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Central Configurations in the Charged Three Body Problem

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. In this paper we find the exact number and type of Central Configurations in the general charged three body problem for any set of parameters (masses and charges). Because of the change of the forces, the number of Central Configurations will be different from the general Newtonian three body problem. Analyzing all the possible changes of sign of the forces, we obtain the different numbers of Central Configurations.

1. Introduction

The Newtonian n-body problem concerns the motion of n point particles with masses $m_j \in R^+$ and positions $\vec{r}_j \in R^3$ $(j = 1, ..., n)$, when moving under the Newtonian law of attraction. The equations of motion are

$$m_j \ddot{\vec{r}}_j = \sum_{i \neq j} \frac{m_i m_j (\vec{r}_i - \vec{r}_j)}{r_{ij}^3} = \frac{\partial U}{\partial \vec{r}_j},$$

where $r_{ij} = |\vec{r}_i - \vec{r}_j|$, and the potential U is given by

$$U = \sum_{i<j} \frac{m_i m_j}{r_{ij}}.$$

By definition

$$\Delta_{ij} = \{(\vec{r}_1, ..., \vec{r}_n) \in R^{3n} \quad | \quad \vec{r}_i = \vec{r}_j\}$$

and

$$\Delta = \bigcup_{i<j} \Delta_{ij},$$

the configuration space is

$$X = R^{3n} \setminus \Delta.$$

Definition 1.1. A point $r_0 \in X$ is a central configuration (CC), if there is some scalar α so that $\alpha r_0 = \bigtriangledown U(r_0)$.

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Defining the matrix $\bar{M} = diag(m_1, m_1, m_1, ..., m_n, m_n, m_n)$ we find that r_0 is a CC iff $\bigtriangledown U(r_0) = \lambda \bar{M} r_0$ where $\lambda = U(r_0)/r_0^t \bar{M} r_0$. Then the study of the central configurations is equivalent to the analysis of the last equation, which can be seen as a problem of Lagrange multipliers, where $r_0^t \bar{M} r_0 = 1$ is a constraint, so if we define

$$\Omega = \{r \in R^{3n} \setminus \Delta \mid m_1 r_1 + ... + m_n r_n = 0,\ r^t \bar{M} r = 1\},$$

then r_0 is a CC iff it is a critical point of the restriction of $U(r)$ to Ω.

For the three body problem, there are five central configurations (or just four if we accept rotations in R^3); which corresponds to three collinear central configurations (Euler's configurations), and two equilateral triangle central configurations (Lagrange's configurations). For all choices of masses m_1, m_2, m_3, the last two CC always corresponds to non-degenerate minimun of the potential U restricted to Ω, so they are stable. Not as much is known about central configurations for the n-body problem when $n > 3$. Even it is not clear whether the number of CC is finite or not.

In this paper we will study the central configurations in the charged three body problem. Here, analyzing the potential U we can see that, depending on the parameters, the forces can have different signs. Then the possible number of CC could change from the general Newtonian three body problem. To get the different numbers of CC, we introduce new parameters, in terms of the original ones, which permits to count the exact number of CC for any set of parameters.

2. Equations for the Central Configurations

Consider three point particles with masses $m_1, m_2, m_3 \in R^+$ and charges $q_1, q_2, q_3 \in R$. Let $\vec{r_i}$ be the position vector of the particle i, and fix the origin at the mass center of the system. The potential U, and the momentum of inertia I are given by

$$U = \frac{m_1 m_2 - q_1 q_2}{r_{12}} + \frac{m_2 m_3 - q_2 q_3}{r_{23}} + \frac{m_1 m_3 - q_1 q_3}{r_{13}} = \frac{\lambda_3}{r_{12}} + \frac{\lambda_1}{r_{23}} + \frac{\lambda_2}{r_{13}},$$

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 = \frac{1}{M} \sum_{i<j} m_i m_j r_{ij}^2.$$

Where $r_{ij} = |\vec{r_i} - \vec{r_j}|$, $|\vec{r_i}| = r_i$, $M = m_1 + m_2 + m_3$, $\lambda_i = m_j m_k - q_j q_k$ where (i, j, k) permutes cyclically in (1,2,3), and units of time have been chosen to set the gravitational constant $G = 1$.

As in the Newtonian three body problem, $\vec{r} \in (R^3)^3$ is a CC iff

$$\bigtriangledown(IU^2) = 0 \text{ at } \vec{r}.$$

Since the center of mass is fixed at the origin, we can introduce Jacobi's coordinates to reduce the number of independent variables.

Let

$$\begin{aligned} \vec{s}_1 &= \vec{r}_2 - \vec{r}_1, \\ \vec{s}_2 &= \vec{r}_3 - \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}. \end{aligned}$$

Using above relationship between $\vec{s}_1, \vec{s}_2$ and $\vec{r}_1, \vec{r}_2, \vec{r}_3$ and the fact that the center of mass is fixed at the origin, that is $m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 = 0$, we have (see fig. 1)

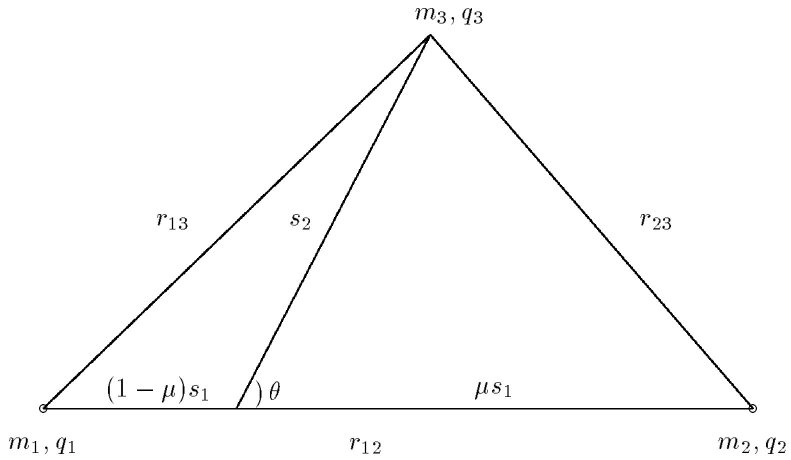

FIGURE 1. The Jacobi Coordinate System J_{12}

$$
\begin{aligned}
\vec{r}_1 &= -\frac{m_2}{m_1+m_2}\vec{s}_1 - \frac{m_3}{m_1+m_2+m_3}\vec{s}_2, \\
\vec{r}_2 &= \frac{m_1}{m_1+m_2}\vec{s}_1 - \frac{m_3}{m_1+m_2+m_3}\vec{s}_2, \\
\vec{r}_3 &= \frac{m_1+m_2}{m_1+m_2+m_3}\vec{s}_2.
\end{aligned}
$$

