ON MODULI OF CONJUGATION FOR SOME N-DIMENSIONAL VECTOR FIELDS

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Abstract. A generalization of a result of Palis for two dimensional vector fields with a heteroclinic connection is made for -dimensional vector fields. The connection is supposed to be 1-dimensional and the same class of moduli is obtained. The class of topological conjugacy of these vector fields can be characterized by one parameter depending on the eigenvalues not related with the connection manifold.

Introduction.

The possibility of characterizing the class of topological conjugacy or equivalence of a vector field X by finitely many parameters is important for the study of the stability of this vector field. When this characterization is possible we say that X has a finite modulus of stability. We will study here a differentiable invariant that arises when we consider topological conjugacies of a certain class of vector fields. This invariant depends on the eigenvalues associated to the saddle points. In 1978 J.Palis [**3**] found an invariant for vector fields with saddle connections in the 2-dimensional case. Here we generalize this invariant to the n-dimensional case.

We will need a few definitions in order to state our results.

Let X, X' be two vector fields on a manifold M, and X_t, X'_t their associated flows. A topological equivalence between X and X' on M is a homeomorphism $H : M \longrightarrow M$ such that H sends orbits of X into orbits of X', preserving time orientation. If, in addition, H preserves time, that is, $HX_t = X'_tH$ holds, then H is called a topological conjugation.

A vector field X is called structurally stable if it is equivalent to any nearby vector field. The moduli of stability may be defined as follows. A vector field is N-stable (has a N modulus of stability), N being a positive integer, if in any small neighbourhood of this vector field there is a N-(real) parameter family of topologically distinct vector fields and every vector field in the neighbourhood is equivalent to one in this family. A structurally stable vector field is 0-stable.

Description of the vector fields.

We consider two vector fields X and X' of class C^r $(r \geq 2)$ defined on a n-dimensional manifold M.

Let p, q and p', q' be hyperbolic critical points for X and X', respectively. We suppose that the dimensions for the stable and unstable manifolds W^s, W^u, are:

$$\begin{aligned} \dim W^s(q) &= 1 & \qquad \dim W^u(q) &= n-1 \\ \dim W^s(p) &= n-1 & \qquad \dim W^u(p) &= 1\,. \end{aligned}$$

Moreover, we suppose that $W^u(q)$ coincides with $W^s(p)$, i.e., the two critical points have a $(n-1)$-dimensional connection, and that this connection is diffeomorphic to S^{n-1}. The same situation holds for X', q' and p'. Finally, let δ and δ' be the negative eigenvalues of the derivatives $DX(q)$, $DX'(q')$, respectively, and μ, μ' the positive eigenvalues of $DX(p)$ and $DX'(p')$ (see fig. 1).

Figure 1

Before stating and proving our main result we would like to give a motivation for the study of this highly degenerate situation. Consider the classical n-body problem of celestial mechanics. It has singularities when two or more bodies collide. It is well known that two body collisions can be regularized. For more than two bodies one can use a device introduced by McGehee [**2**] to produce a blow up of the singularity due to the collision. This method leads to study the flow in a neighbourhood of the manifold which replaces the singularity when the blow up is done (the so called k-ple collision manifold if k bodies collide). For the case of 3 bodies on a line, a fixed level of energy is a 3-dimensional manifold, and the collision manifold is, essentially, a 2-dimensional sphere where the flow is gradient like and has 6 fixed points. Three of them are stable in the transversal direction to the sphere and, when restricted to the sphere, two of them are sources and the other is a saddle. The remaining three are unstable in the transversal direction and, on the sphere, two of them are sinks and the other is a saddle (see [**2**] for the details). A similar situation appears in the isosceles problem but with 10 fixed points (see [**4**], [**5**]) and in more general problems. The eigenvalues at the fixed points and the invariant manifolds on the sphere depend only on the masses of the envolved bodies. Our final goal is to study the possibility of conjugation or, perhaps, equivalence between the vector fields in a neighbourhood of the collision manifold for nearby masses and also, to identify the set of masses for which a bifurcation is produced. Before going into this much more complex problem, we present here a very clean situation which leads to simple results.

We suppose that there exist neighbourhoods U_q, U_p of q and p where the flow X_t of the vector field X can be linearized. In an analogous way we define $U_{q'}$ and $U_{p'}$ for X'. Nevertheless, this hypothesis is not essential. As the computations near the critical points depend only on the normal behaviour one can replace the hypothesis above by the requirement that the codimension one foliations (unstable for q, q' and stable for p, p') can be made C^1.

