CONNECTIONS BETWEEN CRITICAL POINTS IN THE COLLISION MANIFOLD OF THE PLANAR 3-BODY PROBLEM

C.SIMÓ [1] AND A. SUSÍN [2]

[1] Dept. de Matemàtica Aplicada i Anàlisi , Universitat de Barcelona ,
Gran Via 585, 08007 Barcelona, Spain
[2] Dept. de Matemàtica Aplicada I , Universitat Politècnica de Catalunya,
Diagonal 647, 08028 Barcelona, Spain

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Abstract. We study the relative position of the invariant manifolds of the fixed points on the compactification of the so called non rotating triple collision manifold for the planar general three body problem. The results obtained allow to describe all the possible transition from the approach to triple collision to the escape from it. We also describe how these transitions change as a function of the masses of the three bodies in a domain of the space of masses and are completed by numerical simulations in the remainder set of masses.

1. Introduction.

We consider the three-body problem in $\mathbf{R}^3$ with bodies of positive masses m_1, m_2, m_3 which interact according to the mutual gravitational attraction. Let $q_i, p_i \in \mathbf{R}^3$, $i = 1,2,3$ the positions and momenta of the masses and denote by $q = (q_1^T, q_2^T, q_3^T)^T$, $p = (p_1^T, p_2^T, p_3^T)^T$ the vectors in $\mathbf{R}^9$ containing all the coordinates. The Hamiltonian of the system is

$$H(q,p) = \frac{1}{2} p^T M^{-1} p - U(q)\,, \tag{1}$$

where $M = diag(m_1, m_1, m_1, m_2, m_2, m_2, m_3, m_3, m_3)$ and

$$U(q) = \sum_{1 \le i < j \le 3} \frac{m_i m_j}{\| q_i - q_j \|} \qquad (\| \quad \| = \text{euclidean norm}).$$

Let Δ the set of collisions: $\Delta = \{q \in \mathbf{R}^9 : q_i = q_j \text{ for some } i, j\,, i \neq j\}$. Then the phase space of (1) is $(\mathbf{R}^9 - \Delta) \times \mathbf{R}^9$. The equations of motion are obtained from (1) as

$$\begin{aligned} \dot{q} &= M^{-1} p\,, \\ \dot{p} &= \nabla U(q)\,. \end{aligned} \tag{2}$$

They have well known first integrals: the energy integral (H), the center of mass integrals $\left(\sum_{i=1}^{3} m_i q_i, \sum_{i=1}^{3} p_i\right)$ and the angular momentum integral $\left(\sum_{i=1}^{3} q_i \wedge p_i\right)$. In all there are 10 scalar integrals in $\mathbf{R}^3$, 6 if we restrict to $\mathbf{R}^2$ and 3 if we restrict to $\mathbf{R}$.

Singularities appear in(2) if $q \in \Delta$. When approaching Δ the vector field (2) becomes unbounded. If $i, j, k \in \{1, 2, 3\}$ denote different indices then if $q_i = q_j \neq q_k$ the singularity is a double collision which is regularizable in any natural sense ([**3**], [**11**], [**15**]). If $q_1 = q_2 = q_3$ we are faced to a triple collision which is not regularizable ([**6**]). Triple collision requires zero angular momentum ([**16**]) and the motion should be planar. Even one can have very close approaches to triple collision in $\mathbf{R}^3$ from now on we restrict the study to the planar case.

If a singularity is regularizable and we consider initial coordinates leading to the singularity in finite (physical) time t_0 any small variation of the initial conditions, not leading itself to singularity, will produce a small change after passage near the singularity for time $t_1 > t_0$. If the singularity is not regularizable, small initial changes lead to big changes after passage close to the singularity. One losses the continuous dependence with respect to initial conditions at time $t_1 > t_0$.

The main objective of this work is to describe all the possible passages near triple collision when the total angular momentum is zero and how do they depend on the masses of the bodies.

In section 2 we write the equations of motion in some useful coordinates, and the collision manifold, C, and also the non-rotating collision manifold, N, are introduced. Then in section 3 we study the critical points of the flow on N and we give a first analysis of their invariant manifolds.

A useful remark is that the planar problem contain as subproblems two cases which have been widely studied: the collinear problem ([**6**], [**12**]) and the isosceles problem if two of the masses are equal ([**1**], [**2**], [**13**], [**14**]). These subproblems have lower dimensionality, they are easier to analyze and the information obtained is very useful for the planar case. Hence in section 4 we present the known connections ([**6**], [**13**]) between the critical points in these subproblems and some quantitative results useful for the planar case. Also the important results of Moeckel ([**8**]) concerning planar connections are given in this section.

Section 5 contains the main new results about the connections in the planar case. The main result is theorem 5.1 showing that for total angular momentum equal to zero, if two of the values of the masses are close enough all the possible passages from approach to triple collision to escape occur except the case of approaching and escaping in equilateral configuration. Some exceptional masses should be skipped. We recall that if the angular momentum is not zero Moeckel showed [**9**] that approaching and escaping in the same type of equilateral configuration is always possible.

Finally in section 6 we summarize numerical computations showing graphycal displays of the evolution of some curves introduced on section 5 and how the space of masses is partitioned by different behaviours. Part of this numerical information was already given in [**17**].

