse limit is a module over the completed group rings $\mathbf{Z}_p[[\mathbf{Z}_p^\times]]$ arising from the action of the diamond operators, and is endowed with a plethora of extra structures that we now describe.

Hecke operators. The transition maps in (1.16) are compatible with the action of the Hecke operators T_n for all n that are not divisible by p. Of crucial importance for us in this article is Atkin's operator U_p^*, which operates on $H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)$ via the composition

$$U_p^* := \pi_{1*} \pi_2^*$$

arising from the maps in (1.15).

The operator U_p^* is compatible with the transition maps defining $H^1_{\text{ét}}(\bar{X}_\infty^*, \mathbf{Z}_p)$,

Inverse systems of étale sheaves. The cohomology group $H^1_{\text{ét}}(\bar{X}_\infty^*, \mathbf{Z}_p)$ can be identified with the first cohomology group of the base curve X_1 with values in a certain inverse systems of étale sheaves.

For each $r \geq 1$, let

(1.17)
$$\mathcal{L}_r^* := \varpi_{1*}^{r-1} \mathbf{Z}_p$$

be the pushforward of the constant sheaf on X_r via the map

$$\varpi_1^{r-1} : X_r \longrightarrow X_1$$

The stalk of $\mathcal{L}_r^*$ at a geometric point $x = (A, P)$ on X_1 is given by

$$\mathcal{L}_{r,x}^* = \mathbf{Z}_p[A[p^r]\langle P \rangle],$$

where
$$A[p^r]\langle P \rangle := \{Q \in A[p^r] \text{ such that } p^{r-1}Q = P\}.$$

The multiplication by p map on the fibers gives rise to natural homomorphisms of sheaves

$$(1.18) \qquad [p] : \mathcal{L}^*_{r+1} \longrightarrow \mathcal{L}^*_r,$$

and Shapiro's lemma gives canonical identifications

$$H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p) = H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}^*_r),$$

for which the following diagram commutes:

$$
\begin{array}{ccc}
H^1_{\text{ét}}(\bar{X}_{r+1}, \mathbf{Z}_p) & \xrightarrow{\varpi_{1*}} & H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p) \\
\| & & \| \\
H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}^*_{r+1}) & \xrightarrow{[p]} & H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}^*_r).
\end{array}
$$

Let $\mathcal{L}^*_\infty := \varprojlim_r \mathcal{L}^*_r$ denote the inverse system of étale sheaves relative to the maps $[p]$ arising in (1.18). By passing to the limit, we obtain an identification

$$(1.19) \qquad H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p) = \varprojlim_{r \geq 1} H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}^*_r) = H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}^*_\infty).$$

Weight k specialisation maps. Recall the *p*-adic étale sheaves $\mathcal{H}^{k_\circ}$ introduced in (1.4), whose cohomology gave rise to the Deligne representations attached to modular forms of weight $k = k_\circ + 2$ via (1.6). The natural $k_\circ$-th power symmetrisation function

$$A[p^r] \longrightarrow \mathcal{H}^{k_\circ}_r, \qquad Q \mapsto Q^{k_\circ},$$

restricted to $A[p^r]\langle P \rangle$ and extended to $\mathcal{L}^*_{r,x}$ by $\mathbf{Z}_p$-linearity, induces morphisms

$$(1.20) \qquad \text{sp}^*_{k,r} : \mathcal{L}^*_r \longrightarrow \mathcal{H}^{k_\circ}_r$$

of sheaves over X_1 (which are thus compatible with the action of $G_{\mathbf{Q}}$ on the fibers). These specialisation morphisms are compatible with the transition maps $[p]$ in the sense that the diagram

$$
\begin{array}{ccc}
\mathcal{L}^*_{r+1} & \xrightarrow{[p]} & \mathcal{L}^*_r \\
\downarrow{\scriptstyle \text{sp}^*_{k,r+1}} & & \downarrow{\scriptstyle \text{sp}^*_{k,r}} \\
\mathcal{H}^{k_\circ}_{r+1} & \longrightarrow & \mathcal{H}^{k_\circ}_r
\end{array}
$$

commutes, where the bottom horizontal arrow denotes the natural reduction map. The maps $\text{sp}^*_{k,r}$ can thus be pieced together into morphisms

$$(1.21) \qquad \text{sp}^*_k : \mathcal{L}^*_\infty \longrightarrow \mathcal{H}^{k_\circ}.$$

The induced morphism

$$(1.22) \qquad \text{sp}^*_k : H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p) \longrightarrow H^1_{\text{ét}}(\bar{X}_1, \mathcal{H}^{k_\circ}),$$

arising from those on $H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}^*_\infty)$ via (1.19) will be denoted by the same symbol by abuse of notation, and is referred to as the *weight $k = k_\circ + 2$ specialisation map*. The existence of such maps having finite cokernel reveals that the Λ-adic Galois representation $H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p)$ is rich enough to capture the Deligne representations attached to modular forms on X_1 of *arbitrary* weight $k \geq 2$.

For each $a \in 1 + p\mathbf{Z}_p$, the diamond operator $\langle a \rangle$ acts trivially on X_1 and as multiplication by $a^{k_\circ}$ on the stalks of the sheaves $\mathcal{H}^{k_\circ}_r$. It follows that the weight k specialisation map sp^*_k factors through the quotient $H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p) \otimes_{\Lambda, \nu_{k_\circ}} \mathbf{Z}_p$, i.e., one obtains a map

$$\text{sp}^*_k : H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p) \otimes_{\Lambda, \nu_{k_\circ}} \mathbf{Z}_p \quad \longrightarrow \quad H^1_{\text{ét}}(\bar{X}_1, \mathcal{H}^{k_\circ}).$$

Remark 1.3. — *The inverse limit $\mathcal{L}^*_\infty$ of the sheaves $\underline{\mathcal{L}}^*_r$ on X_1 has been systematically studied by G. Kings in [K15, §2.3-2.4], and is referred to as a sheaf of Iwasawa modules. Jannsen introduced in [J88] the étale cohomology groups of such inverse systems of sheaves, and proved the existence of a Hoschild-Serre spectral sequence, Gysin excision exact sequences and cycle map in this context.*

Ordinary projections. Let

$$(1.23) \qquad\qquad\qquad e^* := \lim_{n \to \infty} U^{*n!}_p$$

denote Hida's (anti-)ordinary projector. Since U^*_p commutes with the push-forward maps ϖ_{1*}, this idempotent operates on $H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p)$. While the structure of the Λ-module $H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p)$ seems rather complicated, a dramatic simplification occurs after passing to the quotient $e^* H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p)$, as the following classical theorem of Hida shows.

Theorem 1.2. — [H86, Corollaries 3.3 and 3.7] *The Galois representation $e^* H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p(1))$ is a free Λ-module. For each $\nu_{k_\circ} \in \mathcal{W}$ with $k_\circ \geq 0$, the weight $k = k_\circ + 2$ specialisation map induces maps with bounded cokernel (independent of k)*

$$\text{sp}^*_k \quad : \quad e^* H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p(1)) \otimes_{\nu_{k_\circ}} \mathbf{Z}_p \longrightarrow e^* H^1_{\text{ét}}(\bar{X}_1, \mathcal{H}^{k_\circ}(1)).$$

Galois representations attached to Hida families. The Galois representation $\mathbb{V}_\phi$ of Theorem 1.1 associated by Hida and Wiles to a Hida family ϕ of tame level M and character χ can be realised as a quotient of the Λ-module $e^* H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p(1))$. More precisely, let

$$\xi_\phi : \mathbb{T}_\Lambda \longrightarrow \Lambda_\phi$$

be the Λ-algebra homomorphism from the Λ-adic Hecke algebra $\mathbb{T}_\Lambda$ to the Λ-algebra Λ_ϕ generated by the fourier coefficients of ϕ sending T_ℓ to $a_\ell(\phi)$.

Then we have, much as in (1.7), a quotient map of Λ-adic Galois representations

$$(1.24) \qquad \varpi^*_\phi : e^* H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p(1)) \longrightarrow e^* H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}_\Lambda, \xi_\phi} \Lambda_\phi =: \mathbb{V}_\phi(M),$$

for which the following diagram of $\mathbb{T}_\Lambda[G_{\mathbf{Q}}]$-modules is commutative:

$$(1.25) \qquad \begin{array}{ccc} e^* H^1_{\text{ét}}(\bar{X}^*_\infty, \mathbf{Z}_p(1)) & \xrightarrow{\varpi^*_\phi} & \mathbb{V}_\phi(M) \\ \downarrow{\scriptstyle \text{sp}^*_k} & & \downarrow{\scriptstyle x} \\ e^* H^1_{\text{ét}}(\bar{X}_1, \mathcal{H}^{k_\circ}(1)) & \xrightarrow{\varpi_{\phi_x}} & V_{\phi_x}(Mp), \end{array}$$

for all arithmetic points x of $\mathcal{W}_\phi$ of weight $k = k_\circ + 2$ and trivial character.

As in (1.7), $\mathbb{V}_\phi(M)$ is non-canonically isomorphic to a finite direct sum of copies of a $\Lambda_\phi[G_{\mathbf{Q}}]$-module $\mathbb{V}_\phi$ of rank 2 over Λ_ϕ, satisfying the properties stated in Theorem 1.1.

One can of course work alternatively with the ordinary projection $e := \lim_{n\to\infty} U_p^{n!}$ rather than the anti-ordinary one, in which case one similarly constructs a quotient map of Λ-adic Galois representations

$$(1.26) \qquad \varpi_\phi : e H^1_{\text{ét}}(\bar{X}_\infty, \mathbf{Z}_p(1)) := e \varprojlim_{\varpi_{2*}} H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p(1)) \longrightarrow \mathbb{V}_\phi(M).$$

1.4. Families of Dieudonné modules. — Let $\mathbf{B}_{\text{dR}}$ denote Fontaine's field of de Rham periods, $\mathbf{B}^+_{\text{dR}}$ be its ring of integers and $\log[\zeta_{p^\infty}]$ denote the uniformizer of $\mathbf{B}^+_{\text{dR}}$ associated to a norm-compatible system $\zeta_{p^\infty} = \{\zeta_{p^n}\}_{n\geq 0}$ of p^n-th roots of unity. (cf. e.g.[**BK93**, §1]). For any finite-dimensional de Rham Galois representation V of $G_{\mathbf{Q}_p}$ with coefficients in a finite extension $L_p/\mathbf{Q}_p$, define the de Rham Dieudonné module

$$D(V) = (V \otimes \mathbf{B}_{\text{dR}})^{G_{\mathbf{Q}_p}}.$$

It is an L_p-vector space of the same dimension as V, equipped with a descending exhaustive filtration $\text{Fil}^j D(V) = (V \otimes \log^j[\zeta_{p^\infty}]\mathbf{B}^+_{\text{dR}})^{G_{\mathbf{Q}_p}}$ by L_p-vector subspaces.

Let $\mathbf{B}_{\text{cris}} \subset \mathbf{B}_{\text{dR}}$ denote Fontaine's ring of crystalline p-adic periods. If V is crystalline (which is always the case if it arises as a subquotient of the étale cohomology of an algebraic variety with good reduction at p), then there is a canonical isomorphism

$$D(V) \simeq (V \otimes \mathbf{B}_{\text{cris}})^{G_{\mathbf{Q}_p}},$$

which furnishes $D(V)$ with a linear action of a Frobenius endomorphism Φ.

In [**BK93**] Bloch and Kato introduced a collection of subspaces of the local Galois cohomology group $H^1(\mathbf{Q}_p, V)$, denoted respectively

$$H^1_{\text{e}}(\mathbf{Q}_p, V) \subseteq H^1_{\text{f}}(\mathbf{Q}_p, V) \subseteq H^1_{\text{g}}(\mathbf{Q}_p, V) \subseteq H^1(\mathbf{Q}_p, V),$$

and constructed homomorphisms

$$(1.27) \qquad \log_{\text{BK}} : H^1_{\text{e}}(\mathbf{Q}_p, V) \xrightarrow{\sim} D(V)/\big(\text{Fil}^0 D(V) + D(V)^{\Phi=1}\big)$$

and

$$(1.28) \qquad \exp^*_{\text{BK}} : H^1(\mathbf{Q}_p, V)/H^1_{\text{g}}(\mathbf{Q}_p, V) \xrightarrow{\sim} \text{Fil}^0 D(V)$$

that are usually referred to as the Bloch-Kato logarithm and dual exponential map.

We illustrate the above Bloch-Kato homomorphisms with a few basic examples that shall be used several times in the remainder of this article.

Example 1.4. — *As shown e.g. in* [**BK93**], [**B09**, §2.2], *for any unramified character* ψ *of* $G_{\mathbf{Q}_p}$ *and all* $n \in \mathbf{Z}$ *we have:*

(a) *If* $n \geq 2$, *or* $n = 1$ *and* $\psi \neq 1$, *then* $H^1_{\mathrm{e}}(\mathbf{Q}_p, L_p(\psi\varepsilon^n_{\mathrm{cyc}})) = H^1(\mathbf{Q}_p, L_p(\psi\varepsilon^n_{\mathrm{cyc}}))$ *is one-dimensional over* L_p *and the Bloch-Kato logarithm induces an isomorphism*

$$\log_{\mathrm{BK}} : H^1(\mathbf{Q}_p, L_p(\psi\varepsilon^n_{\mathrm{cyc}})) \xrightarrow{\sim} D(L_p(\psi\varepsilon^n_{\mathrm{cyc}})).$$

(b) *If* $n < 0$, *or* $n = 0$ *and* $\psi \neq 1$, *then* $H^1_{\mathrm{g}}(\mathbf{Q}_p, L_p(\psi\varepsilon^n_{\mathrm{cyc}})) = 0$ *and* $H^1(\mathbf{Q}_p, L_p(\psi\varepsilon^n_{\mathrm{cyc}}))$ *is one-dimensional. The dual exponential gives rise to an isomorphism*

$$\exp^*_{\mathrm{BK}} : H^1(\mathbf{Q}_p, L_p(\psi\varepsilon^n_{\mathrm{cyc}})) \xrightarrow{\sim} \mathrm{Fil}^0 D(L_p(\psi\varepsilon^n_{\mathrm{cyc}})) = D(L_p(\psi\varepsilon^n_{\mathrm{cyc}})).$$

(c) *Assume* $\psi = 1$. *If* $n = 0$, *then* $H^1(\mathbf{Q}_p, L_p)$ *has dimension* 2 *over* L_p, $H^1_{\mathrm{f}}(\mathbf{Q}_p, L_p) = H^1_{\mathrm{g}}(\mathbf{Q}_p, L_p)$ *has dimension* 1 *and* $H^1_{\mathrm{e}}(\mathbf{Q}_p, L_p)$ *has dimension* 0 *over* L_p. *The Bloch-Kato dual exponential map induces an isomorphism*

$$\exp^*_{\mathrm{BK}} : H^1(\mathbf{Q}_p, L_p)/H^1_{\mathrm{f}}(\mathbf{Q}_p, L_p) \xrightarrow{\sim} \mathrm{Fil}^0 D(L_p) = D(L_p) = L_p.$$

Class field theory identifies $H^1(\mathbf{Q}_p, L_p)$ *with* $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^\times, \mathbf{Q}_p) \otimes L_p$, *which is spanned by the homomorphisms* ord_p *and* $\log_p$.

If $n = 1$, *then* $H^1(\mathbf{Q}_p, L_p(1)) = H^1_{\mathrm{g}}(\mathbf{Q}_p, L_p(1))$ *is* 2-*dimensional and* $H^1_{\mathrm{f}}(\mathbf{Q}_p, L_p(1)) = H^1_{\mathrm{e}}(\mathbf{Q}_p, L_p(1))$ *has dimension* 1 *over* L_p. *As proved e.g. in* [**B09**, Prop. 2.9], *Kummer theory identifies the spaces* $H^1_{\mathrm{f}}(\mathbf{Q}_p, L_p(1)) \subset H^1(\mathbf{Q}_p, L_p(1))$ *with* $\mathbf{Z}_p^\times \hat{\otimes} L_p$ *sitting inside* $\mathbf{Q}_p^\times \hat{\otimes} L_p$. *Under this identification, the Bloch-Kato logarithm is the usual* p-*adic logarithm on* $\mathbf{Z}_p^\times$.

Let $\hat{\mathbf{Z}}_p^{\mathrm{nr}}$ denote the ring of integers of the completion of the maximal unramified extension of $\mathbf{Q}_p$. If V is unramified then there is a further canonical isomorphism

$$(1.29) \qquad D(V) \simeq (V \otimes \hat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}.$$

Let ϕ be an eigenform (with respect to the good Hecke operators) of weight $k = k_\circ + 2 \geq 2$, level M and character χ, with fourier coefficients in a finite extension L_p of $\mathbf{Q}_p$. The comparison theorem [**F97**] of Faltings-Tsuji combined with (1.6) asserts that there is a natural isomorphism

$$D(V_\phi(M)) \simeq H^1_{\mathrm{dR}}(X_1(M), \mathcal{H}^{k_\circ}(1))[\phi]$$

of Dieudonné modules over L_p. Note that $D(V_\phi(M))$ is the direct sum of several copies of the two-dimensional Dieudonné module $D(V_\phi)$.

Assume that $p \nmid M$ and ϕ is ordinary at p. Then $V_\phi(M)$ is crystalline and Φ acts on $D(V_\phi(M))$ as

$$(1.30) \qquad \Phi = \chi(p)p^{k_\circ+1}U_p^{-1}.$$

In particular the eigenvalues of Φ on $D(V_\phi(M))$ are $\chi(p)p^{k_\circ+1}\alpha_\phi^{-1} = \beta_\phi$ and $\chi(p)p^{k_\circ+1}\beta_\phi^{-1} = \alpha_\phi$, the two roots of the Hecke polynomial of ϕ at p. For future

reference, recall from [**DR14**, Theorem 1.3] the Euler factors

$$(1.31) \qquad \mathcal{E}_0(\phi) := 1 - \chi^{-1}(p)\beta_\phi^2 p^{1-k} = 1 - \frac{\beta_{\phi_x^\circ}}{\alpha_{\phi_x^\circ}}, \quad \mathcal{E}_1(\phi) := 1 - \chi(p)\alpha_\phi^{-2} p^{k-2}.$$

Let $\phi^* = \phi \otimes \bar\chi \in S_k(M, \bar\chi)$ denote the twist of ϕ by the inverse of its nebentype character. Poincaré duality induces a perfect pairing

$$\langle\,,\,\rangle : D(V_\phi(M)) \times D(V_{\phi^*}(M)) \longrightarrow D(L_p) = L_p.$$

The exact sequence (1.8) induces in this setting an exact sequence of Dieudonné modules

$$(1.32) \qquad 0 \longrightarrow D(V_\phi^+(M)) \stackrel{i}{\longrightarrow} D(V_\phi(M)) \stackrel{\pi}{\longrightarrow} D(V_\phi^-(M)) \longrightarrow 0.$$

Since $V_\phi^-(M$ is unramified, we have $D(V_\phi^-(M)) \simeq (V_\phi^-(M) \otimes \hat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}$. This submodule may also be characterized as the eigenspace $D(V_\phi^-(M)) = D(V_\phi(M))^{\Phi = \alpha_\phi}$ of eigenvalue α_ϕ for the action of frobenius.

The rule $\breve\phi \mapsto \omega_{\breve\phi}$ that attaches to a modular form its associated differential form gives rise to an isomorphism $S_k(M, \chi)_{L_p}[\phi] \stackrel{\sim}{\longrightarrow} \mathrm{Fil}^0(D(V_\phi(M))) \subset D(V_\phi(M))$. Moreover, the map π of (1.32) induces an isomorphism

$$(1.33) \qquad S_k(M, \chi)_{L_p}[\phi] \stackrel{\sim}{\longrightarrow} \mathrm{Fil}^0(D(V_\phi(M))) \stackrel{\pi}{\longrightarrow} D(V_\phi^-(M)).$$

Any element $\omega \in D(V_{\phi^*}^-(M))$ gives rise to a linear map

$$\omega : D(V_\phi^+(M)) \longrightarrow L_p, \quad \eta \mapsto \langle \eta, \pi^{-1}(\omega) \rangle.$$

Similarly, any $\eta \in D(V_{\phi^*}^+(M))$ may be identified with a linear functional

$$\eta : D(V_\phi^-(M)) \longrightarrow L_p, \quad \omega \mapsto \langle \pi^{-1}(\omega), \eta \rangle,$$

and given $\breve\phi \in S_k(M, \chi)_{L_p}[\phi]$ we set $\eta_{\breve\phi} : D(V_{\phi^*}^-(M)) \to L_p,\ \varphi \mapsto \eta_{\breve\phi}(\varphi) = \frac{\langle \breve\phi, \varphi \rangle}{\langle \breve\phi, \breve\phi^* \rangle}$.

Let now $\tilde\Lambda$ be a finite flat extension of the Iwasawa algebra Λ and let $\mathbb{U}$ denote a free $\tilde\Lambda$-module of finite rank equipped with an *unramified* $\tilde\Lambda$-linear action of $G_{\mathbf{Q}_p}$. Define the Λ-adic Dieudonné module

$$\mathbb{D}(\mathbb{U}) := (\mathbb{U} \hat\otimes \hat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}.$$

As shown in e.g. [**O03**, Lemma 3.3], $\mathbb{D}(\mathbb{U})$ is a free module over $\tilde\Lambda$ of the same rank as $\mathbb{U}$.

Examples of such Λ-adic Dieudonné modules arise naturally in the context of families of modular forms thanks to Theorem 1.1. Indeed, let $\boldsymbol\phi$ be a Hida family of tame level M and character χ, and let $\boldsymbol\phi^*$ denote the Λ-adic modular form obtained by twisting $\boldsymbol\phi$ by $\bar\chi$.