From now on and for the sake of simplicity, both in the statement of the theorem and along the proof, we will talk about conjugacy between X and X'. Of course this means conjugacy on some neighbourhood of the $(n-1)$-dimensional connection manifold $W^u(q) = W^s(p)$.

We can now state our result as follows.

Theorem. *The necessary and sufficient condition to have topological conjugacy between two vector fields X and X', in the previous conditions, is* $\dfrac{\delta}{\mu} = \dfrac{\delta'}{\mu'}$.

Proof: The proof of the necessary condition follows Palis [**3**], but for the sufficient condition we need a new result as we will show.

I. *Necessary condition.* First, we suppose that there is a homeomorphism H of M such that $HX_t = X'_tH$, i.e., X and X' are topologically conjugate. To see that the condition $\dfrac{\delta}{\mu} = \dfrac{\delta'}{\mu'}$ is necessary, the idea is to take the vector field X, "to carry downwards" a sequence $x_n \longrightarrow x \in W^s(q)$ following the associated flow X_t, and to transform it into a sequence in the lower part $w_n \longrightarrow w \in W^u(p)$. Finally, because of the topological conjugacy, we can repeat this process for the images $H(x_n) \longrightarrow H(x) \in W^s(q')$ under the vector field X'. The time for both processes must be the same and from this we obtain the necessary condition.

We can divide this process in three steps:

(i) Upper step. We consider the neighbourhood U_q of the hyperbolic critical point q of X, where the flow X_t is linear. We assume that the linearization is done.

In U_q we can identify $W^s(q) \times W^u(q)$ with $R \times R^{n-1}$. Therefore, if $x \in W^s(q) \cap U_q$ we can represent x as $(a, 0)$ where $a \in R$ is the stable coordinate and $0 \in R^{n-1}$ is the unstable one. Let x_n be a sequence in U_q which tends to $x \in W^s(q) \cap U_q$. Using coordinates, $x_n = (a_n, b_n) \longrightarrow (a, 0)$ and hence $a_n \longrightarrow a$ and $b_n \longrightarrow 0$. A consequence of this last condition is that the unstable coordinates will not have importance in the problem. In fact, this is the reason why the topological conjugacy of these vector fields does not depend on the eigenvalues corresponding to the connection manifold. Therefore, we can restrict our attention only to the stable coordinate, or more precisely, to the normal coordinate to the connection manifold $W^u(q)$. Both are equal in U_q.

Now, going back to the sequence $x_n \longrightarrow x$, we start to carry it down following the flow X_t. We consider in U_q a transversal section L_q of $W_u(q)$ through the manifold $C_q = L_q \cap W^u(q)$ diffeomorphic to S^{n-2} (see fig. 2 for the 3-dimensional case) . Note that L_q can be choosen transversal to the flow (see [**6**], p.81).

We denote by y_n the image $X_{t_n}(x_n)$ such that it belongs to L_q. Then we have $t_n \longrightarrow \infty$ when $n \longrightarrow \infty$.

(ii) Intermediate step. To pass from the upper part of the manifold M to the lower part, we must consider a fixed time s big enough to obtain $L_p = X_s(L_q)$ inside the neighbourhood U_p of p, where the flow X_t can be also linearized. We call C_p to the image of $C_q : C_p = X_s(C_q)$; obviously $C_p = L_p \cap W_s(p)$. Then we define $z_n = X_s(y_n)$.

(iii) Lower step. Now, we are in U_p where the flow can be considered linear. To end this process, let us consider $w \in W^u(p) \cap U_p$ and $w_n \longrightarrow w$ defined by $w_n = X_{T_n}(z_n)$ with $T_n \longrightarrow \infty$ when $n \longrightarrow \infty$.

For the sake of definition we shall consider all points w_n with the same normal coordinate: $w_n = (c_n, d_n)$, $c_n = c$, where c_n is the normal coordinate along $W^u(p)$ and d_n along $W^s(p)$.

Figure 2

Now we introduce a function K defined in C_q measuring the magnification of the normal distance when going from a point $y \in C_q$ to the point $X_s(y) = z \in C_q$. Let $(i, f) \in L_q$ with i small and $(g, h) = X_s(i, f) \in L_p$. We know that $g = 0$ if $i = 0$ and that the dependence of g on i, f is C^r.

