Global dynamics of discrete systems through Lie Symmetries

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1. DEFINITION AND FIRST DYNAMICAL INTERPRETATION

Through all the talk $F : \mathcal{U} \subset \mathbb{R}^n \to \mathcal{U}$, will be a diffeomorphism

A vector field X is said to be a **Lie symmetry** of F if it satisfies

$$X(F(\mathbf{x})) = (DF(\mathbf{x}))\, X(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathcal{U}. \tag{1}$$

Which means that $\dot{\mathbf{x}} = X(\mathbf{x})$ is invariant under the change of variables given by F,or in other words

The dynamics of X and F are related in in the following sense:

F maps any orbit of the $\dot{\mathbf{x}} = X(\mathbf{x})$, to another orbit of this system.

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Theorem 1 (Translation–like dynamics).

Let X be a Lie Symmetry of a diffeo $F : \mathcal{U} \to \mathcal{U}$.

Let γ be an orbit of X invariant under F. Then,

$F_{|\gamma}$ is the $\tau-$time map of the flow of X, that is

$$F(\mathbf{p}) = \varphi(\tau, \mathbf{p}).$$

(a) If $\gamma \cong \{p\}$ (isolated) then p is a fixed point of F.

(b) If $\gamma \cong \mathbb{S}^1$, then $F_{|\gamma}$ is conjugated to a rotation, with rotation number $\rho = \tau / T$, where T is the period of γ.

(c) If $\gamma \cong \mathbb{R}$, then $F_{|\gamma}$ is conjugated to a translation of the line.

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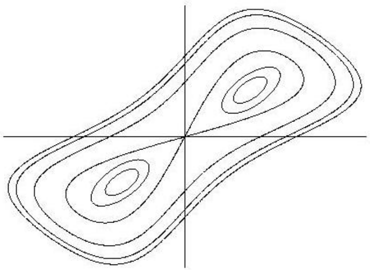
Example: Consider the Gumovski–Mira map:

$$F(x,y) = \left(y, -x + \frac{\alpha + \beta y}{1 + y^2}\right)$$

It has the first integral

$$V(x,y) = x^2y^2 + (x^2 + y^2) - \beta xy - \alpha(x + y)$$

For $\alpha = 0$ and $\beta > 2$, its level sets are



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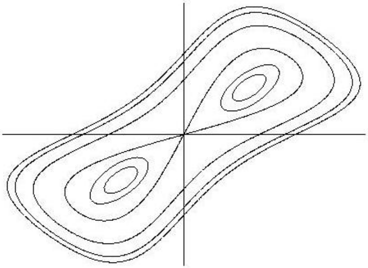

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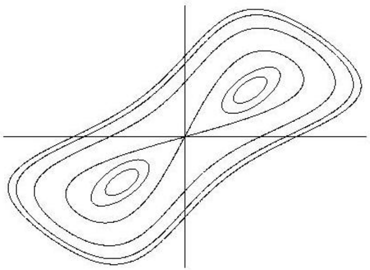

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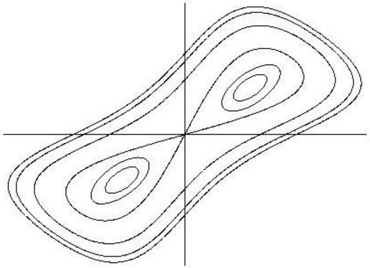

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2. THE INTEGRABLE CASE.

- A diffeo $F:\mathcal{U}\subset\mathbb{R}^n\to\mathcal{U}$ is **INTEGRABLE** if there exist $n-1$ functionally independent first integrals $V_1,\ldots,V_{n-1}$.
- In the integrable case, natural candidates to be Lie Symmetries have the form

$$X_\mu(x)=\mu(x)\left(-\frac{\partial V_1(x)}{\partial x_2},\frac{\partial V_1(x)}{\partial x_1}\right)\text{ if } n=2,\text{ and}$$

$$X_\mu(x)=\mu(x)\left(\nabla V_1(x)\times\nabla V_2(x)\times\cdots\times\nabla V_{n-1}(x)\right)\text{ if } n>2.$$

where $\times$ means the cross product in $\mathbb{R}^n$.