Let $\mathbb{V}_{\boldsymbol\phi}$ and $\mathbb{V}_{\boldsymbol\phi}(M)$ denote the global Λ-adic Galois representations described in (1.24). It follows from (1.12) that to the restriction of $\mathbb{V}_{\boldsymbol\phi}$ to $G_{\mathbf{Q}_p}$ one might associate two natural unramified $\Lambda[G_{\mathbf{Q}_p}]$-modules of rank one, namely

$$\mathbb{V}_{\boldsymbol\phi}^- \simeq \Lambda_{\boldsymbol\phi}(\psi_{\boldsymbol\phi}) \quad \text{and} \quad \mathbb{U}_{\boldsymbol\phi}^+ = \mathbb{V}_{\boldsymbol\phi}^+(\chi^{-1}\varepsilon_{\mathrm{cyc}}\varepsilon_{\mathrm{cyc}}^{-1}).$$

Define similarly the unramified modules $\mathbb{V}_{\boldsymbol\phi}^-(M)$ and $\mathbb{U}_{\boldsymbol\phi}^+(M)$.

Let

$$(1.34) \quad S_\Lambda^{\mathrm{ord}}(M,\chi)[\phi] := \left\{ \breve{\phi} \in S_\Lambda^{\mathrm{ord}}(M,\chi) \quad \text{s.t.} \quad \left| \begin{array}{ll} T_\ell \breve{\phi} = a_\ell(\phi)\breve{\phi}, & \forall \ell \nmid Mp, \\ U_p \breve{\phi} = a_p(\phi)\breve{\phi} \end{array} \right. \right\},$$

For any crystalline arithmetic point $x \in \mathcal{W}_\phi^\circ$ of weight k, the specialization of a Λ-adic test vector $\breve{\phi} \in S_\Lambda^{\mathrm{ord}}(M,\chi)[\phi]$ at x is a classical eigenform $\breve{\phi}_x \in S_k(Mp,\chi)$ with coefficients in $L_p = x(\Lambda_\phi) \otimes \mathbf{Q}_p$ and the same eigenvalues as ϕ_x for the good Hecke operators.

Likewise, define

$$S_\Lambda^{\mathrm{ord}}(M,\bar{\chi})^\vee[\phi] = \left\{ \eta : S_\Lambda^{\mathrm{ord}}(M,\bar{\chi}) \to \Lambda_\phi \left| \begin{array}{ll} \eta \circ T_\ell^* = a_\ell(\phi)\eta, & \forall \ell \nmid Mp, \\ \eta \circ U_p^* = a_p(\phi)\eta \end{array} \right. \right\}$$

Let $\mathcal{Q}_\phi$ denote the field of fractions of Λ_ϕ. Associated to any test vector $\breve{\phi} \in S_\Lambda^{\mathrm{ord}}(M,\chi)[\phi]$, [**DR14**, Lemma 2.19] describes a $\mathcal{Q}_\phi$-linear dual test vector

$$(1.35) \qquad\qquad \breve{\phi}^\vee \in S_\Lambda^{\mathrm{ord}}(M,\bar{\chi})^\vee[\phi] \,\hat{\otimes}\, \mathcal{Q}_\phi$$

such that for any $\varphi \in S_\Lambda^{\mathrm{ord}}(M,\bar{\chi})$ and any point $x \in \mathcal{W}_{\mathbf{f}}^\circ$,

$$x(\breve{\phi}^\vee(\varphi)) = \frac{\langle \breve{\phi}_x, \varphi_x \rangle}{\langle \breve{\phi}_x, \breve{\phi}_x^* \rangle}$$

where $\langle , \rangle$ denotes the pairing induced by Poincaré duality on the modular curve associated to the congruence subgroup $\Gamma_1(M) \cap \Gamma_0(p)$. This way, the specialization of a Λ-adic dual test vector $\breve{\phi}^\vee \in S_\Lambda^{\mathrm{ord}}(M,\bar{\chi})^\vee[\phi]$ at x gives rise to a linear functional

$$\eta_{\breve{\phi}_x} : S_k(Mp,\bar{\chi})[\phi_x^*] \longrightarrow L_p.$$

A natural $\mathcal{Q}_{\mathbf{f}}$-basis of $S_\Lambda^{\mathrm{ord}}(M,\chi)[\phi] \,\hat{\otimes}\, \mathcal{Q}_\phi$ is given by the Λ-adic modular forms $\phi(q^d)$ as d ranges over the positive divisors of M/M_ϕ and it is also obvious that $\{\phi(q^d)^\vee : d \mid \frac{M}{M_\phi}\}$ provides a $\mathcal{Q}_\phi$-basis of $S_\Lambda^{\mathrm{ord}}(M,\bar{\chi})^\vee[\phi] \,\hat{\otimes}\, \mathcal{Q}_\phi$.

The following statement shows that the linear maps described above can be made to vary in families.

Proposition 1.5. — *For any Λ-adic test vector $\breve{\phi} \in S_\Lambda^{\mathrm{ord}}(M,\chi)[\phi]$ there exist homomorphisms of Λ_ϕ-modules*

$$\omega_{\breve{\phi}} : D(\mathbb{U}_{\phi^*}^+(M)) \longrightarrow \Lambda_\phi, \quad \eta_{\breve{\phi}} : D(\mathbb{V}_{\phi^*}^-(M)) \longrightarrow \mathcal{Q}_\phi,$$

whose specialization at a classical point $x \in \mathcal{W}_\phi^\circ$ such that ϕ_x is the ordinary stabilization of an eigenform ϕ_x° of level M are, respectively

1. *$x \circ \omega_{\breve{\phi}} = \mathcal{E}_0(\phi_x^\circ)\, e\, \varpi_1^*(\omega_{\breve{\phi}_x^\circ})$ as functionals on $D(U_{\phi_x^*}^+(Mp))$.*

2. *$x \circ \eta_{\breve{\phi}} = \frac{1}{\mathcal{E}_1(\phi_x^\circ)} \cdot e\, \varpi_1^*(\eta_{\breve{\phi}_x^\circ})$ as functionals on $D(V_{\phi_x^*}^-(Mp))$.*

Proof. — This is essentially a reformulation of [**KLZ17**, Propositions 10.1.1 and 10.1.2], which in turn builds on [**O00**]. Namely, the first claim in Prop. 10.1.2 of loc. cit. asserts that $\omega_{\breve{\phi}}$ exists such that at any $x \in \mathcal{W}_\phi^\circ$ as above, $x \circ \omega_{\breve{\phi}} = \omega_{\breve{\phi}_x} =$

$\mathrm{Pr}^{\alpha*}(\omega_{\breve{\phi}_x^\circ})$ where $\mathrm{Pr}^{\alpha*}$ is the map defined in [**KLZ17**, 10.1.3] sending $\breve{\phi}_x^\circ$ to its ordinary p-stablilization $\breve{\phi}_x$. Note that $\varpi_1^*(\phi_x^\circ) = \frac{\alpha_{\phi_x^\circ}\breve{\phi}_x}{\alpha_{\phi_x^\circ}-\beta_{\phi_x^\circ}} - \frac{\beta_{\phi_x^\circ}\breve{\phi}'_x}{\alpha_{\phi_x^\circ}-\beta_{\phi_x^\circ}}$, where $\breve{\phi}'_x$ denotes the non-ordinary specialization of $\breve{\phi}_x^\circ$. Since $e\omega_{\breve{\phi}'_x} = 0$ and $\mathcal{E}_0(\phi_x^\circ) = \frac{\alpha_{\phi_x^\circ}-\beta_{\phi_x^\circ}}{\alpha_{\phi_x^\circ}}$ the claim follows.

The second part of [**KLZ17**, Proposition 10.1.2] asserts that there exists a Λ-adic functional $\tilde{\eta}_{\breve{\phi}}$ such that for all x as above:

$$x \circ \tilde{\eta}_{\breve{\phi}} = \frac{\mathrm{Pr}^{\alpha*}\eta_{\breve{\phi}_x^\circ}}{\lambda(\phi_x^\circ)\mathcal{E}_0(\phi_x^\circ)\mathcal{E}_1(\phi_x^\circ)}$$

as L_p-linear functionals on $D(V_{\phi_x^*}^-(Mp))$. Here $\lambda(\phi_x^\circ) \in \bar{\mathbf{Q}}^\times$ denotes the pseudo-eigenvalue of ϕ_x°, which we recall is the scalar given by

(1.36) $$W_M(\phi_x^\circ) = \lambda(\phi_x^\circ) \cdot \phi_x^{\circ*},$$

where $W_M : S_k(M, \chi) \to S_k(M, \chi^{-1})$ stands for the Atkin-Lehner operator. Since we are assuming that Λ_ϕ contains the M-th roots of unity (cf. the remark right after Definition 1.1), Prop. 10.1.1 of loc. cit. shows that there exists an element $\lambda(\phi) \in \Lambda_\phi$ interpolating the pseudo-eigenvalues of the classical p-stabilized specializations of ϕ. The claim follows by taking $\eta_{\breve{\phi}} = \lambda(\phi)\tilde{\eta}_{\breve{\phi}}$. The same argument as above yields that for all x as above, $x \circ \eta_{\breve{\phi}} = \mathcal{E}_0(\phi_x^\circ)\frac{e\varpi_1^*\eta_{\breve{\phi}_x^\circ}}{\mathcal{E}_0(\phi_x^\circ)\mathcal{E}_1(\phi_x^\circ)}$, which amounts to the statement of the proposition. $\qquad\square$

2. Generalised Kato classes

2.1. A compatible collection of cycles. — This section defines a collection of codimension two cycles in $X_1(Mp^r)^3$ indexed by elements of $(\mathbf{Z}/p^r\mathbf{Z})^{\times 3}$ and records some of their properties.

We retain the notations that were in force in Section 1.3 regarding the meanings of the curves $X = X_1(M)$, $X_r = X_1(Mp^r)$ and $X_{r,s}$. In addition, let

$$\mathbb{Y}(p^r) := Y \times_{X(1)} Y(p^r), \quad \mathbb{X}(p^r) := X \times_{X(1)} X(p^r)$$

denote the (affine and projective, respectively) modular curve over $\mathbf{Q}(\zeta_r)$ with full level p^r structure. The curve $\mathbb{Y}(p^r)$ classifies triples (A, P, Q) in which A is an elliptic curve with $\Gamma_1(M)$ level structure and (P, Q) is a basis for $A[p^r]$ satisfying $\langle P, Q\rangle = \zeta_r$, where $\langle\ ,\rangle$ denotes the Weil pairing and ζ_r is a fixed primitive p^r-th root of unity. The curve $\mathbb{X}(p^r)$ is geometrically connected but does not descend to a curve over $\mathbf{Q}$, as can be seen by noting that the description of its moduli problem depends on the choice of ζ_r. The covering $\mathbb{X}(p^r)/X$ is Galois with Galois group $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$, acting on the left by the rule

(2.1) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(A, P, Q) = (A, aP + bQ, cP + dQ).$$

Consider the natural projection map

(2.2) $$\varpi_1^r \times \varpi_1^r \times \varpi_1^r : X_r^3 \longrightarrow X^3$$

induced on triple products by the map ϖ_1^r of (1.14). Write $\Delta \subset X^3$ for the usual diagonal cycle, namely the image of X under the diagonal embedding $x \mapsto (x, x, x)$. Let Δ_r be the fiber product $\Delta \times_{X^3} X_r^3$ via the natural inclusion and the map of (2.2), which fits into the cartesian diagram

$$
\begin{array}{ccc}
\Delta_r & \hookrightarrow & X_r^3 \\
\downarrow & & \downarrow \\
\Delta & \hookrightarrow & X^3.
\end{array}
$$

An element of a $\mathbf{Z}_p$-module Ω is said to be *primitive* if it does not belong to $p\Omega$, and the set of such primitive elements is denoted Ω'. Let

$$
\Sigma_r := ((\mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/p^r\mathbf{Z})')^3 \subset ((\mathbf{Z}/p^r\mathbf{Z})^2)^3
$$

be the set of triples of primitive row vectors of length 2 with entries in $\mathbf{Z}/p^r\mathbf{Z}$, equipped with the action of $\mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z})$ acting diagonally by right multiplication.

Lemma 2.1. — *The geometrically irreducible components of Δ_r are defined over $\mathbf{Q}(\zeta_r)$ and are in canonical bijection with the set of left orbits*

$$
\Sigma_r / \mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z}).
$$

Proof. — Each triple

$$
(v_1, v_2, v_3) = ((x_1, y_1), (x_2, y_2), (x_3, y_3)) \in \Sigma_r
$$

determines a morphism

$$
\varphi_{(v_1, v_2, v_3)} : \mathbb{X}(p^r) \longrightarrow \Delta_r \subset X_r^3
$$

of curves over $\mathbf{Q}(\zeta_r)$, defined in terms of the moduli descriptions on $\mathbb{Y}(p^r)$ by

$$
(A, P, Q) \quad \mapsto \quad (\; (A, x_1 P + y_1 Q), \; (A, x_2 P + y_2 Q), \; (A, x_3 P + y_3 Q) \;).
$$

It is easy to see that if two elements (v_1, v_2, v_3) and $(v_1', v_2', v_3') \in \Sigma_r$ satisfy

$$
(v_1', v_2', v_3') = (v_1, v_2, v_3)\gamma, \quad \text{with } \gamma \in \mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z}),
$$

then

$$
\varphi_{(v_1', v_2', v_3')} = \varphi_{(v_1, v_2, v_3)} \circ \gamma,
$$

where γ is being viewed as an automorphism of $\mathbb{X}(p^r)$ as in (2.1). It follows that the geometrically irreducible cycle

$$
\Delta_r(v_1, v_2, v_3) := \varphi_{(v_1, v_2, v_3)*}(\mathbb{X}(p^r))
$$

depends only on the $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$-orbit of (v_1, v_2, v_3).

Since $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$ acts transitively on $(\mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/p^r\mathbf{Z})'$, one further checks that the collection of cycles $\Delta_r(v_1, v_2, v_3)$ for $(v_1, v_2, v_3) \in \Sigma_r/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$ do not overlap on Y_r^3 and cover Δ_r. Hence the irreducible components of Δ_r are precisely $\Delta_r(v_1, v_2, v_3)$ for $(v_1, v_2, v_3) \in \Sigma_r/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$. $\qquad \square$

The quotient $\Sigma_r/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$ is equipped with a natural determinant map

$$D : \Sigma_r/\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z}) \longrightarrow (\mathbf{Z}/p^r\mathbf{Z})^3$$

defined by

$$D\left((x_1 y_1), (x_2, y_2), (x_3, y_3)\right) := \left(\left| \begin{array}{cc} x_2 & y_2 \\ x_3 & y_3 \end{array} \right|, \left| \begin{array}{cc} x_3 & y_3 \\ x_1 & y_1 \end{array} \right|, \left| \begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \end{array} \right| \right).$$

For each $[d_1, d_2, d_3] \in (\mathbf{Z}/p^r\mathbf{Z})^3$, we can then write

$$\Sigma_r[d_1, d_2, d_3] := \{(v_1, v_2, v_3) \in \Sigma_r \text{ with } D(v_1, v_2, v_3) = (d_1, d_2, d_3)\} .$$

The group $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$ operates *simply transitively* on $\Sigma_r[d_1, d_2, d_3]$ if (and only if)

$$(2.3) \qquad\qquad [d_1, d_2, d_3] \in I_r := (\mathbf{Z}/p^r\mathbf{Z})^{\times 3}.$$

In particular, if (v_1, v_2, v_3) belongs to $\Sigma_r[d_1, d_2, d_3]$, then the cycle $\Delta_r(v_1, v_2, v_3)$ depends only on $[d_1, d_2, d_3] \in I_r$ and will henceforth be denoted

$$(2.4) \qquad\qquad \Delta_r[d_1, d_2, d_3] \in \mathrm{CH}^2(X_r^3).$$

A somewhat more intrinsic definition of $\Delta_r[d_1, d_2, d_3]$ as a curve embedded in X_r^3 is that it corresponds to the schematic closure of the locus of points $((A, P_1), (A, P_2), (A, P_3))$ satisfying

$$(2.5) \qquad \langle P_2, P_3 \rangle = \zeta_r^{d_1}, \qquad \langle P_3, P_1 \rangle = \zeta_r^{d_2}, \qquad \langle P_1, P_2 \rangle = \zeta_r^{d_3}.$$

This description also makes it apparent that the cycle $\Delta_r[d_1, d_2, d_3]$ is defined over $\mathbf{Q}(\zeta_r)$ but not over $\mathbf{Q}$. Let $\sigma_m \in \mathrm{Gal}\,(\mathbf{Q}(\zeta_r)/\mathbf{Q})$ be the automorphism associated to $m \in (\mathbf{Z}/p^r\mathbf{Z})^\times$, sending ζ_r to ζ_r^m. The threefold X_r^3 is also equipped with an action of the group

$$(2.6) \qquad \tilde{G}_r := ((\mathbf{Z}/p^r\mathbf{Z})^\times)^3 = \{\langle a_1, a_2, a_3 \rangle, \quad a_1, a_2, a_3 \in (\mathbf{Z}/p^r\mathbf{Z})^\times\}$$

of diamond operators, where the automorphism associated to a triple $(\langle a_1 \rangle, \langle a_2 \rangle, \langle a_3 \rangle)$ has simply been denoted $\langle a_1, a_2, a_3 \rangle$.

Lemma 2.2. — *For all diamond operators $\langle a_1, a_2, a_3 \rangle \in \tilde{G}_r$ and all $[d_1, d_2, d_3] \in I_r$,*

$$(2.7) \qquad \langle a_1, a_2, a_3 \rangle \Delta_r[d_1, d_2, d_3] = \Delta_r[a_2 a_3 \cdot d_1, a_1 a_3 \cdot d_2, a_1 a_2 \cdot d_3].$$

For all $\sigma_m \in \mathrm{Gal}\,(\mathbf{Q}(\zeta_r)/\mathbf{Q})$,

$$(2.8) \qquad\qquad \sigma_m \Delta_r[d_1, d_2, d_3] = \Delta_r[m \cdot d_1, m \cdot d_2, m \cdot d_3].$$

Proof. — Equation (2.7) follows directly from the identity

$$D(a_1 v_1, a_2 v_2, a_3 v_3) = [a_2 a_3, a_1 a_3, a_1 a_2] D(v_1, v_2, v_3).$$

The first equality in (2.8) is most readily seen from the equation (2.5) defining the cycle $\Delta_r[d_1, d_2, d_3]$, since applying the automorphism $\sigma_m \in \mathrm{Gal}\,(\mathbf{Q}(\zeta_r)/\mathbf{Q})$ has the effect of replacing ζ_r by ζ_r^m. $\qquad\square$

Remark 2.3. — *Assume m is a quadratic residue in $(\mathbf{Z}/p^r\mathbf{Z})^\times$, which is the case, for instance, when σ_m belongs to $\mathrm{Gal}\,(\mathbf{Q}(\zeta_r)/\mathbf{Q}(\zeta_1))$. Then it follows from (2.7) and (2.8) that*

$$(2.9) \qquad\qquad \sigma_m \Delta_r[d_1, d_2, d_3] = \langle m, m, m \rangle^{1/2} \Delta_r[d_1, d_2, d_3].$$

Let us now turn to the compatibility properties of the cycles $\Delta_r[d_1, d_2, d_3]$ as the level r varies. Recall the modular curve $X_{r,r+1}$ classifying generalised elliptic curves together with a distinguished cyclic subgroup of order p^{r+1} and a point of order p^r in it. The maps μ, ϖ_1, π_1, ϖ_2 and π_2 of (1.15) induce similar maps on the triple products:

(2.10)

$$
\begin{array}{ccc}
X_{r+1}^3 & & \\
\downarrow{\scriptstyle\mu^3} \quad \searrow{\scriptstyle\varpi_1^3} & & \\
X_{r,r+1}^3 \xrightarrow{\ \pi_1^3\ } X_r^3,
\end{array}
\qquad
\begin{array}{ccc}
X_{r+1}^3 & & \\
\downarrow{\scriptstyle\mu^3} \quad \searrow{\scriptstyle\varpi_2^3} & & \\
X_{r,r+1}^3 \xrightarrow{\ \pi_2^3\ } X_r^3.
\end{array}
$$

A finite morphism $j : V_1 \longrightarrow V_2$ of varieties induces maps

$$j_* : \mathrm{CH}^j(V_1) \longrightarrow \mathrm{CH}^j(V_2), \qquad j^* : \mathrm{CH}^j(V_2) \longrightarrow \mathrm{CH}^j(V_1)$$

between Chow groups, and $j_* j^*$ agrees with the multiplication by $\deg(j)$ on $\mathrm{CH}^j(V_2)$. If j is a Galois cover with Galois group G,

(2.11)
$$j^* j_*(\Delta) = \sum_{\sigma \in G} \sigma\Delta.$$

By abuse of notation we will denote the associated maps on cycles (rather than just on cycle classes) by the same symbols.

Lemma 2.4. — *For all $r \geq 1$ and all $[d_1', d_2', d_3'] \in I_{r+1}$ whose image in I_r is $[d_1, d_2, d_3]$,*

$$
\begin{aligned}
(\varpi_1^3)_* \Delta_{r+1}[d_1', d_2', d_3'] &= p^3 \Delta_r[d_1, d_2, d_3], \\
(\varpi_2^3)_* \Delta_{r+1}[d_1', d_2', d_3'] &= (U_p)^{\otimes 3} \Delta_r[d_1, d_2, d_3].
\end{aligned}
$$

The cycles $\Delta_r[d_1, d_2, d_3]$ also satisfy the distribution relations

$$\sum_{[d_1', d_2', d_3']} \Delta_{r+1}[d_1', d_2', d_3'] = (\varpi_1^3)^* \Delta_r[d_1, d_2, d_3],$$

where the sum is taken over all triples $[d_1', d_2', d_3'] \in I_{r+1}$ which map to $[d_1, d_2, d_3]$ in I_r.

