On periodic solutions of 2–periodic Lyness difference equations

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1. INTRODUCTION

We study the *set of periods* of the 2-periodic Lyness' equations

$$u_{n+2} = \frac{a_n + u_{n+1}}{u_n}, \qquad (1)$$

where

$$a_n = \begin{cases} a & \text{for} \quad n = 2\ell + 1, \\ b & \text{for} \quad n = 2\ell, \end{cases} \qquad (2)$$

and being $(u_1, u_2) \in \mathcal{Q}^+$; $\ell \in \mathbb{N}$ and $a > 0, b > 0$.

This can be done using the *composition map:*

$$F_{b,a}(x,y) := (F_b \circ F_a)(x,y) = \left(\frac{a+y}{x}, \frac{a+bx+y}{xy}\right), \qquad (3)$$

where F_a and F_b are the Lyness maps: $F_\alpha(x,y) = \left(y, \frac{\alpha+y}{x}\right)$. Indeed:

$$(u_1,u_2) \xrightarrow{F_a} (u_2,u_3) \xrightarrow{F_b} (u_3,u_4) \xrightarrow{F_a} (u_4,u_5) \xrightarrow{F_b} (u_5,u_6) \xrightarrow{F_a} \cdots$$

The map $F_{b,a}$:

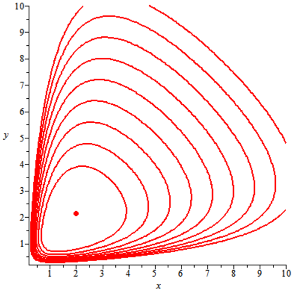
- Is a QRT map whose first integral is (Quispel, Roberts, Thompson; 1989):

$$V_{b,a}(x,y) = \frac{(bx+a)(ay+b)(ax+by+ab)}{xy},$$

see also (Janowski, Kulenović, Nurkanović; 2007) and (Feuer, Janowski, Ladas; 1996).

- Has a unique fixed point $(x_c, y_c) \in \mathcal{Q}^+$, which is the unique global minimum of $V_{b,a}$ in $\mathcal{Q}^+$.
- Setting $h_c := V_{b,a}(x_c, y_c)$, for $h > h_c$ the level sets $\{V_{b,a} = h\} \cap \mathcal{Q}^+$ are the closed curves.

$$\mathcal{C}_h^+ := \{(bx+a)(ay+b)(ax+by+ab) - hxy = 0\} \cap \mathcal{Q}^+ \text{ for } h > h_c.$$



The dynamics of $F_{b,a}$ restricted to $\mathcal{C}_h^+$ is *conjugate to a rotation* with associated *rotation number* $\theta_{b,a}(h)$.

Theorem A

Consider the family $F_{b,a}$ with $a, b > 0$.

(i) *If $(a, b) \neq (1, 1)$, then $\exists\, p_0(a, b) \in \mathbb{N}$, generically computable, s.t. for any $p > p_0(a, b)$ $\exists$ at least an oval $\mathcal{C}_h^+$ filled by p–periodic orbits.*

(ii) *The set of periods arising in the family $\{F_{b,a},\ a > 0, b > 0\}$ restricted to $\mathcal{Q}^+$ contains all prime periods except 2, 3, 4, 6, 10.*

Corollary.

Consider the 2–periodic Lyness' recurrence for $a, b > 0$ and positive initial conditions u_1 and u_2.

(i) *If $(a, b) \neq (1, 1)$, then $\exists\, p_0(a, b) \in \mathbb{N}$, generically computable, s.t. for any $p > p_0(a, b)$ $\exists$ continua of initial conditions giving $2p$–periodic sequences.*

(ii) *The set of prime periods arising when $(a, b) \in (0, \infty)^2$ and positive initial conditions are considered contains all the even numbers except 4, 6, 8, 12, 20.*

If $a \neq b$, then it does not appear any odd period, except 1.

Digression: why to focus on the 2-periodic case?