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Moreover, set $V = (V_1, \ldots, V_{n-1})$.If the number of connected components of $V_p := \{x \mid V(x) = V(p)\}$ is finite then

For each regular orbit of X, $\gamma_p \subset V_p$ of X, there exist m such that γ_p is invariant by F^m,and the dynamics of F^m restricted to γ_p is translation-like.

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Corollary 3 (**integrable** area preserving maps).

Let $F : \mathcal{U} \subset \mathbb{R}^n \to \mathcal{U}$, be an integrable area preserving map, i.e. $\det(DF(x)) \equiv 1$.
The vector field X_μ, with $\mu(x) = \Phi(V_1(x), V_2(x), \ldots, V_{n-1}(x))$, is a Lie symmetry for any any smooth function $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$,

Corollary 4 (a class of **integrable** rational difference equations)

Consider $F(x_1, x_2, \ldots, x_n) = \left(x_2, x_3, \ldots, x_n, \frac{R(x_2, x_3, \ldots, x_n)}{x_1} \right)$, Integrable.

The vector field X_μ, with $\mu(x) = x_1 x_2 \cdots x_n$ is a Lie symmetry.

This result has been the key to study the dynamics of some difference equations of the form $x_{n+k} = \frac{R(x_{n+1}, x_{n+2}, \ldots, x_{n+k-1})}{x_n}$ for $n = 2$ and 3.

3. THE LYNESS' MAPS

The difference equations, and their associated maps:

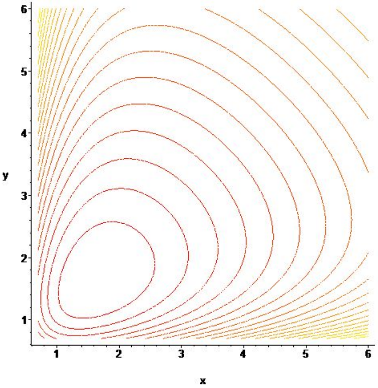
$$y_{n+2} = \frac{a+y_{n+1}}{y_n} \quad \text{with associated map} \quad F_2(x,y) = \left(y, \frac{a+y}{x}\right),$$

$$y_{n+3} = \frac{a+y_{n+1}+y_{n+2}}{y_n} \quad \text{with associated map} \quad F_3(x,y,z) = \left(y, z, \frac{a+y+z}{x}\right)$$

for $a > 0$, are paradigmatic examples of integrable DDS, like the Mc. Millan or the QRT maps...

The Lyness' map $F_2(x,y) = \left(y, \frac{a+y}{x}\right)$, has a first integral

$$V(x,y) = \frac{(x+1)(y+1)(a+x+y)}{xy}.$$



El Corollary 4 gives us the Lie Symmetry

$$X_2 = \left(\frac{(x+1)(a+x-y^2)}{y}\right)\frac{\partial}{\partial x} - \left(\frac{(y+1)(a+y-x^2)}{x}\right)\frac{\partial}{\partial y},$$

- Zeeman (1996, unpublished) and also Bastien and Rogalski (2004) proved (using algebraic geometry) that the dynamics on each closed curve is conjugated to a rotation.

- Beukers and Cushman (1998) From Lie Symmetry

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The third order Lynes–type equation

$$y_{n+3} = \frac{a + y_{n+1} + y_{n+2}}{y_n},$$

is a paradigmatic example of integrable third order difference equations.

It belongs to the list given by Hirota et al. (2001).

The map

$$F_3(x,y,z) = \left(y, z, \frac{a+y+z}{x}\right)$$

has two functionally independent first integrals:

$$V_1(x,y,z) = \frac{(x+1)(y+1)(z+1)(a+x+y+z)}{xyz},$$
$$V_2(x,y,z) = \frac{(1+y+z)(1+x+y)(a+x+y+z+xz)}{xyz}.$$

and Lie symmetry

$$X_3 = \frac{(x+1)(1+y+z)(a+x+y-yz)}{yz}\frac{\partial}{\partial x} + \frac{(y+1)(x-z)(a+x+y+z+xz)}{xz}\frac{\partial}{\partial y} + \frac{(z+1)(1+x+y)(xy-y-a-z)}{xy}\frac{\partial}{\partial x}.$$