Proof. — A direct verification based on the definitions shows that the morphisms μ^3 and π_1^3 of (2.10) induce morphisms

$$\Delta_{r+1}[d_1' d_2', d_3'] \xrightarrow{\ \mu^3\ } \mu_*^3 \Delta_{r+1}[d_1', d_2', d_3'] \xrightarrow{\ \pi_1^3\ } \Delta_r[d_1, d_2, d_3],$$

of degrees 1 and p^3 respectively. Hence the restriction of ϖ_1^3 to $\Delta_{r+1}[d_1', d_2', d_3']$ induces a map of degree p^3 from $\Delta_{r+1}[d_1', d_2', d_3']$ to $\Delta_r[d_1, d_2, d_3]$, which implies the first assertion. It also follows from this that

(2.12)
$$\mu_*^3 \Delta_{r+1}[d_1', d_2', d_3'] = (\pi_1^3)^* \Delta_r[d_1, d_2, d_3].$$

Applying $(\pi_2^3)_*$ to this identity implies that

$$(\varpi_2^3)_* \Delta_{r+1}[d_1', d_2', d_3'] = (U_p)^{\otimes 3} \Delta_r[d_1, d_2, d_3].$$

The second compatibility relation follows. To prove the distribution relation, observe that the sum that occurs in it is taken over the p^3 translates of a fixed $\Delta_{r+1}[d_1', d_2', d_3']$ for the action of the Galois group of X_{r+1}^3 over $X_{r,r+1}^3$, and hence, by (2.11), that

$$\sum_{[d_1', d_2', d_3']} \Delta_{r+1}[d_1', d_2', d_3'] = (\mu^*)^3 \mu_*^3 \Delta_{r+1}[d_1', d_2', d_3'].$$

The result then follows from (2.12). $\qquad\square$

2.2. Galois cohomology classes. — The goal of this section is to parlay the cycles $\Delta_r[d_1, d_2, d_3]$ into Galois cohomology classes with values in $H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)$, essentially by considering their images under the p-adic étale Abel-Jacobi map:

(2.13) $\text{AJ}_{\text{ét}} : \text{CH}^2(X_r^3)_0 \longrightarrow H^1(\mathbf{Q}, H^3_{\text{ét}}(\bar{X}_r^3, \mathbf{Z}_p(2))),$

where

$$\text{CH}^2(X_r^3)_0 := \ker \left(\text{CH}^2(X_r^3) \longrightarrow H^4_{\text{ét}}(\bar{X}_r^3, \mathbf{Z}_p(2)) \right)$$

denotes the kernel of the étale cycle class map, i.e., the group of null-homologous algebraic cycles defined over $\mathbf{Q}$. There are two issues that need to be dealt with. Firstly, the cycles $\Delta_r[d_1, d_2, d_2]$ need not be null-homologous and have to be suitably modified so that they lie in the domain of the Abel Jacobi map. Secondly, these cycles are defined over $\mathbf{Q}(\zeta_r)$ and not over $\mathbf{Q}$, and it is desirable to descend the field of definition of the associated extension classes.

To deal with the first issue, let q be any prime not dividing Mp, and let T_q denote the Hecke operator attached to this prime. It can be used to construct an algebraic correspondence on X_r^3 by setting

$$\theta_q := (T_q - (q+1))^{\otimes 3}.$$

Lemma 2.5. — *The element θ_q annihilates the target $H^4_{\text{ét}}(\bar{X}_r^3, \mathbf{Z}_p)$ of the étale cycle class map on $\text{CH}^2(X_r^3)$.*

Proof. — The correspondence T_q acts as multiplication by $(q+1)$ on $H^2_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)$ and θ_q therefore annihilates all the terms in the Künneth decomposition of $H^4_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)$. $\qquad\square$

The *modified diagonal cycles* in $\text{CH}^2(X_r^3)$ are defined by the rule

(2.14) $\Delta_r^\circ[d_1, d_2, d_3] := \theta_q \Delta_r[d_1, d_2, d_3].$

Lemma 2.5 shows that they are null-homologous and defined over $\mathbf{Q}(\zeta_r)$. Define

$$\kappa_r[d_1, d_2, d_3] \quad := \quad \text{AJ}_{\text{ét}}(\Delta_r^\circ[d_1, d_2, d_3]) \in H^1(\mathbf{Q}(\zeta_r), H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)).$$

To deal with the circumstance that the cycles $\Delta_r^\circ[d_1, d_2, d_3]$ are only defined over $\mathbf{Q}(\zeta_r)$, and hence that the associated cohomology classes $\kappa_r[d_1, d_2, d_3]$ need not (and in fact, do not) extend to $G_{\mathbf{Q}}$, it is necessary to replace the $\mathbf{Z}_p[\tilde{G}_r][G_{\mathbf{Q}}]$-module $H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)$ by an appropriate twist over $\mathbf{Q}(\zeta_r)$. Let G_r denote the Sylow p-subgroup of the group $\tilde{G}_r$ of (2.6), and let $G_\infty := \varprojlim G_r$. Let

$$\Lambda(G_r) := \mathbf{Z}_p[G_r], \qquad \Lambda(G_\infty) = \mathbf{Z}_p[[G_\infty]]$$

be the finite group ring attached to G_r and the associated Iwasawa algebra, respectively.

Let $\Lambda(G_r)(\pm\frac{1}{2})$ denote the Galois module which is isomorphic to $\Lambda(G_r)$ as a $\Lambda(G_r)$-module, and on which the Galois group $G_{\mathbf{Q}(\zeta_1)}$ is made to act via its quotient $\mathrm{Gal}\,(\mathbf{Q}(\zeta_r)/\mathbf{Q}(\zeta_1)) = 1 + p\mathbf{Z}/p^r\mathbf{Z}$, the element σ_m acting as multiplication by the group-like element $\langle m, m, m \rangle^{\mp 1/2}$. Let $\Lambda(G_\infty)(\pm\frac{1}{2})$ denote the projective limit of the $\Lambda(G_r)(\pm\frac{1}{2})$. It follows from the definitions that if

$$\nu_{k_\circ, \ell_\circ, m_\circ} : \Lambda(G_r) \longrightarrow \mathbf{Z}/p^r\mathbf{Z}, \qquad \text{or} \qquad \nu_{k_\circ, \ell_\circ, m_\circ} : \Lambda(G_\infty) \longrightarrow \mathbf{Z}_p$$

is the homomorphism sending $\langle a_1, a_2, a_3 \rangle$ to $a_1^{k_\circ} a_2^{\ell_\circ} a_3^{m_\circ}$, then

$$(2.15) \qquad \Lambda(G_r)(\tfrac{1}{2}) \otimes_{\nu_{k_\circ, \ell_\circ, m_\circ}} \mathbf{Z}/p^r\mathbf{Z} = (\mathbf{Z}/p^r\mathbf{Z})(\varepsilon_{\mathrm{cyc}}^{-(k_\circ + \ell_\circ + m_\circ)/2}),$$

where the tensor product is taken over $\Lambda(G_r)$, and similarly for G_∞. In particular if $k_\circ + \ell_\circ + m_\circ = 2t$ is an even integer,

$$(2.16) \qquad \Lambda(G_\infty)(\tfrac{1}{2}) \otimes_{\nu_{k_\circ, \ell_\circ, m_\circ}} \mathbf{Z}_p = \mathbf{Z}_p(-t)(\omega^t)$$

as $G_{\mathbf{Q}}$-modules. More generally, if Ω is any $\Lambda(G_\infty)$ module, write

$$\Omega(\tfrac{1}{2}) := \Omega \otimes_{\Lambda(G_\infty)} \Lambda(G_\infty)(\tfrac{1}{2}), \qquad \Omega(\tfrac{-1}{2}) := \Omega \otimes_{\Lambda(G_\infty)} \Lambda(G_\infty)(\tfrac{-1}{2}),$$

for the relevant twists of Ω, which are isomorphic to Ω as a $\Lambda(G_\infty)[G_{\mathbf{Q}(\mu_{p^\infty})}]$-module but are endowed with different actions of $G_{\mathbf{Q}}$.

There is a canonical Galois-equivariant $\Lambda(G_r)$-hermitian bilinear, $\Lambda(G_r)$-valued pairing

$$(2.17) \qquad \langle\!\langle\ ,\ \rangle\!\rangle_r : H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}) \ \times \ H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}) \ \longrightarrow \ \Lambda(G_r),$$

given by the formula

$$\langle\!\langle a, b \rangle\!\rangle_r := \sum_{\sigma = \langle d_1, d_2, d_3 \rangle \in G_r} \langle a^\sigma, b \rangle_{X_r} \cdot \langle d_1, d_2, d_3 \rangle,$$

where

$$\langle\ ,\ \rangle_{X_r} : H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2) \times H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1) \longrightarrow H^2_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p(1))^{\otimes 3} = \mathbf{Z}_p$$

arises from the Poincaré duality between $H^3_{\text{ét}}(\bar{X}_r^3, \mathbf{Z}_p)(2)$ and $H^3_{\text{ét}}(\bar{X}_r^3, \mathbf{Z}_p)(1)$. This pairing enjoys the following properties:

— For all $\lambda \in \Lambda(G_r)$,

$$\langle\!\langle \lambda a, b \rangle\!\rangle_r = \lambda^* \langle\!\langle a, b \rangle\!\rangle_r, \qquad \langle\!\langle a, \lambda b \rangle\!\rangle_r = \lambda \langle\!\langle a, b \rangle\!\rangle_r,$$

where $\lambda^* \in \Lambda(G_r)$ is obtained from λ by applying the involution on the group ring which sends every group-like element to its inverse. In particular, the pairing of (2.17) can and will also be viewed as a $\Lambda(G_r)$-valued $*$-hermitian pairing

$$\langle\!\langle\ ,\ \rangle\!\rangle_r : H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2) \times H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1) \longrightarrow \Lambda(G_r).$$

— For all $\sigma \in G_{\mathbf{Q}(\zeta_1)}$, we have $\langle\!\langle \sigma a, \sigma b \rangle\!\rangle_r = \langle\!\langle a, b \rangle\!\rangle_r$.

— The U_p and U_p^* operators are adjoint to each other under this pairing, giving rise to a duality (denoted by the same symbol, by an abuse of notation)

$$\langle\!\langle \ , \ \rangle\!\rangle_r : e^* H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}) \ \times \ e H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}) \ \longrightarrow \ \Lambda(G_r).$$

Define

$$\mathbb{H}^{111}(X_r) \ := \ \text{Hom}_{\Lambda(G_r)}(H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_r)) \simeq H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}),$$

$$\mathbb{H}^{111}_\circ(X_r) \ := \ \text{Hom}_{\Lambda(G_r)}(e H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_r)) \simeq e^* H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}).$$

The above identifications of $\mathbf{Z}_p[G_{\mathbf{Q}(\zeta_1)}]$-modules follow from the pairing (2.17).

To descend the field of definition of the classes $\kappa_r[d_1, d_2, d_3]$, we package them together into elements

$$\boldsymbol{\kappa}_r[a, b, c] \ \in \ H^1(\mathbf{Q}(\zeta_r), \mathbb{H}^{111}(X_r))$$

indexed by triples

(2.18) $$[a, b, c] \in I_1 = (\mathbf{Z}/p\mathbf{Z})^{\times 3} = \mu_{p-1}(\mathbf{Z}_p)^3 \subset (\mathbf{Z}_p^\times)^3.$$

The class $\boldsymbol{\kappa}_r[a, b, c]$ is defined by setting, for all $\sigma \in G_{\mathbf{Q}(\zeta_r)}$ and all $\gamma_r \in H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)$,

(2.19) $$\boldsymbol{\kappa}_r[a, b, c](\sigma)(\gamma_r) = \langle\!\langle \kappa_r[a, b, c](\sigma), \gamma_r \rangle\!\rangle_r,$$

where the elements $a, b, c \in (\mathbf{Z}/p\mathbf{Z})^\times$ are viewed as elements of $(\mathbf{Z}/p^r\mathbf{Z})^\times$ via the Teichmuller lift alluded to in (2.18). Note that there is a natural identification

$$H^1(\mathbf{Q}(\zeta_r), \mathbb{H}^{111}(X_r)) = \text{Ext}^1_{\Lambda(G_r)[G_{\mathbf{Q}(\zeta_r)}]}(H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1), \Lambda(G_r),$$

because $H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1) = H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2})$ *as $G_{\mathbf{Q}(\zeta_r)}$-modules* and the $\Lambda(G_r)$-dual of the latter is $\mathbb{H}^{111}(X_r)$. With these definitions we have

Lemma 2.6. — *The class $\boldsymbol{\kappa}_r[a, b, c]$ is the restriction to $G_{\mathbf{Q}(\zeta_r)}$ of a class*

$$\boldsymbol{\kappa}_r[a, b, c] \in H^1(\mathbf{Q}(\zeta_1), \mathbb{H}^{111}(X_r)) = \text{Ext}^1_{\Lambda(G_r)[G_{\mathbf{Q}(\zeta_1)}]}(H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_r)).$$

Furthermore, for all $m \in \mu_{p-1}(\mathbf{Z}_p)$,

$$\sigma_m \ \boldsymbol{\kappa}_r[a, b, c] = \boldsymbol{\kappa}_r[ma, mb, mc].$$

Proof. — We will prove this by giving a more conceptual description of the cohomology class $\boldsymbol{\kappa}_r[a, b, c]$. Let $|\Delta|$ denote the support of an algebraic cycle Δ, and let

(2.20) $$\Delta_r^\circ[[a, b, c]] := |\Delta_1^\circ[a, b, c]| \times_{X_1^3} X_r^3$$

denote the inverse image in X_r^3 of $|\Delta_1^\circ[a, b, c]|$, which fits into the cartesian diagram

$$\begin{array}{ccc} \Delta_r^\circ[[a, b, c]] & \hookrightarrow & X_r^3 \\ \downarrow & & \downarrow {\scriptstyle (\varpi_1^{r-1})^3} \\ |\Delta_1^\circ[a, b, c]| & \hookrightarrow & X_1^3. \end{array}$$

As in the proof of Lemma 2.1, observe that

$$\Delta_r^\circ[[a,b,c]] = \bigsqcup_{[d_1,d_2,d_3] \in I_r^1} |\Delta_r^\circ[ad_1, bd_2, cd_3]|$$

where I_r^1 denotes the p-Sylow subgroup of I_r. Consider now the commutative diagram of $\Lambda(G_r)[G_{\mathbf{Q}}(\zeta_1)]$-modules with exact rows:

(2.21)

$$\Lambda(G_r)(\tfrac{-1}{2})$$

$$\downarrow{\scriptstyle j}$$

$$H^3_{\text{ét}}(\bar{X}_r^3, \mathbf{Z}_p)(2) \hookrightarrow H^3_{\text{ét}}(\bar{X}_r^3 - \Delta_r^\circ[[a,b,c]], \mathbf{Z}_p)(2) \longrightarrow H^0_{\text{ét}}(\bar{\Delta}_r^\circ[[a,b,c]], \mathbf{Z}_p)_0$$

$$\downarrow{\scriptstyle p}$$

$$H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2),$$

where
— the map j is the inclusion defined on group-like elements by

$$j\left(\langle d_1, d_2, d_3 \rangle\right) = \text{cl}(\Delta_r^\circ[ad_2d_3, bd_1d_3, cd_1d_2]),$$

which is $G_{\mathbf{Q}(\zeta_1)}$-equivariant by Lemma 2.2;
— the middle row arises from the excision exact sequence in étale cohomology (cf. [J88, (3.6)] and [M, p. 108]);
— the subscript of 0 appearing in the rightmost term in the exact sequence denotes the kernel of the cycle class map, i.e.,

$$H^0_{\text{ét}}(\bar{\Delta}_r^\circ[[a,b,c]], \mathbf{Z}_p)_0 := \ker\left(H^0_{\text{ét}}(\bar{\Delta}_r^\circ[[a,b,c]], \mathbf{Z}_p)_0 \longrightarrow H^4_{\text{ét}}(\bar{X}_r^3, \mathbf{Z}_p(2))\right),$$

and the fact that the image of j is contained in $H^0_{\text{ét}}(\bar{\Delta}_r^\circ[[a,b,c]], \mathbf{Z}_p)_0$ follows from Lemma 2.5;
— the projection p is the one arising from the Künneth decomposition.
Taking the pushout and pullback of the extension in (2.21) via the maps p and j yields an exact sequence of $\Lambda(G_r)[G_{\mathbf{Q}}(\zeta_1)]$-modules

(2.22) $$0 \longrightarrow H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2) \longrightarrow E_r \longrightarrow \Lambda(G_r)(\tfrac{-1}{2}) \longrightarrow 0.$$

Taking the $\Lambda(G_r)$-dual of this exact sequence, we obtain

$$0 \longrightarrow \Lambda(G_r)(\tfrac{1}{2}) \longrightarrow \check{E}_r \longrightarrow H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)^* \longrightarrow 0.$$

where M^* means the $\Lambda(G_r)$-module obtained from M by letting act $\Lambda(G_r)$ on it by composing with the involution $\lambda \mapsto \lambda^*$. Twisting this sequence by $(\tfrac{-1}{2})$ and noting that $M^*(\tfrac{-1}{2}) \simeq M(\tfrac{1}{2})^*$ yields an extension

(2.23) $$0 \longrightarrow \Lambda(G_r) \longrightarrow E_r' \longrightarrow H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2})^* \longrightarrow 0.$$

Since

$$H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2})^* = \text{Hom}_{\Lambda(G_r)}(H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}), \Lambda(G_r)),$$

it follows that the cohomology class realizing the extension E'_r is an element of

$$H^1(\mathbf{Q}(\zeta_1), \text{Hom}_{\Lambda(G_r)}(H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_r))) = H^1(\mathbf{Q}(\zeta_1), \mathbb{H}^{111}(X_r)),$$

because the duality afforded by $\langle\!\langle\ ,\ \rangle\!\rangle_r$ is hermitian (and not Λ-linear). When restricted to $G_{\mathbf{Q}(\zeta_r)}$, this class coincides with $\boldsymbol{\kappa}_r[a, b, c]$, and the first assertion follows.

The second assertion is an immediate consequence of the definitions, using the Galois equivariance properties of the cycles $\Delta_r[d_1, d_2, d_3]$ given in the first assertion of Lemma 2.2. $\qquad\square$

Remark 2.7. — *The extension E'_r of (2.23) can also be realised as a subquotient of the étale cohomology group $H^3_c(\bar{X}^3_r{-}\Delta^\circ_r[[a, b, c]], \mathbf{Z}_p)(1)$ with compact supports, in light of the Poincaré duality*

$$H^3_{\text{ét}}(\bar{X}^3_r{-}\Delta^\circ_r[[a, b, c]], \mathbf{Z}_p)(2) \ \times \ H^3_c(\bar{X}^3_r{-}\Delta^\circ_r[[a, b, c]], \mathbf{Z}_p)(1) \longrightarrow \mathbf{Z}_p.$$

2.3. Λ-adic cohomology classes. — Thanks to Lemma 2.6, we now dispose, for each $[a, b, c] \in \mu_{p-1}(\mathbf{Z}_p)^3$, of a system

$$(2.24) \qquad\qquad \boldsymbol{\kappa}_r[a, b, c] \in H^1(\mathbf{Q}(\zeta_1), \mathbb{H}^{111}(X_r))$$

of cohomology classes indexed by the integers $r \geq 1$, so that $e^*\boldsymbol{\kappa}_r[a, b, c] \in H^1(\mathbf{Q}(\zeta_1), \mathbb{H}^{111}_\circ(X_r))$. Let

$$p_{r+1, r} : \Lambda(G_{r+1}) \longrightarrow \Lambda(G_r)$$

be the projection on finite group rings induced from the natural homomorphism $G_{r+1} \longrightarrow G_r$.

Lemma 2.8. — *Let $\gamma_{r+1} \in H^1_{\text{ét}}(\bar{X}_{r+1}, \mathbf{Z}_p)^{\otimes 3}(1)$ and $\gamma_r \in H^1_{\text{ét}}(\bar{X}_r, \mathbf{Z}_p)^{\otimes 3}(1)$ be elements that are compatible under the pushforward by ϖ^3_1, i.e., that satisfy $(\varpi^3_1)_*(\gamma_{r+1}) = \gamma_r$. For all $\sigma \in G_{\mathbf{Q}(\zeta_1)}$,*

$$p_{r+1, r}\left(\boldsymbol{\kappa}_{r+1}[a, b, c](\sigma)(\gamma_{r+1})\right) = \boldsymbol{\kappa}_r[a, b, c](\sigma)(\gamma_r).$$

Proof. — This amounts to the statement that

$$p_{r+1, r}(\langle\!\langle \kappa_{r+1}[a, b, c], \gamma_{r+1} \rangle\!\rangle_{r+1}) = \langle\!\langle \kappa_r[a, b, c], \gamma_r \rangle\!\rangle_r.$$

But the left-hand side of this equation is equal to

$$\sum_{G_r} \langle (\mu^3)^*(\mu^3)_* \kappa_{r+1}[ad'_2 d'_3, bd'_1 d'_3, cd'_1 d'_2], \gamma_{r+1}\rangle_{X_{r+1}} \cdot \langle d_1, d_2, d_3 \rangle,$$

where the sum runs over $\langle d_1, d_2, d_3 \rangle \in G_r$ and $\langle d_1', d_2', d_3' \rangle$ denotes an (arbitrary) lift of $\langle d_1, d_2, d_3 \rangle$ to G_{r+1}. The third assertion in Lemma 2.4 allows us to rewrite this as

$$\sum_{G_r} \langle (\varpi_1^3)^* \kappa_r[ad_2 d_3, bd_1 d_3, cd_1 d_2], \gamma_{r+1} \rangle_{X_{r+1}} \cdot \langle d_1, d_2, d_3 \rangle$$

$$= \sum_{G_r} \langle \kappa_r[ad_2 d_3, bd_1 d_3, cd_1 d_2], (\varpi_1^3)_* \gamma_{r+1} \rangle_{X_r} \cdot \langle d_1, d_2, d_3 \rangle$$

$$= \sum_{G_r} \langle \kappa_r[ad_2 d_3, bd_1 d_3, cd_1 d_2], \gamma_r \rangle_{X_r} \cdot \langle d_1, d_2, d_3 \rangle$$

$$= \langle\!\langle \kappa_r[a, b, c], \gamma_r \rangle\!\rangle_r,$$

and the result follows. $\qquad\square$

Define

$$(2.25) \qquad \mathbb{H}^{111}(X_\infty^*) := \quad \mathrm{Hom}_{\Lambda(G_\infty)}(H^1_{\text{ét}}(\bar{X}_\infty^*, \mathbf{Z}_p)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_\infty))$$

$$= \mathrm{Hom}_{\Lambda(G_\infty)}(H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}_\infty^*)^{\otimes 3}(1)(\tfrac{1}{2}), \Lambda(G_\infty)),$$

where the identification follows from (1.19).

