EQUILIBRIUM CONNECTIONS ON THE TRIPLE COLLISION MANIFOLD

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Abstract. In the three body problem the triple collision manifold plays a fundamental role to describe passages near triple collision. To study the possible transitions from the approach to collision to the escape from it, the invariant submanifolds on that manifold are essential. In this paper we study mainly the connections between the equilateral approaches and escapes

1. Introduction.

We consider the planar 3-body problem with masses m_1, m_2, m_3. If $q_i, p_i \in \mathbf{R}^2$, $i = 1, 2, 3$ are the related positions and momenta, $q = (q_1^T, q_2^T, q_3^T)^T$, $p = (p_1^T, p_2^T, p_3^T)^T$ and $M = \text{diag}\,(m_1, m_1, m_2, m_2, m_3, m_3)$ denotes the mass matrix, the Hamiltonian of the problem is

$$H(q,p) = \frac{1}{2} p^T M p - U(q)\,, \tag{1}$$

where $U(q) = \sum_{1 \le i \le j \le 3} \dfrac{m_i m_j}{\mid q_i - q_j \mid}$ is the potencial energy.

The system has singularities due to double and triple collision. We are interested in the behaviour of the system close to a triple collision where the angular momentum is zero as it is required to have an exact triple collision.

To put the equations of motion in a suitable form, one needs to do several changes (blow up, regularization, etc) which have been described in [**6**], [**4**] and [**5**]. In this way we introduce variables, $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$ essentially related to the normalized form of the configuration triangle which satisfy the relation

$$\sum_{1 \le j \le k \le 3} m_j m_k (\alpha_j^2 + \alpha_k^2)^2 = m_1 + m_2 + m_3 \tag{2}$$

and variables $\pi_1, \pi_2, \pi_3 \in \mathbf{R}$ which play the role of momenta and satisfy the energy relation

$$(3) \quad K(\alpha,\pi) = \frac{1}{8}\pi^T B(\alpha)\pi - \sum_{1\le j\le k\le 3} m_j m_k(\alpha_{j+1}^2+\alpha_{j+2}^2)(\alpha_{k+1}^2+\alpha_{k+2}^2) = 0\,,$$

where $\alpha_s = \alpha_{s-3}$ if $s > 3$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$ and $\pi = (\pi_1, \pi_2, \pi_3)^T$. In (3) $B(\alpha)$ is a symmetric 3×3 matrix with components

$$(4) \quad \begin{aligned} b_{jj} &= \frac{\alpha_k^2+\alpha_\ell^2}{m_j}(\alpha_1^2+\alpha_2^2+\alpha_3^2) + \frac{\alpha_j^2+\alpha_\ell^2}{m_k}\alpha_\ell^2 + \frac{\alpha_j^2+\alpha_k^2}{m_\ell}\alpha_k^2\,, \\ b_{jk} &= -\frac{\alpha_j^2+\alpha_k^2}{m_\ell}\alpha_j\alpha_k\,, \end{aligned}$$

where the indices (j,k,ℓ) are $(1,2,3)$ or a cyclic permutation of it.

If we introduce also $v = \sum_{i=1}^{3} \alpha_i\pi_i$ then the equations of motion are

$$\frac{d\alpha}{d\tau} = \frac{\partial}{\partial\pi}K(\alpha,\pi) - \frac{1}{4}z\cdot\alpha\,, \quad \frac{d\pi}{d\tau} = -\frac{\partial}{\partial\alpha}K(\alpha,\pi)\,.$$

where $z = (\alpha_1^2+\alpha_2^2)(\alpha_2^2+\alpha_3^2)(\alpha_3^2+\alpha_1^2)\cdot v$ and τ is a suitable scaled time (we refer to [**6**], [**4**] again for details).

The set of variables $(\alpha,\pi) \in \mathbf{R}^6$ with the constrains (2), (3) is called the non-rotating triple collision manifold and will be denoted by N.

On N the flow is gradient like with respect to v. The topological type of N may be derived from (2) and (3). The integral (2) can be considered as S^2 in the configuration space and, given α, (3) can be considered as an ellipsoid except when $\alpha_j = \alpha_k = 0$ (binary configurations) where it degenerates into a cylinder. Therefore, N is topologically $S^2 \times S^2$ with 6 holes corresponding to the binary configurations. Adding these 6 points one obtains a compactified manifold that we will denote by $\widetilde{N}$. On the compactified manifold we found the critical points in pairs. We denote them by P^i (respectively P^s) when the corresponding level of v is negative (respect. positive). The critical points of $\widetilde{N}$ are related to the classical central configurations. Then we have 6 Euler's configurations (collinear) denoted by $E_j^{i,s}$, $j = 1,2,3$ and 4 Lagrange's configurations (equilateral) denoted by $L_{+,-}^{i,s}$. We can add also the 6 binary configurations as critical points on $\widetilde{N}$ denoted by $B_j^{i,s}$, $j = 1,2,3$. The corresponding dimensions of the stable and unstable manifolds associated to the critical points are shown on table I.