Expressing U and I in terms of $|\vec{s}_1| = s_1$ and $|\vec{s}_2| = s_2$ we have:

$$U = \frac{\lambda_3}{s_1} + \frac{\lambda_1}{\sqrt{s_2^2 + (\mu s_1)^2 - 2\mu s_1 s_2 \cos\theta}} + \frac{\lambda_2}{\sqrt{s_2^2 + ((1-\mu)s_1)^2 + 2(1-\mu)s_1 s_2 \cos\theta}},$$

$$I = g_1 s_1^2 + g_2 s_2^2.$$

Where θ is the angle between the vectors $\vec{s}_1$ and $\vec{s}_2$ and

$$\mu = \frac{m_1}{m_1+m_2}, \qquad g_1 = \frac{m_1 m_2}{m_1+m_2}, \qquad g_2 = \frac{m_3(m_1+m_2)}{m_1+m_2+m_3}.$$

So we have

$$\frac{1}{g_2} I U^2 = (\beta + \rho^2)\left(\lambda_3 + \frac{\lambda_1}{\sqrt{\rho^2 + \mu^2 - 2\mu\rho\cos\theta}} + \frac{\lambda_2}{\sqrt{\rho^2 + (1-\mu)^2 + 2(1-\mu)\rho\cos\theta}}\right)^2,$$

where $\beta = g_1/g_2$ and $\rho = s_2/s_1$.

Now that IU^2 is in terms of the parameters $\lambda_1, \lambda_2, \lambda_3, \beta, \mu$ and the two independent variables ρ, and θ, the formula for the central configurations of the charged three body problem is

$$
\begin{aligned}
\frac{\partial}{\partial\rho}\left(\frac{1}{g_2} I U^2\right) &= 0, \\
\frac{\partial}{\partial\theta}\left(\frac{1}{g_2} I U^2\right) &= 0\ .
\end{aligned}
\tag{2.1}
$$

The fact that IU^2 remains unchanged under rotation and scalar change of the system requiers the same condition to apply for the central configurations in the charged three body problem.

3. Symmetries

The Jacobi System of the previous section, where the base vector is $\vec{s}_1 = \vec{r}_2 - \vec{r}_1$, is denoted by J_{12}. Denote the selected Jacobi System, where $\vec{s}_1 = \vec{r}_3 - \vec{r}_1$ is the base vector, and $\vec{s}_2$ is the vector which goes from mass center of masses m_1 and m_3 to m_2, as J_{13}. The remain Jacobi System is denoted by J_{21}. Because there exists an one to one correspondence between these Jacobi Systems, all of our analysis is in the Jacobi System J_{12}. An analysis of the collinear central configurations (CCC) in the Jacobi System J_{12} reduces to whether particle m_3, is between the other two, or exterior to them.

Let p_i be the position on a fixed line of particle m_i for $i = 1, 2, 3$. In the next section we prove that it is sufficient to study the case where p_1, p_2 and p_3 are aligned in this order, that will be called Case I. In the system J_{12} this corresponds to $\theta = 0, \rho > \mu$. The other possible CCC correspond to the alignments p_3, p_1, p_2 (Case II), and p_1, p_3, p_2 (Case III).

For Case II, in system J_{12} we have that $\theta = \pi$ and $\rho > 1-\mu$. Which corresponds to the study of the Case I in the system J_{21}.To express the potential U in these variables it is necessary to introduce the changes $\bar{m}_1 = m_2,\ \bar{m}_2 = m_1,\ \bar{m}_3 = m_3,\ \bar{q}_1 = q_2,\ \bar{q}_2 = q_1,\ \bar{q}_3 = q_3$, and with these changes we have $\bar{\lambda}_1 = \lambda_2,\ \bar{\lambda}_2 = \lambda_1,\ \bar{\lambda}_3 = \lambda_3$.

For the Case III, in the System J_{12}, we have two possibities

(i) $\theta = 0, \qquad \rho < \mu$,

(ii) $\theta = \pi, \qquad \rho < 1 - \mu$.

Both possibilities correspond to study the Case I in the system J_{13}, where now the respective changes are $\hat{m}_1 = m_1,\ \hat{m}_2 = m_3,\ \hat{m}_3 = m_2, \hat{q}_1 = q_1,\ \hat{q}_2 = q_3,\ \hat{q}_3 = q_2$, and with these changes we have $\hat{\lambda}_1 = \lambda_1,\ \hat{\lambda}_2 = \lambda_3,\ \hat{\lambda}_3 = \lambda_2$.

4. Noncollinear Central Configurations

Let R_{ij} be the relative distance between p_i and p_j with respect to $s_1 = r_{12}$, *i.e.*

$$R_{13} = \frac{r_{13}}{s_1}, \qquad R_{23} = \frac{r_{23}}{s_1}, \qquad R_{12} = \frac{r_{12}}{s_1} = 1.$$

So

$$\frac{1}{g_2} I U^2 = (\beta + \rho^2)\left[\lambda_3 + \frac{\lambda_1}{R_{23}} + \frac{\lambda_2}{R_{13}}\right]^2. \tag{4.1}$$

From system 2.1 central configurations satisfy

(4.2)
$$\rho\left(\lambda_3 + \frac{\lambda_1}{R_{23}} + \frac{\lambda_2}{R_{13}}\right) + (\rho^2 + \beta)\left[-\frac{\lambda_1(\rho - \mu\cos\theta)}{R_{23}^3} - \frac{\lambda_2(\rho + (1-\mu)\cos\theta)}{R_{13}^3}\right] = 0,$$

$$-\frac{\lambda_1 \rho\mu \sin\theta}{R_{23}^3} + \frac{\lambda_2(\rho(1-\mu)\sin\theta)}{R_{13}^3} = 0. \tag{4.3}$$

Solving the second equation, we have

$$\rho\sin\theta = 0 \qquad \text{or} \qquad \frac{m_2\lambda_2}{R_{13}^3} = \frac{m_1\lambda_1}{R_{23}^3}.$$

In the case of $\rho\sin\theta = 0$, we have $\rho = 0$ or $\sin\theta = 0$ which corresponds to collinear CC. We will study this case later in the next section.