Thanks to Lemma 2.8, the classes $\kappa_r[a, b, c]$ can be packaged into a compatible collection. Namely:

Definition 2.9. — *Set*

$$(2.26) \qquad \kappa_\infty[a, b, c] := (\kappa_r[a, b, c])_{r \geq 1} \in H^1(\mathbf{Q}(\zeta_1), \mathbb{H}^{111}(X_\infty^*)).$$

It will also be useful to replace the classes $\kappa_\infty[a, b, c]$ by elements that are essentially indexed by triples

$$(\omega_1, \omega_2, \omega_3) : (\mathbf{Z}/p\mathbf{Z}^\times)^3 \longrightarrow \mathbf{Z}_p^\times$$

of tame characters of $\tilde{G}_r / G_r$. Assume that the product $\omega_1 \omega_2 \omega_3$ is an *even* character. This assumption is equivalent to requiring that

$$\omega_1 \omega_2 \omega_3 = \delta^2, \quad \text{for some } \delta : (\mathbf{Z}/p\mathbf{Z})^\times \longrightarrow \mathbf{Z}_p^\times.$$

Note that for a given $(\omega_1, \omega_2, \omega_3)$, there are in fact two characters δ as above, which differ by the unique quadratic character of conductor p. With the choices of $\omega_1, \omega_2, \omega_3$ and δ in hand, we set

$$(2.27) \quad \kappa_\infty(\omega_1, \omega_2, \omega_3; \delta) := \frac{p^3}{(p-1)^3} \cdot \sum_{[a,b,c]} \delta^{-1}(abc) \cdot \omega_1(a)\omega_2(b)\omega_3(c) \cdot \kappa_\infty[bc, ac, ab],$$

where the sum is taken over the triples $[a, b, c]$ of $(p-1)$st roots of unity in $\mathbf{Z}_p^\times$. The classes $\kappa_\infty(\omega_1, \omega_2, \omega_3; \delta)$ satisfy the following properties.

Lemma 2.10. — *For all* $\sigma_m \in \mathrm{Gal}(\mathbf{Q}(\zeta_\infty)/\mathbf{Q})$,

$$\sigma_m \kappa_\infty(\omega_1, \omega_2, \omega_3; \delta) = \delta(m)\kappa_\infty(\omega_1, \omega_2, \omega_3; \delta).$$

For all diamond operators $\langle a_1, a_2, a_3 \rangle \in \mu_{p-1}(\mathbf{Z}_p)^3$

$$\langle a_1, a_2, a_3 \rangle \kappa_\infty(\omega_1, \omega_2, \omega_3; \delta) = \omega_{123}(a_1, a_2, a_3) \cdot \kappa_\infty(\omega_1, \omega_2, \omega_3; \delta).$$

Proof. — This follows from a direct calculation based on the definitions, using the compatibilities of Lemma 2.2 satisfied by the cycles $\Delta_r[d_1, d_2, d_3]$.   □

The classes $\boldsymbol{\kappa}_\infty[a, b, c]$ and $\boldsymbol{\kappa}_\infty(\omega_1, \omega_2, \omega_3; \delta)$ are called the Λ-*adic cohomology classes* attached to the triple $[a, b, c] \in \mu_{p-1}(\mathbf{Z}_p)^3$ or the quadruple $(\omega_1, \omega_2, \omega_3; \delta)$. As will be explained in the next section, they are three variable families of cohomology classes parametrised by points in the triple product $\mathcal{W} \times \mathcal{W} \times \mathcal{W}$ of weight spaces, and taking values in the three-parameter family of self-dual Tate twists of the Galois representations attached to the different specialisations of a triple of Hida families.

3. Higher weight balanced specialisations

For every integer $k_\circ \geq 0$ define

$$W_1^{k_\circ} := H^1_{\text{ét}}(\bar{X}_1, \mathcal{H}^{k_\circ})$$

and recall from the combination of (1.19), (1.21) and (1.22) the specialisation map

$$(3.1) \qquad \text{sp}_{k_\circ}^* : H^1_{\text{ét}}(\bar{X}_\infty^*, \mathbf{Z}_p) = H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}_\infty^*) \longrightarrow W_1^{k_\circ}.$$

Fix throughout this section a triple

$$k = k_\circ + 2, \qquad \ell = \ell_\circ + 2, \qquad m = m_\circ + 2$$

of integers ≥ 2 for which $k_\circ + \ell_\circ + m_\circ = 2t$ is even. Let

$$\mathcal{H}^{k_\circ, \ell_\circ, m_\circ} := \mathcal{H}^{k_\circ} \boxtimes \mathcal{H}^{\ell_\circ} \boxtimes \mathcal{H}^{m_\circ}$$

viewed as a sheaf on X_1^3, and

$$W_1^{k_\circ, \ell_\circ, m_\circ} := W_1^{k_\circ} \otimes W_1^{\ell_\circ} \otimes W_1^{m_\circ}(2 - t).$$

As one readily checks, the p-adic Galois representation $W_1^{k_\circ, \ell_\circ, m_\circ}$ is Kummer self-dual, i.e., there is an isomorphism of $G_{\mathbf{Q}}$-modules

$$\text{Hom}_{G_{\mathbf{Q}}}(W_1^{k_\circ, \ell_\circ, m_\circ}, \mathbf{Z}_p(1)) \simeq W_1^{k_\circ, \ell_\circ, m_\circ}.$$

The specialisation maps give rise, in light of (2.16), to the triple product specialisation map

$$(3.2) \qquad \text{sp}_{k_\circ, \ell_\circ, m_\circ}^* := \text{sp}_{k_\circ}^* \otimes \text{sp}_{\ell_\circ}^* \otimes \text{sp}_{m_\circ}^* : \mathbb{H}^{111}(X_\infty^*) \longrightarrow W_1^{k_\circ, \ell_\circ, m_\circ}$$

and to the associated collection of specialised classes

$$(3.3) \qquad \kappa_1(k_\circ, \ell_\circ, m_\circ)[a, b, c] := \text{sp}_{k_\circ, \ell_\circ, m_\circ}(\boldsymbol{\kappa}_\infty[a, b, c]) \in H^1(\mathbf{Q}(\zeta_1), W_1^{k_\circ, \ell_\circ, m_\circ}).$$

Note that for $(k_\circ, \ell_\circ, m_\circ) = (0, 0, 0)$, it follows from the definitions (cf. e.g. the proof of Lemma 2.6) that the class $\kappa_1(k_\circ, \ell_\circ, m_\circ)[a, b, c]$ is simply the image under the étale Abel-Jacobi map of the cycle $\Delta_1^\circ[a, b, c]$.

The main goal of this section is to offer a similar geometric description for the above classes also when (k, ℓ, m) is *balanced* and $k_\circ, \ell_\circ, m_\circ > 0$, which we assume henceforth for the remainder of this section.

In order to do this, it shall be useful to dispose of an alternate description of the extension (2.22) in terms of the étale cohomology of the (open) three-fold $X_1^3 - |\Delta_1^\circ[a,b,c]|$ with values in appropriate sheaves.

Lemma 3.1. — Let $\mathcal{L}_r^{*\boxtimes 3}$ denote the exterior tensor product of $\mathcal{L}_r^*$, over the triple product X_1^3. There is a commutative diagram

$$
\begin{array}{ccccc}
H_{\text{ét}}^3(X_r^3, \mathbf{Z}_p)(2) & \longrightarrow & H_{\text{ét}}^3(X_r^3 - \Delta_r^\circ[[a,b,c]], \mathbf{Z}_p)(2) & \longrightarrow & H_{\text{ét}}^0(\Delta_r^\circ[[a,b,c]], \mathbf{Z}_p) \\
\| & & \| & & \| \\
H_{\text{ét}}^3(\bar{X}_1^3, \mathcal{L}_r^{*\boxtimes 3})(2) & \longrightarrow & H_{\text{ét}}^3(\bar{X}_1^3 - |\Delta_1^\circ[a,b,c]|, \mathcal{L}_r^{*\boxtimes 3})(2) & \longrightarrow & H_{\text{ét}}^0(|\Delta_1^\circ[a,b,c]|, \mathcal{L}_r^{*\otimes 3})),
\end{array}
$$

in which the leftmost maps are injective and the horizontal sequences are exact.

Proof. — Recall from (1.17) that

$$
\mathcal{L}_r^{*\boxtimes 3} = (\varpi_1^{r-1} \times \varpi_1^{r-1} \times \varpi_1^{r-1})_* \mathbf{Z}_p,
$$

where

$$
\varpi_1^{r-1} \times \varpi_1^{r-1} \times \varpi_1^{r-1} : X_r^3 \longrightarrow X_1^3
$$

is defined as in (2.2). The vertical isomorphisms then follow from Shapiro's lemma and the definition of $\Delta_r^\circ[[a,b,c]]$ in (2.20). The horizontal sequence arises from the excision exact sequence in étale cohomology of [J88, (3.6)] and [M, p. 108]. $\square$

Lemma 3.2. — For all $[a,b,c] \in I_1$,

$$
H_{\text{ét}}^0(\bar{\Delta}_1[a,b,c], \mathcal{H}^{k_\circ, \ell_\circ, m_\circ}) = \mathbf{Z}_p(t).
$$

Proof. — The Clebsch-Gordan formula asserts that the space of tri-homogenous polynomials in $6 = 2 + 2 + 2$ variables of tridegree $(k_\circ, \ell_\circ, m_\circ)$ has a unique SL_2-invariant element, namely, the polynomial

$$
P_{k_\circ, \ell_\circ, m_\circ}(x_1, y_1, x_2, y_2, x_3, y_3) = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}^{k_\circ'} \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}^{\ell_\circ'} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}^{m_\circ'},
$$

where

$$
k_\circ' = \frac{-k_\circ + \ell_\circ + m_\circ}{2}, \qquad \ell_\circ' = \frac{k_\circ - \ell_\circ + m_\circ}{2} \qquad m_\circ' = \frac{k_\circ + \ell_\circ - m_\circ}{2}.
$$

Since the triplet of weights is balanced, it follows that $k_\circ', \ell_\circ', m_\circ' \geq 0$. From the Clebsch-Gordan formula it follows that $H_{\text{ét}}^0(\bar{\Delta}_1[a,b,c], \mathcal{H}^{k_\circ, \ell_\circ, m_\circ})$ is spanned by the global section whose stalk at a point $((A, P_1), (A, P_2), (A, P_3)) \in \Delta_1[a,b,c]$ as in (2.5) is given by

$$
(X_2 \otimes Y_3 - Y_2 \otimes X_3)^{\otimes k_\circ'} \otimes (X_1 \otimes Y_3 - Y_1 \otimes X_3)^{\otimes \ell_\circ'} \otimes (X_1 \otimes Y_2 - Y_1 \otimes X_2)^{\otimes m_\circ'},
$$

where (X_i, Y_i), $i = 1, 2, 3$, is a basis of the stalk of $\mathcal{H}$ at the point (A, P_i) in X_1. The Galois action is given by the t-th power of the cyclotomic character because the Weil pairing takes values in $\mathbf{Z}_p(1)$ and $k_\circ' + \ell_\circ' + m_\circ' = t$. $\square$

Write $\mathrm{cl}_{k_\circ, \ell_\circ, m_\circ}(\Delta_1[a,b,c]) \in H^0_{\text{ét}}(|\bar{\Delta}^\circ_1[a,b,c]|, \mathcal{H}^{k_\circ, \ell_\circ, m_\circ})$ for the standard generator given by Lemma 3.2. Define

$$(3.4) \qquad \mathrm{AJ}_{k_\circ, \ell_\circ, m_\circ}(\Delta_1[a,b,c]) \in H^1(\mathbf{Q}(\zeta_1), W^{k_\circ, \ell_\circ, m_\circ}_1)$$

to be the extension class constructed by pulling back by j and pushing forward by p in the exact sequence of the middle row of the following diagram:
(3.5)

$$
\begin{array}{ccc}
& & \mathbf{Z}_p(t) \\
& & \downarrow{\scriptstyle j} \\
H^3_{\text{ét}}(\bar{X}^3_1, \mathcal{H}^{k_\circ, \ell_\circ, m_\circ})(2) \hookrightarrow H^3_{\text{ét}}(\bar{X}^3_1 - \bar{\Delta}, \mathcal{H}^{k_\circ, \ell_\circ, m_\circ})(2) \twoheadrightarrow H^0_{\text{ét}}(\bar{\Delta}, \mathcal{H}^{k_\circ, \ell_\circ, m_\circ}) \\
\downarrow & & \\
W^{k_\circ, \ell_\circ, m_\circ}_1(t), & &
\end{array}
$$

where
— $\Delta = \Delta_1[a,b,c]$;
— the map j is the $G_{\mathbf{Q}(\zeta_1)}$-equivariant inclusion defined by $j(1) = \mathrm{cl}_{k_\circ, \ell_\circ, m_\circ}(\Delta)$;
— the surjectivity of the right-most horizontal row follows from the vanishing of the group $H^4_{\text{ét}}(\bar{X}^3_1, \mathcal{H}^{k_\circ, \ell_\circ, m_\circ})$, which in turn is a consequence of the Künneth formula and the vanishing of the terms $H^2_{\text{ét}}(\bar{X}_1, \mathcal{H}^{k_\circ})$ when $k_\circ > 0$ (cf. [**BDP13**, Lemmas 2.1, 2.2]).

In particular the image of j lies in the image of the right-most horizontal row and this holds regardless whether the cycle is null-homologous or not. The reader may compare this construction with (2.21), where the cycle $\Delta^\circ_r[[a,b,c]]$ is null-homologous and this property was crucially exploited.

Theorem 3.1. — *Set* $\mathrm{AJ}_{k_\circ, \ell_\circ, m_\circ}(\Delta^\circ_1[a,b,c]) = \theta_q \mathrm{AJ}_{k_\circ, \ell_\circ, m_\circ}(\Delta_1[a,b,c])$. *Then the identity*

$$\kappa_1(k_\circ, \ell_\circ, m_\circ)[a,b,c] = \mathrm{AJ}_{k_\circ, \ell_\circ, m_\circ}(\Delta^\circ_1[a,b,c])$$

holds in $H^1(\mathbf{Q}(\zeta_1), W^{k_\circ, \ell_\circ, m_\circ}_1)$.

Proof. — Set $\Delta := \Delta^\circ_1[a,b,c]$ in order to alleviate notations. Thanks to Lemma 3.1, the diagram in (2.21) used to construct the extension E_r realising the class $\boldsymbol{\kappa}_r[a,b,c]$ is *the same* as the diagram
(3.6)

$$
\begin{array}{ccc}
& & \Lambda(G_r)(\frac{-1}{2}) \\
& & \uparrow \\
0 \longrightarrow H^3_{\text{ét}}(\bar{X}^3_1, \mathcal{L}^{*\boxtimes 3}_r)(2) \longrightarrow H^3_{\text{ét}}(\bar{X}^3_1 - |\bar{\Delta}|, \mathcal{L}^{*\boxtimes 3}_r)(2) \longrightarrow H^0_{\text{ét}}(|\bar{\Delta}|, \mathcal{L}^{*\otimes 3}_r) \\
\downarrow & & \\
H^1_{\text{ét}}(\bar{X}_1, \mathcal{L}^*_r)^{\otimes 3}(2). & &
\end{array}
$$

Let

$$\nu_{k_\circ,\ell_\circ,m_\circ} : \Lambda(G_r) \longrightarrow \mathbf{Z}/p^r\mathbf{Z}$$

be the algebra homomorphism sending the group like element $\langle d_1, d_2, d_3 \rangle$ to $d_1^{k_\circ} d_2^{\ell_\circ} d_3^{m_\circ}$, and observe that the moment maps of (1.20) allow us to identify

$$\mathcal{L}_r^{*\boxtimes 3} \otimes_{\nu_{k_\circ,\ell_\circ,m_\circ}} (\mathbf{Z}/p^r\mathbf{Z}) = \mathcal{H}_r^{k_\circ,\ell_\circ,m_\circ}.$$

Tensoring (3.6) over $\Lambda(G_r)$ with $\mathbf{Z}/p^r\mathbf{Z}$ via the map $\nu_{k_\circ,\ell_\circ,m_\circ} : \Lambda(G_r) \longrightarrow \mathbf{Z}/p^r\mathbf{Z}$, yields the specialised diagram which coincides exactly with the mod p^r reduction of (3.5), with $\Delta = \Delta_1^\circ[a,b,c]$. The result follows by passing to the limit with r. $\qquad\square$

Corollary 3.3. — *Let*

$$(3.7) \qquad \Delta_1^\circ(\omega_1,\omega_2,\omega_3;\delta) := \frac{p^3}{(p-1)^3} \cdot \sum_{[a,b,c] \in I_1} \delta^{-1}(abc)\omega_1(a)\omega_2(b)\omega_3(c)\Delta_1^\circ[a,b,c].$$

Then

$$\mathrm{sp}_{k_\circ,\ell_\circ,m_\circ}^*(\boldsymbol{\kappa}_\infty(\omega_1,\omega_2,\omega_3;\delta)) = \mathrm{AJ}_{k_\circ,\ell_\circ,m_\circ}(\Delta_1^\circ(\omega_1,\omega_2,\omega_3;\delta)).$$

Proof. — This follows directly from the definitions. $\qquad\square$

4. Cristalline specialisations

Let $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ be three arbitrary primitive, residually irreducible p-adic Hida families of tame levels M_f, M_g, M_h and tame characters χ_f, χ_g, χ_h, respectively, with associated weight space $\mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{g}}$. Assume $\chi_f\chi_g\chi_h = 1$ and set $M = \mathrm{lcm}(M_f, M_g, M_h)$. Let $(x, y, z) \in \mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}}$ be a point lying above a classical triple $(\nu_{k_\circ,\epsilon_1}, \nu_{\ell_\circ,\epsilon_2}, \nu_{m_\circ,\epsilon_3}) \in \mathcal{W}^3$ of weight space. As in Definition 1.2, the point (x, y, z) is said to be *tamely ramified* if the three characters ϵ_1, ϵ_2 and ϵ_3 are tamely ramified, i.e., factor through the quotient $(\mathbf{Z}/p\mathbf{Z})^\times$ of $\mathbf{Z}_p^\times$, and is said to be *crystalline* if $\epsilon_1\omega^{-k_\circ} = \epsilon_2\omega^{-\ell_\circ} = \epsilon_3\omega^{-m_\circ} = 1$.

Fix such a crystalline point (x, y, z) of balanced weight $(k, \ell, m) = (k_\circ+2, \ell_\circ+2, m_\circ+2)$, and let $(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$ be the specialisations of $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ at (x, y, z). The ordinariness hypothesis implies that, for all but finitely many exceptions, these eigenforms are the p-stabilisations of newforms of level dividing M, denoted f, g and h respectively:

$$\mathbf{f}_x(q) = f(q) - \beta_f f(q^p), \qquad \mathbf{g}_y = g(q) - \beta_g g(q^p), \qquad \mathbf{h}_z(q) = h(q) - \beta_h h(q^p).$$

Since the point (x, y, z) is fixed throughout this section, the dependency of (f, g, h) on (x, y, z) has been suppressed from the notations, and we also write $(f_\alpha, g_\alpha, h_\alpha) := (\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$ for the ordinary p-stabilisations of f, g and h.

Recall the quotient X_{01} of X_1, having $\Gamma_0(p)$-level structure at p, and the projection map $\mu : X_1 \longrightarrow X_{01}$ introduced in (1.15). By an abuse of notation, the symbol $\mathcal{H}^{k_\circ}$ is also used to denote the étale sheaves appearing in (1.4) over any quotient of X_1, such as X_{01}. Let

$$\begin{aligned}
W_1 &:= H_{\text{ét}}^1(\bar{X}_1, \mathcal{H}^{k_\circ}) \otimes H_{\text{ét}}^1(\bar{X}_1, \mathcal{H}^{\ell_\circ}) \otimes H_{\text{ét}}^1(\bar{X}_1, \mathcal{H}^{m_\circ})(2-t), \\
W_{01} &:= H_{\text{ét}}^1(\bar{X}_{01}, \mathcal{H}^{k_\circ}) \otimes H_{\text{ét}}^1(\bar{X}_{01}, \mathcal{H}^{\ell_\circ}) \otimes H_{\text{ét}}^1(\bar{X}_{01}, \mathcal{H}^{m_\circ})(2-t),
\end{aligned}$$

be the Galois representations arising from the cohomology of X_1 and X_{01} with values in these sheaves. They are endowed with a natural action of the triple tensor product of the Hecke algebras of weight $k_\circ$, $\ell_\circ$, $m_\circ$ and level Mp.

Let $W_1[f_\alpha, g_\alpha, h_\alpha]$ denote the $(f_\alpha, g_\alpha, h_\alpha)$-isotypic component of W_1 on which the Hecke operators act with the same eigenvalues as on $f_\alpha \otimes g_\alpha \otimes h_\alpha$. Let $\pi_{f_\alpha, g_\alpha, h_\alpha} : W_1 \to W_1[f_\alpha, g_\alpha, h_\alpha]$ denote the associated projection. Use similar notations for W_{01}.

Recall the family

$$(4.1) \qquad \boldsymbol{\kappa}_\infty(\epsilon_1 \omega^{-k_\circ}, \epsilon_2 \omega^{-\ell_\circ}, \epsilon_3 \omega^{-m_\circ}; 1) = \boldsymbol{\kappa}_\infty(1,1,1;1)$$

that was introduced in (2.27). By Lemma 2.10, this class lies in $H^1(\mathbf{Q}, \mathbb{H}^{111}(X_\infty^*))$.