	B_j^i	E_j^i	$L_{+,-}^i$	$L_{+,-}^s$	E_j^s	B_j^s
$\dim W^n$	4	3	2	2	1	0
$\dim W^s$	0	1	2	2	3	4

Table I

We show on figure 1 the position of the critical points in the configuration space. The equator of this sphere corresponds to the collinear configurations.

Figure 1 Figure 2

On figure 2 we show the relative positions of the equilibrium points on $\widetilde{N}$ with respect to the increasing variable v.

Our goal is to describe all the possible connections between equilibrium points in the planar triple collision manifold and how do they depend on the masses of the bodies.

2. Qualitative description of the $v = 0$ section.

Now we want to give a qualitative description of the equator of $\widetilde{N}$ that is $\widetilde{N} \cap \{v = 0\}$. On table II we give the topological structure of $W^s_{P^s} \cap \{v = d\}$

for different values of $d = v_i$, $i = 0, 1, 2$ where P^s stands for one of the critical points $L^s_{+,-}$, E^s_j or B^s_j. We consider $v_2 > \max_j v(E^s_j)$, $0 < v_1 < \min_j v(E^s_j)$, $v_0 = 0$ (see [**4**]).

	B^s_j	E^s_j	$L^s_{+,-}$
v_2	S^3	$-$	$-$
v_1	$S^2 \times (0,1)$	S^2	$-$
v_0	$E^2 \times S^1$	$S^1 \times (0,1)$	S^1

Table II

On figure 3 we show the relative positions between the stable manifolds $W^s_{P^s} \cap \{v = 0\}$. One can see the circles corresponding to the Lagrange's points and the open cylinders associated to the Euler's points which are the boundaries of the open solid tori corresponding to the binaries.

Figure 3

The full description of the possible transitions between collision and ejection close to triple collision would be given by the intersections of the type $W^u_{P^i} \cap \{v = 0\} \cap W^s_{Q^s}$, P^i (resp. Q^s) being one of the critical points with $v < 0$ ($v > 0$). The main objective will be to know where to put the circles corresponding to the Lagrange's points and in particular when we will obtain connections between these critical points. These connections will classify all the other ones.

3. On the numerical computation of the unstable manifolds.

We consider the simplex of masses T, defined by $m_1 + m_2 + m_3 = 1$. If we denote by λ_1, λ_2 the two positive eigenvalues associated to $W^u_{L^i_+}$ they have the expression

$$\lambda_{1,2} = 8^{-1/2} \cdot \mu^{-3/4} \cdot \left[\sqrt{13 \pm 12\sqrt{1-3\mu}} - 1\right]$$

where $\mu = m_1 m_2 + m_2 m_3 + m_3 m_1$. We obtain $\lambda_1 = \lambda_2$ when we consider equal masses $m_i = 1/3$, $i = 1, 2, 3$.

A linear aproximation of $W^u_{L^i_+}$ does not work because the quotient $\frac{\lambda_1}{\lambda_2}$ can be far away from 1. On figure 4 we display level curves of the quotient $\frac{\lambda_1}{\lambda_2}$ for values of this quocient equal to $1.2^k, k = 0, \ldots, 17$.

Figure 4

To decrease the difficulties we have used a higher order approximation to the invariant manifold. The algorithm to compute this approximation can be described as follows. We consider a new reference taking the Lagrange critical point L^i_+ as origin and a new base made of the eigenvectors e_i, $i = 1, \ldots, 6$, associated to this critical point. Considering that the system of differential equations is polynomial we can write the equations in the new reference as

$$\dot{X} = B^{(1)}X + B^{(2)}(X)^2 + B^{(3)}(X)^3 + \cdots + B^{(9)}(X)^9\,, \tag{5}$$

with $X \in \mathbf{R}^6$. We want to remark that there is not independent term in the equations (of course) and that they are polynomial in 6 variables of degree 9.