Now consider the second case and let

$$\frac{m_2\lambda_2}{R_{13}^3} = \frac{m_1\lambda_1}{R_{23}^3} = z^{-3}.$$

Then we can solve for R_{13} and R_{23} and substitute them into equation 4.2, obtaining

$$z^3 = \frac{1}{m_3\lambda_3}.$$

So the unique solution is

$$\frac{m_1\lambda_1}{R_{23}^3} = \frac{m_2\lambda_2}{R_{13}^3} = \frac{m_3\lambda_3}{R_{12}^3} = m_3\lambda_3. \tag{4.4}$$

Only when $\lambda_1, \lambda_2, \lambda_3$, have the same sign does this solution define an unique central configuration. Physically this means that all charges have the same polarity. Using the above relation we can check that this central configuration is a triangle with one body at each vertex if the parameters satisfy the constraint

$$(m_i\lambda_i)^{1/3} + (m_j\lambda_j)^{1/3} > (m_k\lambda_k)^{1/3}, \tag{4.5}$$

where (i, j, k) permutes cyclically in $(1, 2, 3)$. Note that this triangle could be of any shape depending on the choice of the parameters.

We want to remark that if all λ_i have the same sign and the constraint 4.5 holds, then we can always construct the relation 4.4, which guaranties the existence of non-collinear CC.

Observe that in the Jacobi System J_{12}, if the angle θ is replaced by $-\theta$, we get the same result about the existence of non-collinear CC. This just reflets the symmetry of the potential function U,

$$U(s_1, s_2, \theta) = U(s_1, s_2, -\theta).$$

Because three points in R^3 define a plane, we only consider rotations in the plane. Therefore the existence of one non-collinear CC implies the existence of the symmetric one, which must be considered different because is not obtained by planar rotation of the original one. One special case is when all charges are zero. Then $m_1\lambda_1 = m_2\lambda_2 = m_3\lambda_3 = m_1m_2m_3$. The solution will be

$$\frac{1}{R_{23}^3} = \frac{1}{R_{13}^3} = \frac{1}{R_{12}^3} = 1,$$

or

$$R_{12} = R_{13} = R_{23} = 1,$$

let us observe that in this case the constraint 4.5 holds in a trivial way, so here we get the Lagrange's configurations for the classical Newtonian 3-body problem.

If any two of $\lambda_1, \lambda_2, \lambda_3$ have different signs, then the solution does not define any configuration, which means that non-collinear CC does not exist.

In this way, we can determine the total number of non-collinear CC which can be 0 or 2, according to the values of the parameters (or more precisely, the signs of some parameters must satisfices the conditions 4.4 and 4.5).

5. Collinear Central Configurations

In the last section we obtain two possible conditions associated to the existence of collinear central configurations (CCC). Now we will study these two possibilities.
(i) $\rho = 0$.

The solution for all θ of equation 4.2 becomes

$$\frac{\lambda_1}{\mu^2} = \frac{\lambda_2}{(1-\mu)^2} \tag{5.1}$$

Thus only when the parameters satisfy 5.1 can there be a CC with m_3 sitting at the center of masses m_1 and m_2. This is intuitive because Eq. 5.1 requires the attracting force on m_3 by m_2 (repelling force if λ_1 is negative) to equal the attracting force on m_3 by m_1 , so m_3 can remains at the mass center forever. This corresponds to case III in our symmetries.
(ii) $\sin\theta = 0$

The two possibilities are $\theta = 0$ and $\theta = \pi$.

For $\theta = 0$, we only need to consider $\rho > \mu$, as $0 < \rho < \mu$ corresponds to case III.

Similarly $\theta = \pi$, causes $0 < \rho < 1-\mu$ or $\rho > 1-\mu$ which correspond to case III and case II respectively.

Now we start with the case $\theta = 0, \quad \rho > \mu$ (case I), when $\lambda_i \neq 0, i = 1, 2, 3$. Using $r_{23} = s_2 - \mu s_1$, Eq. 4.2 becomes

$$F(\rho) = \lambda_3\rho - \frac{\lambda_1\mu(\rho + \frac{\beta}{\mu})}{(\rho-\mu)^2} + \frac{\lambda_2(1-\mu)(\rho + \frac{\beta}{1-\mu})}{(\rho+1-\mu)^2} = 0. \tag{5.2}$$

To simplify this function, we introduce new parameters. Let

$$\begin{aligned} \alpha_1 &= m_1\lambda_1, \\ \alpha_2 &= m_2\lambda_2, \\ \alpha_3 &= (m_1+m_2)\lambda_3, \end{aligned}$$

and

$$\begin{aligned} \beta_1 &= \tfrac{\alpha_1}{\alpha_2}, \\ \beta_3 &= \tfrac{\alpha_3}{\alpha_2}. \end{aligned}$$

Also let $x = \rho - \mu$ so in this case, $x > 0$.

