Regularizable and non–regularizable collisions in N–body problems

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Abstract

Collisions appear as singularities in the N–body problem. A binary (resp. total) collision occurs when only two (resp. all) masses collide. Between these cases other types of collisions appear, like triple (TC), simultaneous binary (SBC), etc. We consider a geometrical approach, searching for continuity with respect to i.c. for orbits passing close to a collision. After surveying known results, some recent progress, dealing with finite order of smoothness in the SBC case and with the classification of passages near TC, is presented.

Consider the Newtonian N–body problem in $\mathbb{R}^d$ with punctual masses $m_k > 0$ for $k = 1, \ldots, n$. Let $\mathbf{q}_k, \mathbf{p}_k \in \mathbb{R}^d$ the position and momentum of mass m_k. The motion is governed by

$$\dot{\mathbf{q}}_k = \frac{\mathbf{p}_k}{m_k}, \qquad \dot{\mathbf{p}}_k = \nabla_k U(\mathbf{q}), \quad \text{where} \quad U(\mathbf{q}) = \sum_{1\leq i<j\leq n} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}. \tag{1}$$

The system is defined everywhere except at the *collision set*, $\Delta = \cup_{1\leq i<j\leq n}\Delta_{ij}$, where Δ_{ij} is defined by $\mathbf{q}_i = \mathbf{q}_j$. Approaching Δ the acceleration of some bodies becomes unbounded. A *binary collision* (BC) occurs when only two masses collide. If all the masses coincide at the same point we have a *total collision*. Between these extrema cases other types of collisions can occur, like *triple collision* (TC), *simultaneous binary collisions* (SBC), etc.

Assume a solution $w(t) = (\mathbf{q}(t), \mathbf{p}(t))$ of (1) is defined on a maximal interval (α, ω) with ω finite. When $t \to \omega^-$, $\mathbf{q}(t)$ must approach Δ at a sequence of increasing times, $\{t_j\}$, with $t_j \to \omega$. Otherwise the potential energy is bounded. Hence the kinetic one is also bounded and the motion could be continued beyond ω. A similar thing happens if α is finite and $t \to \alpha^+$ but then we talk about ejections. So, any restriction to the prolongation of solutions is related to collisions or ejections. We shall be interested in solutions such that when $t \to \omega$ the masses tend to a limiting position. This position must be a collision (of some type) and we say that the solution ends at a collision. The standing question is if there is some "natural way" to extend the solution, that is, to match a collision solution with an ejection one. Among the different ways to proceed we choose the one we think has more physical (and numerical) relevance.

Let $t_1 < 0$ be some initial time such that $w(t)$ is defined for $(t_0, 0)$, $t_0 < t_1$ and ends at a collision. Let be $w_c^{(i)} = w(t_1)$. Assume that there are initial conditions (i.c.), $w_i^{(i)}$, in any neighbourhood of $w_c^{(i)}$, not leading to any singularity and such that $w(t; t_1, w_i^{(i)})$ is defined until some fixed $t_3 > 0$. In the case of the collinear problems, $d = 1$, if a BC is produced starting at some i.c., then it is also produced for any nearby i.c. But if we have a triple (at least) collision or a couple (at least) of SBC, a change of i.c. leads to solutions with "separated" BC. As it is well known, the solutions having simple BC can be trivially continued and we shall disregard them. For $d > 1$ the existence of conditions $w_i^{(i)}$ as required follows from the fact that under the Newtonian attraction the manifold of i.c. leading to collision has positive codimension. Let $0 < t_2 < t_3$. Then $w_i^{(f)} = w(t_2; t_1, w_i^{(i)})$ is well defined.

Definition 1 *Let $\{w_i^{(i)}\}$ a sequence of i.c. not leading to collision and such that $\lim_{i\to\infty} w_i^{(i)} = w_c^{(i)}$ and that $\lim_{i\to\infty} w_i^{(f)}$ exists and is independent on the sequence. Then we define $w_c^{(f)}$ as that limit and we identify it with $w(t_2; t_1, w_c^{(i)})$. This holds for any $t_2 \in (0, t_3)$ and we shall denote this extension as the natural or geometric regularization.*

Furthermore, if the passage from $w_i^{(i)}$ to $w_i^{(f)}$ is C^r we shall talk about a C^r–regularization. In particular, if the i.c. are taken on an analytic arc $\hat{w}^{(i)}(l), |l| < l_0, l_0 > 0$ small, such that only $l = 0$ leads to collision (skipping BC in the $d = 1$ case), we can look at the degree of differentiability of the image at $t = t_2 : \hat{w}^{(f)}(l), l \neq 0$. A C^r–regularization implies that $\hat{w}^{(f)}(l)$ is C^r for $|l| < l_0$.

