MODULI OF CONJUGACY IN THE TRIPLE COLLISION COLLINEAR PROBLEM

A. Susín [(1)] and C. Simó [(2)]

[(1)] Dept. de Matemàtica Aplicada I , Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain

[(2)] Dept. de Matemàtica Aplicada i Anàlisi , Universitat de Barcelona , Gran Via 585, 08007 Barcelona, Spain

Abstract. When we consider the triple collision manifold in the collinear problem of three bodies, we are faced to a 3-dimensional manifold which is topologically equivalent to a two-sphere with four holes. The flow on this manifold is gradient-like. We classify the different type of topological conjugation in a neighbourhood of the collision manifold, according to the masses.

The collision manifold was first introduced by McGehee [**1**] using the collinear problem. He shows that the flow on the collision manifold is gradient-like. Essentially the collision manifold is a 3-dimensional manifold wich has two critical points corresponding to collinear central configurations. C. Simó in [**2**] studied the possible connections between these collinear critical points of saddle type. When two of the masses are equal, he found a countable infinity set of mass configurations for which the connection exist. These mass configurations give rise to a countable infinity set of curves in the mass simplex T, defined by $m_1 + m_2 + m_3 = 1$. Here we study of the topological conjugacy between the collision manifolds corresponding to mass configurations in the different regions in T which have the above curves as a boundary. A first step in this study was made in [**3**].

Firstly, we shall introduce the concepts of topological equivalence and topological conjugacy.

Definition. *Let X, X' be two vector fields on two manifolds C and C', respectively, and let φ_t, φ'_t their associated flows. A topological equivalence between X and X' is a homeomorphism $H : C \to C'$ such that H sends orbits of X into orbits of X', preserving time orientation. If in addition H preserves time, that is, if $H \circ \varphi_t = \varphi'_t \circ H$ holds, then H is called a topological conjugation.*

The following theorem allows us to classify the different cases of conjugation that one can find.

Theorem. *If there is not connection between the invariant manifolds of the Euler critical points, there are two classes of topological conjugacy in a neighbourhood of the collision manifolds.*

Proof: Let C and C' be two collision manifolds corresponding to different mass configurations. Let also E^i, E^s and E'^i and E'^s be the respective critical points. We consider a small neighbourhood of the collision manifolds defined by $0 \leq r < \varepsilon$ where ε is taken small enough. We will denote by $\tilde{C}$ and $\tilde{C}'$ the manifolds C and C' adding these neighbourhoods.

The first step to define a homeomorphism between $\tilde{C}$ and $\tilde{C}'$ is to study the flow near the critical points. From Hartman theorem we know that there is a topological conjugacy between the original system and its linear part, in a neighbourhood of these hyperbolic critical points. We can restrict our attention to E^i, the construction would be analogous for the other critical points. Let λ_1, λ_2, λ_3 be the eigenvalues associated to E^i, with λ_1, $\lambda_3 < 0$ and $\lambda_2 > 0$. The eigenvalue λ_3 is the one associated to the direction of the variable r, the one not included in the collision manifold. We always have $|\lambda_3| < |\lambda_1|$, that is, the direction r is slower than the one on C (see figure 1). The same situation can be considered for E'^i with eigenvalues λ'_1, λ'_2, λ'_3.

Figure 1 A neighbourhood of the critical points

Now we can use the fact that one can construct a topological conjugacy between the linear parts of E^i and E'^i. Actually, this topological conjugacy is defined between the invariant unstable manifolds and between the stable ones.This means that there exist two curves $B \subset W^s_{E^i}$ and $B' \subset W^s_{E'^i}$ diffeomorphic to half a circumference (see fig. 2) and a continuous map $\tilde{h}_1 : B \to B'$ that can be extended to a continuous map h_1 defined for every point $x \in W^s_{E^i}$ in a neighbourhood of the critical point using the flow transport. If $y = \varphi_{t_x}(x) \in B$, we define $h_1(x) = \varphi'_{-t_x}(\tilde{h}_1(y)) \in W^s_{E'^i}$. We denote by h_2 the analogous homeomorphism that we can construct between $W^u_{E^i}$ and $W^u_{E'^i}$. Finally we define a topological conjugacy between neighbourhoods U and U' of E^i and E'^i respectively, as follows. If $x \in U$, it can be expressed

as $x = (x_1, x_2)$ with $x_1 \in W^s_{E^i}$ and $x_2 \in W^u_{E^i}$. Hence, we define $h(x) = (h_1(x_1), h_2(x_2))$. In particular, this extends to a homeomorphism between two transversal sections of the flow $\Sigma \subset U$ and $\Sigma' \subset U'$
(see fig. 2), that we will use next.

