On Globally Periodic Maps and Periodic Flows

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1. INTRODUCTION

Globally periodic maps (GP)

A map $F : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$, defined in an open set $\mathcal{U}$, is *globally periodic* if there exists some $p \in \mathbb{N}$ such that

$$F^p(x) = x \text{ for all } x \in \mathcal{U}.$$

Some examples are:

1. The well-known real linear fractional maps
   - $F_1(x, y) = \left(y, \dfrac{1+y}{x}\right)$, 5–periodic.
   - $F_2(x, y, z) = \left(y, z, \dfrac{1+y+z}{x}\right)$, 8–periodic.
   - $F_3(x, y, z) = \left(y, z, \dfrac{-1+y-z}{x}\right)$ 8–periodic.

2. The maps

$$F_1(x,y)=\left(\frac{y}{1+x+y},\frac{-x}{1+x+y}\right)$$

and

$$F_2(x,y)=\left(\frac{-y}{\sqrt[3]{1+4y^3-4x^3}},\frac{x}{\sqrt[3]{1+4y^3-4x^3}}\right).$$

which are 4–periodic.

These maps are given by the flow at time $2\pi/4$ of the planar **isochronous centers**

$$\begin{cases}\dot{x}=-y+x^2,\\ \dot{y}=x(1+y),\end{cases}$$

and

$$\begin{cases}\dot{x}=-y+4x^2y^2,\\ \dot{y}=x+4xy^3\end{cases}$$

respectively.

It is obvious that every T–isochronous flow φ gives rise to **GP** maps of arbitrary period p, via the ***global stroboscopic map***

$$F(x) = \varphi\left(\frac{T}{p}, x\right)$$

But what about the converse?

Periodic flows, periodic vector fields (PVF)

A vector field $\mathbf{X} : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ (with flow φ) is **periodic** if for any $x \in \mathcal{U}$ there exists $T(x) > 0$ such that $\varphi(T(x), x) = x$.

If x is not a singular point, the minimum values $T(x)$ satisfying this property is the period of x.

Isochronous flows

A flow it is called **isochronous**,if there exists $T > 0$ such that $\varphi(T, x) \equiv x$ for all $x \in \mathcal{U}$.

Our main objective is to prove something like

• To prove that for every globally periodic map $F \in \mathcal{C}^1(\mathcal{U})$, **orientation preserving**[a] there exists at least a periodic flow, *in the same phase space*, associated to it, such that

$$F(x) = \varphi(\tau(x), x). \quad (1)$$

• Furthermore, is it possible to find one of them which is isochronous, hence the map is a global stroboscopic map of the flow

$$F(x) = \widetilde{\varphi}(\tau, x)?$$

[a] *I will assume this during the whole talk.*

These are not exactly the same problem although a periodic vector field $\mathbf{X} : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$, it can be reparameterized by

$$\mathbf{Y}(x) = T(x)\mathbf{X}(x) : \quad (2)$$

giving rise to a **1–isochronous** vector field.

To *isochronize* a PVF, $T(x)$ must be **regular** in $\mathcal{U}$.

The regularization problem has been treated by us, but we have knew that the same kind of results have been published by Sergiy Maksymenko at the end of 2009.

Theorem. (Maksymenko... and us, 2009)

Let $\mathbf{X} : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^k(\mathcal{U})$, $k \geq 1$ periodic vector field

Let $\mathcal{S}$ be the set of singular points of $\mathbf{X}$.

If for every point $x \in \mathcal{U} \setminus \mathcal{S}$ there exists a neighborhood $\mathcal{V}_x \subseteq \mathcal{U}$ where the period function $T(x)$ is bounded,

$\Longrightarrow$ the period function $T(x)$ **can be extended regularly** to the whole set $\mathcal{U} \setminus \mathcal{S}$.

The local boundedness hypothesis cannot be removed!

There are examples of periodic flows such that there are periodic orbits s.t. the periods of the surrounding P.O. tend to infinity as they approach to the *singular* periodic orbit Reeb (1952), Epstein (1972), Sullivan (1976), Voght (1977)...