Recall the choice of auxiliary prime q made in the definition of the modified diagonal cycle (2.14). We assume now that q is chosen so that $C_q := (a_q(f) - q - 1)(a_q(g) - q - 1)(a_q(h) - q - 1)$ is a p-adic unit. Note that this is possible because the Galois representations $\varrho_{\mathbf{f}}$, $\varrho_{\mathbf{g}}$ and $\varrho_{\mathbf{h}}$ were assumed to be residually irreducible and hence f, g and h are non-Eisenstein mod p. Let

$$(4.2) \quad \kappa_1(f_\alpha, g_\alpha, h_\alpha) := \frac{1}{C_q} \cdot \pi_{f_\alpha, g_\alpha, h_\alpha} \, \mathrm{sp}^*_{x,y,z} \, \boldsymbol{\kappa}_\infty(1,1,1;1)) \in H^1(\mathbf{Q}, W_1[f_\alpha, g_\alpha, h_\alpha])$$

be the specialisation at the crystalline point (x, y, z) of (4.1), after projecting it to the $(f_\alpha, g_\alpha, h_\alpha)$-isotypic component of W_1 via $\pi_{f_\alpha, g_\alpha, h_\alpha}$. We normalize the class by multiplying it by the above constant in order to remove the dependency on the choice of q.

The main goal of this section is to relate this class to the generalised Gross-Schoen diagonal cycles that were studied in [**DR14**], arising from cycles in Kuga-Sato varieties which are fibered over X^3 and have *good reduction* at p.

The fact that (x, y, z) is a crystalline point implies that the diamond operators in $\mathrm{Gal}\,(X_1/X_{01})$ act trivially on the $(f_\alpha, g_\alpha, h_\alpha)$-eigencomponents, and hence the Hecke-equivariant projection $\mu_*^3 : W_1 \longrightarrow W_{01}$ induces an isomorphism

$$\mu_*^3 : W_1[f_\alpha, g_\alpha, h_\alpha] \longrightarrow W_{01}[f_\alpha, g_\alpha, h_\alpha].$$

The first aim is to give a geometric description of the class

$$\kappa_{01}(f_\alpha, g_\alpha, h_\alpha) := \mu_*^3 \kappa_1(f_\alpha, g_\alpha, h_\alpha)$$

in terms of appropriate algebraic cycles. To this end, recall the cycles $\Delta_1[a, b, c] \in \mathrm{CH}^2(X_1^3)$ introduced in (2.4), and let $p^* := \pm p$ be such that $\mathbf{Q}(\sqrt{p^*})$ is the quadratic subfield of $\mathbf{Q}(\zeta_1)$.

Lemma 4.1. — *The cycle $\mu_*^3 \Delta_1[a, b, c]$ depends only on the quadratic residue symbol $(\frac{abc}{p})$ attached to $abc \in (\mathbf{Z}/p\mathbf{Z})^\times$. The cycles*

$$\Delta_{01}^+ \quad := \quad \mu_*^3 \Delta_1[a, b, c] \quad \text{for any } a, b, c \text{ with } \left(\frac{abc}{p}\right) = 1,$$

$$\Delta_{01}^- \quad := \quad \mu_*^3 \Delta_1[a, b, c] \quad \text{for any } a, b, c \text{ with } \left(\frac{abc}{p}\right) = -1,$$

belong to $\mathrm{CH}^2(X_{01}^3/\mathbf{Q}(\sqrt{p^*}))$ and are interchanged by the non-trivial automorphism of $\mathbf{Q}(\sqrt{p^*})$.

Proof. — Arguing as in Lemma 2.2 shows that for all $(d_1, d_2, d_3) \in I_1 = (\mathbf{Z}/p\mathbf{Z})^{\times 3}$,

$$\langle d_1, d_2, d_3 \rangle \Delta_1[a, b, c] = \Delta_1[d_2 d_3 a, d_1 d_3 b, d_1 d_2 c].$$

The orbit of the triple $[a, b, c]$ under the action of I_1 is precisely the set of triples $[a', b', c']$ for which $(\frac{a'b'c'}{p}) = (\frac{abc}{p})$. Since X_{01} is the quotient of X_1 by the group I_1, it follows that $\mu_*^3 \Delta_1[a, b, c]$ depends only on this quadratic residue symbol, and hence that the classes Δ_{01}^+ and Δ_{01}^- in the statement of Lemma 4.1 are well-defined. Furthermore, Lemma 2.6 implies that, for all $m \in (\mathbf{Z}/p\mathbf{Z})^\times$, the Galois automorphism σ_m fixes Δ_{01}^+ and Δ_{01}^- if m is a square modulo p, and interchanges these two cycle classes otherwise. It follows that they are invariant under the Galois group $\mathrm{Gal}\,(\mathbf{Q}(\zeta_1)/\mathbf{Q}(\sqrt{p^*}))$ and hence descend to a pair of conjugate cycles $\Delta_{01}^\pm$ defined over $\mathbf{Q}(\sqrt{p^*})$, as claimed. $\qquad\square$

It follows from this lemma that the algebraic cycle

$$(4.3) \qquad\qquad \Delta_{01} := \Delta_{01}^+ + \Delta_{01}^- \in \mathrm{CH}^2(X_{01}^3/\mathbf{Q}).$$

is defined over $\mathbf{Q}$. To describe it concretely, note that a triple (C_1, C_2, C_3) of distinct cyclic subgroups of order p in an elliptic curve A admits a somewhat subtle discrete invariant in $(\mu_p^{\otimes 3} - \{1\})$ modulo the action of $(\mathbf{Z}/p\mathbf{Z})^{\times 2}$, denoted $o(C_1, C_2, C_3)$ and called the *orientation* of (C_1, C_2, C_3). This orientation is defined by choosing generators P_1, P_2, P_3 of C_1, C_2 and C_3 respectively and setting

$$o(C_1, C_2, C_3) := \langle P_2, P_3 \rangle \otimes \langle P_3, P_1 \rangle \otimes \langle P_1, P_2 \rangle \in \mu_p^{\otimes 3} - \{1\}.$$

It is easy to check that the value of $o(C_1, C_2, C_3)$ in $\mu_p^{\otimes 3} - \{1\}$ only depends on the choices of generators P_1, P_2 and P_3, up to multiplication by a *non-zero square* in $(\mathbf{Z}/p\mathbf{Z})^\times$. In view of (2.5), we then have

$$(4.4) \qquad \Delta_{01} = \{((A, C_1), (A, C_2), (A, C_3)) \quad \text{with} \quad C_1 \neq C_2 \neq C_3\},$$

and

$$\Delta_{01}^+ = \left\{((A, C_1), (A, C_2), (A, C_3)) \quad \text{with } o(C_1, C_2, C_3) = a\zeta_1^{\otimes 3}, \quad a \in (\mathbf{Z}/p\mathbf{Z})^{\times 2}\right\},$$
$$\Delta_{01}^- = \left\{((A, C_1), (A, C_2), (A, C_3)) \quad \text{with } o(C_1, C_2, C_3) = a\zeta_1^{\otimes 3}, \quad a \notin (\mathbf{Z}/p\mathbf{Z})^{\times 2}\right\}.$$

Recall the natural projections

$$\pi_1, \pi_2 : X_{01} \longrightarrow X, \qquad \varpi_1, \varpi_2 : X_1 \longrightarrow X$$

to the curve $X = X_0(M)$ of prime to p level, and set

$$W_0 := H^1_{\text{ét}}(\bar{X}_0, \mathcal{H}^{k_\circ}) \otimes H^1_{\text{ét}}(\bar{X}_0, \mathcal{H}^{\ell_\circ}) \otimes H^1_{\text{ét}}(\bar{X}_0, \mathcal{H}^{m_\circ})(2 - t),$$

The Galois representation W_0 is endowed with a natural action of the triple tensor product of the Hecke algebras of weight $k_\circ$, $\ell_\circ$, $m_\circ$ and level M. Let $W_0[f, g, h]$ denote the (f, g, h)- isotypic component of W_0, on which the Hecke operators act with the same eigenvalues as on $f \otimes g \otimes h$. Note that the U_p^* operator does not act naturally on W_0 and hence one cannot speak of the $(f_\alpha, g_\alpha, h_\alpha)$-eigenspace of this Hecke module. One can, however, denote by $W_1[f, g, h]$ and $W_{01}[f, g, h]$ the (f, g, h)-isotypic component of these Galois representations, in which the action of the U_p^* operators on the three factors are not taken into account. Thus, $W_{01}[f_\alpha, g_\alpha, h_\alpha]$ is

the image of $W_{01}[f, g, h]$ under the ordinary projection, and likewise for W_1. In other words, denoting by $\pi_{f,g,h}$ the projection to the (f, g, h)-isotypic component on any of these modules, one has

$$\pi_{f_\alpha, g_\alpha, h_\alpha} = e^* \pi_{f,g,h}$$

whenever the left-hand projection is defined.

The projection maps

$$(\pi_1, \pi_1, \pi_1) : X_{01}^3 \longrightarrow X^3, \quad (\varpi_1, \varpi_1, \varpi_1) : X_1^3 \longrightarrow X^3$$

induces push-forward maps

$$\begin{aligned}
(\pi_1, \pi_1, \pi_1)_* &: W_{01}[f_\alpha, g_\alpha, h_\alpha] \longrightarrow W_0[f, g, h], \\
(\varpi_1, \varpi_1, \varpi_1)_* &: W_1[f_\alpha, g_\alpha, h_\alpha] \longrightarrow W_0[f, g, h]
\end{aligned}$$

on cohomology, as well as maps on the associated Galois cohomology groups.

The goal is now to relate the class

(4.5) $$(\varpi_1, \varpi_1, \varpi_1)_*(\kappa_1(f_\alpha, g_\alpha, h_\alpha)) = (\pi_1, \pi_1, \pi_1)_*(\kappa_{01}(f_\alpha, g_\alpha, h_\alpha))$$

to those arising from the diagonal cycles on the curve $X_0 = X$, whose level is prime to p.

To do this, it is key to understand how the maps π_{1*} and $(\pi_1, \pi_1, \pi_1)_*$ interact with the Hecke operators, especially with the ordinary and anti-ordinary projectors e and e^*, which do not act naturally on the target of π_{1*}. Consider the map

$$(\pi_1, \pi_2) : W_{01}^{k_\circ} := H^1_{\text{ét}}(\bar{X}_{01}, \mathcal{H}^{k_\circ}) \longrightarrow W_0^{k_\circ} := H^1_{\text{ét}}(\bar{X}_0, \mathcal{H}^{k_\circ}).$$

It is compatible in the obvious way with the good Hecke operators arising from primes $\ell \nmid Mp$, and therefore induces a map

(4.6) $$(\pi_1, \pi_2) : W_{01}^{k_\circ}[f] \longrightarrow W_0^{k_\circ}[f] \oplus W_0^{k_\circ}[f]$$

on the f-isotypic components for this Hecke action. As before, note that $W_{01}^{k_\circ}[f]$ is a priori larger than $W_{01}^{k_\circ}[f_\alpha]$, which is its ordinary quotient.

Let $\xi_f := \chi_f(p)p^{k-1}$ be the determinant of the frobenius at p acting on the two-dimensional Galois representation attached to f, and likewise for g and h.

Lemma 4.2. — *For the map* (π_1, π_2) *as in* (4.6)*,*

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \circ U_p = \begin{pmatrix} a_p(f) & -1 \\ \xi_f & 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix},$$

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \circ U_p^* = \begin{pmatrix} 0 & p \\ -\xi_f p^{-1} & a_p(f) \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}.$$

Proof. — The definitions π_1 and π_2 imply that, viewing U_p and U_p^* (resp. T_p) as correspondences on a Kuga-Sato variety fibered over X_{01} (resp. over X_0), we have

$$\begin{aligned}
\pi_1 U_p &= T_p \pi_1 - \pi_2, & \pi_1 U_p^* &= p\pi_2 \\
\pi_2 U_p &= p[p]\pi_1 & \pi_2 U_p^* &= -[p]\pi_1 + T_p \pi_2,
\end{aligned}$$

where $[p]$ is the correspondence induced by the multiplication by p on the fibers and on the prime-to-p part of the level structure. The result follows by passing to the f-isotypic parts, using the fact that $[p]$ induces multiplication by $\xi_f p^{-1}$ on this isotypic part. $\qquad\square$

For the next calculations, it shall be notationally convenient to introduce the notations

$$\delta_f = \alpha_f - \beta_f, \qquad \delta_g = \alpha_g - \beta_g, \qquad \delta_h = \alpha_h - \beta_h, \qquad \delta_{fgh} = \delta_f \delta_g \delta_h.$$

Lemma 4.3. — *For (π_1, π_2) as in Lemma 4.2,*

$$\pi_1 \circ e = \frac{\alpha_f \pi_1 - \pi_2}{\delta_f}, \qquad \pi_2 \circ e = \frac{\xi_f \pi_1 - \beta_f \pi_2}{\delta_f} = \beta_f \cdot (\pi_1 \circ e),$$

$$\pi_1 \circ e^* = \frac{-\beta_f \pi_1 + p\pi_2}{\delta_f}, \qquad \pi_2 \circ e^* = \frac{-\xi_f p^{-1}\pi_1 + \alpha_f \pi_2}{\delta_f} = p\alpha_f^{-1} \cdot (\pi_1 \circ e^*).$$

Proof. — The matrix identities

$$\begin{pmatrix} a_p(f) & -1 \\ \xi_f & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \beta_f & \alpha_f \end{pmatrix} \begin{pmatrix} \alpha_f & 0 \\ 0 & \beta_f \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \beta_f & \alpha_f \end{pmatrix}^{-1},$$

$$\begin{pmatrix} 0 & p \\ -\xi_f p^{-1} & a_p(f) \end{pmatrix} = \begin{pmatrix} \beta_f & \alpha_f \\ \xi_f p^{-1} & \xi_f p^{-1} \end{pmatrix} \begin{pmatrix} \alpha_f & 0 \\ 0 & \beta_f \end{pmatrix} \begin{pmatrix} \beta_f & \alpha_f \\ \xi_f p^{-1} & \xi_f(p)p^{-1} \end{pmatrix}^{-1},$$

show that

$$\lim \begin{pmatrix} a_p(f) & -1 \\ \xi_f & 0 \end{pmatrix}^{n!} = \begin{pmatrix} 1 & 1 \\ \beta_f & \alpha_f \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \beta_f & \alpha_f \end{pmatrix}^{-1}$$

$$= \delta_f^{-1} \begin{pmatrix} \alpha_f & -1 \\ \xi_f & -\beta_f \end{pmatrix},$$

$$\lim \begin{pmatrix} 0 & p \\ -\xi_f p^{-1} & a_p(f) \end{pmatrix}^{n!} = \delta_f^{-1} \begin{pmatrix} -\beta_f & p \\ -\xi_f p^{-1} & \alpha_f \end{pmatrix},$$

and the result now follows from Lemma 4.2. $\qquad\square$

Lemma 4.4. — *Let $\kappa \in H^1(\mathbf{Q}, W_{01}[f, g, h])$ be any cohomology class with values in the (f, g, h)-isotypic subspace of W_{01}, and let $e, e^* : H^1(\mathbf{Q}, W_{01}[fgh]) \longrightarrow H^1(\mathbf{Q}, W_{01}[f_\alpha, g_\alpha, h_\alpha])$ denote the ordinary and anti-ordinary projections. Then*

$$
\begin{aligned}
(\pi_1, \pi_1, \pi_1)_*(e\kappa) = {}& \delta_{fgh}^{-1} \times \Big\{ \alpha_f \alpha_g \alpha_h (\pi_1, \pi_1, \pi_1)_* \\
& -\alpha_g \alpha_h (\pi_2, \pi_1, \pi_1)_* - \alpha_f \alpha_h (\pi_1, \pi_2, \pi_1)_* - \alpha_f \alpha_g (\pi_1, \pi_1, \pi_2)_* \\
& +\alpha_f (\pi_1, \pi_2, \pi_2)_* + \alpha_g (\pi_2, \pi_1, \pi_2)_* + \alpha_h \cdot (\pi_2, \pi_2, \pi_1)_* \\
& -(\pi_2, \pi_2, \pi_2)_* \Big\}(\kappa).
\end{aligned}
$$

$$(\pi_1, \pi_1, \pi_1)_* (e^* \kappa) = \delta_{fgh}^{-1} \times \Big\{ - \beta_f \beta_g \beta_h (\pi_1, \pi_1, \pi_1)_*$$
$$+ p\beta_g \beta_h (\pi_2, \pi_1, \pi_1)_* + p\beta_f \beta_h (\pi_1, \pi_2, \pi_1)_* + p\beta_f \beta_g (\pi_1, \pi_1, \pi_2)_*$$
$$- p^2 \beta_f (\pi_1, \pi_2, \pi_2)_* - p^2 \beta_g (\pi_2, \pi_1, \pi_2)_* - p^2 \beta_h (\pi_2, \pi_2, \pi_1)_*$$
$$+ p^3 (\pi_2, \pi_2, \pi_2)_* \Big\} (\kappa),$$

where we recall that $\delta_{fgh} := ((\alpha_f - \beta_f)(\alpha_g - \beta_g)(\alpha_h - \beta_h))$.

Proof. — This follows directly from Lemma 4.3. $\square$

Recall the notations

$$k_\circ := k - 2, \qquad \ell_\circ = \ell - 2, \qquad m_\circ := m - 2, \qquad r := (k_\circ + \ell_\circ + m_\circ)/2.$$

Let $\mathcal{A}$ denote the Kuga-Sato variety over X as introduced in 1.2. In [**DR14**, Definitions 3.1, 3.2 and 3.3], a generalized diagonal cycle

$$\Delta^{k_\circ, \ell_\circ, m_\circ} = \Delta_0^{k_\circ, \ell_\circ, m_\circ} \in \mathrm{CH}^{r+2}(\mathcal{A}^{k_\circ} \times \mathcal{A}^{\ell_\circ} \times \mathcal{A}^{m_\circ}, \mathbf{Q})$$

is associated to the triple $(k_\circ, \ell_\circ, m_\circ)$.

When $k_\circ, \ell_\circ, m_\circ > 0$, $\Delta^{k_\circ, \ell_\circ, m_\circ}$ is obtained by choosing subsets A, B and C of the set $\{1, \dots, r\}$ which satisfy:

$$\#A = k_\circ, \qquad \#B = \ell_\circ, \qquad \#C = m_\circ, \qquad A \cap B \cap C = \emptyset,$$

$$\#(B \cap C) = r - k_\circ, \qquad \#(A \cap C) = r - \ell_\circ, \qquad \#(A \cap B) = r - m_\circ.$$

The cycle $\Delta^{k_\circ, \ell_\circ, m_\circ}$ is defined as the image of the embedding $\mathcal{A}^r$ into $\mathcal{A}^{k_\circ} \times \mathcal{A}^{\ell_\circ} \times \mathcal{A}^{m_\circ}$ given by sending $(E, (P_1, \dots, P_r))$ to $((E, P_A), (E, P_B), (E, P_C))$, where for instance P_A is a shorthand for the $k_\circ$-tuple of points P_j with $j \in A$.

Let also $\Delta_{01}^{k_\circ, \ell_\circ, m_\circ} \in \mathrm{CH}^{r+2}(\mathcal{A}^{k_\circ} \times \mathcal{A}^{\ell_\circ} \times \mathcal{A}^{m_\circ})$ denote the generalised diagonal cycle in the product of the three Kuga-Sato varieties of weights (k, ℓ, m) fibered over X_{01}, defined in a similar way as in (4.4) and along the same lines as recalled above.

More precisely, $\Delta_{01}^{k_\circ, \ell_\circ, m_\circ}$ is defined as the schematic closure in $\mathcal{A}^{k_\circ} \times \mathcal{A}^{\ell_\circ} \times \mathcal{A}^{m_\circ}$ of the set of tuples $((E, C_1, P_A), (E, C_2, P_B), (E, C_3, P_C))$ where P_A, P_B, P_C are as above, and C_1, C_2, C_3 is a triple of pairwise distinct subgroups of order p in the elliptic curve E.

Since the triple $(k_\circ, \ell_\circ, m_\circ)$ is fixed throughout this section, in order to alleviate notations in the statements below we shall simply denote $\Delta^\sharp$ and $\Delta_{01}^\sharp$ for $\Delta^{k_\circ, \ell_\circ, m_\circ}$ and $\Delta_{01}^{k_\circ, \ell_\circ, m_\circ}$ respectively.

Lemma 4.5. — *The following identities hold in* $\mathrm{CH}^{r+2}(\mathcal{A}^{k_\circ} \times \mathcal{A}^{\ell_\circ} \times \mathcal{A}^{m_\circ})$:

$$
\begin{aligned}
(\pi_1, \pi_1, \pi_1)_*(\Delta_{01}^\sharp) &= (p+1)p(p-1)(\Delta^\sharp), \\
(\pi_2, \pi_1, \pi_1)_*(\Delta_{01}^\sharp) &= p(p-1) \times (T_p, 1, 1)(\Delta^\sharp), \\
(\pi_1, \pi_2, \pi_1)_*(\Delta_{01}^\sharp) &= p(p-1) \times (1, T_p, 1)(\Delta^\sharp), \\
(\pi_1, \pi_1, \pi_2)_*(\Delta_{01}^\sharp) &= p(p-1) \times (1, 1, T_p)(\Delta^\sharp), \\
(\pi_1, \pi_2, \pi_2)_*(\Delta_{01}^\sharp) &= (p-1) \times ((1, T_p, T_p)(\Delta^\sharp) - p^{r-k_\circ} D_1) \\
(\pi_2, \pi_1, \pi_2)_*(\Delta_{01}^\sharp) &= (p-1) \times ((T_p, 1, T_p)(\Delta^\sharp) - p^{r-\ell_\circ} D_2) \\
(\pi_2, \pi_2, \pi_1)_*(\Delta_{01}^\sharp) &= (p-1) \times ((T_p, T_p, 1)(\Delta^\sharp) - p^{r-m_\circ} D_3) \\
(\pi_2, \pi_2, \pi_2)_*(\Delta_{01}^\sharp) &= (T_p, T_p, T_p)(\Delta^\sharp) - p^{r-k_\circ} E_1 - p^{r-\ell_\circ} E_2 - p^{r-m_\circ} E_3 \\
&\quad - p^r(p+1)\Delta^\sharp,
\end{aligned}
$$

where the cycles D_1, D_2 and D_3 satisfy

$$
([p], 1, 1)_*(D_1) = p^{k_\circ}(T_p, 1, 1)_*(\Delta^\sharp), \qquad (1, [p], 1)_*(D_2) = p^{\ell_\circ}(1, T_p, 1)_*(\Delta^\sharp),
$$
$$
(1, 1, [p])_*(D_3) = p^{m_\circ}(1, 1, T_p)(\Delta^\sharp),
$$

the cycles E_1, E_2 and E_3 satisfy

$$
([p], 1, 1)_*(E_1) = p^{k_\circ}(T_{p^2}, 1, 1)(\Delta^\sharp), \qquad (1, [p], 1)_*(E_2) = p^{\ell_\circ}(1, T_{p^2}, 1)(\Delta^\sharp),
$$
$$
(1, 1, [p])_*(E_3) = p^{m_\circ}(1, 1, T_{p^2})(\Delta^\sharp),
$$

and $T_{p^2} := T_p^2 - (p+1)[p]$.

