# On Poncelet type maps

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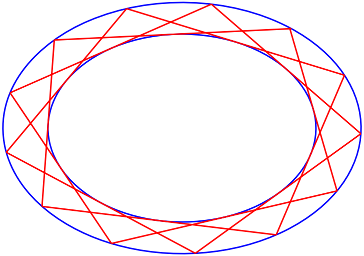
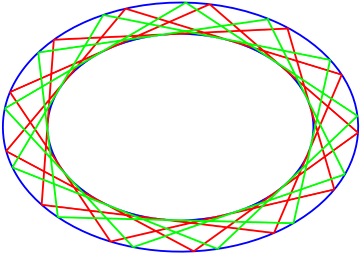
## PONCELET'S PORISM

**Poncelet's Porism**

*Given one ellipse inside another, if there exists one n-gon simultaneously inscribed in the outer and circumscribed on the inner, then any point on the boundary of the outer ellipse is the vertex of some n-gon.*



A star 13–gon



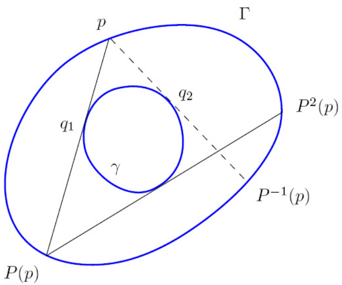
Another 13–gon.

All points have the same property

# PONCELET TYPE MAPS

Set Γ and $\gamma$ two $\mathcal{C}^r$–closed curves such that $\gamma \subset \mathrm{Int}(\Gamma)$.

Given $p \in \Gamma$ there are two points $q_1, q_2$ in $\gamma$ such that $\overline{pq_1}$, $\overline{pq_2}$ are tangent to $\gamma$.




The *Poncelet map*, $P : \Gamma \to \Gamma$, associated to $\gamma, \Gamma$ is

$$P_{\Gamma,\gamma}(p) = \overline{pq_1} \cap \Gamma,$$

where $\overline{pq_1} \cap \Gamma$ is the first point in $\{\overline{pq_1} \cap \Gamma, \overline{pq_2} \cap \Gamma\}$ starting from $p$, following Γ counterclockwise

## PONCELET'S PORISM IN THE DISCRETE DYNAMICAL SYSTEM SETTING

By construction $P$ can be seen as a $\mathcal{C}^r$ diffeomorphism of the topological circle and has associated a *rotation number*

$$\rho = \rho(P_{\Gamma,\gamma}) \in (0, 1/2).$$

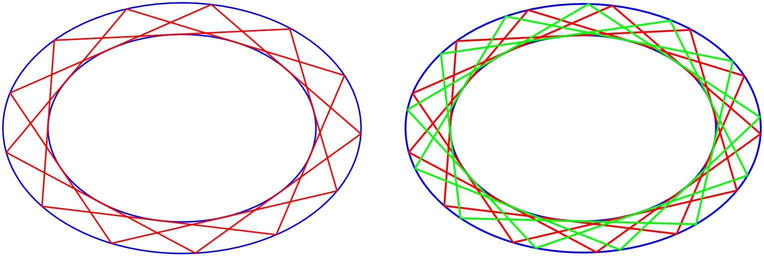
Notice that if $\rho \in \mathbb{R} \setminus \mathbb{Q}$ then it is well known (from the DDS theory) that $P_{\Gamma,\gamma}$ is conjugated to an irrational rotation.

With the above notation *Poncelet's Porism* asserts that

**Poncelet's Porism**

*If $\gamma$ and $\Gamma$ are ellipses and $\rho \in \mathbb{Q}$ then the Poncelet map $P_{\Gamma,\gamma}$ is conjugated to a the rational rotation of angle $2\pi\rho$ in $\mathbb{S}^1$. $\Rightarrow$ Each orbit is periodic.*

## Example 1:



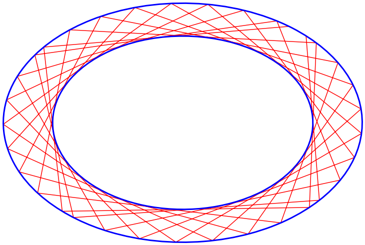
A 13–periodic orbit

Another 13–periodic orbit.