Epstein's example (1972), defined on $\mathbb{R}^2 \times \mathbb{S}^1$:

$$\left\{\begin{array}{l} \dot{x} = -y(x^2+y^2-1), \\ \dot{y} = x(x^2+y^2-1), \\ \dot{z} = y, \ \text{mod } 2\pi \end{array}\right. \Longrightarrow \left\{\begin{array}{l} \dot{r} = 0, \\ \dot{\theta} = r^2-1, \\ \dot{z} = r\sin(\theta), \ \text{mod } 2\pi. \end{array}\right.$$

It can be integrated giving rise to

$$\begin{array}{l} r(t) = r_0, \\ \theta(t) = \theta_0 + (r_0^2-1)t, \ \text{mod } 2\pi \\ z(t) = \left\{\begin{array}{ll} z_0 + \frac{r_0}{r_0^2-1}\left[\cos(\theta_0) - \cos(\theta_0+(r_0^2-1)t)\right], \ \text{mod } 2\pi & \text{if } r_0 \neq 1 \\ z_0 + \sin(\theta_0 t), \ \text{mod } 2\pi & \text{if } r_0 = 1 \end{array}\right. \end{array}$$

The circles $\{r=1, \theta=0\}$; $\{r=1, \theta=\pi\}$; $\{r=0\}$ are full of singular points, the other orbits are periodic of minimal period:

$$T(r_0,\theta_0,z_0) = \left\{\begin{array}{ll} \frac{2\pi}{|r_0^2-1|} & \text{if } r_0 \neq 1 \\ \frac{2\pi}{|\theta_0|} & \text{if } r_0 = 1 \ (\theta_0 \neq 0, \pi) \end{array}\right. \quad \text{so} \quad \lim_{r_0 \to 1} T(r_0,\theta_0,z_0) = +\infty$$

We have recently known a beautiful polynomial example in $\mathbb{R}^3$ due to Daniel Peralta–Salas (2009).

With respect a singular points x_0 of $\mathbf{X}$, of course $\lim_{x \to x_0} T(x)$ can tend to ∞ *(e.g. degenerate planar centers)*.

Proposition

If $T(x)$ extends regularly to a singular point x_0 not necessarily isolated, then

($\Rightarrow$) (i) $\text{Dim}(\text{Ker}(D\mathbf{X}(x_0))) = \text{Dim}(\mathcal{S})$.

(ii) $n - \text{Dim}(\text{Ker}(D\mathbf{X}(x_0)))$ is even

($\Leftarrow$) The converse is not true, as the example of Peralta–Salas evidences.

2. LINEARIZATIONS

Linearization

Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set homeo– or diffeomorphic to $\mathbb{R}^n$.

A map GP map $F : \mathcal{U} \to \mathcal{U}$, $\mathcal{C}^r$**–linearizes** if there exists a $\mathcal{C}^r$–diffeomorphism ψ : such that

$$\begin{array}{ccccc} & \mathcal{U} & \stackrel{F}{\longrightarrow} & \mathcal{U} & \\ \psi & \downarrow & \searrow & \downarrow & \psi \\ & \mathcal{V} & \underset{L}{\longrightarrow} & \mathcal{V} & \end{array}$$

commutes, that is $\psi \circ \mathbf{F} = \mathbf{L} \circ \psi$

Where L is a linear map.

We talk about linearization in $\mathcal{U} \cong \mathbb{R}^n$ not in all $\mathbb{R}^n$.

Cima, Gasull, and F. Mañosas (2009) have proved that $(n-3)$–periodic Coxeter maps (Coxeter, 1972): $F(x_1, \ldots, x_n) = \left(x_2, x_3, \ldots, x_n, 1 - \frac{x_{n-1}}{1-\frac{x_{n-2}}{1-\frac{x_{n-3}}{1-\cdots}}} \right)$, have $\left[\frac{n+2}{2}\right]$ fixed points each of them locally conjugate with **different** linear models.

In $\mathbb{R}$ and $\mathbb{R}^2$ our objective is achieved!

Theorem (Kerékjártó, 1919)

Let $\mathcal{U} \subseteq \mathbb{R}^2$ be homeomorphic to $\mathbb{R}^2$, and let F be a p–periodic map in $\mathcal{U}$. Then F is $\mathcal{C}^0$–linearizable

Corollary.

For every globally periodic map F in $\mathcal{U} \cong \mathbb{R}^2$, there exists an isochronous periodic flow φ such that

$$F(x) = \varphi(\theta, x).$$

Proof. Kerékjártó's Theorem $\Rightarrow \exists$ a homeo ψ s.t. $\psi \circ F = L \circ \psi$ where L is a periodic linear map. We can consider that

$$L(x) = A\,x, \text{ where } A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $\rho = \frac{\theta}{2\pi} \in \mathbb{Q}$ Hence L is the "time-θ" stroboscopic map of the isochronous flow φ_L associated to the harmonic oscillator $\dot{u} = -v,\ \dot{v} = u$. Therefore

$$\mathbf{F}(\mathbf{x}) = \psi^{-1} \circ \mathbf{L} \circ \psi(\mathbf{x}) = \psi^{-1} \circ \varphi_{\mathbf{L}}(\theta, \psi(\mathbf{x})) =: \varphi(\theta, \mathbf{x})$$

∎

In general it is an open question to know if a GP map linearizes,

At this point we give summary of some results some of which encourage to keep this way and some other do not.