2. The equations of motion.

The system (2) in the planar case has dimension 12. A classical reduction to dimension 6 can be found in [**20**] but it destroyes the symmetry of the problem and it seems very difficult to use for our purposes. Another more symmetrical reduction was proposed by Waldvogel in [**19**] using ideas of Murnaghan [**10**] and Lemaître [**5**] and combining them with the blow up method of McGehee [**6**]. We summarize the changes of coordinates.

First we consider $q_j \in \mathbf{R}^2$, $j = 1, 2, 3$ as complex numbers ([**10**]). It is not restrictive to consider that the origin is located at the center of masses of the system. Let

$$q_\ell - q_k = a_j e^{i\varphi_j}, \tag{3}$$

where $a_j \in \mathbf{R}$, $a_j \geq 0$, $\varphi_j \in [0, 2\pi]$ and (j, k, ℓ) is $(1, 2, 3)$ or a cyclical permutation.

Then let $\varphi = \frac{1}{3}(\varphi_1 + \varphi_2 + \varphi_3)$ and

$$p_j = P_k e^{i\varphi_k} - P_\ell e^{i\varphi_\ell} + \frac{P_\varphi}{3} i \left(\frac{1}{a_k} e^{i\varphi_k} - \frac{1}{a_\ell} e^{i\varphi_\ell} \right), \tag{4}$$

P_φ and P_j , $j = 1, 2, 3$ being real variables.

The new variables a_1, a_2, a_3 (that is the sides of the triangle formed by the three masses at a given time t), φ (which is related to the rotation of the triangle) and the associated momenta P_1, P_2, P_3, P_φ , introduced in (3), (4) allow to recover the old ones.

The momentum P_φ turns out to be the angular momentum of the system and it is equal to zero by hypothesis. The associate coordinate φ is a cyclic variable and its time evolution can be recovered from the evolution of the remaining variables (elimination of the node).

The second step is the regularization of the binary collisions (**[5]**). To this end we introduce new variables $\widetilde{\alpha}_j$, π_j , $j=1,2,3$ and a new time $\widetilde{\tau}$ by

$$
(5)\qquad
\begin{aligned}
a_j &= \widetilde{\alpha}_k^2 + \widetilde{\alpha}_\ell^2 \,, \\
P_j &= \frac{1}{4}\left(-\frac{\pi_j}{\widetilde{\alpha}_j} + \frac{\pi_k}{\widetilde{\alpha}_k} + \frac{\pi_\ell}{\widetilde{\alpha}_\ell}\right) , \\
dt &= a_1 a_2 a_3 d\widetilde{\tau} \,,
\end{aligned}
$$

where (j,k,ℓ) is again the triplet $(1,2,3)$ or a cyclical permutation of it.

The final step is a blow up (**[6]**) introducing the moment of inertia of the system, I, new coordinates α_j , $j=1,2,3$ and a new time τ but keeping the π_j variables:

$$
(6)\qquad
\begin{aligned}
r^2 &= I = \Sigma m_j \parallel q_i \parallel^2 , \\
\alpha_j &= \widetilde{\alpha}_j r^{-1/2} , \quad j = 1,2,3 , \\
d\tau &= r^{3/2} d\widetilde{\tau} \,.
\end{aligned}
$$

From (2) and using the changes (3–6) it is easy to obtain the equations for $\dfrac{dr}{d\tau}$, $\dfrac{d\alpha_j}{d\tau}$, $\dfrac{d\pi_j}{d\tau}$, $j=1,2,3$. These equations are analytical for $r>0$ and can be extended analytically to $r=0$. The set $r=0$ is called the triple collision manifold with regularized binary collisions, C . From (5) it follows that several points in this manifold should be identified. More concretely, each set $(\alpha_1,\alpha_2,\alpha_3,\pi_1,\pi_2,\pi_3)$ has to be identified with the one which one obtains by changing the sign of two of the α's (say α_j, α_k) and, simultaneously, the sign of π_j , π_k .

We remark that for general values of P_φ (not forcely zero) one should scale also this variable introducing $\widetilde{P}_\varphi$ by $P_\varphi = \widetilde{P}_\varphi r^{1/2}$. For $r=0$ one has $P_\varphi = 0$ but $\widetilde{P}_\varphi$ needs not to be zero, and this is a variable which also appears in C . As the variables α, π satisfy the constraints

$$
(7)\qquad \sum_{1\le j<k\le 3} m_j m_k (\alpha_j^2 + \alpha_k^2)^2 = \sum_{i=1}^{3} m_i \quad \text{(normalization)}
$$

and

$$
(8)\qquad K(\alpha,\pi) = 0 \quad \text{(energy integral)},
$$

the manifold C is 5-dimensional.

The differential equation for $\widetilde{P}_\varphi$ shows that $\widetilde{P}_\varphi = 0$ is invariant. This defines on C a 4-dimensional invariant submanifold called the non-rotating collision manifold, N, already introduced by Waldvogel in [**18**]. If we consider motion in the physical space $(r > 0)$ with $P_\varphi = 0$ one has also $\widetilde{P}_\varphi = 0$. Then it is enough to restrict the study to N if we would learn about passages near triple collision with zero angular momentum.