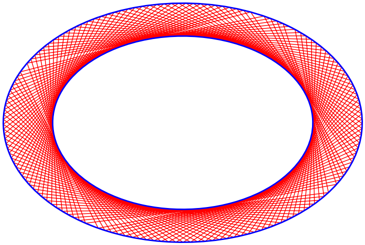
All points are PERIODIC because $P_{\Gamma,\gamma}$ is conjugated to a rational rotation

with rotation number $\rho(P_{\Gamma,\gamma}) = \dfrac{2}{13}$.

## Example 2:



35 iterates of $P_{\Gamma,\gamma}$



150 iterates of $P_{\Gamma,\gamma}$

In this case the map $P_{\Gamma,\gamma}$ is an irrational rotation

Each orbit fills densely $\Gamma$.

## IS THE PONCELET PROPERTY TRUE FOR OTHER ALGEBRAIC OVALS?

# NO

### Theorem 1

Fix $\gamma = \{x^2 + y^2 = 1\}$. Then for any $m \in \mathbb{N}$, $m > 2$, there is an algebraic curve of degree $m$, containing a convex oval $\Gamma$, such that the Poncelet's map associated to $\gamma$ and $\Gamma$ has a RATIONAL rotation number and it is NOT conjugated to a rotation.

This result is consequence of

### Proposition 2

Consider

$$\gamma = \{x^{2n} + y^{2n} = 1\} \quad \text{and} \quad \Gamma = \{x^{2m} + y^{2m} = 2\} \quad \text{with} \quad n, m \in \mathbb{N},$$

and let $P$ be the Poncelet's map associated to them. Then $\rho_{n,m}(P) = 1/4$. Moreover, the map is conjugated a rotation if and only if $n = m = 1$.

Proof of Proposition 2: For any $n$ and $m$, the Poncelet map $P$ has the periodic orbit of period 4, given by

$$\mathcal{O} = \{(1,1),(-1,1),(-1,-1),(1,-1)\}.$$

Hence $\rho_{n,m}(P) = 1/4$.

The rest of the proof follows by straightforward computations imposing that $p_1 = (0, \sqrt{2m}) \in \Gamma$ must be a 4–periodic point. $\blacksquare$

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Proof of Theorem 1: If $m$ is even is a corollary of the Proposition 2.

If $m \geq 3$ is odd, notice that the sets $\{x^{2m} + y^{2m} - 2 = 0\}$ and $\{(x+10)\left(x^{2m} + y^{2m} - 2\right) = 0\}$ coincide in $\{x > -10\}$

So in both cases the Poncelet's maps coincide. $\blacksquare$

## WHICH KIND OF DYNAMICS CAN APPEAR? AN EXAMPLE

### Proposition 3

Consider

$$\gamma = \{x^2 + y^2 = 1\} \quad \text{and} \quad \Gamma = \{x^{2m} + y^{2m} = 2\} \quad \text{with} \quad m \in \mathbb{N}, m > 1.$$

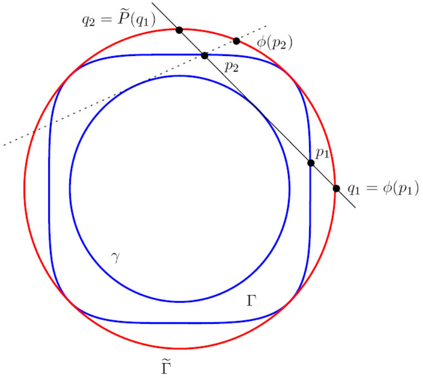
and let $P$ be the Poncelet map associated to them. Then

- $\mathcal{O} = \{(1,1), (-1,1), (-1,-1), (1,-1)\}$ is a 4-periodic orbit of $P$
- $\rho(P) = 1/4$, and
- $\mathcal{O}$ is the $\alpha$ and $\omega$ limit set of all the orbits of $P$.

# Proof of Proposition 3:

Recall that:

- $\gamma = \{x^2 + y^2 = 1\}$ and
- $\Gamma = \{x^{2m} + y^{2m} = 2\}$.
- Set $\tilde{\Gamma} = \{x^2 + y^2 = 2\}$
- Let $\tilde{P} = \tilde{P}_{\tilde{\Gamma},\gamma}$ and $P = P_{\Gamma,\gamma}$ be the associated Poncelet maps.