With these new parameters, we have

$$F(\rho) = \frac{f(x)\alpha_2}{m_1+m_2},$$

where

$$f(x) = \beta_3 x + \beta_3\mu - \frac{\beta_1(x + 1 + \frac{m_2}{m_3})}{x^2} + \frac{x - \frac{m_1}{m_3}}{(x+1)^2}. \tag{5.3}$$

By choosing m_3 as the unit of mass, we can let $m_3 = 1$. So the above equation becomes

$$f(x) = \beta_3 f_3(x) - \beta_1 f_1(x) + f_2(x) = 0, \tag{5.4}$$

where

$$f_3(x) = x + \frac{m_1}{m_1+m_2} > 0,$$

$$f_1(x) = \frac{x+1+m_2}{x^2} > 0,$$

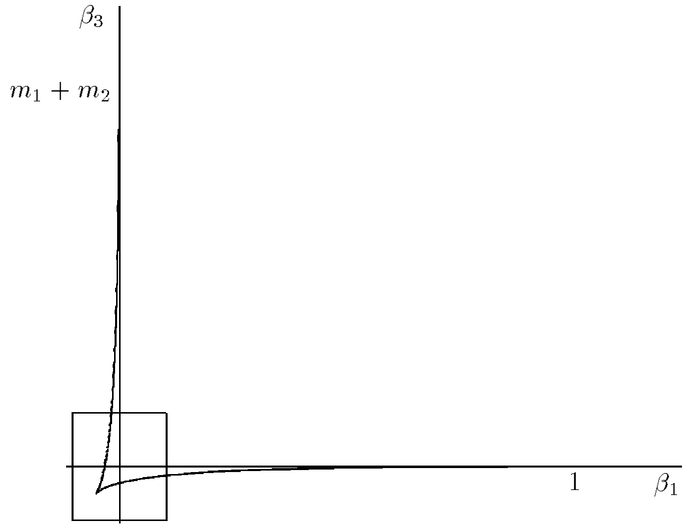

FIGURE 2. The Bifurcation Curve for $m_1 = m_2 = m_3 = 1$.

$$f_2(x) = \frac{x - m_1}{(x+1)^2}.$$

From 5.4 we have

$$\beta_3 = \beta_1 g_1(x) + g_2(x), \tag{5.5}$$

where

$$g_1(x) = \frac{f_1(x)}{f_3(x)},$$
$$g_2(x) = -\frac{f_2(x)}{f_3(x)}.$$

In the $\beta_1\beta_3$ plane Eq. 5.5 becomes a line for a fixed x and fixed masses. Here $g_1(x)$ is the slope and $g_2(x)$ is the β_3-intercept. For any fixed point (β_1, β_3), the number of lines passing through this point when x is moving from zero to infinity, will determine the number of CCC for that set of parameters.

We now examine how the line changes as x goes from 0 to $+\infty$. First of all,

$$g_1'(x) = -\frac{2x^2 + [3(m_2+1) + \frac{m_1}{m_1+m_2}]x + \frac{2m_1(m_2+1)}{m_1+m_2}}{x^3(x + \frac{m_1}{m_1+m_2})^2} < 0,$$

and $g_1(0) = +\infty, g_1(+\infty) = 0$. So the slope of the line monotonely decreases from $+\infty$ to 0 as x goes from 0 to $+\infty$.

Secondly,

$$g_2'(x) = -\frac{-2x^2 + (3m_1 - \frac{m_1}{m_1+m_2})x + \frac{m_1(3m_1+m_2)}{m_1+m_2} + \frac{m_1}{m_1+m_2}}{(x+1)^3(x + \frac{m_1}{m_1+m_2})^2},$$

and $g_2(0) = m_1 + m_2, \quad g_2(+\infty) = 0$. So we will have

$$\begin{cases} g_2'(x) & < 0 \text{ when } 0 < x < x_+, \\ g_2'(x) & > 0 \text{ when } x_+ < x, \end{cases}$$

where

$$x_+ = \frac{(3m_1 - \frac{m_1}{m_1+m_2}) + \sqrt{(3m_1 - \frac{m_1}{m_1+m_2})^2 + 8(\frac{m_1(3m_1+m_2)}{m_1+m_2} + \frac{m_1}{m_1+m_2})}}{4}.$$

So the β_3-intercept of the line decreases from $m_1 + m_2$ when $0 < x < x_+$ and then increases to 0 when $x > x_+$.

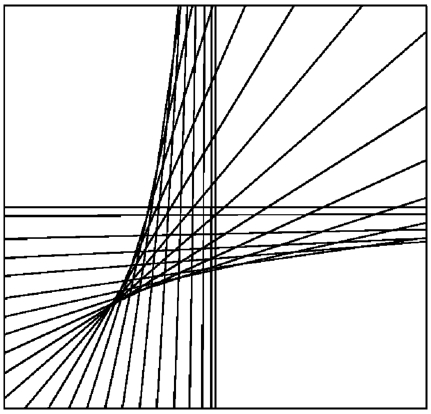

FIGURE 3. Detail of the family of lines associated to the Bifurcation Curve for $m_1 = m_2 = m_3 = 1$.

Based on the analysis of $g_1(x)$ and $g_2(x)$, we can sketch in the $\beta_1\beta_3$ plane the changes of the line as x goes from 0 to $+\infty$. Thus we get some regions in the $\beta_1\beta_3$ plane and in each region the number of times the line passes through any point in that region is the same, *i.e.* the number of CCC remains fixed (see Fig.3).

We can also get the bifurcation curve between regions computing the envelope of the family of lines given by the equation 5.5. For this, we derivate 5.5 with respect to x, obtaining the parametric form of the bifurcation curve

$$\begin{aligned} \beta_1(x) &= -\frac{g_2'(x)}{g_1'(x)}, \\ \beta_3(x) &= -\frac{g_2'(x)}{g_1'(x)} g_1(x) + g_2(x). \end{aligned} \tag{5.6}$$

which is a piecewise smooth curve. It is easy to check that $\lim_{x\to\infty} \beta_1(x) = 1$ and $\lim_{x\to\infty} \beta_3(x) = 0$. The bifurcation curve given by 5.6, and the coordinate axis in the $\beta_1\beta_3$ plane determine the regions where we get the same number of CCC, this number could be 0,1,2 or 3 (see fig.4).

