REGULARIZATION OF THE SPATIAL THREE-BODY PROBLEM. TRIPLE COLLISION

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Abstract. We follow D.C.Heggie to obtain a regularized coordinate system to study the spatial three body problem. Basically we use symmetric variables and the regularization of Kustaanheimo-Stiefel obtainig a global regularization of the problem. Using these coordinates we study the triple collision in the spatial problem introducing the non-rotating triple collision manifold and studing its topology

Let $\bar{q}_i$, $\bar{p}_i$, $i = 1,2,3$ the coordinates and momenta respectively of three particles in some 3-dimensional inertial frame. We can write the Hamiltonian $\bar{H}$ as follows

$$\bar{H} = \sum_{i=1}^{3} \frac{1}{2m_i}\bar{p}_i^2 - \frac{m_2 m_3}{\mid \bar{q}_2 - \bar{q}_3 \mid} - \frac{m_3 m_1}{\mid \bar{q}_3 - \bar{q}_1 \mid} - \frac{m_1 m_2}{\mid \bar{q}_1 - \bar{q}_2 \mid} \tag{1}$$

where m_i, $i = 1,2,3$ are the three masses.

The corresponding hamiltonian equations are

$$\begin{aligned} \dot{\bar{q}}_i &= \frac{1}{m_i}\bar{p}_i \\ \dot{\bar{p}}_i &= -\frac{m_i m_j(\bar{q}_i - \bar{q}_j)}{\mid \bar{q}_i - \bar{q}_j \mid^3} - \frac{m_i m_k(\bar{q}_i - \bar{q}_k)}{\mid \bar{q}_i - \bar{q}_k \mid^3} \end{aligned} \tag{2}$$

where a dot denotes differentiation with respect to time, t. In the equation above and throughout this paper the other equations can be obtained interchanging the subindex (i,j,k) over the cyclic permutations of (1,2,3).

Now, we introduce new variables q_i, $i = 1,2,3$ and $< q >$ defined by

$$q_i = \bar{q}_j - \bar{q}_k \quad \text{and} \quad < q > = \frac{1}{M}\sum_{i=1}^{3} m_i \bar{q}_i \tag{3}$$

where $M = m_1 + m_2 + m_3$.

We have to note that q_i are not independent variables because they satisfy the relation $q_1 + q_2 + q_3 = 0$.

Their conjugate momenta p_i, $i = 1,2,3$ and $< p >$ are introduced by the relation (see [**3**])

$$\bar{p}_i = -p_j + p_k + \frac{m_i}{M} < p > . \tag{4}$$

Substituing (3) and (4) into (1) we obtain a new Hamiltonian

$$H = \sum \frac{1}{2\mu_{jk}} p_i^2 + \frac{1}{2M} < p >^2 - \sum \frac{1}{m_i} p_j^T p_k - \sum \frac{m_j m_k}{\mid q_i \mid} \tag{5}$$

where all the summations run over the cyclic permutations of $(1,2,3)$, and $\mu_{ij} = m_i m_j/(m_i + m_j)$.

Our goal is to obtain a global regularization of the three body problem, so we must remove the terms $\dfrac{m_j m_k}{\mid q_i \mid}$ in the Hamiltonian H.

Firstly we adopt the center of mass frame and that way the term in $< p >$ in H can be dropped. Then we introduce the regularizing transformation of Kustaanheimo and Stiefel (see [**8**]) to each of the 3-vectors $q_i = (q_{i1}, q_{i2}, q_{i3})$ obtaining the 4-vectors $u_i = (u_{i1}, u_{i2}, u_{i3}, u_{i4})$ for $i = 1,2,3$. (see [**1**]).

The relation between q_i and u_i is given by $q_i = L(u_i)u_i$ where the 4x4 matrix $L(u_i)$ is defined by

$$L(u_i) = \begin{pmatrix} u_{i1} & -u_{i2} & -u_{i3} & u_{i4} \\ u_{i2} & u_{i1} & -u_{i4} & -u_{i3} \\ u_{i3} & u_{i4} & u_{i1} & u_{i2} \\ u_{i4} & -u_{i3} & u_{i2} & -u_{i1} \end{pmatrix}$$

The 3-vectors q_i are considered vectors of $\mathbf{R}^4$ with a zero fourth component. We also introduce the matrix of partial derivatives (trasposed)

$$A_i = 2 \begin{pmatrix} u_{i1} & u_{i2} & u_{i3} \\ -u_{i2} & u_{i1} & u_{i4} \\ -u_{i3} & -u_{i4} & u_{i1} \\ u_{i4} & -u_{i3} & u_{i2} \end{pmatrix}$$

The new 4-momentum z_i, $i = 1,2,3$ is related to the old 3-momentum by the equation

$$p_i = \frac{1}{4s_i} A_i^T z_i \quad \text{where} \quad s_i = \mid q_i \mid = u_i^T u_i\,, \quad i = 1,2,3\,.$$

Because s_i are the lenght of the sides of the triangle formed by the three masses, they must satisfy the triangular inequality $s_i \leq s_j + s_k$.