The final equations on N are almost Hamiltonian, given by

$$\begin{aligned} \frac{d\alpha}{d\tau} &= \frac{\partial}{\partial\pi}K(\alpha,\pi) - \frac{1}{4}z\alpha\,, \\ \frac{d\pi}{d\tau} &= -\frac{\partial}{\partial\pi}K(\alpha,\pi)\,, \end{aligned} \tag{9}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$, $\pi = (\pi_1, \pi_2, \pi_3)^T$, $z = (\alpha_1^2+\alpha_2^2)(\alpha_2^2+\alpha_3^2)(\alpha_3^2+\alpha_1^2)v$, v being the inner product $v = (\alpha, \pi)$, and K is the function

$$K(\alpha,\pi) = \frac{1}{8}\pi^T B(\alpha)\pi - \sum_{1\le j<k\le 3} m_j m_k (\alpha_{j+1}^2 + \alpha_{j+2}^2)(\alpha_{k+1}^2 + \alpha_{k+2}^2)\,, \tag{10}$$

where $\alpha_s = \alpha_{s-3}$ if $s > 3$. The components of the symmetric matrix $B(\alpha)$ which appear in (10) are

$$\begin{aligned} b_{jj}(\alpha) &= \frac{\alpha_k^2+\alpha_\ell^2}{m_j}(\alpha_1^2+\alpha_2^2+\alpha_3^2) + \frac{\alpha_j^2+\alpha_\ell^2}{m_k}\alpha_\ell^2 + \frac{\alpha_j^2+\alpha_k^2}{m_\ell}\alpha_k^2\,, \\ b_{jk}(\alpha) &= -\frac{\alpha_j^2+\alpha_k^2}{m_\ell}\alpha_j\alpha_k\,. \end{aligned} \tag{11}$$

As before, in (11) the indices (j,k,ℓ) are $(1,2,3)$ or a cyclic permutation of it.

On N the variable v is gradient like $(\frac{dv}{d\tau} \geq 0)$ and it increases everywhere except at the critical points of (9). We remark also that the equations (9) are polynomial in (α, π).

3. Critical points and the geometry of the collision manifold.

As it is known ([**2**], [**19**]) the critical points of (9) are related to the central configurations, that is, going back to (2), if the origin is placed at the center of masses, one has $\ddot{q}_j(t) = -\lambda(t)q_j(t)$ for all time. Taking into account the identifications there are 10 equilibrium points, 5 of them with $v > 0$ and the other 5 with $v < 0$. For $v > 0$ (resp. $v < 0$) there are 2 equilateral or Lagrange configurations L^s_+, L^s_- (resp. L^i_+, L^i_-). In the case of L^s_+ (resp. L^s_-) the position of m_2 is obtained from the position of m_1 by a rotation of $\frac{2\pi}{3}$ (resp. $-\frac{2\pi}{3}$) with respect to the center of the triangle.

The other 3 configurations for $v > 0$ (resp. $v < 0$) are the so called collinear or Euler configurations E^s_j (resp. E^i_j), with the mass m_j located between the other two masses, for $j = 1, 2, 3$. From (9) it follows that the critical points satisfy $B(\alpha)\pi = z\alpha$ and the related value of v is $v = \pm(8\mathcal{U})^{1/2}$, where
$\mathcal{U} = \sum_{1\le k\le \ell\le 3} \dfrac{m_k m_\ell}{\alpha_k^2 + \alpha_\ell^2}$.

The value of v at $L^s_{+,-}$ is less than the value at any of the Euler points. This will be useful later.

The critical points are hyperbolic ([**2**]) and the dimensions of the invariant manifolds are given in table I

	$L^i_{+,-}$	$L^s_{+,-}$	E^i_j	E^s_j	B^i_j	B^s_j
$\dim W^u$	2	2	3	1	4	0
$\dim W^s$	2	2	1	3	0	4

Table I

This table contains also information on some points $B^{i,s}_j$ to be described later.

At $L^i_{+,-}$ the positive eigenvalues are

$$\frac{3}{4} \mid v \mid (\alpha_1^2 + \alpha_2^2)(\alpha_2^2 + \alpha_3^2)(\alpha_3^2 + \alpha_1^2)\sigma_{1,2} \, , \tag{12}$$

where $\sigma_{1,2} = \frac{1}{6}((13 \pm 12(1-3\mu)^{1/2})^{1/2} - 1)$. Here μ denotes the ratio $(m_1m_2 + m_2m_3 + m_3m_1)/(m_1 + m_2 + m_3)^2$.

Before going to the connections between the critical points we describe the geometry of N . The configuration space, i.e., the allowed values of $(\alpha_1, \alpha_2, \alpha_3)$, is given by (7) with the additional constrains and the symmetries allow to consider only $\alpha_2 \ge 0$, $\alpha_3 \ge 0$. But then also points like $(\alpha_1, \alpha_2, 0)$ (resp. $(\alpha_1, 0, \alpha_3)$) should be identified to $(-\alpha_1, \alpha_2, 0)$ (resp. $(-\alpha_1, 0, \alpha_3)$).

Therefore we have a two dimensional sphere, were the points on the equator denote the collinear configurations for which at least one of the α's is zero. This includes the binary collisions B_j (for which two α's are zero) and the Euler configurations as special cases. The points on the northern (resp. southern) hemispheres correspond to triangles with direct (resp. retrograde) orientation when one goes from m_1 to m_2 and then to m_3. The poles can be thought as the Lagrange points (see fig. 1 and de § 5 for the meaning of the curve S). A similar representation can be found in [**4**].