6. Number of Collinear Central Configurations

In order to get the total number of CCC we need to work with different Jacobi systems. To use the bifurcation curve (Eq. 5.6) obtained in the above section when changing

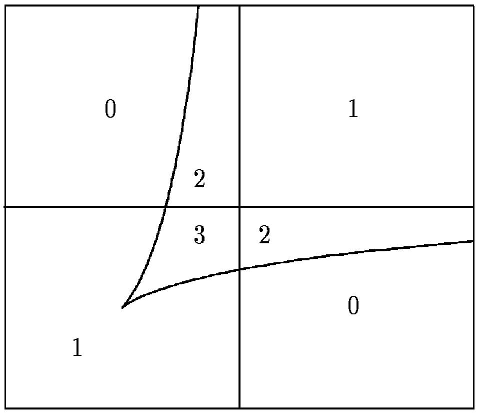

FIGURE 4. Different regions according to the number of CCC.

the Jacobi system we need that all the parameters $\lambda_i \neq 0, \quad i = 1, 2, 3$. In order to point out the relation between two different Jacobi systems, we introduce the rate charge-mass $\delta_i = \frac{q_i}{m_i}, i = 1, 2, 3$. So

$$\lambda_i = m_j m_k (1 - \delta_j \delta_k) \text{ where } (i, j, k) \text{ permutes cyclically in } (1,2,3).$$

With this notation

$$\beta_1 = \frac{1 - \delta_2\delta_3}{1 - \delta_1\delta_3}, \qquad \beta_3 = \left(\frac{m_1 + m_2}{m_3}\right) \frac{1 - \delta_1\delta_2}{1 - \delta_1\delta_3}. \tag{6.1}$$

We denote by $\bar{\beta}_1$ and $\bar{\beta}_3$ the analogous variables in the Jacobi system J_{21} (case II), and by $\hat{\beta}_1$ and $\hat{\beta}_3$ the respective ones in the Jacobi system J_{13} (case III). By the symmetries given in section 3, we get

$$\bar{\beta}_1 = \frac{1}{\beta_1}, \qquad \hat{\beta}_1 = \left(\frac{m_1 + m_2}{m_3}\right) \frac{\beta_1}{\beta_3}, \tag{6.2}$$

$$\bar{\beta}_3 = \frac{\beta_3}{\beta_1}, \qquad \hat{\beta}_3 = \frac{(m_1 + m_2)(m_1 + m_3)}{m_2 m_3} \frac{1}{\beta_3}. \tag{6.3}$$

Let us observe that in order to get the $\bar{\beta}$s and $\hat{\beta}$s we need $\lambda_i \neq 0$ for $i = 1, 2, 3$. Therefore we split the study of the number of CCC in two subsections depending on $\lambda_1\lambda_2\lambda_3 \neq 0$ or not.

6.1. All the λ's do not vanish. In the above section we have studied the case $\theta = 0$ and $\rho > \mu$ in the Jacobi system J_{12}. Using the symmetries given in section 3, we will obtain the total number of CCC according to the different signs of the parameters λ_i.

We will denote by $1Q, 2Q, 3Q$ and $4Q$ the first, second, third and fourth quadrants respectively, corresponding to the plane $\beta_1\beta_3$ associated to the Jacobi system J_{12}. Analogously we use the same notation with ($\bar{}$) or ($\hat{}$) for the Jacobi systems J_{21} and J_{13}. And we will also denote by $BC, \bar{BC}, \hat{BC}$ to the bifurcation curve in the respective Jacobi system.

(i) $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$ (resp. $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_3 < 0$).

In this case we have $\beta_1 > 0$ and $\beta_3 > 0$, then according to the bifurcation diagram given in the above section, we only have one CCC, because the point $(\beta_1, \beta_3) \in 1Q$ in the Jacobi system J_{12}. Now using the relations given by 6.2 and 6.3 the points $(\bar{\beta}_1, \bar{\beta}_3) \in 1\bar{Q}$ and $(\hat{\beta}_1, \hat{\beta}_3) \in 1\hat{Q}$ (the first quadrants of the respective bifurcation diagrams).

Therefore, in this case we have exactely 3 CCC.

(ii) $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$ (resp. $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_3 > 0$).

In this case we have :
$(\beta_1, \beta_3) \in 2Q \setminus BC$, so we can have 0 or 2 CCC, or 1 if $(\beta_1, \beta_3) \in BC$.
$(\bar{\beta}_1, \bar{\beta}_3) \in 3\bar{Q} \setminus \bar{B}\bar{C}$, so we can have 1 or 3 CCC, or 2 if $(\bar{\beta}_1, \bar{\beta}_3) \in \bar{B}\bar{C}$.
$(\hat{\beta}_1, \hat{\beta}_3) \in 2\hat{Q} \setminus \hat{B}\hat{C}$, so we can have 0 or 2 CCC or 1 if $(\hat{\beta}_1, \hat{\beta}_3) \in \hat{B}\hat{C}$.
Therefore, in this case we can have 1, 2, 3, 4, 5, 6 or 7 CCC.

(iii) $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_3 < 0$ (resp. $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 > 0$).

In this case we have :
$(\beta_1, \beta_3) \in 3Q \setminus BC$, so we can have 1 or 3 CCC, or 2 if $(\beta_1, \beta_3) \in BC$.
$(\bar{\beta}_1, \bar{\beta}_3) \in 2\bar{Q} \setminus \bar{B}\bar{C}$, so we can have 0 or 2 CCC, or 1 if $(\bar{\beta}_1, \bar{\beta}_3) \in \bar{B}\bar{C}$.
$(\hat{\beta}_1, \hat{\beta}_3) \in 4\hat{Q} \setminus \hat{B}\hat{C}$, so we can have 0 or 2 CCC, or 1 if $(\hat{\beta}_1, \hat{\beta}_3) \in \hat{B}\hat{C}$.
Therefore, in this case we can have 1, 2, 3, 4, 5, 6 or 7 CCC.

(iv) $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_3 > 0$ (resp. $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 < 0$).

In this case we have :
$(\beta_1, \beta_3) \in 4Q \setminus BC$, so we can have 0 or 2 CCC, or 1 if $(\beta_1, \beta_3) \in BC$.
$(\bar{\beta}_1, \bar{\beta}_3) \in 4\bar{Q} \setminus \bar{B}\bar{C}$, so we can have 0 or 2 CCC or 1 if $(\bar{\beta}_1, \bar{\beta}_3) \in \bar{B}\bar{C}$.
$(\hat{\beta}_1, \hat{\beta}_3) \in 3\hat{Q} \setminus \hat{B}\hat{C}$, so we can have 1 or 3 CCC or 2 if $(\hat{\beta}_1, \hat{\beta}_3) \in \hat{B}\hat{C}$.
Therefore, in this case we can have 1, 2, 3, 4, 5, 6 or 7 CCC.