Moreover, from the definition of p_i, we have a new relation between coordinates and momenta. This relation is called bilinear relation (see [**8**]) and its expression is

$$u_{i4} \cdot z_{i1} - u_{i3} \cdot z_{i2} + u_{i2} \cdot z_{i3} - u_{i1} \cdot z_{i4} = 0\,, \quad i = 1,2,3\,. \tag{6}$$

Let $u, z \in \mathbf{R}^{12}$ defined by $u = (u_1, u_2, u_3)$, $z = (z_1, z_2, z_3)$, substituted in the Hamiltonian (5) together with the time scaling $d\tau = \dfrac{dt}{s_1 s_2 s_3}$. We obtain a new

Hamiltonian restricted to a fixed level of energy h, $K(u,z) = H_o(u,z) - h \cdot s_1 s_2 s_3$, explicitely

$$
\begin{aligned}
K =& \frac{1}{8}\left(\frac{s_2 s_3}{\mu_{23}} z_1^T z_1 + \frac{s_3 s_1}{\mu_{31}} z_2^T z_2 + \frac{s_1 s_2}{\mu_{12}} z_3^T z_3\right) \\
& - \frac{1}{16}\left(\frac{s_1}{m_1} z_2^T A_2 A_3^T z_3 + \frac{s_2}{m_2} z_3^T A_3 A_1^T z_1 + \frac{s_3}{m_3} z_1^T A_1 A_2^T z_2\right) \\
& - m_2 m_3 s_2 s_3 - m_3 m_1 s_3 s_1 - m_1 m_2 s_1 s_2 - h \cdot s_1 s_2 s_3 \,.
\end{aligned} \tag{7}
$$

As we can see, that is a polynomial Hamiltonian of degree six in the regularized coordinates and degree two in the conjugate momenta. Therefore all the collisions between two bodies are regularized.

Blow up and scaling time.

Our goal was to study the triple collision but, because of the above change of time, this is not possible yet. We need infinite time to reach triple collision. Then we introduce a new change of variables and a new time by means of the moment of inertia $I = \dfrac{1}{M}\sum m_i m_j \mid q_k \mid^2$. This change of variables was first introduced by McGehee, [4], for the collinear problem, (see also [**2**], [**6**], [**10**]).

A typical distance $r \geq 0$ in the three body system is introduced by $r^2 = I$. If we approach triple collision, then $r \longrightarrow 0$. We use appropriate powers of r as scale factors in order to define normalized variables with finite limit as $r \longrightarrow 0$, denoted by the symbol $(\sim)$:

$$
u_i = \widetilde{u}_i r^{1/2}\,, \qquad z_i = \widetilde{z}_i\,, \qquad d\tau = r^{-3/2} d\widetilde{\tau}\,. \tag{8}
$$

We have also

$$
s_i = \widetilde{s}_i \cdot r\,, \qquad A_i = \widetilde{A}_i \cdot r^{1/2}\,.
$$

Then we can write the new system of differential equations as

$$
\begin{aligned}
\frac{dr}{d\widetilde{\tau}} &= \frac{r}{2}\widetilde{s}_1\widetilde{s}_2\widetilde{s}_3\widetilde{v}\,, \\
\frac{d\widetilde{u}_i}{d\widetilde{\tau}} &= \frac{\partial H_o}{\partial z_i}(\widetilde{u}, z) - \frac{1}{4}\widetilde{s}_1\widetilde{s}_2\widetilde{s}_3\widetilde{v}\widetilde{u}_i\,, \\
\frac{dz_i}{d\widetilde{\tau}} &= \frac{\partial H_o}{\partial \widetilde{u}_i}(\widetilde{u}, z) + \widetilde{h}\frac{\partial}{\partial \widetilde{u}_i}(\widetilde{s}_1\widetilde{s}_2\widetilde{s}_3)\,,
\end{aligned} \tag{9}
$$

where $\widetilde{v} = \sum \widetilde{u}_i \cdot z_i$.

They admit the integrals

$$
\begin{aligned}
H_o(\widetilde{u}, z) - \widetilde{h}\widetilde{s}_1\widetilde{s}_2\widetilde{s}_3 &= 0 \qquad \text{(energy)} \\
\sum s_i m_j \widetilde{s}_k^2 &= M \qquad \text{(normalization)}.
\end{aligned} \tag{10}
$$