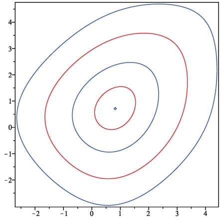
Because of computational issues, and because is one of the few *integrable* ones. For each k, the *composition maps* are

$$F_{[k]} := F_{a_k,\ldots,a_2,a_1} = F_{a_k} \circ \cdots \circ F_{a_2} \circ F_{a_1} \tag{4}$$

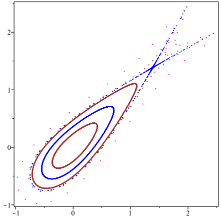
where

$$F_{a_i}(x,y) = \left(y, \frac{a_i + y}{x}\right) \text{ and } a_1, a_2, \ldots, a_k \text{ are a k-cycle.}$$

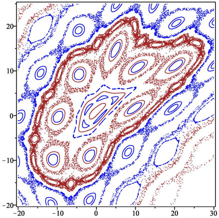
The figure summarizes the situation.



All k



$k = \dot{5}$

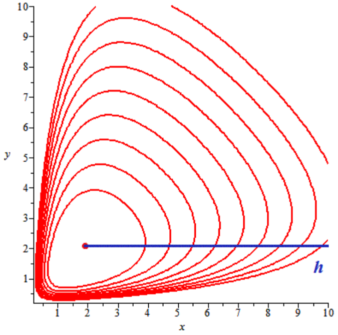


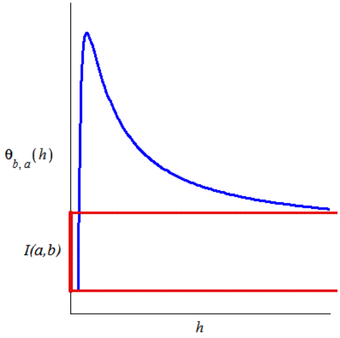
$k \notin \{1, 2, 3, 5, 6, 10\}$(*)

The cases 1,2,3 and 6 have first integrals given by $V(x,y) = \dfrac{P_3(x,y)}{xy}$ *(Cima, Gasull, M; 2012b)*.

(*) This phase portraits are the ones of the DDS associated with the recurrence obtained after the change $z_n = \log(u_n)$.

2. THE STRATEGY: analysis of the asymptotic behavior of $\theta_{b,a}(h)$.





The main issues that allow us to compute the allowed periods are:

1. The fact that the rotation number function $\theta_{b,a}(h)$ is continuous in $[h_c, +\infty)$.
2. The fact that *generically* $\theta_{b,a}(h_c) \neq \lim_{h \longrightarrow +\infty} \theta_{b,a}(h) \Longrightarrow \exists\, I(a,b)$, a *rotation interval*.

$\forall \theta \in I(a,b)$, $\exists$ at least an oval $\mathcal{C}_h^+$ s.t. $F_{b,a}$ restricted to the this oval is conjugate to a rotation, with a rotation number $\theta_{b,a}(h) = \theta$

In particular, for all the irreducible $q/p \in I(a,b)$, $\exists$ periodic orbits of $F_{b,a}$ of prime period p.

Proof of Theorem A (i).

Proposition B.

$$\lim_{h\to+\infty}\theta_{b,a}(h)=\frac{2}{5}$$

$$\lim_{h\to h_c}\theta_{b,a}(h)=\sigma(a,b)=\frac{1}{2\pi}\arccos\left(\frac{1}{2}\left[-2+\frac{1}{x_c\,y_c}\right]\right).$$

Theorem C.

Set $I(a,b):=\left\langle\sigma(a,b),\frac{2}{5}\right\rangle.$

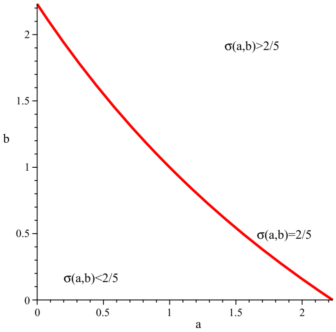
If $\sigma(a,b)\neq 2/5$, *for any fixed* $a,b>0$, *and any* $\theta\in I(a,b)$, $\exists$ *at least an oval* $\mathcal{C}_h^+$ *s.t.* $F_{b,a}(\mathcal{C}_h^+)$ *is conjugate to a rotation, with a rotation number* $\theta_{b,a}(h)=\theta$.