Figure 2 Conjugation between the linear parts

The second step will be to use the homeomorphism h to obtain a homeomorphism between the equators of the manifolds $\widetilde{C}$ and $\widetilde{C}'$. We denote these equators, $\widetilde{C} \cap \{v = 0\}$ and $\widetilde{C}' \cap \{v = 0\}$ by Q and Q', respectively. Following the points of Σ by the flow, we obtain in Q a subset Ω diffeomorphic to Σ. Analogously we obtain a subset $\Omega' \subset Q'$ diffeomorphic to Σ' . We can define $H' : \Omega \to \Omega'$ an homeomorphism such that, if $p \in \Omega$ and $q = \varphi_{-t_p}(p) \in \Sigma$ and if $\varphi'_{t'_{h(q)}}(h(q)) \in \Omega'$ we define

$$H'(p) = (\varphi'_{t'_{h(q)}} \circ h \circ \varphi_{-t_p})(p) = p' \in \Omega'.$$

Looking now to the whole equator Q, we have four similar compact sets Ω_k, $k = 1, 2, 3, 4$, corresponding to the two branches of $W^u_{E^i}$ and the ones of $W^s_{E^s}$, respectively. They can be considered disjoints because Σ_k can be chosen small enough to satisfy this (recall that we do not have connection from the lower to the upper equilibrium points). Our goal is to extend the four homeomorphisms $H'_k : \Omega_k \to \Omega'_k$ to all points in Q.

Let R_i be the intersection point of the right branch of $W^u_{E^i}$ with Q, and L_i the one corresponding to the left branch. We denote by R_s, L_s the ones associated to $W^s_{E^s}$. Each one of these four points belong to one of the subsets Ω_k , $k = 1, 2, 3, 4$. These points also belongs to the equator of the collision manifold $Q_0 = Q \cap C$, which is diffeomorphic to S^1. Therefore, to define an homeomorphism between Q_0 and Q'_0, the location of these points on Q_0 must be congruent with the one in Q'_0. This location distinguishes one class of topological conjugacy from the other one. In figure 3 we display these two cases, each one with two different possibilities which are equivalent.

Figure 3 The two classes of topological conjugation

Now, assuming that both Q_0 and Q'_0 belong to the same class (I) or (II), displayed in figure 3, we can proceed with the construction of the homeomorphism between Q and Q'.

We will carry out this construction for the case Q_0 and Q'_0 in class (I), but of different type. We will suppose them to be parametrized by an angle θ such that the points of the invariant manifolds are associated to $\theta_n = (n-1)\frac{\pi}{2}$, $n = 1, 2, 3, 4$. Thus, for Q_0 we have that L_s is associated to θ_1, R_i to θ_2, R_s to θ_3 and L_i to θ_4. For Q'_0, in the case that we are considering, we have that R'_s is associated to θ_1, R'_i to θ_2, L'_s to θ_3 and L'_i to θ_4. The intersection of each Ω_k with Q_0 defines an arc containing these points. We denote these arcs by $A_k = [\theta_k - \delta_k, \theta_k + \delta_k]$, $k = 1, 2, 3, 4$. Analogously we denote by A'_k the arcs on Q'_0.

After the above considerations, we will first perform a homeomorphism between Q_0 and Q'_0. For that, we map each point of the invariant manifolds in Q_0 with the analogous in Q'_0. Using the parametrization, it can be expressed as a function f such that $f(\theta_1) = \theta_3$, $f(\theta_2) = \theta_2$, $f(\theta_3) = \theta_1$, $f(\theta_4) = \theta_4$. Hence, to preserve the continuity we need to do a symmetrical map with respect to the vertical axis of figure 3, that is, $f(\theta) = \pi - \theta$. We suppose the parametrization is chosen in such a way that $f(A_1) = A'_3$, $f(A_2) = A'_2$, $f(A_3) = A'_1$ and $f(A_4) = A'_4$. In this way we have constructed an homeomorphism between Q_0 and Q'_0.