Now we will analyze if all these CCC are possible. From 5.3 we have

(6.4)
$$\beta_3 x^3 (x+1)^2 + \beta_3 \mu x^2 (x+1)^2 - \beta_1 (x + 1 + \frac{m_2}{m_3})(x+1)^2 + (x - \frac{m_1}{m_3})x^2 = 0.$$

This polynomial can be written as

$$p(x) = a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0, \tag{6.5}$$

where

$$\begin{aligned}
a_5 &= \beta_3, \\
a_4 &= \beta_3 (2 + \mu), \\
a_3 &= \beta_3 (1 + 2\mu) - \beta_1 + 1, \\
a_2 &= \beta_3 \mu - \beta_1 (3 + \frac{m_2}{m_3}) - \frac{m_1}{m_3}, \\
a_1 &= -\beta_1 (3 + 2\frac{m_2}{m_3}), \\
a_0 &= -\beta_1 (1 + \frac{m_2}{m_3}).
\end{aligned}$$

The CCC correspond to the positive roots of 6.5, in this form we can prove in a different way the above statements about the number of possible CCC. Now we will prove that is impossible to have the cases with 6 or 7 CCC.

We start analyzing the case

(ii) $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$ (resp. $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_3 > 0$).

The idea is to prove that $(\beta_1, \beta_3) \in 2Q \setminus BC$ and $(\hat\beta_1, \hat\beta_3) \in 2\hat{Q} \setminus \hat{B}\hat{C}$ can not be simultaneously in the region of 2 CCC.

Let us observe that in this case a_5, a_4, a_3, a_1 and a_0 are positive, and the same is true for the $\hat{a}_i, i = 5, 4, 3, 1, 0$, so in order to have (β_1, β_3) and $(\hat\beta_1, \hat\beta_3)$ in the same region of 2 CCC is necessary by the Descartes' rule of signs to find the positive roots of $p(x)$ that, $a_2 < 0$ and $\hat{a}_2 < 0$, that is

$$\beta_3\mu - \beta_1(3 + \frac{m_2}{m_3}) - \frac{m_1}{m_3} < 0,$$

$$\hat\beta_3\hat\mu - \hat\beta_1(3 + \frac{\hat m_2}{\hat m_3}) - \frac{\hat m_1}{\hat m_3} < 0.$$

Now using the symmetries studied in the part 3 of this paper we can write the above equations in terms of $\beta_1, \beta_3, \mu, m_1, m_2$, and m_3, so

$$\beta_3\mu - \beta_1(3 + \frac{m_2}{m_3}) - \frac{m_1}{m_3} < 0, \tag{6.6}$$

$$\frac{m_1(m_1+m_2)}{m_2 m_3} - \beta_1(3 + \frac{m_3}{m_2})(\frac{m_1+m_2}{m_3}) - \frac{m_1}{m_2}\beta_3 < 0. \tag{6.7}$$

Multipying the first equation 6.6 by $\frac{m_1}{m_2}$, the second by μ and adding we get

$$-\beta_1\left[(3 + \frac{m_2}{m_3})\frac{m_1}{m_2} + (\frac{m_1+m_2}{m_3})(3 + \frac{m_3}{m_2})\mu\right] < 0, \tag{6.8}$$

which is a contradiction. The analysis in the cases $iii)$ and $iv)$ is similar. Therefore we have proved the following result.

Proposition 6.1. *If all the λ_i's do not vanish, is possible to have 1, 2, 3, 4 or 5 CCC.*

6.2. Some λ_i vanish. Different possibilities. Now we will study the non-generic cases, that is when $\lambda_i = 0$ for some $i = 1, 2, 3$, and therefore, we can not use the bifurcation diagram given in section 5. Moreover, we can not have non-collinear CC. We will start our study with the simplest cases.

(i) $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

This is a totally degenerate situation. Here the potential function $U \equiv 0$ therefore we do not have any CCC.

(ii) $\lambda_1 = \lambda_2 = 0$, $\lambda_3 \neq 0$.

Here the potential U takes the form $U = \frac{\lambda_3}{r_{12}}$. Thus, the charged three body problem is reduced to a Kepler problem, attractive or repulsive, depending on the sign of λ_3, and it is well known that there is not any CCC. The other two cases when only one $\lambda_i \neq 0$ give the same result.

(iii) $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

When only one λ_i vanishes we will use the same symmetries given in section 3. Therefore, it is enough to study the case when $\theta = 0$ and $\rho > \mu$.

¿From the equation 5.2 we have

$$F(\rho) = \lambda_3 \rho + \frac{\lambda_2 (1-\mu)(\rho + \frac{\beta}{1-\mu})}{(\rho + 1 - \mu)^2} = 0. \tag{6.9}$$

Let $x = \rho - \mu > 0$. Then the above equation takes the form

$$\lambda_2 = G(x)\lambda_3,$$

where

$$G(x) = -\frac{(x+\mu)(x+1)^2}{(x+\mu)(1-\mu)+\beta}.$$

Let us observe that

$$-1 < G(0) = -\frac{\mu}{\mu(1-\mu)+\beta} < 0, \quad \lim_{x \to +\infty} G(x) = -\infty \quad \text{and} \quad G'(x) < 0 \quad \forall x > 0.$$

Therefore, when x goes from 0 to $+\infty$, we have the following bifurcation diagram (see fig. 5). As a consequence we will have 0 or 1 CCC.

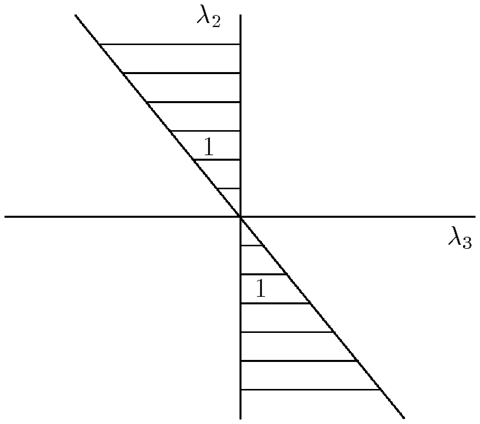

FIGURE 5. Bifurcation diagram for $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$.

(iv) $\lambda_1 \neq 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$.