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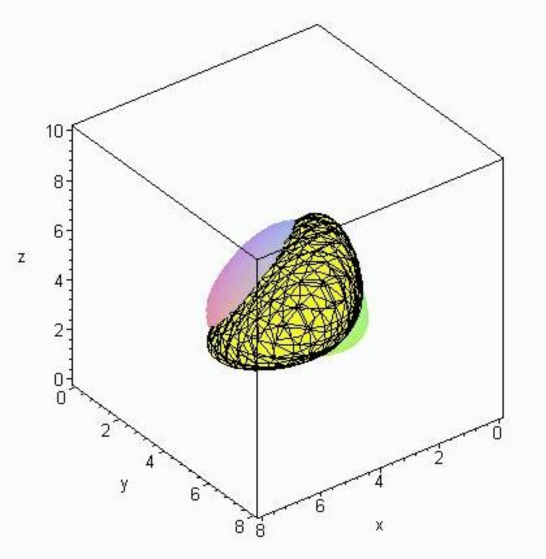
$$V_1(x, y, z) = \frac{(x+1)\,(y+1)\,(z+1)\,(a+x+y+z)}{xyz},$$
$$V_2(x, y, z) = \frac{(1+y+z)(1+x+y)(a+x+y+z+xz)}{xyz}.$$

and Lie symmetry

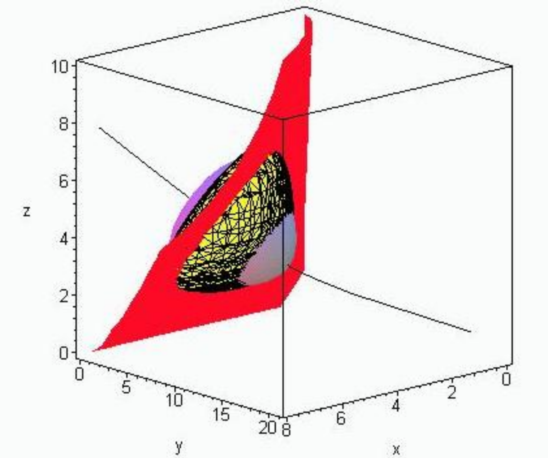
$$X_3 = \frac{(x+1)(1+y+z)(a+x+y-yz)}{yz}\frac{\partial}{\partial x} + \frac{(y+1)(x-z)(a+x+y+z+xz)}{xz}\frac{\partial}{\partial y} + \frac{(z+1)(1+x+y)(xy-y-a-z)}{xy}\frac{\partial}{\partial x}.$$

Generic intersection of two energy levels of F_3



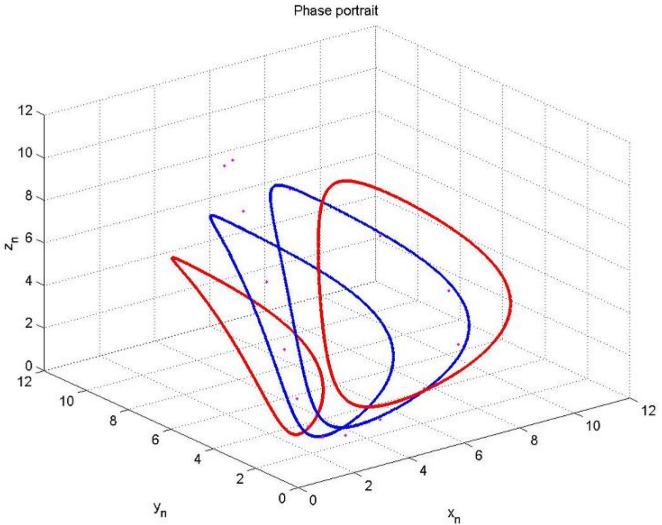
We will denote $I_{k,h} = \{V_1 = k\} \cap \{V_2 = h\}$.

Invariant sets of F_3 and F_3^2



$\mathcal{G} := \{(x, y, z) \in O^+$ such that $G(x, y, z) = 0\}$, and
$\mathcal{L} := \{(x, (x + a)/(x - 1), x) \in \mathbb{R}^3$ such that $x > 1\}$

Orbits of the map F_3^2



Observe a 15–periodic orbit in $\mathcal{G}$.