In case the collision is not regularizable the relevant question is whether it is possible to classify the *possible transitions* from an approach to collision to an escape from ejection. That is, which kind of ejections can be expected from a given collision. Even if this is possible, it can depend on the values of the masses m_k.

At this point we comment on simple BC. It is well known that gravitational BC are regularizable (in all the senses used till present) and that the regularization is obtained by elastic bounce. Reducing the problem to a fixed center problem (in the case of $n = 2$) if $\mathbf{q}(t) = \mathbf{q}_2(t) - \mathbf{q}_1(t)$ and $\mathbf{q}(0) = 0$ then $\mathbf{q}(t) = \mathbf{u}\varphi(t)$ where $\mathbf{u}$ is a fixed vector and $\varphi(t)$ is an analytic function in $t^{1/3}$, starting with terms in $t^{2/3}$ and one has $\varphi(t) = \varphi(-t)$. The same regularization holds for any dimension d. This is in contrast with other kinds of two–body problem or fixed center problem. If we consider $\ddot{\mathbf{q}} = -\mathbf{q}/|\mathbf{q}|^{\gamma+1}, \gamma \geq 1$, different cases occur. If $d = 1$ and we consider an ejection orbit with $q(0) = 0$ then a solution $q(t) = \varphi(t)$ with $\varphi(t) = at^b + \ldots, b = 2/(\gamma + 1)$ exists for $\gamma > 1$. This can be naturally continued to $t < 0$ if b is an irreducible rational with odd denominator. But when the problem is considered in $\mathbb{R}^d, d \geq 2$, elementary blow up techniques show that it is only regularizable if $\gamma = 3 - 2/m$ for some natural $m > 1$. If m is even the regularization is obtained by bouncing ($\varphi(t) = \varphi(-t)$), while m odd leads to regularization by passage "across the singularity" ($\varphi(t) = -\varphi(-t)$). If $\gamma \geq 3$ and $d \geq 2$, the arrival to collision acts as a "black hole". All nearby i.c. go also to collision, which acts as an attractor (concerning $\mathbf{q}$; of course $\mathbf{p}$ becomes unbounded). So, if $\gamma = 4$, for instance, the $d = 1$ case is regularizable (the solution is analytic in $t^{2/5}$), while for $d > 1$ is not in the sense defined above, simply because it is not possible to find nearby i.c. escaping from collision.

The case $\gamma = 1$ (or logarithmic potential) becomes regularizable, but solutions going to 0 as t goes to 0 have $\varphi(t) = t\sqrt{\ln t^{-2}} + \ldots$ These problems as well as modifications like the case of non central forces (like anisotropic problems, Kepler problem with the inclusion of different harmonics of the potential, etc) lead to an interesting zoo of different behaviours.

Now we return to (1). A comment worth to be made is that some phase space and time variables can be misleading concerning regularity. Consider, for instance three bodies on the line with $m_k = 1, k = 1, 2, 3$, and assume that at $t = 0$, $q_1 = q_2$ while $q_3 - q_1 = 1$. As

$$\ddot{q}_3 = -(q_3 - q_1)^{-2} - (q_3 - q_2)^{-2},$$

and the dominant terms in q_1, q_2 around $t = 0$ are $-q_1 = q_2 = \frac{1}{2}(3t)^{2/3} + \ldots$, we have

$$\ddot{q}_3 = -2 + 4\beta t - \frac{3}{2}(3t)^{4/3} + \ldots,$$

around $t=0$, where β is the velocity of m_3 with respect to the center of mass of m_1 and m_2 at $t=0$. There appear terms in $t^{10/3}$ in the expression of q_3. This means nothing against analytic regularization, which in fact occurs as can be shown by using $t^{1/3}$ as new time. In a similar way, some formulations can lead to vector fields which are just C^{r_1}. This do not prevents a C^{r_2}–regularization with $r_2>r_1$. Cancellations of unwanted terms can be relevant.