Proof. — The first four identities are straightforward: the map $\pi_1 \times \pi_1 \times \pi_1$ induces a finite map from $\Delta_{01}^\sharp$ to $\Delta^\sharp$ of degree $(p+1)p(p-1)$, which is the number of possible choices of an ordered triple of distinct subgroups of order p on an elliptic curve, and likewise $\pi_2 \times \pi_1 \times \pi_1$ induces a map of degree $p(p-1)$ from $\Delta_{01}^\sharp$ to $(T_p, 1, 1)\Delta^\sharp$. The remaining identities follow from combinatorial reasonings based on the explicit description of the cycles $\Delta_{01}^\sharp$ and $\Delta^\sharp$. For the 5th identity, observe that $(\pi_1, \pi_2, \pi_2)_*$ induces a degree $(p-1)$ map from $\Delta_{01}^\sharp$ to the variety X parametrising triples $((E, P_A), (E', P'_B), (E'', P''_C))$ for which there are distinct cyclic p-isogenies $\varphi' : E \longrightarrow E'$ and $\varphi' : E \longrightarrow E''$, the system of points $P'_B \subset E'$ and $P''_C \subset E''$ indexed by the sets B and C satisfy

$$
\varphi'(P_{A \cap B}) = P'_{A \cap B}, \quad \varphi''(P_{A \cap C}) = P''_{A \cap C},
$$

and for which there exists points $Q_{B \cap C} \subset E$ indexed by $B \cap C$ satisfying

$$
\varphi'(Q_{B \cap C}) = P'_{B \cap C}, \qquad \varphi''(Q_{B \cap C}) = P''_{B \cap C}.
$$

On the other hand, $(1, T_p, T_p)$ parametrises triples of the same type, in which E' and E'' need not be distinct. It follows that

$$
(4.7) \qquad\qquad (1, T_p, T_p)(\Delta^\sharp) = X + p^{r-k_\circ} D_1,
$$

where the closed points of D_1 correspond to triples $((E, P_A), (E', P'_B), (E', P'_C))$ for which there is a cyclic p-isogeny $\varphi' : E \longrightarrow E'$ satisfying

$$\varphi'(P_{A \cap B}) = P'_{A \cap B}, \qquad \varphi'(P_{A \cap C}) = P'_{A \cap C}.$$

The coefficient of $p^{r-k_\circ}$ in (4.7) arises because each closed point of D_1 comes from $p^{\#(B \cap C)}$ distinct closed points on $(1, T_p, T_p)(\Delta^\sharp)$, obtained by translating the points $P_j \in P_{B \cap C}$ with $j \in B \cap C$ by arbitrary elements of $\ker(\varphi)$. The fifth identity now follows after noting that the map $([p], 1, 1)$ induces a map of degree $p^{k_\circ}$ from D_1 to $(T_p, 1, 1)_* \Delta^\sharp$. The 6th and 7th identity can be treated with an identical reasoning by interchanging the three factors in $W^{k_\circ} \times W^{\ell_\circ} \times W^{m_\circ}$. As for the last identity, the map (π_2, π_2, π_2) induces a map of degree 1 to the variety Y consisting of triples (E', E'', E''') of elliptic curves which are p-isogenous to a common elliptic curve E and distinct. But it is not hard to see that

$$(T_p, T_p, T_p)(\Delta^\sharp) = Y + p^{r-k_\circ} E_1 + p^{r-\ell_\circ} E_2 + p^{r-m_\circ} E_3 + p^r(p+1)\Delta^\sharp$$

where the additional terms on the right hand side account for triples (E', E'', E''') where $E' \neq E'' = E'''$, where $E'' \neq E' = E'''$, where $E''' \neq E' = E''$, and where $E' = E'' = E'''$ respectively. $\qquad \square$

Assume for the remainder of the section that $k_\circ, \ell_\circ, m_\circ > 0$. Recall the projectors $\epsilon_{k_\circ}$ of (1.5). It was shown in [**DR14**, §3.1] that $(\epsilon_{k_\circ}, \epsilon_{\ell_\circ}, \epsilon_{m_\circ})\Delta^{k_\circ, \ell_\circ, m_\circ}$ is a null-homologous cycle and we may define

$$(4.8) \qquad \kappa(f, g, h) := \pi_{f,g,h} \, \mathrm{AJ}_{\text{ét}}((\epsilon_{k_\circ}, \epsilon_{\ell_\circ}, \epsilon_{m_\circ})\Delta^{k_\circ, \ell_\circ, m_\circ}) \in H^1(\mathbf{Q}, W_0[f, g, h])$$

as the image of this cycle under the p-adic étale Abel-Jacobi map, followed by the natural projection from $H^{2c-1}_{\text{ét}}(\bar{\mathcal{A}}^{k_\circ} \times \bar{\mathcal{A}}^{\ell_\circ} \times \bar{\mathcal{A}}^{m_\circ}, \mathbf{Q}_p(c))$ to $W_0^{k_\circ, \ell_\circ, m_\circ}$ induced by the Künneth decomposition and the projection $\pi_{f,g,h}$.

It follows from [**DR14**, (66)], (1.5) and the vanishing of the terms $H^i_{\text{ét}}(\bar{X}_1, \mathcal{H}^{k_\circ})$ for $i \neq 1$ when $k_\circ > 0$, that the class $\kappa(f, g, h)$ is realized by the (f, g, h)-isotypic component of the same extension class as in (3.5), after replacing X_1 by the curve $X = X_0$ and $\Delta = \Delta^{0,0,0}$ is taken to be the usual diagonal cycle in X^3. In the notations of (3.4), this amounts to

$$(4.9) \qquad \kappa(f, g, h) = \pi_{f,g,h} \mathrm{AJ}_{k_\circ, \ell_\circ, m_\circ}(\Delta).$$

Similar statements holds over the curve X_{01}. Namely, we also have the following:

Proposition 4.6. — *The cycle* $(\epsilon_{k_\circ}, \epsilon_{\ell_\circ}, \epsilon_{m_\circ})\Delta_{01}^{k_\circ, \ell_\circ, m_\circ}$ *is null-homologous and the following equality of classes holds in* $H^1(\mathbf{Q}, W_{01}[f_\alpha, g_\alpha, h_\alpha])$:

$$(4.10) \qquad \kappa_{01}(f_\alpha, g_\alpha, h_\alpha) = p^3 \cdot \pi_{f_\alpha, g_\alpha, h_\alpha} \, \mathrm{AJ}_{\text{ét}}((\epsilon_{k_\circ}, \epsilon_{\ell_\circ}, \epsilon_{m_\circ})\Delta_{01}^{k_\circ, \ell_\circ, m_\circ}).$$

Proof. — Corollary 3.3 together with (4.2) imply that

$$\kappa_1(f_\alpha, g_\alpha, h_\alpha) = \frac{1}{C_q} \cdot \pi_{f_\alpha, g_\alpha, h_\alpha} \, \mathrm{AJ}_{k_\circ, \ell_\circ, m_\circ}(\Delta_1^\circ(1, 1, 1; \delta)),$$

in which $\delta = 1$ is the trivial character of $(\mathbf{Z}/p\mathbf{Z})^\times$. Since μ^3 induces a finite map of degree $(p-1)^3$ from the support of $\Delta_1(1,1,1;\delta)$ to Δ_{01}, it follows from the convention adopted in (3.7) that

$$\kappa_{01}(f_\alpha, g_\alpha, h_\alpha) := \mu_*^3 \kappa_1(f_\alpha, g_\alpha, h_\alpha) = \frac{p^3}{C_q} \cdot \pi_{f_\alpha, g_\alpha, h_\alpha} \mathrm{AJ}_{k_\circ, \ell_\circ, m_\circ}(\Delta_{01}^\circ),$$

where $\mathrm{AJ}_{k_\circ, \ell_\circ, m_\circ}(\Delta_{01}^\circ)$ is defined to be the class realized by the same extension class as in (3.5), after replacing X_1 by the curve X_{01} and replacing Δ by the cycle Δ_{01}° arising from (4.4). Since $\Delta_{01}^{k_\circ, \ell_\circ, m_\circ}$ is fibered over Δ_{01}, the same argument as in (4.9) then shows that

$$\mathrm{AJ}_{k_\circ, \ell_\circ, m_\circ}(\Delta_{01}) = \mathrm{AJ}_{\text{ét}}((\epsilon_{k_\circ}, \epsilon_{\ell_\circ}, \epsilon_{m_\circ})\Delta_{01}^{k_\circ, \ell_\circ, m_\circ}).$$

Since $\pi_{f_\alpha, g_\alpha, h_\alpha}(\Delta_{01}) = \frac{1}{C_q}\pi_{f_\alpha, g_\alpha, h_\alpha}(\Delta_{01}^\circ)$, the proposition follows. $\qquad\square$

Theorem 4.1. — *With notations as before, letting $c = r + 2$, we have*

$$(\varpi_1, \varpi_1, \varpi_1)_* \, \kappa_1(f_\alpha, g_\alpha, h_\alpha) = \frac{\mathcal{E}^{\mathrm{bal}}(f_\alpha, g_\alpha, h_\alpha)}{\mathcal{E}(f_\alpha)\mathcal{E}(g_\alpha)\mathcal{E}(h_\alpha)} \times \kappa(f, g, h),$$

where

$$\mathcal{E}^{\mathrm{bal}}(f_\alpha, g_\alpha, h_\alpha) = (1 - \alpha_f \beta_g \beta_h p^{-c})(1 - \beta_f \alpha_g \beta_h p^{-c})(1 - \beta_f \beta_g \alpha_h p^{-c})(1 - \beta_f \beta_g \beta_h p^{-c}),$$

and

$$\mathcal{E}(f_\alpha) = 1 - \chi_f^{-1}(p)\beta_f^2 p^{1-k}, \quad \mathcal{E}(g_\alpha) = 1 - \chi_g^{-1}(p)\beta_g^2 p^{1-\ell}, \quad \mathcal{E}(h_\alpha) = 1 - \chi_h^{-1}(p)\beta_h^2 p^{1-m}.$$

Proof. — In view of (4.5), (4.8) and (4.10), it suffices to prove the claim for the cycles $\Delta^{k_\circ, \ell_\circ, m_\circ}$ and $(\pi_1, \pi_1, \pi_1)_* \, e^* \Delta_{01}^{k_\circ, \ell_\circ, m_\circ}$. Since $k_\circ, \ell_\circ, m_\circ$ are fixed throughout the discussion, we again denote $\Delta^\sharp = \Delta^{k_\circ, \ell_\circ, m_\circ}$ and $\Delta_{01}^\sharp = \Delta_{01}^{k_\circ, \ell_\circ, m_\circ}$ to lighten notations.

When combined with Lemma 4.4, Lemma 4.5 equips us with a completely explicit formula for comparing $(\pi_1, \pi_1, \pi_1)_* e^*(\Delta_{01}^\sharp)$ with the generalised diagonal cycle $\Delta^\sharp$. Namely, since the correspondences $([p], 1, 1)$, $(1, [p], 1)$ and $(1, 1, [p])$ induce multiplication by $p^{k_\circ}$, $p^{\ell_\circ}$ and $p^{m_\circ}$ respectively on the (f, g, h)-isotypic parts, while $(T_p, 1, 1)$, $(1, T_p, 1)$, and $(1, 1, T_p)$ induce multiplication by $a_p(f)$, $a_p(g)$, and $a_p(h)$ respectively, it follows that, with notations as in the proof of Lemma 4.5,

$$\begin{aligned}
\pi_{f,g,h}(D_1) &= a_p(f)\pi_{f,g,h}(\Delta^\sharp), \\
\pi_{f,g,h}(D_2) &= a_p(g)\pi_{f,g,h}(\Delta^\sharp), \\
\pi_{f,g,h}(D_3) &= a_p(h)\pi_{f,g,h}(\Delta^\sharp),
\end{aligned}$$

and that

$$\begin{aligned}
\pi_{f,g,h}(E_1) &= (a_p^2(f) - (p+1)p^{k_\circ})\pi_{f,g,h}(\Delta^\sharp), \\
\pi_{f,g,h}(E_2) &= (a_p^2(g) - (p+1)p^{\ell_\circ})\pi_{f,g,h}(\Delta^\sharp), \\
\pi_{f,g,h}(E_3) &= (a_p^2(h) - (p+1)p^{m_\circ})\pi_{f,g,h}(\Delta^\sharp).
\end{aligned}$$

By projecting the various formulae for $(\pi_1, \pi_1, \pi_1)_*(\Delta_{01}^\sharp)$ that are given in Lemma 4.5 to the (f, g, h)-isotypic component and substituting them into Lemma 4.4, one

obtains a expression for $e_{f,g,h}(\pi_1, \pi_1, \pi_1)_* e^*(\Delta_{01}^\sharp)$ as a multiple of $\pi_{f,g,h}(\Delta^\sharp)$ by an explicit factor, which is a rational function in α_f, α_g and α_h. This explicit factor is somewhat tedious to calculate by hand, but the identity asserted in Theorem 4.1 is readily checked with the help of a symbolic algebra package. $\square$

5. Triple product *p*-adic *L*-functions and the reciprocity law

Let $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ be a triple of *p*-adic Hida families of tame levels M_f, M_g, M_h and tame characters χ_f, χ_g, χ_h as in the previous section. Let also $(\mathbf{f}^*, \mathbf{g}^*, \mathbf{h}^*) = (\mathbf{f} \otimes \bar\chi_f, \mathbf{g} \otimes \bar\chi_g, \mathbf{h} \otimes \bar\chi_h)$ denote the conjugate triple. As before, we assume $\chi_f \chi_g \chi_h = 1$ and set $M = \mathrm{lcm}(M_f, M_g, M_h)$.

Let $\Lambda_\mathbf{f}$, $\Lambda_\mathbf{g}$ and $\Lambda_\mathbf{h}$ be the finite flat extensions of Λ generated by the coefficients of the Hida families $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$, and set $\Lambda_\mathbf{fgh} = \Lambda_\mathbf{f} \hat\otimes_{\mathbf{Z}_p} \Lambda_\mathbf{g} \hat\otimes_{\mathbf{Z}_p} \Lambda_\mathbf{h}$. Let also $\mathcal{Q}_\mathbf{f}$ denote the fraction field of $\Lambda_\mathbf{f}$ and define

$$\mathcal{Q}_{\mathbf{f},\mathbf{gh}} := \mathcal{Q}_\mathbf{f} \hat\otimes \Lambda_\mathbf{g} \hat\otimes \Lambda_\mathbf{h}.$$

Let $\mathcal{W}_\mathbf{fgh}^\circ := \mathcal{W}_\mathbf{f}^\circ \times \mathcal{W}_\mathbf{g}^\circ \times \mathcal{W}_\mathbf{h}^\circ \subset \mathcal{W}_\mathbf{fgh} = \mathrm{Spf}(\Lambda_\mathbf{fgh})$ denote the set of triples of *crystalline* classical points, at which the three Hida families specialize to modular forms with trivial nebentype at p (and may be either old or new at p). This set admits a natural partition, namely

$$\mathcal{W}_\mathbf{fgh}^\circ = \mathcal{W}_\mathbf{fgh}^f \sqcup \mathcal{W}_\mathbf{fgh}^g \sqcup \mathcal{W}_\mathbf{fgh}^h \sqcup \mathcal{W}_\mathbf{fgh}^\mathrm{bal}$$

where

— $\mathcal{W}_\mathbf{fgh}^f$ denotes the set of points $(x, y, z) \in \mathcal{W}_\mathbf{fgh}^\circ$ of weights (k, ℓ, m) such that $k \geq \ell + m$.
— $\mathcal{W}_\mathbf{fgh}^g$ and $\mathcal{W}_\mathbf{fgh}^h$ are defined similarly, replacing the role of $\mathbf{f}$ with $\mathbf{g}$ (resp. $\mathbf{h}$).
— $\mathcal{W}_\mathbf{fgh}^\mathrm{bal}$ is the set of *balanced* triples, consisting of points (x, y, z) of weights (k, ℓ, m) such that each of the weights is strictly smaller than the sum of the other two.

Each of the four subsets appearing in the above partition is dense in $\mathcal{W}_\mathbf{fgh}$ for the rigid-analytic topology.

Recall from (1.34) the spaces of Λ-adic test vectors $S_\Lambda^\mathrm{ord}(M, \chi_f)[\mathbf{f}]$. For any choice of a triple

$$(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}) \in S_\Lambda^\mathrm{ord}(M, \chi_f)[\mathbf{f}] \times S_\Lambda^\mathrm{ord}(M, \chi_g)[\mathbf{g}] \times S_\Lambda^\mathrm{ord}(M, \chi_h)[\mathbf{h}]$$

of Λ-adic test vectors of tame level M, in [**DR14**, Lemma 2.19 and Definition 4.4] we constructed a *p*-adic *L*-function $\mathscr{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ in $\mathcal{Q}_\mathbf{f} \hat\otimes \Lambda_\mathbf{g} \hat\otimes \Lambda_\mathbf{h}$, giving rise to a meromorphic rigid-analytic function

(5.1) $$\mathscr{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}) : \mathcal{W}_\mathbf{fgh} \longrightarrow \mathbf{C}_p.$$

As shown in [**DR14**, §4], this *p*-adic *L*-function is characterized by an interpolation property relating its values at classical points $(x, y, z) \in \mathcal{W}_\mathbf{fgh}^f$ to the central critical value of Garrett's triple-product complex *L*-function $L(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z, s)$ associated to the triple of classical eigenforms $(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$. The fudge factors appearing in the interpolation property depend heavily on the choice of test vectors: cf. [**DR14**, §4] and

[**DLR15**, §2] for more details. In a recent preprint, Hsieh [**H17**] has found an explicit choice of test vectors, which yields a very optimal interpolation formula which shall be useful for our purposes. We describe it below:

Proposition 5.1. — *for every* $(x, y, z) \in \mathcal{W}_{\mathbf{fgh}}^f$ *of weights* (k, ℓ, m) *we have*

$$(5.2) \qquad \mathscr{L}_p{}^f(\check{\mathbf{f}}, \check{\mathbf{g}}, \check{\mathbf{h}})^2(x, y, z) = \frac{\mathfrak{a}(k, \ell, m)}{\langle \mathbf{f}_x^\circ, \mathbf{f}_x^\circ \rangle^2} \cdot \mathfrak{e}^2(x, y, z) \cdot \prod_{v | N\infty} C_v \times L(\mathbf{f}_x^\circ, \mathbf{g}_y^\circ, \mathbf{h}_z^\circ, c)$$

where

i) $c = \frac{k+\ell+m-2}{2}$,

ii) $\mathfrak{a}(k, \ell, m) = (2\pi i)^{-2k} \cdot \left(\frac{k+\ell+m-4}{2}\right)! \cdot \left(\frac{k+\ell-m-2}{2}\right)! \cdot \left(\frac{k-\ell+m-2}{2}\right)! \cdot \left(\frac{k-\ell-m}{2}\right)!$,

iii) $\mathfrak{e}(x, y, z) = \mathcal{E}(x, y, z)/\mathcal{E}_0(x)\mathcal{E}_1(x)$ *with*

$$\mathcal{E}_0(x) \quad := \quad 1 - \chi_f^{-1}(p)\beta_{\mathbf{f}_x}^2 p^{1-k},$$
$$\mathcal{E}_1(x) \quad := \quad 1 - \chi_f(p)\alpha_{\mathbf{f}_x}^{-2} p^{k_\circ},$$
$$\mathcal{E}(x, y, z) \quad := \quad \left(1 - \chi_f(p)\alpha_{\mathbf{f}_x}^{-1}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z} p^{\frac{k-\ell-m}{2}}\right) \times \left(1 - \chi_f(p)\alpha_{\mathbf{f}_x}^{-1}\alpha_{\mathbf{g}_y}\beta_{\mathbf{h}_z} p^{\frac{k-\ell-m}{2}}\right)$$
$$\times \left(1 - \chi_f(p)\alpha_{\mathbf{f}_x}^{-1}\beta_{\mathbf{g}_y}\alpha_{\mathbf{h}_z} p^{\frac{k-\ell-m}{2}}\right) \times \left(1 - \chi_f(p)\alpha_{\mathbf{f}_x}^{-1}\beta_{\mathbf{g}_y}\beta_{\mathbf{h}_z} p^{\frac{k-\ell-m}{2}}\right).$$

iv) *The local constant* $C_v \in \mathbf{Q}(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$ *depends only on the admissible representations of* $\mathrm{GL}_2(\mathbf{Q}_v)$ *associated to* $(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$ *and on the local components at* v *of the test vectors.*

Moreover, there exists a distinguished choice of test vectors $(\check{\mathbf{f}}, \check{\mathbf{g}}, \check{\mathbf{h}})$ *(as specified by Hsieh in [**H17**, §3]) for which* $\mathscr{L}_p{}^f(\check{\mathbf{f}}, \check{\mathbf{g}}, \check{\mathbf{h}})$ *lies in* $\Lambda_{\mathbf{fgh}}$ *and the local constants may be taken to be* $C_v = 1$ *at all* $v \mid N\infty$.