From Corollary 4, F_3 has the following Lie Symmetry

$$X_3 = \mu \cdot (\nabla V_1 \times \nabla V_2)$$

where

$$\mu(x, y, z) = xyz$$

$$X_3 = \frac{(x+1)(1+y+z)(a+x+y-yz)}{yz}\frac{\partial}{\partial x} + \frac{(y+1)(x-z)(a+x+y+z+xz)}{xz}\frac{\partial}{\partial y} + \frac{(z+1)(1+x+y)(xy-y-a-z)}{xy}\frac{\partial}{\partial x}.$$

Using the Lie Symmetry X_3, we have obtained the following results (among others)

Theorem A

Except at the fixed point, and a curve $\mathcal{L}$, filled of 2-periodic points of F, we have:

- The restriction of F^2 on $I_{k,h} \cap \{G > 0\}$ or on $I_{k,h} \cap \{G < 0\}$ is conjugated to a rotation on the circle.
- The restriction of F on $I_{k,h} \cap \{G = 0\}$ is conjugated to a rotation on the circle. If there exists a periodic orbit in O^+ of odd period, it must be contained in $\{G = 0\}$.

Theorem B

Set $\rho_a := \frac{1}{2\pi} \arccos\left(\frac{(1-a)\sqrt{1+a}}{2(1+\sqrt{1+a})(a+1+\sqrt{1+a})}\right)$ $a > 0$. For each $a \neq 1$ there are initial conditions outside $\mathcal{G}$.

$\rho_{F^2}(p) \in \left(\frac{1}{4}, \rho_a\right)$, if $a > 1$, and $\rho_{F^2}(p) \in \left(\rho_a, \frac{1}{4}\right)$, if $0 < a < 1$.

At this point it is possible to determine the possible periods applying a finite algorithm.

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HIGHER DIMENSIONAL CASES

$$F_k(x_1,\ldots,x_k) = \left(x_2,\ldots,x_k,\frac{a+\sum_{i=2}^k x_i}{x_1}\right), \text{ with } \quad a \geq 0.$$

It has the following functionally independent first integrals

$$V_1(\mathbf{x}) = \left(a+\sum_{i=1}^k x_i\right)\left(\prod_{i=1}^k (x_i+1)\right)/(x_1\cdots x_k)$$

and

$$V_2(\mathbf{x}) = \left(a+\sum_{i=1}^k x_i + x_1x_k\right)\left(\prod_{i=1}^{k-1}(1+x_i+x_{i+1})\right)/(x_1\cdots x_k).$$

Gao et al. (2004) have given a third functionally independent first integral for $k \geq 5$, but

for $k > 3$, F_k seems to be not–integrable anymore

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Theorem 5 (Lie Symmetry in the general case).

For $k \geq 3$, the vector field $\mathbf{X}_k = \sum_{i=i}^{k} X_i \dfrac{\partial}{\partial x_i}$, is a Lie symmetry for the k-dimensional Lyness' map, where

$$X_1(\mathbf{x}) = \frac{(x_1+1)\left[\prod_{i=2}^{k-1}(1+x_i+x_{i+1})\right]\left(a+\sum_{i=1}^{k-1} x_i - x_2x_k\right)}{\prod_{i=2}^{k} x_i},$$

$$X_m(\mathbf{x}) = \frac{(x_m+1)\left[\prod_{i=1,i\neq m-1,m}^{k-1}(1+x_i+x_{i+1})\right]\left(a+\sum_{i=1}^{k} x_i + x_1x_k\right)(x_{m-1}-x_{m+1})}{\prod_{i=1,i\neq m}^{k} x_i},$$

for all $2 \leq m \leq k-1$, and

$$X_k(\mathbf{x}) = -\frac{(x_k+1)\left[\prod_{i=1}^{k-2}(1+x_i+x_{i+1})\right]\left(a+\sum_{i=2}^{k} x_i - x_1x_{k-1}\right)}{\prod_{i=1}^{k-1} x_i}.$$

Remember that for $k > 3$, F_k seems to be not–integrable anymore

Conjecture 1 (Number of first integrals), GKI–CGM

Both F_k and their associated Lie Symmetries X_k have exactly $E(\frac{k+1}{2})$ first integrals.

Conjecture 2 (Topology and Dynamics), CGM-BR

- For $k = 2\ell$, most of the orbits lie on invariant manifolds which are diffeomorphic to ℓ–dimensional tori, $S^1 \times \overset{\ell)}{\cdots} \times S^1$.
- For $k = 2\ell + 1$, most of the orbits lie on two diffeomorphic ℓ–dimensional tori $S^1 \times \overset{\ell)}{\cdots} \times S^1$, separated by the invariant set $\mathcal{G}$. Moreover these orbits jump from one of these tori to the other one and viceversa.