Like in case (iii), from the equation 5.2 we have

$$F(\rho) = \lambda_3 \rho - \frac{\lambda_1 \mu (\rho + \frac{\beta}{\mu})}{(\rho - \mu)^2} = 0, \tag{6.10}$$

and doing $x = \rho - \mu > 0$ we have

$$\lambda_1 = G(x)\lambda_3,$$

where now

$$G(x) = \frac{(x+\mu)x^2}{\mu(x+\mu)+\beta},$$

here

$$G(0)=0, \qquad \lim_{x\to+\infty} G(x)=+\infty, \quad \text{and} \quad G'(x)>0 \quad \forall x>0.$$

So the bifurcation diagram is given in Figure 6. Here we will also have 0 or 1 CCC.

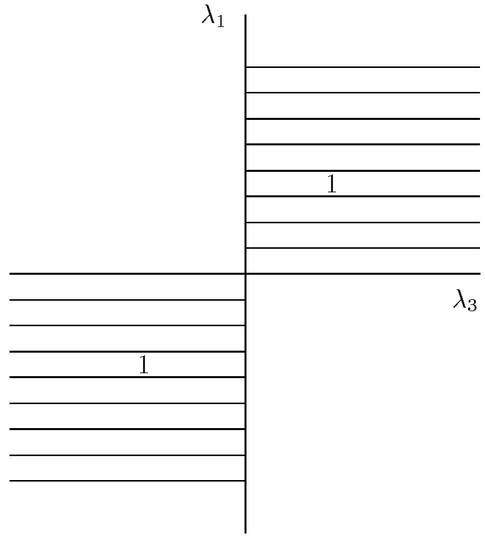

FIGURE 6. Bifurcation diagram for $\lambda_1 \neq 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$.

(v) $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$.

As we will show, this is the more difficult case. From the equation 5.2, we have

$$F(\rho)=\frac{\lambda_2(1-\mu)(\rho+\frac{\beta}{1-\mu})}{(\rho+1-\mu)^2}-\frac{\lambda_1\mu(\rho+\frac{\beta}{\mu})}{(\rho-\mu)^2}=0, \tag{6.11}$$

and in terms of $x=\rho-\mu>0$ we have

$$\lambda_1 = G(x)\lambda_2,$$

where now

$$G(x)=\frac{x^2\left[(x+\mu)(1-\mu)+\beta\right]}{(x+1)^2\left[\mu(x+\mu)+\beta\right]}$$

Let us observe that

$$G(0)=0, \qquad \lim_{x\to+\infty} G(x)=\frac{1-\mu}{\mu}, \quad \text{and} \quad G(x)<1 \quad \forall x>0.$$

The sign of the derivative of $G(x)$ is given by the sign of its numerator which is the polynomial

$$q(x)=xp(x)=x(ax^3+bx^2+cx+d),$$

where

$$\begin{aligned}
a &= \beta(1-2\mu)+2\mu(1-\mu),\\
b &= 2(1-\mu)(2\beta+\mu+2\mu^2),\\
c &= 3(1-\mu)(\beta+\mu^2)+(\beta+\mu(1-\mu))(2\beta+\mu+2\mu^2),\\
d &= 2(\mu^2+\beta)(\beta+\mu(1-\mu)).
\end{aligned}$$

We point out that the coefficients b, c and d are always positive for any choice of the parameters, while the coefficient a can have any sign. Writing a in terms of the mass parameters, we obtain

$$a = \frac{m_1 m_2}{(m_1+m_2)^2}\left[\frac{M}{m_3}\left(\frac{m_2-m_1}{m_1+m_2}\right)+2\right],$$

and then $a \geq 0$ iff $0 < \mu \leq \frac{m_3}{M} + \frac{1}{2} \equiv \mu_0$. In this case $G'(x) > 0 \quad \forall x > 0$. Therefore the bifurcation diagram is shown in Figure 7. So we can have 0 or 1 CCC.

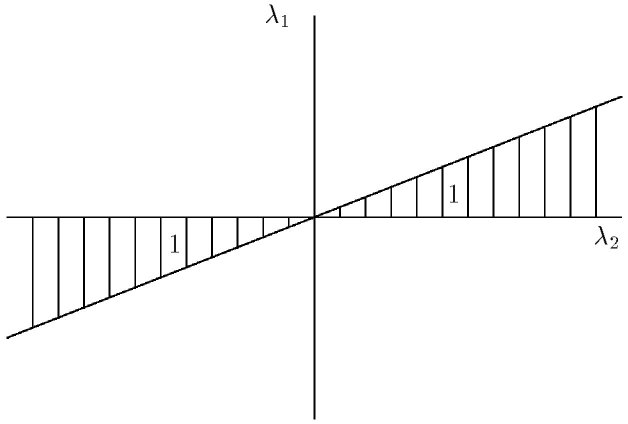

FIGURE 7. Bifurcation diagram for $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$ and $0 < \mu \leq \mu_0$.

When $a < 0$, that is $\mu_0 < \mu < 1$, by a straightforward but tedious computation, and using the limit behaviour of $G(x)$ at 0 and $+\infty$, we get that $p(x)$ has an unique positive root x_0. So $G(x)$ is a monotone increasing function in $(0, x_0)$, and then monotonely decreases until $\frac{1-\mu}{\mu}$ in the interval $(x_0, +\infty)$.

So the bifurcation diagram is shown in Figure 8. We point out that now we can have 0,1 or 2 CCC.

6.3. Some λ_i's vanish. Number of collinear central configurations. In section 3 we have seen that

$$\bar\lambda_1 = \lambda_2\ ,\bar\lambda_2 = \lambda_1 \ \text{ and } \bar\lambda_3 = \lambda_3\ ,$$

and also that

$$\hat\lambda_1 = \lambda_1\ ,\hat\lambda_2 = \lambda_3 \ \text{ and } \hat\lambda_3 = \lambda_2\ .$$

In order to count all the CCC we divide our analysis in several cases. By the symmetries of the respective bifurcation diagrams with respect to the origen, in any case is enough to analyze the first two quadrants.

(1.a) $\lambda_1 = 0$, $\lambda_2 > 0$, $\lambda_3 > 0$.
$(\lambda_3, \lambda_2) \in 1Q$ so we always have 0 CCC.
$(\bar\lambda_3, \bar\lambda_1) \in 1\bar Q$ so we always have 1 CCC.
$(\hat\lambda_3, \hat\lambda_2) \in 1\hat Q$ so we always have 0 CCC.
Therefore, the only possible number of CCC is 1.