Which are the periods of a particular $F_{b,a}$? $\Leftrightarrow$ Which are the irreducible fractions in $I(a,b)$?

- If $\sigma(a,b)\neq 2/5$, it is possible to obtain *constructively* a value p_0 s.t. for any $r>p_0$ $\exists$ an irreducible fraction $q/r\in I(a,b)$.
- A finite checking determines which values of $p\leq p_0$ are s.t. $\exists\ q/p\in I(a,b)$.
- Still the forbidden periods must be detected. Since $I(a,b)\subseteq\mathrm{Image}\left(\theta_{b,a}\left(h_c,+\infty\right)\right)$.

Generically?

Set $\mathcal{P} := \{(a,b),\ a,b>0\}$:



The curve $\sigma(a,b) = 2/5$ for $a, b > 0$ is given by

$$\Gamma := \{\sigma(a,b) = 2/5,\ a,b>0\} = \left\{(a,b) = \left(\frac{t^3-\phi^2}{t}, \frac{\phi^4-t^3}{t^2}\right),\ t \in (\phi^{\frac{2}{3}}, \phi^{\frac{4}{3}})\right\} \subset \mathcal{P}.$$

Of course $\mathcal{P} \setminus \Gamma$ is open and dense in $\mathcal{P}$

3. The periods of the family $F_{b,a}$. Proof of Theorem A (ii)

Using the previous results with the family $a = b^2$ we found that:

$$\bigcup_{b>0} I(b^2, b) = \left(\frac{1}{3}, \frac{1}{2}\right) \subset \bigcup_{a>0,\, b>0} I(a, b) \subset \bigcup_{a>0,\, b>0} \text{Image}\left(\theta_{b,a}\left(h_c, +\infty\right)\right).$$

Proposition D.

For each θ in $(1/3, 1/2)$ $\exists$ $a, b > 0$ and at least an oval $\mathcal{C}_h^+$, s.t. $F_{b,a}(\mathcal{C}_h^+)$ is conjugate to a rotation with rotation number $\theta_{b,a}(h) = \theta$.

In particular, $\forall$ irreducible $q/p \in (1/3, 1/2)$, $\exists$ periodic orbits of $F_{b,a}$ of prime period p.

We'll know <u>some</u> periods of $\{F_{b,a},\ a, b > 0\}$

$\Leftrightarrow$

We know which are the irreducible fractions in $(1/3, 1/2)$

Lemma (Cima, Gasull, M; 2007)

Given (c,d); Let $p_1=2, p_2=3, p_3,\ldots,p_n,\ldots$ be all the prime numbers.

- *Let p_{m+1} be the smallest prime number satisfying that $p_{m+1}>\max(3/(d-c),2)$,*
- *Given any prime number p_n, $1\le n\le m$, let s_n be the smallest natural number such that $p_n^{s_n}>4/(d-c)$.*
- *Set $p_0:=p_1^{s_1-1}p_2^{s_2-1}\cdots p_m^{s_m-1}$.*

Then, for any $p>p_0$ $\exists$ an irreducible fraction q/p s.t. $q/p\in(c,d)$.

Proof of Theorem A (ii):

- We apply the above result to $(1/3,1/2)$. $\forall p\in\mathbb{N}$, s.t. $p>p_0$

$$p_0:=2^4\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17=12\,252\,240,$$

$\exists$ an irreducible fraction $q/p\in(1/3,1/2)$.

- A finite checking determines which values of $p\le p_0\in(1/3,1/2)$, resulting that there appear irreducible fractions with all the denominators except 2, 3, 4, 6 and 10.
- Proposition C $\Longrightarrow\exists\ a,b>0$ s.t. $\exists$ an oval with rotation number $\theta_{b,a}(h)=q/p$, thus giving rise to p–periodic orbits of $F_{b,a}$ for all allowed p.
- Still it must be proved that 2, 3, 4, 6 and 10 are forbidden, since