Proof. — This follows from [**H17**, Theorem A], after spelling out explicitly the definitions involved in Hsieh's formulation.

Let us remark that throughout the whole article [**DR14**], it was implicitly assumed that $\mathbf{f}_x$, $\mathbf{g}_\ell$ and $\mathbf{h}_m$ are all old at p, and note that the definition we have given here of the terms $\mathcal{E}_0(x)$, $\mathcal{E}_1(x)$ and $\mathcal{E}(x, y, z)$ is exactly the same as in [**DR14**] in such cases, because $\beta_{\mathbf{f}_x} = \chi_f(p)\alpha_{\mathbf{f}_x}^{-1}p^{k-1}$ when $\mathbf{f}_x$ is old at p.

In contrast with loc. cit., in the above proposition we also allow any of the eigenforms $\mathbf{f}_x$, $\mathbf{g}_\ell$ and $\mathbf{h}_m$ to be new at p (which can only occur when the weight is 2); in such case, recall the usual convention adopted in §1.2 to set $\beta_\phi = 0$ when p divides the primitive level of an eigenform ϕ. With these notations, the current formulation of $\mathcal{E}(x, y, z)$, $\mathcal{E}_0(x)$ and $\mathcal{E}_1(x)$ is the correct one, as one can readily verify by rewriting the proof of [**DR14**, Lemma 4.10]. $\qquad\qquad\square$

5.1. Perrin-Riou's regulator. — Recall the Λ-adic cyclotomic character $\underline{\varepsilon}_{\mathrm{cyc}}$ and the unramified characters $\Psi_{\mathbf{f}}$, $\Psi_{\mathbf{g}}$, $\Psi_{\mathbf{h}}$ of $G_{\mathbf{Q}_p}$ introduced in Theorem 1.1. As a piece of notation, let $\underline{\varepsilon}_{\mathbf{f}} : G_{\mathbf{Q}_p} \longrightarrow \Lambda_{\mathbf{f}}^\times$ denote the composition of $\underline{\varepsilon}_{\mathrm{cyc}}$ and the natural inclusion $\Lambda^\times \subset \Lambda_{\mathbf{f}}^\times$, and likewise for $\underline{\varepsilon}_{\mathbf{g}}$ and $\underline{\varepsilon}_{\mathbf{h}}$. Expressions like $\Psi_{\mathbf{f}}\Psi_{\mathbf{g}}\Psi_{\mathbf{h}}$ or $\underline{\varepsilon}_{\mathbf{f}}\underline{\varepsilon}_{\mathbf{g}}\underline{\varepsilon}_{\mathbf{h}}$

are a short-hand notation for the $\Lambda_{\mathbf{fgh}}^{\times}$-valued character of $G_{\mathbf{Q}_p}$ given by the tensor product of the three characters.

Let $\mathbb{V}_{\mathbf{f}}$, $\mathbb{V}_{\mathbf{g}}$ and $\mathbb{V}_{\mathbf{h}}$ be the Galois representations associated to $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ in Theorem 1.1.

The purpose of this section is describing in precise terms the close connection between the diagonal cycles constructed above and the three-variable triple-product *p*-adic *L*-function. In order to do that, let us introduce the $\Lambda_{\mathbf{fgh}}$-modules

$$(5.3) \qquad \mathbb{V}_{\mathbf{fgh}}^{\dagger} := \mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}}(-1)(\tfrac{1}{2}) = \mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}}(\varepsilon_{\mathrm{cyc}}^{-1}\underline{\varepsilon}_{\mathbf{f}}^{-1/2}\underline{\varepsilon}_{\mathbf{g}}^{-1/2}\underline{\varepsilon}_{\mathbf{h}}^{-1/2}).$$

and

$$(5.4) \qquad \mathbb{V}_{\mathbf{fgh}}^{\dagger}(M) := \mathbb{V}_{\mathbf{f}}(M) \otimes \mathbb{V}_{\mathbf{g}}(M) \otimes \mathbb{V}_{\mathbf{h}}(M)(-1)(\tfrac{1}{2}).$$

The pairing defined in (2.17) yields an identification $\mathbb{H}^{111}(X_{\infty}^{*}) = H_{\text{ét}}^{1}(\bar{X}_{\infty}, \mathbf{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2})$. As explained in (1.26), $\mathbb{V}_{\mathbf{fgh}}^{\dagger}(M)$ is isomorphic to the direct sum of several copies of $\mathbb{V}_{\mathbf{fgh}}^{\dagger}$ and there are canonical projections $\varpi_{\mathbf{f}}$, $\varpi_{\mathbf{g}}$, $\varpi_{\mathbf{h}}$ which assemble into a $G_{\mathbf{Q}}$-equivariant map

$$\varpi_{\mathbf{f},\mathbf{g},\mathbf{h}} : \mathbb{H}^{111}(X_{\infty}^{*}) = H_{\text{ét}}^{1}(\bar{X}_{\infty}, \mathbf{Z}_p)^{\otimes 3}(2)(\tfrac{1}{2}) \longrightarrow \mathbb{V}_{\mathbf{fgh}}^{\dagger}(M).$$

Recall the three-variable Λ-adic global cohomology class

$$\boldsymbol{\kappa}_{\infty}(\epsilon_1\omega^{-k_{\circ}}, \epsilon_2\omega^{-\ell_{\circ}}, \epsilon_3\omega^{-m_{\circ}}; 1) = \boldsymbol{\kappa}_{\infty}(1,1,1;1) \in H^1(\mathbf{Q}, \mathbb{H}^{111}(X_{\infty}^{*}))$$

introduced in (4.1).

Set $C_q(\mathbf{f}, \mathbf{g}, \mathbf{h}) := (a_q(\mathbf{f}) - q - 1)(a_q(\mathbf{g}) - q - 1)(a_q(\mathbf{h}) - q - 1)$. Note that $C_q(\mathbf{f}, \mathbf{g}, \mathbf{h})$ is a unit in $\Lambda_{\mathbf{fgh}}$, because its classical specializations are *p*-adic units (cf. (4.2)).

Definition 5.2. — *Define*

$$\boldsymbol{\kappa}(\mathbf{f}, \mathbf{g}, \mathbf{h}) := \frac{1}{C_q(\mathbf{f}, \mathbf{g}, \mathbf{h})} \cdot \varpi_{\mathbf{f},\mathbf{g},\mathbf{h}*}(\boldsymbol{\kappa}_{\infty}(\epsilon_1\omega^{-k_{\circ}}, \epsilon_2\omega^{-\ell_{\circ}}, \epsilon_3\omega^{-m_{\circ}}; 1)) \in H^1(\mathbf{Q}, \mathbb{V}_{\mathbf{fgh}}^{\dagger}(M))$$

to be the projection of the above class to the $(\mathbf{f}, \mathbf{g}, \mathbf{h})$*-isotypical component.*

In the above definition, we normalize $\boldsymbol{\kappa}(\mathbf{f}, \mathbf{g}, \mathbf{h})$ by the constant $C_q(\mathbf{f}, \mathbf{g}, \mathbf{h})$ so that the classical specializations of $\boldsymbol{\kappa}(\mathbf{f}, \mathbf{g}, \mathbf{h})$ at classical points coincide with the classes $\kappa_1(f_{\alpha}, g_{\alpha}, h_{\alpha})$ introduced in (4.2).

Let

$$\mathrm{res}_p : H^1(\mathbf{Q}, \mathbb{V}_{\mathbf{fgh}}^{\dagger}(M)) \to H^1(\mathbf{Q}_p, \mathbb{V}_{\mathbf{fgh}}^{\dagger}(M))$$

denote the restriction map to the local cohomology at *p* and set

$$\boldsymbol{\kappa}_p(\mathbf{f}, \mathbf{g}, \mathbf{h}) := \mathrm{res}_p(\boldsymbol{\kappa}(\mathbf{f}, \mathbf{g}, \mathbf{h})) \in H^1(\mathbf{Q}_p, \mathbb{V}_{\mathbf{fgh}}^{\dagger}(M)).$$

The main result of this section asserts that the *p*-adic *L*-function $\mathscr{L}_p^{f}(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ introduced in §5 can be recast as the image of the Λ-adic class $\boldsymbol{\kappa}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$ under a suitable three-variable Perrin-Riou regulator map whose formulation relies on a choice of families of periods which depends on the test vectors $\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}$.

The recipe we are about to describe depends solely only on the projection of $\kappa_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$ to a suitable sub-quotient of $\mathbb{V}_{\mathbf{fgh}}^{\dagger}$ which is free of rank one over $\Lambda_{\mathbf{fgh}}$, and whose definition requires the following lemma.

Lemma 5.3. — *The Galois representation $\mathbb{V}_{\mathbf{fgh}}^{\dagger}$ is endowed with a four-step filtration*

$$0 \subset \mathbb{V}_{\mathbf{fgh}}^{++} \subset \mathbb{V}_{\mathbf{fgh}}^{+} \subset \mathbb{V}_{\mathbf{fgh}}^{-} \subset \mathbb{V}_{\mathbf{fgh}}^{\dagger}$$

by $G_{\mathbf{Q}_p}$-stable $\Lambda_{\mathbf{fgh}}$-submodules of ranks 0, 1, 4, 7 and 8 respectively.

The group $G_{\mathbf{Q}_p}$ acts on the successive quotients for this filtration (which are free over $\Lambda_{\mathbf{fgh}}$ of ranks 1, 3, 3 and 1 respectively) as a direct sum of one dimensional characters,

$$\mathbb{V}_{\mathbf{fgh}}^{++} = \eta^{\mathbf{fgh}}, \qquad \frac{\mathbb{V}_{\mathbf{fgh}}^{+}}{\mathbb{V}_{\mathbf{fgh}}^{++}} = \eta_{\mathbf{f}}^{\mathbf{gh}} \oplus \eta_{\mathbf{g}}^{\mathbf{fh}} \oplus \eta_{\mathbf{h}}^{\mathbf{fg}}, \qquad \frac{\mathbb{V}_{\mathbf{fgh}}^{-}}{\mathbb{V}_{\mathbf{fgh}}^{+}} = \eta_{\mathbf{gh}}^{\mathbf{f}} \oplus \eta_{\mathbf{fh}}^{\mathbf{g}} \oplus \eta_{\mathbf{fg}}^{\mathbf{h}}, \qquad \frac{\mathbb{V}_{\mathbf{fgh}}^{\dagger}}{\mathbb{V}_{\mathbf{fgh}}^{-}} = \eta_{\mathbf{fgh}},$$

where

$$\eta^{\mathbf{fgh}} = (\Psi_{\mathbf{f}} \Psi_{\mathbf{g}} \Psi_{\mathbf{h}} \times \varepsilon_{\mathrm{cyc}}^{2} (\underline{\varepsilon}_{\mathbf{f}} \underline{\varepsilon}_{\mathbf{g}} \underline{\varepsilon}_{\mathbf{h}})^{1/2}, \qquad \qquad \eta_{\mathbf{fgh}} = \Psi_{\mathbf{f}} \Psi_{\mathbf{g}} \Psi_{\mathbf{h}} \times \varepsilon_{\mathrm{cyc}}^{-1} (\underline{\varepsilon}_{\mathbf{f}} \underline{\varepsilon}_{\mathbf{g}} \underline{\varepsilon}_{\mathbf{h}})^{-1/2},$$

$$\eta_{\mathbf{f}}^{\mathbf{gh}} = \chi_{f}^{-1} \Psi_{\mathbf{f}} \Psi_{\mathbf{g}}^{-1} \Psi_{\mathbf{h}}^{-1} \times \varepsilon_{\mathrm{cyc}} (\underline{\varepsilon}_{\mathbf{f}}^{-1} \underline{\varepsilon}_{\mathbf{g}} \underline{\varepsilon}_{\mathbf{h}})^{1/2}, \quad \eta_{\mathbf{gh}}^{\mathbf{f}} = \chi_{f} \Psi_{\mathbf{f}}^{-1} \Psi_{\mathbf{g}} \Psi_{\mathbf{h}} \times (\underline{\varepsilon}_{\mathbf{f}} \underline{\varepsilon}_{\mathbf{g}}^{-1} \underline{\varepsilon}_{\mathbf{h}}^{-1})^{1/2},$$

$$\eta_{\mathbf{g}}^{\mathbf{fh}} = \chi_{g}^{-1} \Psi_{\mathbf{g}} \Psi_{\mathbf{f}}^{-1} \Psi_{\mathbf{h}}^{-1} \times \varepsilon_{\mathrm{cyc}} (\underline{\varepsilon}_{\mathbf{f}} \underline{\varepsilon}_{\mathbf{g}}^{-1} \underline{\varepsilon}_{\mathbf{h}})^{1/2}, \quad \eta_{\mathbf{fh}}^{\mathbf{g}} = \chi_{g} \Psi_{\mathbf{f}} \Psi_{\mathbf{h}} \Psi_{\mathbf{g}}^{-1} \times (\underline{\varepsilon}_{\mathbf{f}}^{-1} \underline{\varepsilon}_{\mathbf{g}} \underline{\varepsilon}_{\mathbf{h}}^{-1})^{1/2},$$

$$\eta_{\mathbf{h}}^{\mathbf{fg}} = \chi_{h}^{-1} \Psi_{\mathbf{h}} \Psi_{\mathbf{f}}^{-1} \Psi_{\mathbf{g}}^{-1} \times \varepsilon_{\mathrm{cyc}} (\underline{\varepsilon}_{\mathbf{f}} \underline{\varepsilon}_{\mathbf{g}} \underline{\varepsilon}_{\mathbf{h}}^{-1})^{1/2}, \quad \eta_{\mathbf{fg}}^{\mathbf{h}} = \chi_{h} \Psi_{\mathbf{f}} \Psi_{\mathbf{g}} \Psi_{\mathbf{h}}^{-1} \times (\underline{\varepsilon}_{\mathbf{f}}^{-1} \underline{\varepsilon}_{\mathbf{g}}^{-1} \underline{\varepsilon}_{\mathbf{h}})^{1/2}.$$

Proof. — Let ϕ be a Hida family of tame character χ as in §1.3. Let ψ_{ϕ} denote the unramified character of $G_{\mathbf{Q}_p}$ sending a Frobenius element Fr_p to $\mathbf{a}_p(\phi)$ and recall from (1.12) that the restriction of $\mathbb{V}_{\phi}$ to $G_{\mathbf{Q}_p}$ admits a filtration

$$0 \to \mathbb{V}_{\phi}^{+} \to \mathbb{V}_{\phi} \to \mathbb{V}_{\phi}^{-} \to 0$$

with

$$\mathbb{V}_{\phi}^{+} \simeq \Lambda_{\phi}(\psi_{\phi}^{-1} \chi \varepsilon_{\mathrm{cyc}}^{-1} \underline{\varepsilon}_{\mathrm{cyc}}), \qquad \mathbb{V}_{\phi}^{-} \simeq \Lambda_{\phi}(\psi_{\phi}).$$

Set

$$\mathbb{V}_{\mathbf{fgh}}^{++} = \mathbb{V}_{\mathbf{f}}^{+} \otimes \mathbb{V}_{\mathbf{g}}^{+} \otimes \mathbb{V}_{\mathbf{h}}^{+} (\varepsilon_{\mathrm{cyc}}^{-1} \underline{\varepsilon}_{\mathbf{f}}^{-1/2} \underline{\varepsilon}_{\mathbf{g}}^{-1/2} \underline{\varepsilon}_{\mathbf{h}}^{-1/2}),$$

$$\mathbb{V}_{\mathbf{fgh}}^{+} = \left(\mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}}^{+} \otimes \mathbb{V}_{\mathbf{h}}^{+} + \mathbb{V}_{\mathbf{f}}^{+} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}}^{+} + \mathbb{V}_{\mathbf{f}}^{+} \otimes \mathbb{V}_{\mathbf{g}}^{+} \otimes \mathbb{V}_{\mathbf{h}} \right) (\varepsilon_{\mathrm{cyc}}^{-1} \underline{\varepsilon}_{\mathbf{f}}^{-1/2} \underline{\varepsilon}_{\mathbf{g}}^{-1/2} \underline{\varepsilon}_{\mathbf{h}}^{-1/2})$$

$$\mathbb{V}_{\mathbf{fgh}}^{-} = \left(\mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}}^{+} + \mathbb{V}_{\mathbf{f}} \otimes \mathbb{V}_{\mathbf{g}}^{+} \otimes \mathbb{V}_{\mathbf{h}} + \mathbb{V}_{\mathbf{f}}^{+} \otimes \mathbb{V}_{\mathbf{g}} \otimes \mathbb{V}_{\mathbf{h}} \right) (\varepsilon_{\mathrm{cyc}}^{-1} \underline{\varepsilon}_{\mathbf{f}}^{-1/2} \underline{\varepsilon}_{\mathbf{g}}^{-1/2} \underline{\varepsilon}_{\mathbf{h}}^{-1/2}).$$

It follows from the definitions that these three representations are $\Lambda_{\mathbf{fgh}}[G_{\mathbf{Q}_p}]$-submodules of $\mathbb{V}_{\mathbf{fgh}}^{\dagger}$ of ranks 1, 4, 7 as claimed. Moreover, since $\chi_f \chi_g \chi_h = 1$, the rest of the lemma follows from (1.12).   □

A one-dimensional character $\eta : G_{\mathbf{Q}_p} \longrightarrow \mathbf{C}_p^{\times}$ is said to be *of Hodge-Tate weight* $-j$ if it is equal to a finite order character times the j-th power of the cyclotomic character. The following is an immediate corollary of Lemma 5.3.

Corollary 5.4. — *Let $(x, y, z) \in \mathcal{W}_{\mathbf{fgh}}^{\circ}$ be a triple of classical points of weights (k, ℓ, m). The Galois representation $V_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}^{\dagger}$ is endowed with a four-step $G_{\mathbf{Q}_p}$-stable filtration*

$$0 \subset V_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}^{++} \subset V_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}^{+} \subset V_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}^{-} \subset V_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}^{\dagger},$$

and the Hodge-Tate weights of its successive quotients are:

Subquotient	Hodge-Tate weights
$V_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}^{++}$	$\frac{-k-\ell-m}{2} + 1$
$V_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}^{+}/V_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}^{++}$	$\frac{k-\ell-m}{2}, \frac{-k+\ell-m}{2}, \frac{-k-\ell+m}{2}$
$V_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}^{-}/V_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}^{+}$	$\frac{-k+\ell+m}{2} - 1, \frac{k-\ell+m}{2} - 1, \frac{k+\ell-m}{2} - 1$
$V_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}/V_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}^{-}$	$\frac{k+\ell+m}{2} - 2$

Corollary 5.5. — *The Hodge-Tate weights of $V_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}^{+}$ are all strictly negative if and only if (k, ℓ, m) is balanced.*

Let $\mathbb{V}_{\mathbf{f}}^{\mathbf{gh}}$ and $\mathbb{V}_{\mathbf{f}}^{\mathbf{gh}}(M)$ be the subquotient of $\mathbb{V}_{\mathbf{fgh}}^{\dagger}$ (resp. of $\mathbb{V}_{\mathbf{fgh}}^{\dagger}(M)$) on which $G_{\mathbf{Q}_p}$ acts via (several copies of) the character

$$(5.5) \qquad\qquad \eta_{\mathbf{f}}^{\mathbf{gh}} := \Psi_{\mathbf{f}}^{\mathbf{gh}} \times \Theta_{\mathbf{f}}^{\mathbf{gh}}$$

where

— $\Psi_{\mathbf{f}}^{\mathbf{gh}}$ is the unramified character of $G_{\mathbf{Q}_p}$ sending Fr_p to $\chi_f^{-1}(p)\mathbf{a}_p(\mathbf{f})\mathbf{a}_p(\mathbf{g})^{-1}\mathbf{a}_p(\mathbf{h})^{-1}$, and

— $\Theta_{\mathbf{f}}^{\mathbf{gh}}$ is the $\Lambda_{\mathbf{fgh}}$-adic cyclotomic character whose specialization at a point of weight (k, ℓ, m) is $\varepsilon_{\mathrm{cyc}}^{t}$ with $t := (-k + \ell + m)/2$.

The classical specializations of $\mathbb{V}_{\mathbf{f}}^{\mathbf{gh}}$ are

$$(5.6) \qquad V_{\mathbf{f}_x}^{\mathbf{g}_y \mathbf{h}_z} := V_{\mathbf{f}_x}^{-} \otimes V_{\mathbf{g}_y}^{+} \otimes V_{\mathbf{h}_z}^{+}(\frac{-k-\ell-m+4}{2}) \simeq L_p(\chi_f^{-1}\psi_{\mathbf{f}_x}\psi_{\mathbf{g}_y}^{-1}\psi_{\mathbf{h}_z}^{-1})(t),$$

where the coefficient field is $L_p = \mathbf{Q}_p(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$. Note that $t > 0$ when $(x, y, z) \in \mathcal{W}_{\mathbf{fgh}}^{\mathrm{bal}}$, while $t \leq 0$ when $(x, y, z) \in \mathcal{W}_{\mathbf{fgh}}^{f}$.

Recall now from §1.4 the Dieudonné module $D(V_{\mathbf{f}_x}^{\mathbf{g}_y \mathbf{h}_z}(Mp))$ associated to (5.6). As it follows from loc. cit., every triple

$$(\eta_1, \omega_2, \omega_3) \in D(V_{\mathbf{f}_x}^{+}(Mp)) \times D(V_{\mathbf{g}_y}^{-}(Mp)) \times D(V_{\mathbf{h}_z}^{-}(Mp))$$

gives rise to a linear functional $\eta_1 \otimes \omega_2 \otimes \omega_3 : D(V_{\mathbf{f}_x}^{\mathbf{g}_y \mathbf{h}_z}(Mp)) \longrightarrow L_p$.

In order to deal with the *p*-adic variation of these Dieudonné modules, write $\mathbb{V}_{\mathbf{f}}^{\mathbf{gh}}(M)$ as

$$\mathbb{V}_{\mathbf{f}}^{\mathbf{gh}}(M) = \mathbb{U}(\Theta_{\mathbf{f}}^{\mathbf{gh}})$$

where $\mathbb{U}$ is the unramified $\Lambda_{\mathbf{fgh}}$-adic representation of $G_{\mathbf{Q}_p}$ given by (several copies of) the character $\Psi_{\mathbf{f}}^{\mathbf{gh}}$.