Concerning TC, McGehee [3] studied it in the collinear three–body problem using a transformation which blows up the collision to an invariant manifold called the "triple collision manifold", $\mathcal{C}$. The flow on $\mathcal{C}$ has several fixed points. Collision/ejection orbits are asymptotic (for positive/negative time) to one of them. The study of the flow on the $\mathcal{C}$ allows to prove that TC cannot be regularized except, at most, for a zero measure set of masses. This is a consequence of the fact that the W^u of one rest point on $\mathcal{C}$ do not coincides with W^s of another rest point. This has been a useful method to prove non regularizability of collision singularities, mainly TC in the three–body problem. Anyhow the McGehee transformation allows to study the behaviour of the solutions passing near TC. Many authors used this approach to study singularities due to collision [1], [8], [7].

In the four–body problem besides binary, triple and total collision, SBC can happen. In [9] Simó and Lacomba studied SBC by looking at the solutions near the singularity as perturbations of the solutions of m uncoupled two–body problems. They proved that regularization is possible, but without giving any geometric insight and using only continuity arguments.

In this note we present some results concerning SBC in the four–body problem for some different cases and on the TC for the planar three–body problem. It should be stressed that blow up techniques are quite useful from a geometric point of view, to understand the underlying structure of the phase space. But the analytic computations can become cumbersome, as noted in [2]. Furthermore when this technique is used in the SBC and TC cases, it remains still the task to carry out the regularization of the single BC inside the collision manifold. Despite this is feasible, it can lead to the use of different charts. Instead, a direct use of cartesian coordinates or the use of a number of variables larger than the dimension of the phase space but satisfying some constraints, can produce all the desired information.

Fist we consider the SBC problem. Let m_1-m_2 and m_3-m_4 the masses which collide at $t=0$. Let $\mathbf{q}_j, j=1,\ldots,4$ denote the respective positions. Let us introduce

$$\mu_1=m_1+m_2,\ \ \mu_2=m_3+m_4,\ \ M=\mu_1+\mu_2,\ \ \alpha_1=\frac{m_1}{\mu_1},\ \ \alpha_2=\frac{m_2}{\mu_1},\ \ \alpha_3=\frac{m_3}{\mu_2},\ \ \alpha_4=\frac{m_4}{\mu_2},$$

$\mathbf{Q}_1=\mathbf{q}_2-\mathbf{q}_1,\ \ \mathbf{Q}_2=\mathbf{q}_4-\mathbf{q}_3,\ \ \mathbf{Q}=\mathbf{q}_{34}-\mathbf{q}_{12},$ where $\mathbf{q}_{12}=\alpha_1\mathbf{q}_1+\alpha_2\mathbf{q}_2,\ \ \mathbf{q}_{34}=\alpha_3\mathbf{q}_3+\alpha_4\mathbf{q}_4.$
The Hamiltonian is a function of $\mathbf{Q}_1,\mathbf{Q}_2,\mathbf{Q}$ and the related momenta $\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}$:

$$H=\frac{1}{2}\left(|\mathbf{P}|^2+\frac{\mu_1^2\mu_2}{M}\frac{1}{m_1m_2}|\mathbf{P}_1|^2+\frac{\mu_1\mu_2^2}{M}\frac{1}{m_3m_4}|\mathbf{P}_2|^2\right)-\frac{M}{\mu_1\mu_2}\left(\frac{m_1m_2}{|\mathbf{Q}_1|}+\frac{m_3m_4}{|\mathbf{Q}_2|}+\frac{\mu_1\mu_2}{|\mathbf{Q}|}+\mathcal{F}\right),$$

where

$$\begin{aligned}\mathcal{F} &= \mu_1\mu_2\left(\frac{\alpha_2\alpha_3}{|\mathbf{Q}-\alpha_1\mathbf{Q}_1-\alpha_4\mathbf{Q}_2|}+\frac{\alpha_2\alpha_4}{|\mathbf{Q}-\alpha_1\mathbf{Q}_1+\alpha_3\mathbf{Q}_2|}+\frac{\alpha_1\alpha_3}{|\mathbf{Q}+\alpha_2\mathbf{Q}_1-\alpha_4\mathbf{Q}_2|}+\right.\\ &\left.\qquad\frac{\alpha_1\alpha_4}{|\mathbf{Q}+\alpha_2\mathbf{Q}_1+\alpha_3\mathbf{Q}_2|}-\frac{1}{|\mathbf{Q}|}\right)\end{aligned}$$