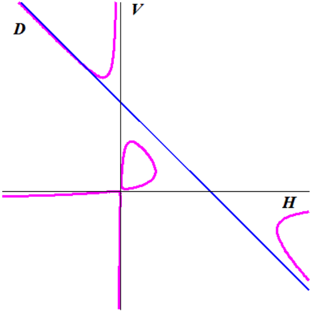
$I(a,b)\subseteq\mathrm{Image}\left(\theta_{b,a}\left(h_c,+\infty\right)\right)$ ∎

4. Back to the rotation number: an algebraic-geometric approach.

The curves $\mathcal{C}_h$, in homogeneous coordinates $[x : y : t] \in \mathbb{C}P^2$, are

$$\widetilde{\mathcal{C}}_h = \{(bx + at)(ay + bt)(ax + by + abt) - hxyt = 0\}.$$

The points $H = [1 : 0 : 0]$; $V = [0 : 1 : 0]$; $D = [b : -a : 0]$ are common to all curves



Proposition

If $a > 0$ and $b > 0$, and for all $h > h_c$, the curves $\widetilde{\mathcal{C}}_h$ are elliptic.

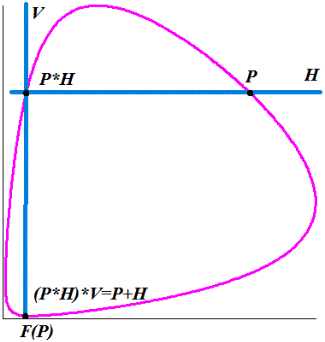
$F_{b,a}$ extends to $\mathbb{C}P^2$ as $\widetilde{F}_{b,a}([x:y:t]) = \left[ayt + y^2 : at^2 + bxt + yt : xy\right]$.

Lemma. Relation between the dynamics of $F_{b,a}$ and the *group structure* of $\mathcal{C}_h$ (*)

For each h s.t. $\widetilde{\mathcal{C}}_h$ is elliptic,

$$\widetilde{F}_{b,a|_{\widetilde{\mathcal{C}}_h}}(P) = P + H$$

Where $+$ is the addition of the group law of $\widetilde{\mathcal{C}}_h$ taking the infinite point V as the zero element.



Observe that

$$F^n(P) = P + nH,$$

so $\widetilde{\mathcal{C}}_h$ is full of p-periodic orbits iff

$$pH = V$$

i.e. H is a *torsion point* of $\widetilde{\mathcal{C}}_h$.

(*) Birational maps preserving elliptic curves can be explained using its group structure (Jogia, Roberts, Vivaldi; 2006).

How to prove that $\lim_{h\to\infty} \theta_{b,a}(h) = 2/5$?

Instead of looking to a normal form for F we look for a normal form for $\widetilde{\mathcal{C}}_h$.

$$\begin{array}{ccc} \left(\widetilde{\mathcal{C}}_h, +, V\right) & \xrightarrow{\cong} & \left(\widehat{\mathcal{E}}_L, +, \widehat{V}\right) \\ \widetilde{F}_{|_{\widetilde{\mathcal{C}}_h}} : P \mapsto P + H & \longrightarrow & \widehat{G}_{|_{\mathcal{E}_L}} : P \mapsto P + \widehat{H} \end{array}$$

Where $\widehat{\mathcal{E}}_L$ is the *Weierstrass Normal Form*:

$$\widehat{\mathcal{E}}_L = \{[x : y : t],\ y^2 t = 4\,x^3 - g_2\,xt^2 - g_3\,t^3\},$$

WHY?

1. Because we can *parameterize* it using the Weierstrass $\wp$ function...
2. ...that gives an *integral expression* for the *rotation number function*.
3. The *asymptotics* of this integral expression can be studied.

This scheme was used in (Bastien, Rogalski; 2004).

Proof of Proposition B.