As in §1.4, define the Λ-adic Dieudonné module

$$\mathbb{D}(\mathbb{U}) := (\mathbb{U}\hat{\otimes}\hat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}.$$

In view of (1.29), for every $(x, y, z) \in \mathcal{W}_{\mathbf{fgh}}^{\circ}$ there is a natural specialisation map

$$\nu_{x,y,z} : \mathbb{D}(\mathbb{U}) \longrightarrow D(U_{\mathbf{f}_x}^{\mathbf{g}_y \mathbf{h}_z})$$

where $U_{\mathbf{f}_x}^{\mathbf{g}_y \mathbf{h}_z} := \mathbb{U} \otimes_{\Lambda_{\mathbf{fgh}}} \mathbf{Q}_p(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z) \simeq V_{\mathbf{f}_x}^{\mathbf{g}_y \mathbf{h}_z}(Mp)(-t).$

Proposition 5.6. — *For any triple of test vectors*

$$(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}) \in S_\Lambda^{\mathrm{ord}}(M, \chi_f)[\mathbf{f}] \times S_\Lambda^{\mathrm{ord}}(M, \chi_g)[\mathbf{g}] \times S_\Lambda^{\mathrm{ord}}(M, \chi_h)[\mathbf{h}],$$

there exists a homomorphism of $\Lambda_{\mathbf{fgh}}$-modules

$$\langle \ \ , \eta_{\breve{\mathbf{f}}^*} \otimes \omega_{\breve{\mathbf{g}}^*} \otimes \omega_{\breve{\mathbf{h}}^*} \rangle : \mathbb{D}(\mathbb{U}) \longrightarrow \mathcal{Q}_{\mathbf{f},\mathbf{gh}}$$

such that for all $\boldsymbol{\lambda} \in \mathbb{D}(\mathbb{U})$ and all $(x, y, z) \in \mathcal{W}_{\mathbf{fgh}}^{\circ}$ such that $\mathbf{f}_x$ is the ordinary stabilization of an eigenform $\mathbf{f}_x^{\circ}$ of level M:

$$\nu_{x,y,z}\big(\langle \boldsymbol{\lambda}, \eta_{\breve{\mathbf{f}}^*} \otimes \omega_{\breve{\mathbf{g}}^*} \otimes \omega_{\breve{\mathbf{h}}^*} \rangle\big) = \frac{1}{\mathcal{E}_0(\mathbf{f}_x^{\circ})\mathcal{E}_1(\mathbf{f}_x^{\circ})} \times \langle \nu_{x,y,z}(\boldsymbol{\lambda}), \eta_{\breve{\mathbf{f}}_x^*} \otimes \omega_{\breve{\mathbf{g}}_y^*} \otimes \omega_{\breve{\mathbf{h}}_z^*} \rangle.$$

Recall from (1.31) that

$$\mathcal{E}_0(\mathbf{f}_x^{\circ}) = 1 - \chi^{-1}(p)\beta_{\mathbf{f}_x^{\circ}}^2 p^{1-k}, \quad \mathcal{E}_1(\mathbf{f}_x^{\circ}) = 1 - \chi(p)\alpha_{\mathbf{f}_x^{\circ}}^{-2} p^{k-2}.$$

Proof. — Since $\mathbb{U}$ is isomorphic to the unramified twist of $\mathbb{V}_{\mathbf{f}}^{-} \otimes \mathbb{V}_{\mathbf{g}}^{+} \otimes \mathbb{V}_{\mathbf{h}}^{+}$, this follows from Proposition 1.5 because $\mathcal{E}_0(\mathbf{f}_x^{\circ}) = \mathcal{E}_0(\mathbf{f}_x^{\circ *})$ and $\mathcal{E}_1(\mathbf{f}_x^{\circ}) = \mathcal{E}_1(\mathbf{f}_x^{\circ *})$. $\qquad\square$

It follows from Example 1.4 (a) and (b) that the Bloch-Kato logarithm and dual exponential maps yield isomorphisms

$$\log_{\mathrm{BK}} : H^1(\mathbf{Q}_p, V_{\mathbf{f}_x}^{\mathbf{g}_y \mathbf{h}_z}) \xrightarrow{\sim} D(V_{\mathbf{f}_x}^{\mathbf{g}_y \mathbf{h}_z}), \qquad \text{if } t > 0,$$

$$\exp_{\mathrm{BK}}^* : H^1(\mathbf{Q}_p, V_{\mathbf{f}_x}^{\mathbf{g}_y \mathbf{h}_z}) \xrightarrow{\sim} D(V_{\mathbf{f}_x}^{\mathbf{g}_y \mathbf{h}_z}), \qquad \text{if } t \le 0.$$

Define

$$(5.7) \qquad \mathcal{E}^{\mathrm{PR}}(x, y, z) = \frac{1 - p^{\frac{k-\ell-m}{2}}\alpha_{\mathbf{f}_x}^{-1}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}}{1 - p^{\frac{\ell+m-k-2}{2}}\alpha_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}^{-1}\alpha_{\mathbf{h}_z}^{-1}} = \frac{1 - p^{-c}\beta_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}}{1 - p^{-c}\alpha_{\mathbf{f}_x}\beta_{\mathbf{g}_y}\beta_{\mathbf{h}_z}}.$$

The following is a three-variable version of Perrin-Riou's regulator map constructed in [**PR95**] and [**LZ14**].

Proposition 5.7. — *There is a homomorphism*

$$\mathcal{L}_{\mathbf{f},\mathbf{gh}} : H^1(\mathbf{Q}_p, \mathbb{V}_{\mathbf{f}}^{\mathbf{gh}}(M)) \longrightarrow \mathbb{D}(\mathbb{U})$$

such that for all $\boldsymbol{\kappa}_p \in H^1(\mathbf{Q}_p, \mathbb{V}_{\mathbf{f}}^{\mathbf{gh}}(M))$ the image $\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p)$ satisfies the following interpolation properties:

(i) *For all balanced points $(x, y, z) \in \mathcal{W}_{\mathbf{fgh}}^{\mathrm{bal}}$,*

$$\nu_{x,y,z}\big(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p)\big) = \frac{(-1)^t}{t!} \cdot \mathcal{E}^{\mathrm{PR}}(x, y, z) \cdot \log_{\mathrm{BK}}(\nu_{x,y,z}(\boldsymbol{\kappa}_p)),$$

(ii) *For all points $(x, y, z) \in \mathcal{W}_{\mathbf{fgh}}^{f}$,*

$$\nu_{x,y,z}\big(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\boldsymbol{\kappa}_p)\big) = (-1)^t \cdot (1 - t)! \cdot \mathcal{E}^{\mathrm{PR}}(x, y, z) \cdot \exp_{\mathrm{BK}}^*(\nu_{x,y,z}(\boldsymbol{\kappa}_p)).$$

Proof. — This follows by standard methods as in [**KLZ17**, Theorem 8.2.8], [**LZ14**, Appendix B], [**DR17**, §5.1]. □

Proposition 5.8. — *The class* $\boldsymbol{\kappa}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$ *belongs to the image of* $H^1(\mathbf{Q}_p, \mathbb{V}^+_{\mathbf{fgh}}(M))$ *in*
$H^1(\mathbf{Q}_p, \mathbb{V}^\dagger_{\mathbf{fgh}}(M))$ *under the map induced from the inclusion* $\mathbb{V}^+_{\mathbf{fgh}}(M) \hookrightarrow \mathbb{V}^\dagger_{\mathbf{fgh}}(M)$.

Proof. — Let $(x, y, z) \in \mathcal{W}^\circ_{\mathbf{fgh}}$ be a triple of classical points of weights (k, ℓ, m). By the results proved in §4, the cohomology class $\kappa_p(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$ is proportional to the image under the *p*-adic étale Abel-Jacobi map of the cycles appearing in (4.8), that were introduced in [**DR14**, §3]. The purity conjecture for the monodromy filtration is known to hold for the variety $\mathcal{A}^{k_\circ} \times \mathcal{A}^{\ell_\circ} \times \mathcal{A}^{m_\circ}$ by the work of Saito (cf. [**S97**], [**N98**, (3.2)]). By Theorem 3.1 of loc.cit., it follows that the extension $\kappa_p(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$ is crystalline. Hence $\kappa_p(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$ belongs to $H^1_{\mathrm{f}}(\mathbf{Q}_p, V^\dagger_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}(Mp)) \subset H^1(\mathbf{Q}_p, V^\dagger_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}(Mp))$.

Since (k, ℓ, m) is balanced, Corollary 5.5 implies that $V^+_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}$ is the subrepresentation of $V^\dagger_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}$ on which the Hodge-Tate weights are all *strictly negative*. As is well-known (cf. [**F90**, Lemma 2, p. 125], [**LZ19**, §3.3] for similar results), the finite Bloch-Kato local Selmer group of an ordinary representation can be recast à la Greenberg [**G89**] as

$$H^1_{\mathrm{f}}(\mathbf{Q}_p, V^\dagger_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}) = \ker\left(H^1(\mathbf{Q}_p, V^\dagger_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}) \longrightarrow H^1(I_p, V^\dagger_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}/V^+_{\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z}) \right),$$

where I_p denotes the inertia group at *p*.

Since the set of balanced classical points is dense in $\mathcal{W}_{\mathbf{fgh}}$ for the rigid-analytic topology, it follows that the Λ-adic class $\kappa_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$ belongs to the kernel of the natural map

$$H^1(\mathbf{Q}_p, \mathbb{V}^\dagger_{\mathbf{fgh}}(M)) \longrightarrow H^1(I_p, \mathbb{V}^\dagger_{\mathbf{fgh}}(M)/\mathbb{V}^+_{\mathbf{fgh}}(M)).$$

Since the kernel of the restriction map

$$H^1(\mathbf{Q}_p, \mathbb{V}^\dagger_{\mathbf{fgh}}(M)/\mathbb{V}^+_{\mathbf{fgh}}(M)) \longrightarrow H^1(I_p, \mathbb{V}^\dagger_{\mathbf{fgh}}(M)/\mathbb{V}^+_{\mathbf{fgh}}(M))$$

is trivial by Lemma 5.3, the result follows. □

Thanks to Lemma 5.3 and Proposition 5.8, we are entitled to define

$$(5.8) \qquad\qquad \boldsymbol{\kappa}^f_p(\mathbf{f}, \mathbf{g}, \mathbf{h})^- \quad \in \quad H^1(\mathbf{Q}_p, \mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}(M))$$

as the projection of the local class $\boldsymbol{\kappa}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$ to $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}(M)$.

Theorem 5.1. — *For any triple of Λ-adic test vectors* $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$, *the following equality holds in the ring* $\mathcal{Q}_{\mathbf{f}, \mathbf{gh}}$:

$$\langle \mathcal{L}_{\mathbf{f}, \mathbf{gh}}(\boldsymbol{\kappa}^f_p(\mathbf{f}, \mathbf{g}, \mathbf{h})^-), \eta_{\breve{\mathbf{f}}^*} \otimes \omega_{\breve{\mathbf{g}}^*} \otimes \omega_{\breve{\mathbf{h}}^*} \rangle = \mathscr{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}).$$

Proof. — It is enough to prove this equality for a subset of classical points that is dense for the rigid-analytic topology, and we shall do so for all balanced triple of crystalline classical points $(x, y, z) \in \mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$ such that $\mathbf{f}_x$, $\mathbf{g}_\ell$ and $\mathbf{h}_m$ are respectively the ordinary stabilization of an eigenform $f := \mathbf{f}^\circ_x$, $g := \mathbf{g}^\circ_y$ and $h := \mathbf{h}^\circ_z$ of level M.

Set $\kappa_p^- := \kappa_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})^-$ and $\mathcal{L} = \langle \mathcal{L}_{\mathbf{f},\mathbf{gh}}(\kappa_p^-), \eta_{\breve{\mathbf{f}}^*} \otimes \omega_{\breve{\mathbf{g}}^*} \otimes \omega_{\breve{\mathbf{h}}^*} \rangle$ for notational simplicity. Proposition 5.6 asserts that the following identity holds in L_p:

$$\nu_{x,y,z}(\mathcal{L}) = \langle \nu_{x,y,z}(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\kappa_p^-)), \eta_{\breve{\mathbf{f}}_x^*} \otimes \omega_{\breve{\mathbf{g}}_y^*} \otimes \omega_{\breve{\mathbf{h}}_z^*} \rangle.$$

Recall also from Proposition 1.5 that

$$\eta_{\breve{\mathbf{f}}_x^*} = \frac{1}{\mathcal{E}_1(f)} e\varpi_1^*(\eta_{\breve{f}^*}), \quad \omega_{\breve{\mathbf{g}}_y^*} = \mathcal{E}_0(g) e\varpi_1^*(\omega_{\breve{g}^*}), \quad \omega_{\breve{\mathbf{h}}_z^*} = \mathcal{E}_0(h) e\varpi_1^*(\omega_{\breve{h}^*})$$

and

$$\nu_{x,y,z}(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\kappa_p^-)) = \frac{(-1)^t}{t!} \cdot \mathcal{E}^{\mathrm{PR}}(x,y,z) \log_{\mathrm{BK}}(\nu_{x,y,z}(\kappa_p^-))$$

by Proposition 5.7.

Recall the class $\kappa(f,g,h) = \kappa(\mathbf{f}_x^\circ, \mathbf{g}_y^\circ, \mathbf{h}_z^\circ)$ introduced in (4.8) arising from the generalized diagonal cycles of [DR14]. As in (5.8), we may define $\kappa_p^f(f,g,h)^- \in H^1(\mathbf{Q}_p, \mathbb{V}_f^{gh}(M))$ as the projection to $V_f^{gh}(M)$ of the restriction at p of the global class $\kappa(f,g,h)$.

It follows from Theorem 4.1 that

$$(\varpi_1, \varpi_1, \varpi_1)_* \nu_{x,y,z}(\kappa_p^-) = \frac{\mathcal{E}^{\mathrm{bal}}(x,y,z)}{(1 - \beta_f/\alpha_f)(1 - \beta_g/\alpha_g)(1 - \beta_h/\alpha_h)} \times \kappa_p^f(f,g,h)^-$$

where

$$\mathcal{E}^{\mathrm{bal}}(x,y,z) = (1 - \alpha_f\beta_g\beta_h p^{-c})(1 - \beta_f\alpha_g\beta_h p^{-c})(1 - \beta_f\beta_g\alpha_h p^{-c})(1 - \beta_f\beta_g\beta_h p^{-c}).$$

The combination of the above identities shows that the value of $\mathcal{L}$ at the balanced triple (x,y,z) is

$$\nu_{x,y,z}(\mathcal{L}) = \frac{(-1)^t \cdot \mathcal{E}^{\mathrm{bal}}(x,y,z)\mathcal{E}^{\mathrm{PR}}(x,y,z)}{t! \cdot \mathcal{E}_0(f)\mathcal{E}_1(f)} \times \langle \log_{\mathrm{BK}}(\kappa_p^f(f,g,h)), \eta_{\breve{f}^*} \otimes \omega_{\breve{g}^*} \otimes \omega_{\breve{h}^*} \rangle$$

Besides, since the syntomic Abel-Jacobi map appearing in [DR14] is the composition of the étale Abel-Jacobi map and the Bloch-Kato logarithm, the main theorem of loc. cit. asserts in the present notations that

$$\nu_{x,y,z}\big(\mathscr{L}_p^{f}(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})\big) = \frac{(-1)^t}{t!} \frac{\mathcal{E}^{\mathrm{f}}(x,y,z)}{\mathcal{E}_0(f)\mathcal{E}_1(f)} \langle \log_{\mathrm{BK}}(\kappa_p^f(f,g,h)^-), \eta_{\breve{f}^*} \otimes \omega_{\breve{g}_y^*} \otimes \omega_{\breve{h}^*} \rangle$$

where

$$\mathcal{E}^{\mathrm{f}}(x,y,z) = \big(1 - \beta_f\alpha_g\alpha_h p^{-c}\big)\big(1 - \beta_f\alpha_g\beta_h p^{-c}\big)\big(1 - \beta_f\beta_g\alpha_h p^{-c}\big)\big(1 - \beta_f\beta_g\beta_h p^{-c}\big).$$

Since

$$\mathcal{E}^{\mathrm{f}}(x,y,z) = \mathcal{E}^{\mathrm{bal}}(x,y,z) \times \mathcal{E}^{\mathrm{PR}}(x,y,z)$$

and the sign and factorial terms also cancel, we have

$$\nu_{x,y,z}(\mathcal{L}) = \nu_{x,y,z}\big(\mathscr{L}_p^{f}(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})\big),$$

as we wanted to show. The theorem follows. $\square$

References

[BDP13] Massimo Bertolini, Henri Darmon, and Kartik Prasanna, *Generalised Heegner cycles and p-adic Rankin L-series*, Duke Math J. **162**, No. 6, (2013) 1033–1148. 4, 5, 27

[B09] Joel Bellaiche, *An introduction to the conjecture of Bloch and Kato*, available at `http://www.people.brandeis.edu/ jbellaic/BKHawaii5.pdf`. 12

[BK93] Spencer Bloch and Kazuya Kato, *L-functions and Tamagawa numbers of motives*, in The Grothendieck Festschrift I, Progr. Math. **108**, 333-400 (1993), Birkhauser. 11, 12

[BSVa] Massimo Bertolini, Marco Seveso, and Rodolfo Venerucci, *Reciprocity laws for balanced diagonal classes*. In this volume. 3

[CS20] Francesc Castella and Ming-Lun Hsieh, *On the non-vanishing of generalized Kato classes for elliptic curves of rank two*, preprint available at `https://web.math.ucsb.edu/~castella`. 3

[DLR15] Henri Darmon, Alan Lauder, and Victor Rotger, *Stark points and p-adic iterated integrals attached to modular forms of weight one.* Forum of Mathematics, Pi, (2015), Vol. 3, e8, 95 pages. 38

[DR14] Henri Darmon and Victor Rotger, *Diagonal cycles and Euler systems I: a p-adic Gross-Zagier formula*, Annales Scientifiques de l'Ecole Normale Supérieure **47**, n. 4 (2014), 779–832. 2, 13, 14, 29, 33, 35, 37, 38, 43, 44

[DR17] Henri Darmon and Victor Rotger, *Diagonal cycles and Euler systems II: the Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin L-series*, Journal of the American Mathematical Society **30** Vol. 3, (2017) 601–672. 2, 43

[DRa] Henri Darmon and Victor Rotger, *Stark-Heegner points and generalised Kato classes*, in this volume.

[F97] Gerd Faltings, *Crystalline cohomology of semistable curves –the $\mathbf{Q}_p$-theory*, J. Alg. Geom. **6** (1997), 1–18. 12

[F90] Matthias Flach, *A generalisation of the Cassels-Tate pairing*, J. reine angew. Math. **412** (1990), 113–127. 43

[G89] Ralph Greenberg, *Iwasawa theory for p-adic representations*, Adv. Studies Pure Math. **17** (1989), 97–137. 43

[GS93] Ralph Greenberg and Glenn Stevens, *p-adic L-functions and p-adic periods of modular forms.* Invent. Math. **111** (1993), 407–447.

[H86] Haruzo Hida, *Iwasawa modules attached to congruences of cusp forms*, Ann. Sci. École Norm. Sup. **19** (1986), 231–273. 10

[H17] Ming-Lun Hsieh, *Hida families and p-adic triple product L-functions*, American J. Math., to appear. 38

[J88] Uwe Jannsen, *Continuous étale cohomology*, Math. Annalen **280** (1988), 207–245. 10, 22, 26

[K15] Guido Kings, *Eisenstein classes, elliptic Soulé elements and the ℓ-adic elliptic polylogarithm.* London Math. Soc. Lecture Note Ser., **418**, Cambridge Univ. Press, 2015. 10

[KLZ17] Guido Kings, David Loeffler, and Sarah Zerbes, *Rankin-Eisenstein classes and explicit reciprocity laws*, Cambridge J. Math. 5 (2017), no. 1, 1–122. 14, 15, 43

[Li] David Lilienfeldt, McGill Ph.D Thesis, 2021. 3

[LZ14] David Loeffler and Sarah Zerbes, *Iwasawa theory and p-adic L-functions over $\mathbf{Z}_p^2$-extensions*, Int. J. Number Theory **10** (2014), no. 8, 2045–2095. 42, 43

[LZ19] David Loeffler and Sarah Zerbes, *Iwasawa theory for the symmetric square of a modular form*, J. Reine Angew. Math. **752** (2019), 179–210. 43

[M] James Milne, *Étale cohomology*, on-line notes. 22, 26

[MT90] Barry Mazur and Jacques Tilouine. Représentations galoisiennes, différentielles de Kähler et conjectures principales, Inst. Hautes Études Sci. Publ. Math. **71** (1990), 65–103. 7

[N98] Jan Nekovář, *p-adic Abel-Jacobi maps and p-adic heights*, in *The Arithmetic and Geometry of Algebraic Cycles* (Banff, Canada, 1998), CRM Proc. Lect. Notes **24** (2000), 367–379. 43

[O03] Tadashi Ochiai, *A generalization of the Coleman map for Hida deformations*, Amer. J. Math., **125**, (2003). 13

[O00] Masami Ohta, *Ordinary p–adic étale cohomology groups attached to towers of elliptic modular curves II*, Math. Annalen **318** (2000), no. 3, 557–583. 14

[PR95] Bernadette Perrin-Riou, *Fonctions L p-adiques des représentations p-adiques*, Astérisque **229** (1995), 1–198. 42

[S97] Takeshi Saito, *Modular forms and p-adic Hodge theory*, Invent. Math. **129** (1997), 607–620. 43

Henri Darmon and Victor Rotger, H. D.: Montreal, Canada
E-mail : darmon@math.mcgill.ca • V. R.: IMTech, UPC and Centre de Recerca Matemàtiques, C. Jordi Girona 1-3, 08034 Barcelona, Spain • *E-mail :* victor.rotger@upc.edu