We can express $W^u_{L^i_+}$ up to order n as

$$\bar{X}^{(n)} = C_1^{(1)}s_1e_1 + C_2^{(1)}s_2e_2 + \sum_{p=2}^{n}\sum_{k=1}^{6}\sum_{i+j=p} C^{(p)}_{i,j,k}s_1^i s_2^j e_k, \tag{6}$$

where $s_1 = e^{\lambda_1\tau}$ and $s_2 = e^{\lambda_2\tau}$, being τ the time variable. We can compute recursively the coefficients of $W^u_{L^i_+}$ of order n as a function of their coefficients up to order $n-1$. The idea is to impose the invariance of the manifold (6) when we substitute in the differential system (5). If we do that, we obtain the following expression for the coefficients of order n:

$$\sum_{k=1}^{6}\sum_{i+j=n} C^{(n)}_{i,j,k}(i\lambda_1 + j\lambda_2)s_1^i s_2^j e_k =$$

$$= \sum_{k=1}^{6}\sum_{i+j=n} C^{(n)}_{i,j,k}\lambda_k s_1^i s_2^j e_k + \sum_{k=1}^{6}\sum_{i+j=n} d^{(n)}_{i,j,k}s_1^i s_2^j e_k,$$

where $d^{(n)}_{i,j,k}$ denote the coefficients of the terms of order n in s_1, s_2 obtained from the substitution $X = X^{(n-1)}$ in the nonlinear terms of (5), that is

$$B^{(2)}(\bar{X}^{(n-1)})^2 + B^{(3)}(\bar{X}^{(n-1)})^3 + \cdots + B^{(n)}(\bar{X}^{(n-1)})^9\,.$$

Finally the expression for the coefficients of order n is

$$C^{(n)}_{i,j,k} = \frac{d^{(n)}_{i,j,k}}{i\lambda_1 + j\lambda_2 - \lambda_k}\,. \tag{7}$$

This has been implemented up to order 20 by means of and ad hoc algebraic manipulator. Let us discuss the presence of resonances in (7). As λ_1,λ_2,λ_5,$\lambda_6 > 0$,λ_3,$\lambda_4 < 0$, $i + j \geq 2$, $\lambda_1 > \lambda_2$, $\lambda_5 = \lambda_6$, $\lambda_1 > \lambda_5$, the only possible resonances are of the form $j\lambda_2 = \lambda_5 = \lambda_6$ with $j \geq 2$. There is a decreasing sequence of values of μ, $\mu_j > \mu_{j+1}$, $j \geq 2$ for which a resonance appears. One has $\mu_2 0.001984$. Hence the problem of resonance occurs only for small values of μ

When we have this local approximation we can start the numerical computations far away enough from the critical point to avoid the troubles we mention before. We can choose the optimal distance to start the computations comparing the numerical results obtained, when we cut the unstable manifold for differents values of v equal constant, whith the analytical results predicted for these levels by the local approximation. In other words, we found a practical radius of convergence of our expansion. Once we have decided this radius r we start the computations from a circle of radius r until we get the points on the equator of the manifold, $v = 0$. These points on the equator are obtained controlling the maximum distance between two consecutive points and controlling also the maximum angle between three of them. That way, we obtain a sufficiently regular curve on $v = 0$ which is compared to symmetrical copies of it (because of the symmetry of the problem) to found the possible connections between Lagrange's points.

4. Some numerical results.

We summarize the Lagrange's connections we have found at the moment in figure 5, where we show only one sixth of T.

Figure 5

In figure 5 we can see a family of curves beginning in an isosceles configuration of masses (in this case $m_1 = m_2$) and also three other curves not belonging to this family.

The difference between them is that the curves corresponding to the family do not give an isosceles connection when we consider the points of these curves corresponding to an isosceles configuration of masses (see [**3**]).

On the curves of this family we can distinguish two differents subfamilies when we look to the birth of the curves in the points with isosceles configuration of masses. We found alternatively connection points which give rise to one or two different curves and they correspond also to different type of connections.

On the table III we give the values of the masses of these points. We remark that they satisfy $m_1 = m_2$, $m_1 + m_2 + m_3 = 1$. We use $+ \longrightarrow +$ (resp. $-$) to denote Lagrange's connections between $L^i_+ \longrightarrow L^s_+$ (resp. L^s_-).

m_3	num. curves	Type of connection
0.632912	1	$+ \longrightarrow +$ non isosceles
0.5710061	1	$+ \longrightarrow -$ isosceles
0.1597719	1	$+ \longrightarrow -$ non isosceles
0.1591555	1	$+ \longrightarrow +$ isosceles
0.1081336	2	$+ \longrightarrow +$ non isosceles
0.0808633	1	$+ \longrightarrow -$ non isosceles
0.0638546	2	$+ \longrightarrow +$ non isosceles
0.0522381	1	$+ \longrightarrow -$ non isosceles

Table III

The shape of the curves corresponding to the intersection $W^u_{L^i_+} \cap \{v = 0\}$ is shown in figures 6,7,8. In the last section we will give an explanation to the increase of the intersections in the curves observed when the mass parameter m_3 decreases.