Then, we define $K(f) = \left.\dfrac{\partial g}{\partial i}\right|_{(i=0,f)}$. It is clear that K can be obtained from the first order variational equations and that K is C^{r-1}.

By compactness K is bounded in $C_q : 0 < K_1 < K(f) < K_2$. Using successively the 3 steps with the notation $x_n = (a_n, b_n)$, $y_n = (i_n, f_n)$, $z_n = (g_n, h_n)$, $w_n = (c_n, d_n)$ and recalling that $r \geq 2$ we have:

$$i_n = a_n \cdot e^{\delta t_n}\,; \quad g_n = K(f_n) \cdot i_n + O(i_n^2)\,; \quad c_n = g_n \cdot e^{\mu T_n}\,.$$

Hence, $c = a_n \cdot e^{\delta t_n} \cdot K(f_n) \cdot (1 + O(e^{\delta t_n})) \cdot e^{\mu T_n}$.

We should have a similar relation with identical time intervals, t_n, s, T_n, for the flow X'_t. So, we obtain

$$c' = a'_n \cdot e^{\delta' t_n} \cdot K'(f'_n) \cdot (1 + O(e^{\delta' t_n})) \cdot e^{\mu' T_n}\,,$$

where we recall that K' has also bounds $0 < K'_1 < K'(f'_n) < K'_2$. Then we have

$$\frac{1}{T_n}(\delta t_n + \mu T_n) = \frac{1}{T_n} \log\left[c/(a_n \cdot K(f_n) \cdot (1 + O(e^{\delta t_n})))\right]$$

and, going to the limit, we obtain $\lim_{n\to\infty} \delta \dfrac{t_n}{T_n} + \mu = 0$. In a similar way, $\lim_{n\to\infty} \delta' \dfrac{t_n}{T_n} + \mu' = 0$ and then $\dfrac{\delta}{\mu} = \dfrac{\delta'}{\mu'}$ as desired.

II. *Sufficient condition.* Let X, X' be two vector fields in the conditions of the theorem. Now, we assume the equality $\dfrac{\delta}{\mu} = \dfrac{\delta'}{\mu'}$. We want to show that this is a sufficient condition to have topological conjugacy between X and X'. First we need the following result:

LEMMA. *Let X, X' be two vector fields in the conditions of the theorem, and suppose $\dfrac{\delta}{\mu} = \dfrac{\delta'}{\mu'}$. Then, we can define a homeomorfism F between two manifolds $C_q \subset W^u(q) \cap U_q$ and $C_{q'} \subset W^u(q') \cap U_{q'}$, both diffeomorphic to S^{n-2} and transversal to the respective flows, such that, for every point $f \in C_q$ the following condition holds:*

$$\frac{\log K'(F(f))}{\log K(f)} = \frac{\delta'}{\delta} = \frac{\mu'}{\mu}, \tag{1}$$

where K and K' are the functions introduced in the proof of the necessary condition.

PROOF OF THE LEMMA: Without loss of generality , we can take the same coordinates for the two vector fields and therefore, we can identify $q = q'$. Let $U_q = U_{q'}$, $U_p = U_{p'}$ be neighbourhoods of q, p where the flows X_t and X'_t can be both linearized. For the moment, s will denote a certain fixed time, big enough to make all the following constructions. At the end of the proof, we will find a lower bound of s.

We have certain values of K and K' measuring the normal magnification between the points in C_q and the points in the images by the flows $C_p = X_s(C_q)$ and $\bar{C}_p = X'_s(C_q)$. In general, they do not satisfy condition (1). Then, we can modify C_q by the flow X'_t to obtain new manifolds $C_{q'}$ and $C_{p'} = X'_s(C_{q'})$. The idea is to make the construction in such a way that for the manifolds C_q, C_p, $C_{q'}$ and $C_{p'}$ the condition (1) will hold.

We are going to define the manifold $C_{q'}$ from the points of C_q. For each point $f \in C_q$, we can define its image denoted by $f' = F(f)$ where $f' = X'_{t_f}(f)$. We need $f' \in U_q \cap W^u(q)$, and this will be possible if we take C_q and the time s in a suitable way.