Definition 2 *We say that the four–body problem is 1D–reducible if $\mathbf{Q}_1,\mathbf{Q}_2$ and $\mathbf{Q}$ have constant direction along the motion. That is, there exist unitary fixed vectors, $\mathbf{u_1},\mathbf{u_2}$ and $\mathbf{u}$ such that $\mathbf{Q}_j(t)=\mathbf{u_j}\varphi_j(t), j=1,2, \mathbf{Q}(t)=\mathbf{u}\varphi(t)$, for some scalar functions $\varphi_1,\varphi_2,\varphi$.*

We consider first 4 1D–reducible cases. Later it is extended to general 1D–reducible cases.

(a) Collinear subproblem: The bodies have arbitrary masses and coordinates of the form $x_1 \leq x_2 \leq x_3 \leq x_4$ for all time. SBC is produced if $x_1 = x_2 < x_3 = x_4$.

(b) Biisosceles subproblem: The bodies move on the plane with coordinates of the form $(\mp x, y_1)$, $(0, y_3)$, $(0, y_4)$ for all time, with $y_3 \leq y_4$. Furthermore $m_1 = m_2$ is required, m_3 and m_4 arbitrary. SBC is produced for $x = 0, y_1 \neq y_3, y_3 = y_4$.

(c) Trapezoidal subproblem: Again a planar case with coordinates of the masses of the form $(\pm x_1, y_1), (\pm x_3, y_3)$ and $m_1 = m_2, m_3 = m_4$. SBC appears for $x_1 = 0, x_3 = 0, y_1 \neq y_3$.

(d) Tetrahedron subproblem: Now the bodies move in $\mathbb{R}^3$ with masses $m_1 = m_2, m_3 = m_4$, as in the trapezoidal problem, with coordinates of the form $(\pm x_1, 0, z_1), (0, \pm y_3, z_3)$ for all time. SBC are obtained for $x_1 = 0, y_3 = 0, z_1 \neq z_3$.

Theorem 1 *In subproblems (a) to (d) of the 1D–reducible four–body problem the SBC is regularizable and the regularization is $C^{8/3-\epsilon}$ for any $\epsilon > 0$, but it is not $C^{8/3}$.*

A study of the behavior of these collisions inside the general problem or of other SBC in the four–body problem has not been done. A SBC in that case is characterized by the limit energies of the binaries and by the limit directions, that is, by the scalar products $(\mathbf{u_1}, \mathbf{u}), (\mathbf{u_2}, \mathbf{u}), (\mathbf{u_1}, \mathbf{u_2})$ when $t \to 0$. Guided by preliminary analysis we state the following

Conjecture 1 *For all limit $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u}$ and all $m_j > 0$, except by sets of zero measure, the SBC is $C^{8/3-\epsilon}$ regularizable in the four–body problem. The exceptional cases have a higher regularity.*

Full details and additional relevant bibliography appear in [6], as well as the extension to the 1D–reducible cases in general. A proof of the regularizability based on geometric ideas, and numerical evidence of the finite order regularity (with the exact bound) appears in [5]. The key idea, beyond the use of suitable variables and several technicalities, consists in looking for the dominant "coupling" terms between different binaries. The effect of them on the binary energies after passing close to a SBC, involves integrals of the form $\int_{-t}^{t} \tau^{1/3}(\tau - t_2)^{4/3} d\tau$, where t_2 is the delay between individual BC. From this, contributions of the form $\operatorname{sign}(t_2) t_2^{8/3}$ appear, which become the obstruction to regularity.