1. CAREFUL! there are examples of GP homeomorphisms (thus $\mathcal{C}^0$) which do NOT linearize. The first was given by Bing (1952).
2. BUT the Montgomery–Bochner (1955) Theorem states that any periodic $\mathcal{C}^1$–diffeomorphism with a fixed point locally linearizes. The linearization is

$$\psi = \frac{1}{p}\sum_{i=0}^{p-1}(DF(x_0))^{-i}\cdot F^i \qquad (*)$$

3. HOWEVER although any periodic $\mathcal{C}^1$–diffeomorphism in $\mathbb{R}^n$, $n \leq 6$ must have a fixed point, there are periodic $\mathcal{C}^1$–diffeomorphisms in $\mathbb{R}^n$, $n \geq 7$ without fixed points.

   (*) MB cannot be used to obtain a general proof but it can be used to construct linearizations in a "case by case" analysis.

4. BUT There are periodic $\mathcal{C}^0$–flows without singular points, HOWEVER the period function is NOT locally bounded.

3. LIE SYMMETRIES VIA INTEGRABILITY OF GP MAPS

A first integral for a map F is a function V such that $V(F) = V$.

Integrability and complete integrability

A map $F : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ will be called

- **integrable** if there exists $n-1$ *functionally independent* first integrals in $\mathcal{U}$, and
- **completely integrable** if there exists n *functionally independent* first integrals in $\mathcal{U}$.

There is a strong relation between globally periodic maps and complete integrability.

Theorem. (Cima, Gasull, M)

Let $F : \mathcal{U} \subseteq \mathbb{R}^n \to \mathcal{U}$ be an injective $\mathcal{C}^\omega$ map defined in an open set $\mathcal{U}$. The following statements hold:

($\Rightarrow$) **If F is globally periodic, then the DDS generated by F is completely integrable.**

($\Leftarrow$) True with an extra condition.

Some remarks

- The proof is constructive, i.e. given F GP $\Rightarrow$ we obtain $V_1, V_2, \ldots, V_n$.
- The statement is for $\mathcal{C}^\omega$ maps because we use strongly this condition to prove that the system of first integrals is *functionally independent.*

  However the method of construction gives functionally independent integrals even if the regularity is relaxed.

Back to vector fields: Lie Symmetries

Lie Symmetry of a map

A **Lie Symmetry** of a map F is a **vector field X** such that F maps any orbit of the $\dot{x} = \mathbf{X}(x)$, to another orbit of this system (not necessarily the same).

Lie Symmetry of a map (formal definition)

A **Lie Symmetry** of a map F is a **vector field X** such that the differential equation

$$\dot{x} = \mathbf{X}(x),$$

is invariant under the change of variables $\mathbf{u} = F(\mathbf{x})$.

Why to look at Lie Symmetries?

Theorem. (Cima, Gasull, M)

If $F : \mathcal{U} \to \mathcal{U}$ is a diffeomorphism having a Lie symmetry $\mathbf{X}$,

and such F preserves a solution γ of the differential equation $\dot{x} = \mathbf{X}(x)$,

- then $F(x) = \varphi(\tau(\gamma), x)$,
- and the dynamics of $F|_\gamma$ is either conjugated to a rotation, conjugated to a translation of the line, or constant, according whether γ is homeomorphic to $\mathbb{S}^1$, $\mathbb{R}$, or a point, respectively.

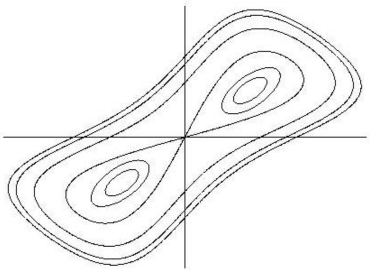
Example: the Gumovski–Mira map. Not periodic!

$$F(x,y) = \left(y, -x + \frac{\alpha + \beta y}{1+y^2} \right).$$

This map has the first integral

$$V(x,y) = x^2y^2 + (x^2+y^2) - \beta xy - \alpha(x+y)$$

$X = V \cdot \left(-V_y \frac{\partial}{\partial x} + V_x \frac{\partial}{\partial y} \right)$ is a Lie Symmetry of F. Such that F preserves any solution γ of the associated differential equation.



Energy levels for $\alpha = 0$ and $\beta > 2$.

Integrable maps preserving a measure have Lie Symmetries

Theorem (Cima, Gasull, M)

An integrable map, with first integrals given by $V_1, \ldots, V_{n-1}$, such that it preserves a measure

$$m_\nu(B) = \int_B \nu \, dx,$$

where $\nu \in \mathcal{C}^1(\mathcal{U})$ and nonvanishing except a zero measure set.