Figure 1

To obtain the non-rotating triple collision manifold we put in each point of the configuration space a fiber which is S^2 unless we are at B_j , $j = 1, 2, 3$ where we put a cylinder. This follows from (8). Then we obtain $S^2 \times S^2$ with 6 holes (after the usual identification). If we compactify the holes by adding a point to each one we get the compact manifold $\overline{N} = S^2 \times S^2$. The holes in the region $v > 0$ (resp. $v < 0$) are denoted by B_j^s (resp. B_j^i), $j = 1, 2, 3$ and they can be considered as "hard" binaries ([**7**]) where the two bodies forming the binary are really at the same point. It is natural to say that $\dim W^u_{B^i_j} = \dim W^s_{B^s_j} = 4$, as already pointed out in Table I. In $\overline{N}$ the set $v = 0$ is topologically equivalent to S^3 .

4. Connections between critical points. I.

In this section we describe some results already known and some preliminary results to be used in the next section. First we recall that the masses m_1 , m_2 , m_3 , can be normalized in such a way that the sume is equal to 1. Then the values of the positive masses can be thought as the barycentric coordinates of a point of a triangle. We refer loosely to this triangle as the triangle of masses. First we give some results for the subproblems and also for the planar problem.

(a) Collinear problem ([**6**]).

We consider the mass m_3 located between the masses m_1 and m_2 . In this case $C = N$ and topologically it is equivalent to a sphere with 4 holes. They can be filled up by compactification, what is equivalent to add the hard binaries. In C two critical points appear (one in $v < 0$, the other on $v > 0$) which are related to the Euler equilibrium points $E_3^{i,s}$ (see fig. 2). The following results

Figure 2

are known ([**6**], [**12**]):

(i) For all masses one branch of $W^u_{E_3^s}$ goes to B_1^s , the other to B_2^s .

(ii) For an open dense set of masses one branch of $W^u_{E_3^i}$ goes to B_1^s , the other to B_2^s .

(iii) For a codimension 1 set of masses one branch of $W^u_{E_3^i}$ goes to E_3^s and the other either to B_1^s or to B_2^s .

(iv) For a codimension 1 set of masses (roughly one half of the set of case (iii)) both branches of $W^u_{E_3^i}$ go to E_3^s .

Symmetrical results hold for the branches of $W^s_{E_3^i}$, $W^s_{E_3^s}$.

(b) Isosceles problem ([**1**], [**13**], [**14**] and references therein).

We consider $m_1 = m_2$, $m_3/m_1 = \varepsilon$ and initial positions and velocities such that the form of the triangle spanned by the 3 masses is isosceles for any time. That is, the coordinates of q_1 , q_2 , q_3 are of the form $(-x(t), -\frac{\varepsilon}{2}y(t))$, $(x(t), -\frac{\varepsilon}{2}y(t))$, $(0, y(t))$, respectively. Now $N = C$ is again a sphere with 4 holes that can be compactified as in (a). But there are 6 critical points (3 for $v < 0$, 3 for $v > 0$) related to the Euler configuration and to the two (direct and retrograde) Lagrange configurations. All of them are hyperbolic.

According to the possible connections between the invariant manifolds of the critical points there are 5 cases (I to V).

Cases I, III and V are shown in figure 3.

Figure 3

Case II and IV are bifurcation cases. For case II exactly one branch of $W^u_{L^i_+}$ coincides with one of $W^s_{L^s_+}$ (and by symmetry one of $W^u_{L^i_-}$ with one of $W^s_{L^s_-}$) and for case IV exactly one branch of $W^u_{L^i_+}$ coincides with one $W^s_{L^s_-}$ (and one of $W^u_{L^i_-}$ with one of $W^s_{L^s_+}$, again by symmetry).

The existence of values of ε realizing the 5 cases has been proved in [**13**]. There is numerical evidence that cases II and IV occur for only one value of ε : $\varepsilon = \varepsilon_1 (\simeq 0.378532\ldots)$ and $\varepsilon = \varepsilon_2 (\simeq 2.661993\ldots)$.

Then, by analizing the limit cases $\varepsilon \searrow 0$, $\varepsilon \nearrow \infty$ one has from this numerical evidence:

(i) Case I occurs for $0 < \varepsilon < \varepsilon_1$.

(ii) Case III occurs for $\varepsilon_1 < \varepsilon < \varepsilon_2$.

(iii) Case V occurs for $\varepsilon > \varepsilon_2$.

It is desirable to have a proof of what happens when $m_1 = m_2 = m_3$, ($\varepsilon = 1$).

Lemma 4.1. *If the three masses are equal then we are in case III.*

Proof: By using the symmetries it is enough to see at what point the branches of $W^u_{L^i_+}$ reach the level $v = 0$. Taking into account the equation of N the differential equation on the triple collision manifold ([**1**], [**13**]) can be put in the form

$$\frac{dv}{d\theta} = \pm\sqrt{\frac{\sqrt{2}}{4\cos\theta} + \frac{2}{(2+4\sin^2\theta)^{1/2}} - \frac{v^2}{4}} \tag{13}$$

where the sign should be changed when one reaches $\theta = \pm\frac{\pi}{2}$. The starting point L^i_+ , has coordinates $v = -\sqrt{6}$, $\theta = \frac{\pi}{4}$, one branch of $W^u_{L^i_+}$ starting with positive sign in (13), the other with negative one.