The problem is the rigth election of the time t_f to define the point f' satisfying the required condition (1). For that, we need to make the following process. Let $\bar{h}' = X'_s(f)$ and $h' = X'_{t_f}(\bar{h}')$ (see fig. 3). We want to find the expression of the distortion of the normal component of the flow X'_t in the point f', denoted by $K'(f')$, as a function of the distortion K'_i in the original point f. Using the linearization of the flow X'_t in U_q and U_p, one easily obtains

$$K'(f') = K'_i(f) \cdot e^{(\delta' - \mu')t_f f} \qquad \text{(see fig. 3).}$$

Figure 3

We wish to have the equality

$$\frac{\log K'(f')}{\log K(f)} = \frac{\mu'}{\mu} = \frac{\delta'}{\delta}\,.$$

This is obtained picking up

$$t_f = \frac{\mu' \log K(f) - \mu \log K_i'(f)}{\mu(\delta' - \mu')}. \tag{2}$$

Then we define $f' = X'_{t_f}(f)$ and the homeomorphism F by $F(f) = f'$.

The function F is continuous from the continuity of t_f, $K(f)$ and $K_i'(f)$. Moreover, we can define its inverse taking the time $-t_f$. Both conditions show us that F is a homeomorphism (in fact F is even C^{r-1}).

Now we must be sure that the previous constructions can be done in such a way that $C_{q'}$ or $C_{p'}$ do not leave the neighbourhoods where the linearization is possible.

First, we select a previous value $\bar{s}$ of s such that $\bar{C}_p = X_{\bar{s}}(C_q) \subset U_p$ and also $\bar{C}_{p'} = X'_{\bar{s}}(C_q) \subset U_p$. For every $f \in C_q$ we compute t_f from (2). By the continuity of K and K_i' and the compactness of C_q we have $T_m \leq t_f \leq T_M$. As it is possible that either $X'_{T_M}(f)$ leaves U_q or $X'_{T_m^{+s}}(f)$ leaves U_p we shall modify C_q and s keeping K constant (in some sense) to avoid this problem. Let $T = \max(| T_m |, | T_M |) \cdot \max\left(1, \frac{\mu}{| \delta |}\right)$. Then, instead of C_q we take $\widetilde{C}_q = \{X - T(f), f \in Cq\}$ and we use $s = \bar{s} + T\left(1 + \frac{| \delta |}{\mu}\right)$. It is clear that both $\widetilde{C}_q \subset U_q$ and $\widetilde{C}_p = X_s(\widetilde{C}_q) \subset U_p$. Furthermore as $X_s = X_{\frac{T\,| \delta |}{\mu}} \circ \frac{X}{s} \circ XT$, the new values $\widetilde{K}$ of K are obtained from the old ones multiplying by $e^{\delta T} \cdot e^{\mu T |\delta| / \mu} = 1$. So, K is invariant, that is, $\widetilde{K}(X_{-T}(f)) = K(f)$.

Let us see what happens under the flow X'. Using the same argument it is seen that K_i' is invariant and hence t_f does not change.

As $t_f - T < 0$, one has $X'_{t_f}(X'_{-T}(f)) \in U_q$. As $T\dfrac{|\delta|}{\mu} + t_f > 0$ one has, in a similar way, $X'_{t_f}(X'_s(X'_{-T}(f))) \in U_p$ because $t_f + s - T = t_f + \bar{s} + T\dfrac{|\delta|}{\mu} > \bar{s}$.

We define $\widetilde{C}'_q = \{X'_{t_f}(X' - T(f)),\ f \in C_q\}$ and the final homeomorphism $\widetilde{F}$ goes from $\widetilde{C}_q$ to $\widetilde{C}_{q'}$ by $\widetilde{F}(f) = X'_{t_f}(X'_{-T}(X_T(f)))$ for $f \in \widetilde{C}_q$.

Finally $\widetilde{C}_{p'} = X'_s(\widetilde{C}_{q'})$. From now on we rename $\widetilde{C}_q$, $\widetilde{C}_p$, $\widetilde{C}_{q'}$, $\widetilde{C}_{p'}$ and $\widetilde{F}$ as C_q, C_p, $C_{q'}$, $C_{p'}$ and F respectively.

This ends the proof of the lemma.

Using the lemma we can proof the sufficient condition of the theorem. We will use here the same notation as in the necessary condition.