$\Rightarrow$ F **has a Lie Symmetry with the same set of functionally independent first integrals.**

Furthermore the Lie Symmetry is given by

$$\begin{cases} \mathbf{X}(p) &= \mu(p)\,(-V_{1,y}, V_{1,x}) \text{ if } n=2, \text{and} \\ \mathbf{X}(p) &= \mu(p)\,(\nabla V_1(p) \times \nabla V_2(p) \times \cdots \times \nabla V_{n-1}(p))\,, \text{ if } n>2, \end{cases}$$

where

$$\mu = \frac{1}{\nu}.$$

Any globally periodic map has an invariant measure:

Lemma.

If F is GP it preserves the measure

$$m(B) = \sum_{i=0}^{p-1} \int_{F^i(B)} dx \qquad (3)$$

where dx is the Lebesgue measure.

Since

$$m(B) = \sum_{i=0}^{p-1} \int_{F^i(B)} dx = \int_B \sum_{i=0}^{p-1} |DF^i(x)| dx$$

we can take

$$\nu(x) = \sum_{i=0}^{p-1} |DF^i(x)| \Longrightarrow \mu(x) = \frac{1}{\sum_{i=0}^{p-1} |DF^i(x)|}. \qquad (4)$$

This is only one possible μ.

In summary

1. GP maps are complete integrable.

2. An integrable map with an invariant measure has a Lie Symmetry.

3. GP posses an invariant measure.

HENCE

Corollary. Every complete integrable GP map admits **n independent** Lie symmetries

Given GP map defined having $V_1, \ldots, V_n$ functionally independent $\mathcal{C}^1$–first integrals, then there exist n **linearly independent** Lie Symmetries given by

$$X_k(x) = \mu(x)\left(\vec{\nabla} V_1(x) \times \vec{\nabla} V_2(x) \times \cdots \times \widehat{\vec{\nabla} V_k(x)} \times \cdots \times \vec{\nabla} V_n(x)\right),$$

for all $k = 1, \ldots, n$, and where μ can be chosen as in equation (4).

There are $n-1$ of the Lie Symmetries which are periodic!

Proposition

Let $F : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a GP map, having $n-1$ first integrals (namely $V_1, \ldots, V_{n-1}$), with

$$V_1(x) = \sum_{i=0}^{p-1} |F^i(x)|^2$$

$\Longrightarrow$ the induced foliation given by the transversal energy levels $L_{h_1,\ldots,h_p} := \pitchfork_{i=1}^{n-1} \{V_i = h_i\}$ are diffeomorphic to a finite disjoint union of $\mathbb{S}^1$.

Proof.

• The nonempty level sets $L_{h_1} = \{V_1 = h_1\}$ are compact

• Since $V_1, \ldots V_{n-1}$ are functionally independent, the locus where they intersect transversally

$$\widetilde{L}_{h_1,\ldots,h_{n-1}} := \bigcap_{i=1}^{n-1} \{V_i = h_i\}$$

is a compact 1–dimensional manifold $\Rightarrow$ are diffeomorphic to a finite disjoint union of $\mathbb{S}^1$ (Guillemin–Pollack's book Page 208 for instance). $\blacksquare$

Observe that the locus of non–transversality of the energy levels will correspond to singular points of the associated Lie Symmetries.

$\Rightarrow$ The associated Lie symmetries are periodic vector fields.

Obstruction: Although it seems that we have a lot of periodic vector fields how to guarantee that we can obtain a periodic one such that F preserves any solution γ of the differential equation $\dot{x} = \mathbf{X}(x)$?

We have not succeeded... yet.

Example. The map

$$F(x,y,z)=\left(z,\frac{1+y+z}{x},\frac{1+x+y+z+xz}{xy}\right)$$

is globally periodic in its good set. The Lie symmetry

$$\mathbf{X}(\mathbf{x},\mathbf{y},\mathbf{z})=\mu(x,y,z)\cdot\left(\vec{\nabla}V_1\times\vec{\nabla}W\right)(x,y,z)$$

where

$$W=\frac{(x+1)(z+1)}{y}\quad\text{and}$$

and

$$V_1=\frac{(x+1)(y+1)(z+1)(1+x+y+z)}{xyz},$$

is periodic and it preserves any solution γ of the differential equation $\dot{x}=\mathbf{X}(x)$ (Cima, Gasull, M, (2008); Cima, Mañosas 2009).

$\Longrightarrow$ F can seen as the global stroboscopic map of an isochronous flow on $\mathbb{R}^3$.

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THANK YOU!