One should show that the right (left) branch reaches $\theta = \frac{\pi}{2}$ ($\theta = -\frac{\pi}{2}$) for some $v < 0$ and then reaches $v = 0$ for some $\theta > 0$ ($\theta < 0$). First we proceed with the right branch (the easiest case). For $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$ one has from (13)

$$\frac{dv}{d\theta} = \frac{1}{\sqrt{\cos\theta}}\sqrt{\frac{\sqrt{2}}{4} + \frac{2\cos\theta}{(2+4\sin^2\theta)^{1/2}} - \frac{v^2}{4}\cos\theta} \leq \sqrt{\frac{3\sqrt{2}}{4}\frac{1}{\cos\theta}}. \tag{14}$$

By integration of (14) we obtain

$$v(\theta = \frac{\pi}{2}) - (-\sqrt{6}) < \sqrt{\frac{3\sqrt{2}}{4}}\int_{\pi/4}^{\pi/2}\frac{d\theta}{\sqrt{\cos\theta}} =$$

$$= \sqrt{\frac{3\sqrt{2}}{4}}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\sin\theta}} < \sqrt{\frac{3\pi}{8}}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\theta}} = \pi\sqrt{\frac{3}{8}},$$

where we have used $\sin\theta \geq \frac{\sqrt{8}}{\pi}\theta$ in $[0,\pi/4]$. Hence $v(\theta = \frac{\pi}{2}) < \pi\sqrt{\frac{3}{8}} - \sqrt{6} < 0$, as desired.

To finish the proof for the right branch we note first that $v(\theta = \frac{3\pi}{8}) > -\sqrt{6}$. Going from $\theta = \frac{3\pi}{8}$ to $\theta = \frac{\pi}{2}$ we can bound the vector field by

$$\frac{dv}{d\theta} > \sqrt{\frac{\sqrt{2}}{4\cos\theta} - \left(\frac{6}{4} - \frac{2}{(4+\sqrt{2})^{1/2}}\right)}.$$

Hence

$$v(\theta = \frac{\pi}{2}) > -\sqrt{6} + \int_{3\pi/8}^{\pi/2}\frac{dv}{d\theta} > -\sqrt{6} + \int_0^{\pi/8}\sqrt{\frac{\sqrt{2}}{4\theta} - b},$$

where b stands for $\frac{3}{2} - \frac{2}{(4+\sqrt{2})^{1/2}}$. Using

$$\int\sqrt{\frac{a}{x} - b} = \sqrt{x(a-bx)} + \frac{a}{\sqrt{b}}\arctan\sqrt{\frac{bx}{a-bx}},$$

one has

$$v(\theta = \frac{\pi}{2}) > -\sqrt{6} + \sqrt{\frac{\pi}{8}\left(\frac{\sqrt{2}}{4} - b\frac{\pi}{8}\right)} + \frac{\sqrt{2}/4}{\sqrt{b}} \arctan\sqrt{\frac{b\pi/8}{\sqrt{2}/4 - b\pi/8}} > -1.80593\,.$$

We introduce $v_1 = -1.80593$. Now θ goes from $\frac{\pi}{2}$ to the left. As before we have in $\left[\frac{3\pi}{8}, \frac{\pi}{2}\right]$

$$\left|\frac{dv}{d\theta}\right| > \sqrt{\frac{\sqrt{2}}{4\cos\theta} + \left(\frac{2}{(4+\sqrt{2})^{1/2}} - \frac{v_1^2}{4}\right)}\,.$$

Then $v(\theta = \frac{3\pi}{8}) > v_1 + \int_0^{\pi/8} \sqrt{\frac{\sqrt{2}}{4\theta} + b}$, where now b stands for $\frac{2}{(4+\sqrt{2})^{1/2}} - \frac{v_1^2}{4} > 0$. Using

$$\int \sqrt{\frac{a}{x} + b} = \sqrt{(a+bx)x} + \frac{a}{\sqrt{b}} \log(\sqrt{b(a+bx)} + b\sqrt{x})\,, \tag{15}$$

one has

$$v(\theta = \frac{3\pi}{8}) >$$

$$v_1 + \sqrt{\left(\frac{\sqrt{2}}{4}b + \frac{\pi}{8}\right)\frac{\pi}{8}} + \frac{\sqrt{2}/4}{\sqrt{b}}\left[\log\left(\sqrt{b\left(\frac{\sqrt{2}}{4} + b\frac{\pi}{8}\right)} + b\sqrt{\frac{\pi}{8}}\right) - \log\sqrt{\frac{\sqrt{2}}{4}b}\right] >$$

$$-1.05466 = v_2\,.$$

Between $\theta = \frac{3\pi}{8}$ and $\theta > 0$ one can use the simple bound $\left|\frac{dv}{d\theta}\right| > \sqrt{\frac{3}{2} - \frac{v^2}{4}}$. Then

$$v(\theta = 0) > \sqrt{6}\sin\left[\frac{3\pi}{16} + \arcsin\frac{v(\theta = 3\pi/8)}{\sqrt{6}}\right] > \sqrt{6}\sin\left(\frac{3\pi}{16} + \arcsin\frac{v_2}{\sqrt{6}}\right) > 0.35134\,.$$

Therefore the right branch of $W^u_{L^i_+}$ reaches $v = 0$ before θ reaches the value 0 .