Figure 6

Figure 7 Figure 8

As it can be seen we do not display the termination of the curves on figure 5. This is due to the numerical troubles we mentioned before. Moreover we can not reach the level $m_3 = 0$ because the equations become unbounded when one of the masses vanishes. As a conjecture we think that these curves will finish at the corner point corresponding to the masses configuration $m_2 = m_3 = 0$, $m_1 = 1$. We base our conjecture on the relation, that we will comment later, between the planar and the collinear problem. For this last problem we have the family of curves given in figure 9 (see [**2**], [**4**]).

Figure 9

5. Theoretical considerations.

As a final point, we want to give an explanation of why we conjecture the existence of an infinity of curves giving Lagrange's connections. In [**4**] we gave a complete description of the different connections between critical points known up to that moment. We can give here a summary of the connections known for the planar problem (see [**1**])

(i) For all masses there are orbits connecting $L^s_{+,-}$ with E^s_i and B^s_i, $i = 1, 2, 3$. There are also the symmetric connections in the $v < 0$ region. All these connections are transversal.

(ii) If there is an orbit connecting E^i_j to L^s_+ transversally, then there are also orbits connecting E^i_j to L^s_- and to E^s_k, B^s_k, $k = 1, 2, 3$.

(iii) If the index of the projection of the curve $W^u_{L^i_+} \cap \{v = c+\varepsilon\}$ on the configuration space with respect to the projections of L^s_+ or L^s_- is different from zero, then all the connections given in table IV occur. Here $c = v(L^s_+)$ and $\varepsilon > 0$. This condition holds for the equal masses configuration

	B^i_k	E^i_k	$L^i_{+,-}$	$L^s_{+,-}$	E^s_k	B^s_k
B^i_j		$\times(*)$	$\times$	$\times$	$\times$	$\times$
E^i_j			$\times$	$\times$	$\times$	$\times$
$L^i_{+,-}$					$\times$	$\times$

$j, k = 1, 2, 3$
$(*) k \neq j$

Table IV

The connections we have shown in this paper are connections between Lagrange's points. When we have an isosceles masses configuration, $m_1 = m_2$, two sufficient conditions to assure that one of these connections exist are

$$(\alpha, \alpha, 0, -\pi, \pi, \pi') \in W^u_{L^i_+} \cap \{v = 0\}\,, \tag{8}$$

$$(0, \alpha, 0, \pi, 0, \pi') \in W^u_{L^i_+} \cap \{v = 0\}\,. \tag{9}$$

Condition (8) corresponds to connections $L^i_{+,-} \longrightarrow L^s_{-,+}$ and (9) to $L^i_{+,-} \longrightarrow L^i_{+,-}$. They are related to an Euler and a Binary configuration respectively, both associated to points in the equator of the configuration space (see fig. 1). We may have in mind that this equator corresponds to the collinear configurations and also that the collinear problem is a subproblem of the one we are studing. As one can see there is a similitude between figures 5 and 9, which correspond to the planar and the collinear problem respectively. Considering the collinear problem included in the planar one, the invariant manifold $W^u_{E^i_3}$ (3–dimensional) contains the submanifold corresponding to the collinear problem (1–dimensional) and it lies on the equator of the configuration space (see fig. 1) between B_1 and B_2. The collinear submanifold has two branches (see [2]) which get successive binary configurations. That is, on the configuration space they go from B_1 to B_2. The curves corresponding to $W^u_{L^i_+} \cap \{v = \text{ctant.}\}$ have points in the equator of the configuration space and they lie between the two branches mentioned before. The behaviour of these branches (see fig.7,8,9) forces the mouvement of the points of the curve. In particular for the isosceles configurations it means that we will find points with configuration corresponding to (8) and (9).Therefore, we can conjecture that we will find an infinity of curves giving Lagrange connections.

References

1. Moeckel, R., , Arch. Rat. Mechanics and Analysis (1989), 37–70.
2. Simó, C., , Celestial Mech. (1980), 25–36.
3. ________, , in "Classical Mechanics and Dynamical Systems," Ed.R.L. Devaney and Z. Nitecki, Marcel Dekker, 1981, pp. 203–224.
4. Simó, C.; Susín, A., , To appear Proceed. Workshop on the Geometry of Hamiltonian Systems, Berkeley (1989).
5. Susín, A., , in "Long-Term Dynamical Behaviour of Natural and Artificial N-Body Systems," Ed. A.E.Roy. Reidel, 1988, pp. 505–513.
6. Waldvogel, J., , Celestial Mech. (1982), 69–82.