In the above cases F (resp F^2), are conjugated to

$$R(z_1, \ldots, z_\ell) = \left(z_1 e^{2\pi i \rho_1}, \ldots, z_\ell e^{2\pi i \rho_\ell}\right)$$

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Construction of the Lie symmetry

$$X_k(F_k) = DF_k \cdot X_k$$

writes as

$$\begin{pmatrix} X_1(F) \\ X_2(F) \\ \vdots \\ X_k(F) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ -\frac{a+\sum_{i=2}^{k} x_i}{x_1^2} & \frac{1}{x_1} & & \cdots & & \frac{1}{x_1} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix}.$$

Hence it is necessary that

$$X_{i+1} = X_i(F), \text{ for } i = 1, \ldots, k-1,$$

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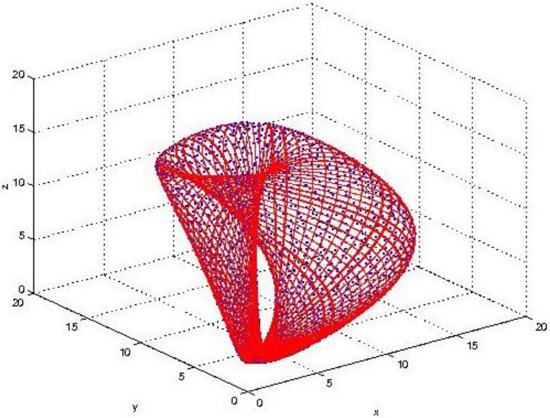
If we assume that both first integrals intersect transversally on $C_{h,k}$, a connected component of $I_{h,k}$,**(*)** then $C_{h,k}$ **is diffeomorphic to a torus**.

(*) But we have failed to proof that this happens.



Projections into $\mathbb{R}^3$ of the flow of the Lie symmetry $\mathbf{X}_4$, and the orbit of the Lyness' map, F_4.

Numerics evidence that we cannot apply Theorem 1.

- From Bastien & Rogalski (2008) each connected component of $I_{h,k}$ (namely $C_{h,k}$) is compact.
- If $\{V_1 = h\}$ and $\{V_2 = k\}$ intersect transversally on $C_{h,k}$, then for all points

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- This fact implies that the dual 2–form associated to the 2–field $\nabla V_1 \wedge \nabla V_2$ is nonzero at every point of $C_{h,k}$, and therefore it is orientable.
- The unique equilibrium point of $\mathbf{X}_4$ in $\mathcal{Q}^+$ is the fixed point of F.

  Hence $\mathbf{X}_4|C_{h,k}$ has no equilibrium points, and therefore the Poincaré–Hopf formula gives:

$$0 = i(\mathbf{X}_4|C_{h,k}) = \chi(C_{h,k}) = 2 - 2g \quad \Rightarrow \quad g = 1.$$

An orientable, compact, connected surface of genus one is a TORUS, as we wanted to prove.

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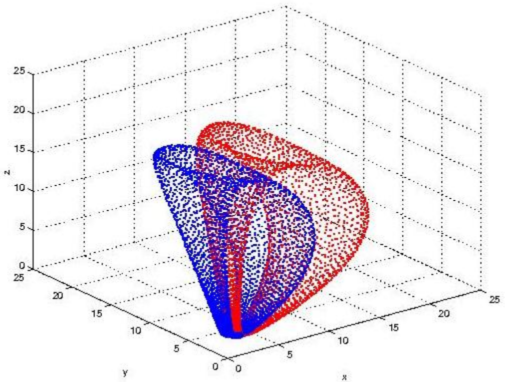
Para $k = 5$ existe una nueva integral primera V_3.

If we assume that the three first integrals intersect transversally on $C_{h,k,\ell}$, a connected component of $I_{h,k,\ell}$, then $C_{h,k,\ell}$ **is diffeomorphic to a (two–dimensional) torus**.



Projections into $\mathbb{R}^3$ of and orbit F_5, giving rise to two orbits of F_5^2.

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THANK YOU!