Next we shift to the TC in the planar three–body problem. To this end we have to introduce new variables. Let $\mathbf{q}_j \in \mathbb{R}^2$. We identify $\mathbb{R}^2$ and $\mathbb{C}$. Assume the origin is located at the center of mass and let $\mathbf{q}_l - \mathbf{q}_k = a_j e^{i\varphi_j}$, where $a_j > 0, \varphi_j \in \mathbb{S}^1$ and (j, k, l) is $(1, 2, 3)$ or a cyclical permutation of it. An index m greater that 3 has to be seen as equivalent to the index $m - 3$.

Let $\varphi = \frac{1}{3}(\varphi_1 + \varphi_2 + \varphi_3)$ and

$$\mathbf{p}_j = P_k e^{i\varphi_k} - P_l e^{i\varphi_l} + \frac{P_\varphi}{3} i \left(\frac{1}{a_k} e^{i\varphi_k} - \frac{1}{a_l} e^{i\varphi_l} \right), \quad P_\varphi \in \mathbb{R}, \quad P_j \in \mathbb{R}, \quad j = 1, 2, 3.$$

The variables $a_1, a_2, a_3, \varphi, P_1, P_2, P_3$ and P_φ allow to recover the old coordinates. Now we introduce new variables $\tilde{\alpha}_j, \Pi_j, j = 1, 2, 3$ and a new time $\tilde{\tau}$ by

$$a_j = \tilde{\alpha}_k^2 + \tilde{\alpha}_l^2, \quad \mathbf{P}_j = \frac{1}{4} \left(-\frac{\Pi_j}{\tilde{\alpha}_j} + \frac{\Pi_k}{\tilde{\alpha}_k} + \frac{\Pi_l}{\tilde{\alpha}_l} \right), \quad dt = a_1 a_2 a_3 d\tilde{\tau},$$

with the triple (j,k,l) as before. Finally let

$$r^2=\sum m_j|\mathbf{q}_j|^2,\qquad \alpha_j=\tilde\alpha_j r^{-1/2},\qquad j=1,2,3,\qquad d\tau=r^{3/2}d\tilde\tau.$$

Note that each set $(\alpha_1,\alpha_2,\alpha_3,\Pi_1,\Pi_2,\Pi_3)$ has to be identified with the one obtained by changing the sign of two of the α's (say α_j,α_k) and, simultaneously, the sign of Π_j,Π_k. We introduce $\tilde P_\varphi=P_\varphi r^{-1/2}$. On $r=0$ P_φ becomes 0 but $\tilde P_\varphi$ does not, in general. However $\tilde P_\varphi=0$ is invariant and this leads to the so called non rotating collision manifold, N. It is 4 dimensional and described by the variables (α,Π) with the constraints

$$\sum_{1\le j<k\le 3} m_jm_k(\alpha_j^2+\alpha_k^2)=\sum_{i=1}^{3} m_i=1\qquad\text{(normalization)},\qquad K(\alpha,\Pi)=0,$$

$$\begin{aligned}
K(\alpha,\Pi)&=\frac18\Pi^TB(\alpha)\Pi-\sum_{1\le j<k\le 3} m_jm_k(\alpha_{j+1}^2+\alpha_{j+2}^2)(\alpha_{k+1}^2+\alpha_{k+2}^2),\qquad B(\alpha)=(b_{jk}(\alpha)),\\
b_{jj}(\alpha)&=\frac{\alpha_k^2+\alpha_l^2}{m_j}(\alpha_1^2+\alpha_2^2+\alpha_3^2)+\frac{\alpha_j^2+\alpha_l^2}{m_k}\alpha_l^2+\frac{\alpha_j^2+\alpha_k^2}{m_l}\alpha_k^2,\qquad b_{jk}(\alpha)=-\frac{\alpha_j^2+\alpha_k^2}{m_l}\alpha_j\alpha_k.
\end{aligned}$$

The equations of motion on N are

$$\frac{d\alpha_j}{d\tau}=\frac{\partial}{\partial\Pi_j}K(\alpha,\Pi)-\frac14 z\alpha_j,\qquad \frac{d\Pi_j}{d\tau}=-\frac{\partial}{\partial\alpha_j}K(\alpha,\Pi),\qquad j=1,2,3,$$

where $z=(\alpha_1^2+\alpha_2^2)(\alpha_2^2+\alpha_3^2)(\alpha_3^2+\alpha_1^2)v$, $v=(\alpha,\Pi)$. On N, v is gradient like ($\frac{dv}{d\tau}\ge 0$). Note that the equations above are polynomial. We refer to [10] for further details on the derivation of the equations, meaning of the variables and related references.