- $\phi(p_1)$ is a bijection between $\Gamma$ and $\tilde{\Gamma}$.
- Notice that $\arg(q_2) > \arg(\phi(p_2))$. That is $\arg(\tilde{P}(q_1)) > \arg(\phi(P(p_1)))$

This can be understood as a "delay" of $P$ with respect to $\tilde{P}$ which propagates. ∎

Recall that $V$ is a FIRST INTEGRAL of $F$ in a set of $\mathbb{R}^2$ if

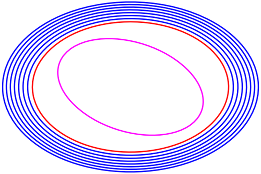
$$V(F(p)) = V(p), \text{ for all } p \text{ in } \mathcal{U}.$$

This means that the orbits of $F$ lie in the level sets of $V$.

In this talk a planar map is called *INTEGRABLE* if it has a first integral

Given two closed regular curves $\gamma$ and $\Gamma_{k_0} = \{V(x,y) = k_0\}$, we can extend the construction of $P_{\Gamma_{k_0},\gamma}$ to the curves

$$\Gamma_k = \{V(x,y) = k\}$$



So we can consider PLANAR Poncelet type maps which are INTEGRABLE

### Theorem 4

Let $F : \mathcal{U} \to \mathcal{U}$ be a diffeo. defined on an open set $\mathcal{U} \subset \mathbb{R}^2$ s.t.

(a) It has a smooth first integral $V : \mathcal{U} \to \mathbb{R}$, having its level sets $\Gamma_k =: \{p \in \mathcal{U} \ : \ V(p) = k\}$ as simple closed curves,

(b) There exists a smooth function $\mu : \mathcal{U} \to \mathbb{R}^+$ such that for any $p \in \mathcal{U}$,

$$\mu(F(p)) = \det(DF(p))\,\mu(p). \qquad (1)$$

Then the map $F$ restricted to each $\Gamma_k$ is conjugated to a rotation[a]

[a] This is because $F$ is the stroboscopic map of the flow of $\dot{p} = \mu(p)\left(-\frac{\partial V(p)}{\partial p_2}, \frac{\partial V(p)}{\partial p_1}\right)$.

It provides a way to check whether integrable planar maps $F$ of the circle are conjugated to rotations or not, by studying the existence of solutions $\mu$ of (1)

## A new proof (yet another) of Poncelet's Porism:

- Outer ellipse $\Gamma = \{x^2 + y^2 = 1\}$ (this assumption is not restrictive).
- Inner ellipse $\gamma = \{Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\}$.

The PLANAR Poncelet map is defined in the open set of $\mathbb{R}^2$, and it has the form

$$P(x,y) = \left( \frac{-N_1 N_2 - 4N_3\sqrt{\Delta}}{M}, \frac{-N_1 N_3 + 4N_2\sqrt{\Delta}}{M} \right)$$

Where $N_1, N_2, N_3, M$ and $\Delta$ are polynomials in $A, B, C, D, E, F, x$ and $y$.

P is integrable with invariant $V(x,y) = x^2 + y^2$, and

$$\mu(x,y) = \sqrt{(x^2+y^2)(Ax^2 + Bxy + Cy^2 + Dx + Ey + F)}$$

is a solution of $\mu(F(p)) = \det(DP(p))\,\mu(p) \Rightarrow$ (by Theorem 4) $P$ is conjugated to a rotation. ∎

How to find $\mu$? The existence $\mu$ for a map $F$ is related with the existence of an invariant measure absolutely continuous with respect to the Lebesgue one $m(B) = \int_B \nu$

$m$ is an invariant measure of $F$ if and only if $m(F^{-1}(B)) = m(B)$ (*)

By using the change of variables formula (*) can be rewritten as,

$$m(F^{-1}(B)) = \int_{F^{-1}(B)} \nu = \int_B \nu = m(B) \Leftrightarrow \int_B \nu(F)\mathrm{det}(DF) = \int_B \nu \quad (**)$$

If $\mu(F) = \mathrm{det}(DF)\mu$, setting $\nu = \dfrac{1}{\mu} \Rightarrow m(B) = \displaystyle\int_B \frac{1}{\mu}$ is an invariant measure of $F$.

From King, 1994 we learned an invariant arclenght measure for the Poncelet maps in the cases of ellipses but it is not a planar measure.

By tuning king's arclenght measure we obtained a PLANAR measure and then we obtained $\mu$.

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