The Weierstrass normal form of C_h is

$$\mathcal{E}_L = \{\, y^2 = 4\,x^3 - g_2(\alpha,\beta,L)\,x - g_3(\alpha,\beta,L) \,\}$$

where

$$g_2 = \frac{1}{192}\left(L^8 + \sum_{i=4}^{7} p_i(\alpha,\beta)L^i\right) \text{ and } g_3 = \frac{1}{13824}\left(-L^{12} + \sum_{i=6}^{11} q_i(\alpha,\beta)L^i\right),$$

being

$$\begin{aligned}
p_7(a,b) &= -4\,(\alpha+\beta+1)\,,\\
p_6(a,b) &= 2\left(3(\alpha-\beta)^2 + 2(\alpha+\beta)+3\right),\\
p_5(a,b) &= -4\,(\alpha+\beta-1)\left(\alpha^2-4\beta\alpha+\beta^2-1\right),\\
p_4(a,b) &= (\alpha+\beta-1)^4\,.
\end{aligned}$$

and

$$\begin{aligned}
q_{11}(a,b) &= 6\,(\alpha+\beta+1)\,,\\
q_{10}(a,b) &= 3\left(-5\alpha^2+2\alpha\beta-5\beta^2-6\alpha-6\beta-5\right)\\
q_9(a,b) &= 4\left(5\alpha^3-12\alpha^2\beta-12\alpha\beta^2+5\beta^3+3\alpha^2-3\alpha\beta+3\beta^2+3\alpha+3\beta+5\right)\\
q_8(a,b) &= 3\left(-5\alpha^4+16\alpha^3\beta-30\alpha^2\beta^2+16\alpha\beta^3-5\beta^4+4\alpha^3\right.\\
&\quad \left.-12\alpha^2\beta-12\alpha\beta^2+4\beta^3+2\alpha^2-8\alpha\beta+2\beta^2+4\alpha+4\beta-5\right)\\
q_7(a,b) &= 6\left(\alpha^2-4\alpha\beta+\beta^2-1\right)(\alpha+\beta-1)^3\\
q_6(a,b) &= -\,(\alpha+\beta-1)^6
\end{aligned}$$

where $\alpha = a/b^2$ and b/a^2 and $L \to +\infty \Leftrightarrow h \to +\infty$.

Step 1: parametrization.

$$\mathcal{E}_L = \{\, y^2 = 4\,x^3 - g_2\,x - g_3 \,\}$$

$\exists\ \omega_1$ and ω_2 depending on α, β and L and a lattice in $\mathbb{C}$

$$\Lambda = \{2n\omega_1 + 2m\, i\omega_2 \ \text{such that}\ (n,m) \in \mathbb{Z}^2\} \subset \mathbb{C},$$

such that the *Weierstrass* $\wp$ *function relative to* Λ

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right]$$

gives a parametrization of $\mathcal{E}_L$. This is because the map

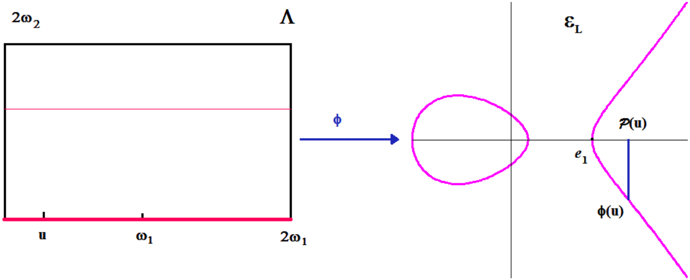
$$\begin{array}{cccc} \phi: & \mathbb{T}^2 = \mathbb{C}/\Lambda & \longrightarrow & \widehat{\mathcal{E}}_L \\ & z & \longrightarrow & \left\{ \begin{array}{lll} [\wp(z) : \wp'(z) : 1] & \text{if} & z \notin \Lambda, \\ \ [0:1:0] = \widehat{V} & \text{if} & z \in \Lambda, \end{array} \right. \end{array}$$

is an holomorphic homeomorphism, and therefore

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

• The oval $\mathcal{C}_h$ corresponds with the bounded branch of $\mathcal{E}_L$.

• The parametrization is s.t. $[0, \omega_1]$ is projected onto the real unbounded semi-branch of $\mathcal{E}_L$ with *negative y–coordinates*: so $\wp(\omega_1) = e_1$ and $\lim_{u \to 0} \wp(u) = +\infty$:



Integrating the differential equation $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ on $[0, u]$:

$$\mathbf{u} = \int_{\wp(\mathbf{u})}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4s^3 - g_2 s - g_3}} = \int_{\wp(u)}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}} \quad (5)$$

Step 2: towards an integral expression.