Introducing now the radial component, we can consider the points of Q and Q' parametrized by the angle θ, introduced above, and by a radius ρ such that, $\rho = 0$ corresponds to points on Q_0. We use f to define the image of a generic point (ρ, θ) of Q, $F(\rho, \theta) = (\rho, f(\theta))$. Of course, for the points on Q_0, F coincides with f. Therefore, we have defined a continuous map F between Q and Q'. We can suppose the parametrization has been taken in such a way that F, restricted to the subsets Ω_k, coincides with the homeomorphism H'_k defined previously, but introducing the symmetry with respect to the vertical axis for the subsets Ω_2 and Ω_4. This can be thought as a change in the role of the invariant manifolds of the Binary points B_1^i and B_2^i.

However, this is not a problem for the continuity of our construction.

Now we need to extend the homeomorphism F to all the points in $\widetilde{C}$, for that, we use the flow transport. That way, for a point $x \in \widetilde{C}$ we denote by $\overline{x}$ the corresponding point in the equator, such that $\overline{x} = \varphi_{t_x}(x) \in Q$. Finally, we define a continuous map G between $\widetilde{C}$ and $\widetilde{C}'$ as $G(x) = (\varphi'_{-t_x} \circ F \circ \varphi_{t_x})(x)$. This map G is not, for the moment, a topological conjugacy. The problem are the points for which the associated orbit pass near the critical points, that is, for points x with $\overline{x} \in \Omega_k$ for some k. At these points, F takes the same values as H'_k and the image of a point $\overline{x}$ can spend different time going from Ω'_k to Σ'_k than the time spent by the original point $\overline{x}$ going from Ω_k to Σ_k. This problem can be overcome modifying slightly the definition of F in the points of Ω_k. Let us define $\tau_{\overline{x}} = t_{\overline{x}} - t'_{\overline{x}}$ where $t_{\overline{x}}$ is the time spent to go from Ω_k to Σ_k and $t'_{\overline{x}}$ the one corresponding to go from Ω'_k to Σ'_k. Hence, we introduce a new function $H''_k : \Omega_k \to \Omega''_k$ defined by $H''_k(\overline{x}) = (\varphi'_{-\tau_{\overline{x}}} \circ H'_k)(\overline{x})$. Thus, Ω'' will not be in general included in Q', but this is not essential.

Finally we can define a new function $\widetilde{F}$ in the equator that agrees with F in all the points not belonging to one of the subsets Ω_k, in which $\widetilde{F}$ will coincide with H''_k. Let U_k be an open set with $\Omega_k \subset U_k$, $k = 1, 2, 3, 4$. Using Urysohn Lemma, we know that there exists a continuous function $f_k : Q \to [0, 1]$ such that $f_k(x) = 1$ for $x \in \Omega_k$ and $f_k(x) = 0$ if $x \in Q \setminus U_k$. Now, if we define the function $\widetilde{\tau}_x = (f_1 + f_2 + f_3 + f_4)(t_{\overline{x}} - t'_{\overline{x}})$, where $\overline{x}$ is the point giving minimum distance from x to all the points $y \in \Omega_k$, $k = 1, 2, 3, 4$. Finally, we define the modification of the function F as follows, $\widetilde{F} = \varphi'_{-\tau_x} \circ F$, which fulfills the desired conditions. The topological conjugacy $\widetilde{G} : \widetilde{C} \to \widetilde{C}'$ can be defined as
$\widetilde{G}(x) = (\varphi'_{-t_x} \circ \widetilde{F} \circ \varphi_{t_x})(x)$.

The conjugacy between the other different cases of situation (II) on figure 3, can be constructed in an analogous way. ∎

Using the simplex of masses T, we can show the different regions where one can find the two classes of conjugacy. The boundaries of these regions are the connection curves between the collinear critical points. We may point out that the curves corresponding to connections between the two branches of the invariant manifolds simultaneously, make not change the class of conjugacy because they correspond to passages from one picture of figure 3 to the other, but in the same class.

An open question is if there is a topological conjugacy when there is at least one connection between the invariant manifolds. The construction seems to be much more complex than in the case we have studied, and probably we will need some more conditions to establish the conjugation.

References

1. McGehee, R., , Invent. Math. (1974), 191–227.

Figure 4 Regions of different class of topological conjugation in the simplex of masses

2. Simó, C., , Celestial Mech. (1980), 25–36.
3. Susín, A.; Simó, C., , Journal of Differential Equations vol 79 (1989), 168–177.