Since for the same value of θ the left branch of $W^u_{L^i_+}$ has a value of v larger than the left branch of $W^u_{L^i_-}$, and using the symmetry, we have that the left branch of $W^u_{L^i_+}$ reaches $v = 0$ for some negative value of θ . The only thing to prove is that it reaches $\theta = -\frac{\pi}{2}$ for some negative value of v .

We write (13) as $\left|\frac{dv}{d\theta}\right| = \sqrt{a(\theta) - \frac{v^2}{4}}$, where $a(\theta) = \frac{\sqrt{2}}{4\cos\theta} + \frac{2}{(2+4\sin^2\theta)^{1/2}}$.

In an interval $J_n = [\theta_n, \theta_{n-1}]$ we define $a_n = \max_{\theta \in J_n} a(\theta)$. Then $\left|\frac{dv}{d\theta}\right| < \sqrt{a_n - \frac{v^2}{4}}$ and therefore

$$v(\theta = \theta_n) < 2\sqrt{a_n} \sin\left(\frac{1}{2} \mid \theta_n - \theta_{n-1} \mid + \arcsin\left(\frac{v(\theta = \theta_{n-1})}{2\sqrt{a_n}}\right)\right) . \tag{16}$$

Due to the behaviour of $a(\theta)$ one has $a_n = a(\theta_n)$ (resp. $a_n = a(\theta_{n-1})$) if $J_n \subset [0, \pi/4] \cup [-\pi/2, -\pi/4]$ (resp. $J_n \subset [-\pi/4, 0]$). Starting at $v_0 = -\sqrt{6}$, $\theta_0 = \frac{\pi}{4}$ and using $\theta_n - \theta_{n-1} = -\frac{\pi}{48}$ for $n = 1, \ldots, 32$ the iterated application of (16) gives $v_{32} = v(\theta = -\frac{5\pi}{12}) < -0.6738$. The computations are done with a pocket calculator taking into account the rounding error.

Now we consider the interval $J = \left[-\frac{\pi}{2}, -\frac{5\pi}{12}\right]$. Here we use the bound

$$\left|\frac{dv}{d\theta}\right| < \sqrt{\frac{\sqrt{2}}{4\cos\theta} + \frac{2}{(2 + 4\sin^2(5\pi/12))^{1/2}}} < \tag{17}$$
$$< \sqrt{\frac{1}{\theta + \pi/2} \, \frac{\sqrt{2}\pi/12}{4\sin(\pi/12)} + \frac{2}{(2 + 4\sin^2(5\pi/12))^{1/2}}} ,$$

where we have used $\cos\theta = \sin(\theta + \frac{\pi}{2}) > (\theta + \frac{\pi}{2})\frac{\sin(\pi/12)}{\pi/12}$ in J . The bound (17) is of the form $\sqrt{\frac{\alpha}{x} + \beta}$ with $x = \theta + \frac{\pi}{2}$, $x \in [0, \frac{\pi}{12}]$. The integral has been already given in (15). Suppose $v(\theta = -\frac{\pi}{2}) = 0$. Then, by integration of (16) one obtains $v(\theta = -\frac{5\pi}{12}) > -0.6696$. This is in contradiction with the upper bound of v_{32} found before. Hence $v(\theta = -\frac{\pi}{2}) < 0$. ∎

c) Planar problem ([**8**]).

The following connections are known to exist:

i) For all masses there are orbits connecting $L^s_{+,-}$ with E^s_i and B^s_i , $i = 1, 2, 3$. Symmetrical connections occur in the $v < 0$ region. All these connections are transversal.

ii) If there is an orbit connecting E^i_j with L^s_+ transversally, then there are also orbits connecting E^i_j to L^s_- and to E^s_k , B^s_k , $k = 1, 2, 3$. A similar statement is obtained by symmetry by changing the sign of v .

To prove these results one needs a lemma which will be also used in next section. First we introduce some definitions.

Let $c = v(L^s_{+,-})$, $N_\lambda = \{(\alpha,\pi) \in N : v(\alpha,\pi) \geq \lambda\}$, which is a positively invariant set, and let S_λ the projection of N_λ on the configuration space: $S_\lambda = \text{proj } N_\lambda$. The set $S_{c+\varepsilon}$ is S^2 minus two small neighbourhoods of the poles if ε is small.

LEMMA 4.2. *([8]). Let* $\alpha : S^1 \longrightarrow N_{c+\varepsilon}$ *be an analytical closed curve such that proj* $\alpha : S^1 \longrightarrow S_{c+\varepsilon}$ *is not contractible in* $S_{c+\varepsilon}$ *. Then* $\alpha(S^1)$ *intersects the stable manifolds of* E^s_i , B^s_i *for* $i = 1,2,3$.