We can express the energy integral in compact form introducing vectorial notation, that way we obtain

$$H_o(\tilde{u}, z) = \frac{1}{8} z^T B(\tilde{u}) \cdot z - \sum m_j m_k \tilde{s}_j \tilde{s}_k$$

where $B(\tilde{u})$ is a 12×12 symmetric block matrix defined by

$$B(\tilde{u}) = \begin{pmatrix} \dfrac{\tilde{s}_2\tilde{s}_3}{\mu_{23}}\ \mathrm{Id} & -\dfrac{1}{4}\dfrac{\tilde{s}_3}{m_3} A_1 A_2^T & -\dfrac{1}{4}\dfrac{\tilde{s}_2}{m_2} A_1 A_3^T \\ -\dfrac{1}{4}\dfrac{\tilde{s}_3}{m_3} A_2 A_1^T & \dfrac{\tilde{s}_3\tilde{s}_1}{\mu_{31}}\ \mathrm{Id} & -\dfrac{1}{4}\dfrac{\tilde{s}_1}{m_1} A_2 A_3^T \\ -\dfrac{1}{4}\dfrac{\tilde{s}_2}{m_2} A_3 A_1^T & -\dfrac{1}{4}\dfrac{\tilde{s}_1}{m_1} A_3 A_2^T & \dfrac{\tilde{s}_1\tilde{s}_2}{\mu_{12}}\ \mathrm{Id} \end{pmatrix}$$

where Id is the 4×4 identity matrix.

Triple Collision.

The system of differential equations (9) permit a complete description of the physical problem of 3-bodies when $r > 0$. For the case $r = 0$ these equations give us fictitious solutions because the real time t does not advance, as follows from equation (8). We can see from equations (9) that the relation $r = 0$ is invariant, so that the solutions of (9) with $r = 0$ are said to form the triple collision manifold C of the 3-body problem. This manifold was first introduced by McGehee [**4**] for the collinear 3-body problem.

On the triple collision manifold C, $\dfrac{d\tilde{v}}{d\tau} \geq 0$, as a consequence of the Sundman inequality. This shows us that the flow on C is "gradient-like", where $\tilde{v}$ plays the role of the Lyapounov function ($\tilde{v}$ does never decrease).

We can study now the angular momentum integral. Recalling the definition of $\tilde{w}$, $w = r^{1/2}\tilde{w}$, we know that w necessarily vanishes on C, but not $\tilde{w}$. We obtain a differential equation for $\tilde{w}$

$$\frac{d\tilde{w}}{d\tilde{\tau}} = -\frac{1}{4}\tilde{s}_1\tilde{s}_2\tilde{s}_3\tilde{v}\tilde{w}\,. \tag{11}$$

We observe that $\tilde{w} = 0$ is an invariant relation on C. The corresponding submanifold of C, defined by $r = 0$, $\tilde{w} = 0$, was first introduced by Waldvogel [**9**] and was called the non-rotating triple collision manifold $N \subset C$.

The dimension of C can be obtained from the dependence of the variables, $q_1 + q_2 + q_3 = 0$, the center of masses integrals, the normalization and bilinear relation (10), (6), the freedom of choosing p_i and also u_i in the corresponding fiber and finally using the condition that define C, that is $r = 0$. All these conditions give us a 10-dimensional manifold, C. To obtain N, we add the condition $\tilde{w} = 0$ and therefore N is a 7-dimensional manifold. Another important reduction of the dimensions of the problem can be made considering that the spatial three-body problem is a mechanical system with symmetry (see [**7**]) given by the Lie group $SO(3)$ acting diagonally. Under this action there are analytic quotient manifolds C_0 and N_0 of dimensions 7 and 4 respectively. We restrict our attention to these new manifolds.

On N_0, with $r = 0$, $\widetilde{w} = 0$ and, as a consequence, $\widetilde{h} = rh = 0$, the differential equations (9) of the system can be writen in vectorial form as

$$\begin{aligned} \frac{d\widetilde{u}}{d\widetilde{\tau}} &= \frac{1}{4}B(\widetilde{u})z - \frac{1}{4}\widetilde{s}_1\widetilde{s}_2\widetilde{s}_3\widetilde{v}\widetilde{u} \\ \frac{d\widetilde{z}}{d\widetilde{\tau}} &= -\frac{\partial H_o(\widetilde{u}, z)}{\partial\widetilde{u}} \end{aligned} \tag{12}$$

with the two integrals

$$\frac{1}{8}z^T B(\widetilde{u})z - \sum m_j m_k \widetilde{s}_j\widetilde{s}_k = 0 \tag{13}$$

$$\sum m_i m_j \widetilde{s}_k^2 = M\,. \tag{14}$$

From this two integrals we can derive the topological type of N_0 . The normalization integral (14) represents an algebraic surface D of the topological type S^2 in the space of the configurations $\widetilde{s}_k$, or a surface of order 4 in the configurations $\widetilde{u}_k$ (see [**10**]). But, because of the triangular inequality, only a little spheric triangle on the first quadrant is the set of posible configurations $\widetilde{s}_k$.

The integral (13) is a quadratic form in z with the symmetric matrix $B(\widetilde{u})$. In general equation (13) represents an ellipsoid because $B(\widetilde{u})$ is positive definite on D. In the points corresponding to the binary escapes, $\widetilde{s}_i = 0$, $\widetilde{s}_j = \widetilde{s}_k$, the ellipsoid degenerates into a cylinder.

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