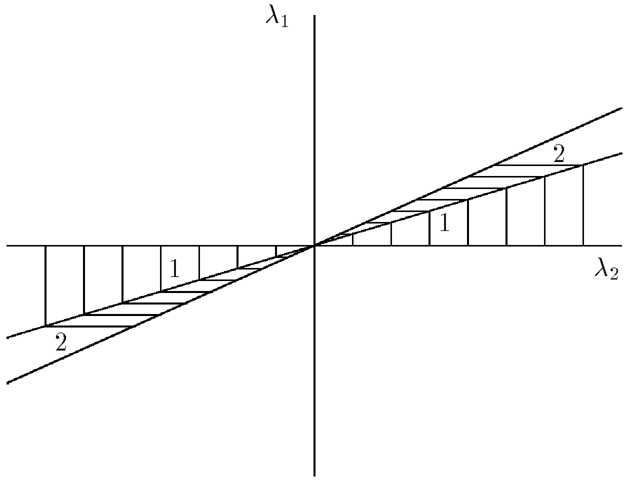

FIGURE 8. Bifurcation diagram for $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$ and $\mu_0 < \mu < 1$.

(1.b) $\lambda_1 = 0$, $\lambda_2 > 0$, $\lambda_3 < 0$.
$(\lambda_3, \lambda_2) \in 2Q$ so we can have 0 or 1 CCC.
$(\bar\lambda_3, \bar\lambda_1) \in 2\bar Q$ so we always have 0 CCC.
$(\hat\lambda_3, \hat\lambda_2) \in 4\hat Q$ so we can have 0 or 1 CCC.
Therefore, the possible number of CCC is 0,1 or 2.

(2.a) $\lambda_1 > 0$, $\lambda_2 = 0$, $\lambda_3 > 0$.
$(\lambda_3, \lambda_1) \in 1Q$ so we always have 1 CCC.
$(\bar\lambda_3, \bar\lambda_2) \in 1\bar Q$ so we always have 0 CCC.
$(\hat\lambda_2, \hat\lambda_1) \in 1\hat Q$ so we can have 0 or 1 CCC if $0 < \mu \leq \mu_0$ or we can have 0,1 or 2 CCC if $\mu_0 < \mu < 1$.
Therefore, the possible number of CCC is 1,2 or 3.

(2.b) $\lambda_1 > 0$, $\lambda_2 = 0$, $\lambda_3 < 0$.
$(\lambda_3, \lambda_1) \in 2Q$ so we always have 0 CCC.
$(\bar\lambda_3, \bar\lambda_2) \in 2\bar Q$ so we can have 0 or 1 CCC.
$(\hat\lambda_2, \hat\lambda_1) \in 2\hat Q$ so we always have 0 CCC.
Therefore, the possible number of CCC is 0 or 1.

(3.a) $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 = 0$.
Here we have two possible bifurcation diagrams depending on $\mu \in (0, \mu_0)$ or $\mu \in (\mu_0, 1)$. Let us observe that when we change J_{12} to J_{21}, the parameter $\bar\mu$ corresponds to $1 - \mu$, so is possible that we have to change the respective bifurcation diagrams.
(i) $0 < \mu \leq \mu_0$.
$(\lambda_2, \lambda_1) \in 1Q$ so we can have 0 or 1 CCC.

$(\bar{\lambda}_2, \bar{\lambda}_1) \in 1\bar{Q}$ so we can have 0,1 or 2 CCC if the previous one is 0; or 0 CCC if the previous one is 1.

$(\hat{\lambda}_3, \hat{\lambda}_1) \in 1\hat{Q}$ so we always have 1 CCC.

Therefore, the possible number of CCC is 1,2 or 3.

(ii) $\mu_0 < \mu < 1$.

$(\lambda_2, \lambda_1) \in 1Q$ so we can have 0,1 or 2 CCC.

$(\bar{\lambda}_2, \bar{\lambda}_1) \in 1\bar{Q}$ so we can have 0 or 1 CCC if the previous one is 0; or 0 CCC if the previous one is 1 or 2.

$(\hat{\lambda}_3, \hat{\lambda}_1) \in 1\hat{Q}$ so we always have 1 CCC.

Therefore, the possible number of CCC is 1,2 or 3.

(3.b) $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 = 0$.

In this case it is not necessary to distinguish the different possibilities for the value of μ , because we always work in the second or fourth quadrants.

$(\lambda_2, \lambda_1) \in 2Q$ so we always have 0 CCC.

$(\bar{\lambda}_2, \bar{\lambda}_1) \in 4\bar{Q}$ so we always have 0 CCC.

$(\hat{\lambda}_3, \hat{\lambda}_1) \in 2\hat{Q}$ so we always have 0 CCC.

Therefore here we do not have any CCC.

7. Total number of Central Configurations

After our previous analysis, now we can count all the possible central configurations (CC) obtaining the following result.

THEOREM 7.1. *Given $m_1, m_2, m_3 \in R^+$ and $q_1, q_2, q_3 \in R$ we can construct the respective λ_i, $i = 1, 2, 3$. So*

(i) If at least two λ_i vanish, we have 0 CC.

(ii) If just one λ_i vanishes, then we can have 0,1,2 or 3 CC.

(iii) If all the λ_i have the same sign, and $(m_i\lambda_i)^{1/3} + (m_j\lambda_j)^{1/3} > (m_k\lambda_k)^{1/3}$ then we have exactely 5 CC (2 non-collinear and 3 collinear), in the other cases we only have the three collinear CC.

(iv) If the λ_i's do not vanish and do not have the same sign, then we can have 1, 2, 3, 4 or 5 CC (all of them are collinear).

REMARK 7.2. The limit case when the inequality in $iii)$ is an equality, we have a CCC, in this case because all the λ_i have the same sign, we only have the well known three collinear central configurations like in the Newtonian case. So the limit of non-collinear central configurations is a classical collinear central configurations. The big difference between the Newtonian Three Body Problem and the Charged Three Body Problem with respect to central configurations is that in the last case we can have 5 collinear central configurations.

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