The figure 4 summarizes the known connections in the $v > 0$ region (similar ones exist on $v < 0$). As stated on Table I $\dim W^u_{E^s_j} = 1$, $j = 1,2,3$. These manifolds end on B^j_k , $k \neq j$ and are already found in the linear problem

Figure 4 Figure 5

The connections $E^s_j \longrightarrow B^s_k$ consist of just one orbit. It is not known if the connections $L^s_{+,-} \longrightarrow E^s_j$ are made of just one orbit. Assuming that this is true one can obtain the topological structure of $W^s_{P^s} \cap \{v = d\}$, for different values of d where P^s stands for one of the critical points $L^s_{+,-}$, E^s_j or B^s_j . Let $v_2 > \max v(E^s_j)$, $0 < v_1 < \min v(E^s_j)$, $v_0 = 0$. Table II gives the topology of $W^s_{P^s} \cap \{v = d\}$ for the different P^s and $d = v_i$, $i = 0,1,2$.

$d \backslash P^s$	B^s_j	E^s_j	$L^s_{+,-}$
v_2	S^3	$--$	$--$
v_1	$S^2 \times (0,1)$	S^2	$--$
v_0	$E^2 \times S^1$	$S^1 \times (0,1)$	S^1

Table II

In particular, for $v = 0$ one has that $\overline{N} \cap \{v = 0\} = S^3$ has been partitioned in 3 open solid tori going to the hard binaries. Each one of the three two-dimensional tori in the boundary of the three $E^2 \times S^1$ is the union of two S^1 and two open cylinders $S^1 \times (0,1)$. The cylinders go to E^s_j and the circles to $L^s_{+,-}$ (see fig. 5). Similar results are obtained for $W^u_{P^i} \cap \{v=0\}$ .The full description of the possible transitions close to triple collision would be given by the intersections of the type $W^u_{P^i} \cap \{v=0\} \cap W^s_{Q^s}$, P^i (resp. Q^s) being one of the critical points on $v<0$ ($v>0$) .

The next section gives a partial answer to this question.

5. Connections between critical points II.

We consider $W^u_{L^i_+} \cap \{v=-c+\varepsilon\}$, $c=v(L^s_+)$, ε positive small enough. This set is topologically equivalent to S^1 . Then we study the evolution of the curve $\mathcal{C}_k$ obtained by projecting in the configuration space S^2 (see fig. 1) the set of N defined by $W^u_{L^i_+} \cap \{v=k\}$ when k ranges from $-c+\varepsilon$ to $c+\varepsilon$. In order that this projection be always homeomorphic to S^1 we require, first, that there are not connections $L^i_+ \longrightarrow L^s_{+,-}$.

First of all we consider the case $m_1=m_2$. Let S be the meridian of the figure 1 passing through L_+ , B_3 , L_- , E_3 .

Then, due to the symmetry of (9) $\mathcal{C}_k$ is symmetrical with respect to S for any k . For $k=-c+\varepsilon$, $\mathcal{C}_k$ is a small circle around L_+ . The curve S intersects $\mathcal{C}_{-c+\varepsilon}$ in just two points, R (right) and L (left), R between L_+ and B_3 , and L between L_+ and E_3 . Due to the symmetry, when k increases the projections on S^2 of the points obtained by flow transport on N of the preimages r and ℓ of R and L are still on S , and they correspond to the evolution of the right and left branches of the unstable manifold of L^i_+ on the isosceles problem. For this problem one always has $\alpha_1=\alpha_2$, $\pi_1=\pi_2$.

Let $t \in [0,2\pi]$ a parameter of $\mathcal{C}_k$. We denote by $\mathcal{C}_k(t)$ the point of $\mathcal{C}_k$ with parameter t . It can be choosen in such a way that $t=0$ ($t=\pi$) corresponds to the point obtained by evolution on $R(L)$ when k increases and points with parameter t and $2\pi-t$ are symmetrical with respect to S for all k . The parameter t is also choosen such that when $k=-c+\varepsilon$ the curve $\mathcal{C}_k(t)$, $t\in[0,2\pi]$ has index $+1$ around L_+ . It is possible (in fact it occurs, see § 6) that for some k there are points $\mathcal{C}_k(t_0) \in S$, $t_0 \neq 0,\pi$. However then also $\mathcal{C}_k(2\pi-t_0)\in S$ and, again by the symmetry, the vectors $\frac{d}{dt}\mathcal{C}_k\big|_{t=t_0}$, $\frac{d}{dt}\mathcal{C}_k\big|_{t=2\pi-t_0}$ point towards the same of the two hemispheres defined by S . Of course the vectors $\frac{d}{dt}\mathcal{C}_k\big|_{t=0,\pi}$ are always ortogonal to S , except when the related points coincide with B_3 where these vectors become zero. A local analysis shows that if for some k one has $\mathcal{C}_k(0) = B_3$ then $\frac{d}{dt}\mathcal{C}_{k_1}(0)$ and $\frac{d}{dt}\mathcal{C}_{k_2}(0)$, where $k_1 \in (k-\delta, k)$, $k_2 \in (k, k+\delta)$, point to opposite hemispheres. But this fact will not be used here.

Now we look at the final positions of R and L on S when $k = c + \varepsilon$ in the cases I, III and IV of the isosceles problem (§ 4, b). The situation is depicted on fig. 6, where the positions of R and L are shown on S .

Figure 6

In cases I and V, and taking into account that extra crossings of the curve S at $L_{+,-}$ for some $k \in (-c+\varepsilon, c+\varepsilon)$ and $t_0 \neq 0, \pi$ change the index with respect L_+ or L_- in multiples of 2 , the final curve has odd index and therefore is not contractible.