Since

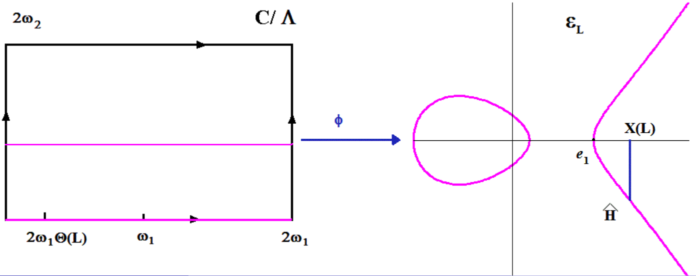
$$\widehat{G}_{|\mathcal{E}_L} : \widehat{V} \mapsto \widehat{V} + \widehat{H} = \widehat{H} \quad \text{is a rotation of } \textit{rot. num.} \ \Theta(L) \in \left[0, \frac{1}{2}\right),$$

and since $\widehat{H}$ has *negative ordinate*, it corresponds with a parameter u such that

$$u = 2\omega_1 \Theta(L).$$

The abscissa of $\widehat{H}$ is then given by

$$X(L) = \wp(2\omega_1\Theta(L)).$$



Since

$$X(L) = \wp(2\omega_1\Theta(L)),$$

using the integral expression (5):

$$u = \int_{\wp(u)}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}} \Rightarrow 2\omega_1\Theta(L) = \int_{X(L)}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}}$$

hence, since $e_1 = \wp(\omega_1)$, using again (5):

$$2\Theta(L) = \frac{\displaystyle\int_{X(L)}^{+\infty} \frac{\mathrm{d}s}{\sqrt{(s-e_1)(s-e_2)(s-e_3)}}}{\displaystyle\int_{e_1}^{+\infty} \frac{\mathrm{d}s}{\sqrt{(s-e_1)(s-e_2)(s-e_3)}}}$$

Step 4: asymptotic analysis.

Using $\left\{ \begin{array}{l} s = e_1 + 1/r^2 \text{ and} \\ r\sqrt{e_1 - e_3} = u \end{array} \right. \quad \Rightarrow \quad 2\Theta(L) = \dfrac{\displaystyle\int_0^{\sqrt{\frac{e_1-e_3}{\nu}}} \frac{\mathrm{d}u}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}}{\displaystyle\int_0^{+\infty} \frac{\mathrm{d}u}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}}$

Studying the asymptotics of $e_1 - e_3$, $\nu := X(L) - e_1$, and $\varepsilon := (e_1 - e_2)/(e_1 - e_3)$, the main computational obstruction, we can apply...

Lemma (Bastien, Rogalski; 2004)

Let $\lambda, \varepsilon, \gamma$ be positive numbers. For any map $\phi(\varepsilon)$ such that $\lim_{\varepsilon \to 0} \phi(\varepsilon) = 0$, and $\lambda + \phi(\varepsilon) > 0$, set

$$N(\varepsilon, \lambda, \gamma) = \int_0^{\frac{\lambda+\phi(\varepsilon)}{\varepsilon^\gamma}} \frac{\mathrm{d}u}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}, \text{ and } D(\varepsilon) = \int_0^{+\infty} \frac{\mathrm{d}u}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}.$$

Then $D(\varepsilon) \sim (1/2)\ln(1/\varepsilon)$, and if $\gamma < 1/2$ we have $N(\varepsilon, \lambda, \gamma) \sim \gamma \ln(1/\varepsilon)$, where $\sim$ denotes the equivalence with the leading term of the asymptotic development at zero.

...obtaining

$$2\Theta(L) = \frac{N(\varepsilon, A, 2/5)}{D(\varepsilon)} \sim \frac{\frac{2}{5}\ln(1/\varepsilon)}{\frac{1}{2}\ln(1/\varepsilon)} = \frac{4}{5} \Rightarrow \lim_{L\to\infty} \Theta(L) = 2/5 \qquad \blacksquare$$

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THANK YOU!