We must define a homeomorphism $H : M \longrightarrow M$ such that $HX_t = X'_tH$. First, we will define H for the neighbourhoods U_q and $U_{q'}$ where the vector fields can be linearized. Let $x = (a, 0)$, $x' = (a', 0)$ be two points on $W^s(q)$ and $W^s(q')$ respectively, and C_q, $C_{q'}$ the two manifolds satisfying the condition of the lemma with $F : C_q \longrightarrow C_{q'}$ a homeomorphism between them. We define $H(q) = q'$, $H(x) = x'$ and $H(y) = F(y) \in C_{q'}$ for $y \in C_q$. For any point $(\underline{a}, \underline{b}) \in U_q$ we have $(\underline{a}, 0) = X_{t_1}(a, 0)$ and $(0, \underline{b}) = X_{t_2}(0, b)$ where $(0, b) \in C_q$, and we define $H(\underline{a}, \underline{b}) = (\pi_1(X'_{t_1}(a', 0)), \pi_2(X'_{t_2}(0, b')))$ where π_1, π_2 are the natural projections, and $(0, b') = H(0, b)$.

For the points v near $W^u(q)$ but not included in U_q, we can define $H(v)$ by the flow as follows, $H(v) = X'_t(H(u))$ where $u = X_{-t}(v)$ with $u \in L_q$ (see the upper step in the proof of the necessity).

We have to show that H can be extended continously to a conjugacy between $W^u(p)$ and $W^u(p')$ since this definition is not possible for the points of these sets. For that, we consider the time s defined in the Lemma.

Let w_n be a sequence approaching $w \in W^u_p$. We shall go backwards under the flow X_t. Using coordinates along the stable and unstable manifolds both in the upper and lower part we have $w_n = (c_n, d_n)$, $w = (c, 0)$, with $c_n \longrightarrow c$, $d_n \longrightarrow 0$ when $n \to \infty$.

Let $z_n = (g_n, h_n)$ such that $X_{T_n}(z_n) = w_n$. Then $(0, h_n) \in C_p$ and $g_n e^{\mu T_n} = c_n$ and $T_n \longrightarrow \infty$ for $n \to \infty$. Furthermore, it exists $\widetilde{y}_n = (i_n, \widetilde{f}_n) \in L_q$ such that $X_s(\widetilde{y}_n) = z_n$. There is some point $y_n = (0, f_n)$ such that $X_s(y_n) = (0, h_n)$. Then $g_n = K(y_n) \cdot i_n + O(i_n^2)$.

Select some value of a. Then there exists $x_n = (a_n, b_n)$ with $a_n = a$, such that $X_{t_n}(x_n) = \widetilde{y}_n$ and $b_n \longrightarrow 0$, $t_n \longrightarrow \infty$ for $n \to \infty$. We have also $i_n = ae^{\delta t_n}$.

We can write the deformation of the normal coordinates as follows:

$$c_n = e^{\mu T_n} \cdot K(y_n) \cdot a \cdot e^{\delta t_n}(1 + O(e^{\delta t_n})) .$$

For the flow X' one has, in a similar way

$$c'_n = e^{\mu' T_n} \cdot K'(y'_n) \cdot a' \cdot e^{\delta' t_n}(1 + O(e^{\delta' t_n})) .$$

Using the relation (1), that is $\dfrac{\log K'(f')}{\log K(f)} = \dfrac{\delta'}{\delta} = \dfrac{\mu'}{\mu} = R$ we have $c'_n = \left(\dfrac{c_n}{\delta t_n}\right)^R \cdot a' \cdot (1 + O(e^{\delta' t_n}))$.

When n goes to infinity one has that c'_n goes to $\left(\dfrac{c}{a}\right)^R a'$, showing that H is well defined, by continuity on $W^u(p)$.

This ends the proof of the theorem.

A consequence of this theorem like in the 2-dimensional case is that any two such vector fields X and X' in the conditions of the theorem, are always topologically equivalent near the connection manifold, even if $\dfrac{\delta'}{\delta} \neq \dfrac{\mu'}{\mu}$. To show that, we consider $\widetilde{X} = g \cdot X$ such that, this moduli condition is satisfied for $\widetilde{X}$ and X', where g is a positive real function with $g = 1$ outside of a neighbourhood of a hyperbolic critical point for X either q or p. Since X and $\widetilde{X}$ are equivalent and $\widetilde{X}$ and X' are conjugate, X and X' are equivalent.

There are many others papers refering to different moduli of stability, in different kinds of vector fields [**7**]. For example, in the three dimensional case, for Shilnikov systems one can see Togawa [**8**] and for connections between a hyperbolic critical point and a periodic orbit one can see Beloqui [**1**].

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