In the case III we use the fact that it contains the case of equal masses (Lemma 4.1). Then 3 isosceles problems (obtained by $\pm\frac{2\pi}{3}$ rotation in the configuration space) are included as subproblems. By symmetry the changes of index should occur in multiples of 3. Hence, also the final curve is not contractible.

This implies that there are connections from L^i_+ to E^s_j , B^s_j , $j = 1, 2, 3$. If a connection $L^i_+ \longrightarrow L^s_{+,-}$ occurs, we can deformate the initial curve on $v = -c + \varepsilon$ slightly out of $W^u_{L^i_+}$ and the same conclusion follows with the eventual exception that the connection $L^i_+ \longrightarrow E^s_j$ can fail. Similar connections occur starting at L^i_- .

From the previous facts and (ii) in§ 4, c) it follows that for $m_1 = m_2$ if there are not connections of the type $L^i_{+,-} \longrightarrow L^s_{+,-}$ the connections given in table III occur.

	B^i_k	E^i_k	$L^i_{+,-}$	$L^s_{+,-}$	E^s_k	B^s_k	
B^i_j		$\times\,(*)$	$\times$	$\times$	$\times$	$\times$	$j,k=1,2,3$
E^i_j			$\times$	$\times$	$\times$	$\times$	$(*)\,k \neq j$
$L^i_{+,-}$					$\times$	$\times$	

Table III

Continuing with $m_1 = m_2$, for the masses of case II of the isosceles problem, i.e., $\varepsilon = \varepsilon_1$ or, equivalently, $m_1 + m_2 + m_3 = 1$, $m_3 = 0.159145\ldots$, there are isosceles connections $L^i_+ \longrightarrow L^s_+$, $L^i_- \longrightarrow L^s_-$. For the masses of case IV of the isosceles problem, i.e., $\varepsilon = \varepsilon_2$ or, equivalently $m_3 = 0.570999\ldots$, there are isosceles connections $L^i_+ \longrightarrow L^s_-$, $L^i_- \longrightarrow L^s_+$. Of course, it is not excluded that for $m_1 = m_2$ connections between L^i_+ or L^i_- and L^s_+ or L^s_- occur which are not contained in the isosceles subproblem. Numerical simulations show that there are two symmetrical (with respect to S say), non isosceles connections of each one of the types $L^i_+ \longrightarrow L^s_+$ and $L^i_- \longrightarrow L^s_-$ for $m_1+m_2+m_3 = 1$, $m_3 = 0.632911\ldots$. There is some numerical evidence that these are the only non isosceles connections when two of the masses are equal.

The index of the curve $\mathcal{C}_{c+\varepsilon}$ is a locally continuous constant function of the masses. We summarize the result obtained as follows.

THEOREM 5.1. *If two of the values of the masses of the planar three body problem are close enough and there are not connections from the equilateral points on $v<0$ to the equilateral ones on $v>0$ then the connections of table III occur.*

As a consequence for the masses for which 5.1 holds any type of approach to collision can give rise to any type of ejection, except the approaches and escapes which are simultaneously of equilateral type.

6. Some numerical simulations.

To show the evolution of $\mathcal{C}_v$ when v changes increasing from $-c+\varepsilon$ we show in figure 7 the projection of the curves contained on S^2 (the configuration space, see

Figure 7

fig. 1) into the equilateral plane, for the case of equal masses $m_1 = m_2 = m_3 = 1/3$. In this case $c = 1.2409\ldots$ and figure 7 shows the behaviour for a selection of values of v , the last one with $v > c$.

Each one of the plots contains the position of R and L , the orientation of $\mathcal{C}_v$ and the value of the index (when it is defined). The projection is shown as a continuous curve in the arcs contained in the northern hemisphere (the one which contains L_+) and as a dotted curve in the arcs contained in the southern one. We remark that the final index is -2 . In particular this implies that there are at least two connections from L_+^i to each one of the E_j^s , $j = 1, 2, 3$.

The complexity of $\mathcal{C}_k$ when k reaches values slightly larger than $v(L_{+,-}^s)$ even for the case $m_1 = m_2$ is displayed in figure 8 for $m_1 + m_2 + m_3 = 1$ and the following values of $m_3 : 0.1, 0.2, 0.4, 0.6$ and 0.7 . These curves are given always as dotted curves disregarding whether the corresponding point is in the northern or southern hemisphere. The points R and L as well as the final index are also shown. The numbers along the curves allow to follow how the curve is described when the parameter t (see § 5) increases from 0 to 2π . The point R always correspond to the initial and final numbers.

To complete the study for all the masses one should check that there are not masses such that the final index of $\mathcal{C}_k$ (when it is defined) around $L_{+,-}$ is zero. The numerical evidence that we have at this moment using a mesh on the space of masses shows that there are not such masses, but a finer mesh would be desirable and it is being done.

It is also interesting to display the curves on the triangle of masses for which there are connections between equilateral points. Furthermore these curves separate regions with different final index of $\mathcal{C}_k$. Using a continuation method starting at the masses for which these connections occur in the isosceles problem we have obtained so far the results shown in figure 9. In this figure we display only one sixth of the full triangle of masses, the other ones being obtained by symmetry. The masses (m_1, m_2, m_3) associated to each vertex are given and also the final index in each component of the complementary of the curves for which an equilateral connection occurs.

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Figure 8

Figure 8 (Continuation)

Figure 9

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