The equations above, taking into account the identifications, have 10 critical points, 5 of them on $v<0$ (collisions) and 5 (symmetrical to the previous ones) on $v>0$ (ejections). These five points on $v<0$ correspond to the central configurations of the three–body problem: two of them are triangular, L_+^i,L_-^i and the other three are collinear, $E_j^i, j=1,2,3$. The superscript i means inferior (they are on $v<0$). In a similar way we have the related points with superscript s, superior (on $v>0$). The subscript $+/-$ on $L^{i,s}$ refers to the orientation, that is, to the sign of $\alpha_1\alpha_2\alpha_3$. In the $E_j^{i,s}$ points the mass j is located between the other two. Then $\alpha_j=0$.

Furthermore it is convenient to think of other "points" on N obtained by compactification. They correspond to the "hard" binaries (see [4]), that is, two of the α's are 0. Then $\Pi^TB(\alpha)\Pi=$ *ctant* becomes a cylinder, which can be compactified to $\mathbb{S}^2$. The points obtained in this way will be denoted as $B_j^{i,s}$ if $\alpha_k=\alpha_l=0$. The superscript refers to the sign of v as before. The compactified manifold is denoted as $\bar N$.

Hence we have 16 relevant objects in $\bar N$, all of them hyperbolic. The dimensions of the related invariant manifolds are:

$$\begin{aligned}
&\dim W^{u,s}(L_{+,-}^{i,s})=2,\quad \dim W^u(E_j^i)=\dim W^s(E_j^s)=3,\quad \dim W^s(E_j^i)=\dim W^u(E_j^s)=1,\\
&\dim W^u(B_j^i)=\dim W^s(B_j^s)=4,\qquad j=1,2,3.
\end{aligned}$$

Inside $\bar N$ the B_j^i points behave as sources and B_j^s as sinks.

It is worth to consider the space of shapes of triangles as an sphere $\hat S$. This is the suitable configuration space. The northern hemisphere corresponds to direct triangles (we go from m_1 to

m_2 and then to m_3 counterclockwise), the southern one to retrograde triangles and the equator to collinear situations. The north pole represents the equilateral triangle L_+, the south pole being related to L_-. On the equator, points $B_1E_3B_2E_1B_3E_2$ are found in this order. The passage from the vicinity of one of the X^i to one of the X^s points on the physical space $v > 0$ takes place close to a heteroclinic connection on the collision manifold. We shall put $X^i \to X^s$ to denote such connection. The first result is

Theorem 2 *At least if two of the values of the masses are close enough and there are not connections from points $L^i_{+,-}$ to points $L^s_{+,-}$ then points $B^i_j, E^i_j, j = 1,2,3$ connect to all points of the type X^s, and points $L^i_{+,-}$ connect to $B^s_j, E^s_j, j = 1,2,3$.*

By considering the dimensions of the manifolds one has that the most common behavior is a transition $B^i_j \to B^s_k$, where j can be equal to k or not. Quantitative information on the different probabilities and the dependence on the masses is relevant for the applications.

Conjecture 2 *The result above is true for all positive masses.*

Hence, to decide about the possible transitions one has to locate the masses for which connections $L^i_{+,-} \to L^s_{+,-}$ occur. Normalizing $\sum m_i = 1$ the values of the masses can be seen as the barycentric coordinates of a point in a triangle. We refer to it as the *mass triangle.*

Some of the connections are obtained by symmetry. For instance, if $L^i_+ \to L^s_-$ occurs then also $L^i_- \to L^s_+$ occurs. If two of the masses are equal and $L^i_+ \to L^s_+$ then also $L^i_- \to L^s_-$.

If two of the masses are equal some simplifications occur. In that case one obtains:

Theorem 3 *Let be $m_3 = \epsilon, m_1 = m_2 = (1-\epsilon)/2$. Then the set of values of ϵ for which a connection $L^i_{+,-} \to L^s_{+,-}$ occurs is countable. Furthermore, for n large $\epsilon(n) = \pi^2/n^2 + O(n^{-3})$.*

From those values of $\epsilon(n)$ one can do continuation by breaking the symmetry. The values of $\epsilon(n)$, for $n \to \infty$, correspond alternatively to $L^i_+ \to L^s_-$ and to $L^i_+ \to L^s_+$ connections. For $m_2 = m_3$ this means that also the connections $L^i_- \to L^s_+$ and $L^i_- \to L^s_-$ occur, respectively, for these values of ϵ. From the points with the $+ \to +$ connection two curves emerge, one related to $+ \to +$, the other to $- \to -$ when we move away from $m_2 = m_1$. From the points with the $+ \to -$ only one curve emerges, the connection $+ \to -$ implying also $- \to +$.

Guided by that result and several numerical experiments we are lead to

Conjecture 3 *In the triangle of masses there is a countable set of lines for which connection $L^i \to L^s$ occurs. For all other masses these connection do not occur.*

Note that beyond the curves emerging from $m_1 = m_2 = (1-\epsilon(n))/2, m_3 = \epsilon(n)$ (and the symmetrical ones by permutation of indices) there is the numerical evidence that only 9 exceptional curves (in 3 groups of 3) appear. Figure 1 displays 1/6 of the mass triangle, part of 3 exceptional curves and some arcs of the curves bifurcating from the points mentioned in theorem 3.

Full proofs of the results above for the TC can be found in [10], [11] and [12]. Here we only sketch some ideas. To prove theorem 2 one has to consider how $W^u(L^i_+)$ intersects the levels $v = w$ of the gradient function v (the case of L^i_- being similar). It is relevant to consider the projections of these intersections into $\hat{S}$. We shall denote them as $\mathcal{C}_w$. Locally (for $w = v(L^i_+) + \delta, \delta > 0$ small) they are circles. If we study first the case of two equal masses (say

$m_1 = m_2$), then the isosceles problem appears as a subproblem and it also has $L^{i,s}_{+,-}$ as rest points. For the isosceles case it is possible to follow the evolution of $\mathcal{C}_w$ when w increases. In particular one obtains information about the index (on $\hat{S}$) of $\mathcal{C}_w$ with respect to L_+. An analysis of the changes of index when increasing w allows to derive that, if connections $L^i_+ \to L^s_{+,-}$ do not occur, $\mathcal{C}_w$ reaches values of w larger than $v(L^s_{+,-})$ being a non contractible curve inside the set $\{s \in \hat{S} | U(s) > v(L^s_{+,-})\}$ (the hard binaries being included in that set), where U is the potential energy (restricted now to $\hat{S}$). Then, using a lemma of Moeckel [7], the result follows for two equal masses. Theorem 2 is obtained by deformation.

The proof of theorem 3 is much more involved. First one derives conditions on the points of $\mathcal{C}_{w=0}$ implying a connection $L^i_+ \to L^s_{+,-}$. Next, assuming $m_1 = m_2 = (1-\epsilon)/2$ and ϵ small, suitable coordinates are introduced such that the equations of motion tend to a limit, when $\epsilon \to 0$, on some finite domain which contains the rest points L^i_+ and E^i_3 as well as some relevant part of their unstable manifolds. Bounds on the location of these manifolds are obtained. It turns out that the suitable orbit in $W^u(L^i_+)$ (i.e., the orbit candidate for the heteroclinic connection) comes very close to the collinear problem. Then an analysis of that problem must be carried out. This is followed by the study of the variational equations (along orbits of the collinear problem) of the full problem in N. Then a matching of a perturbation of the limit problem above mentioned and a perturbation of the collinear solutions is done. This allows to prove the existence of Lagrange–Lagrange connections and provides all the required estimates.

Figure 1: Part of the mass triangle with some of the arcs related to $L^i \to L^s$ heteroclinic connections. The two points marked as crosses belong also to the set of points predicted by theorem 3. The other parts of the triangle are obtained by using the symmetries.

Acknowledgements

This work has been supported by DGICYT grant PB 94-0215 (Spain) and CIRIT grant 1998S0 GR–00042 (Catalonia).

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