

# Ovoids in the cyclic presentation of $\text{PG}(3, q)$

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## Abstract

We consider the cyclic presentation of  $\text{PG}(3, q)$  whose points are in the finite field  $\mathbb{F}_{q^4}$  and describe the known ovoids therein. We revisit the set  $\mathcal{O}$ , consisting of  $(q^2 + 1)$ -th roots of unity in  $\mathbb{F}_{q^4}$ , and prove that it forms an elliptic quadric within the cyclic presentation of  $\text{PG}(3, q)$ . Additionally, following the work of Glauberman on Suzuki groups, we offer a new description of Suzuki-Tits ovoids in the cyclic presentation of  $\text{PG}(3, q)$ , characterizing them as the zeroes of a polynomial over  $\mathbb{F}_{q^4}$ .

Keywords: Finite geometries, Cyclic presentation, Ovoids, Elliptic quadrics, Suzuki-Tits ovoids, Projective polynomials

# 1 Introduction

In the projective space  $\text{PG}(3, q)$  with  $q > 2$ , an *ovoid* is a set of  $q^2 + 1$  points meeting every line in at most 2 points. The classical example of an ovoid is an elliptic quadric, whose points come from a non-degenerate elliptic quadratic form. Ovoids have been an interesting topic in finite geometry with important applications in coding theory as it has been shown (for example in [3, 8]) that error-correcting codes from ovoids have optimal parameters.

The only known ovoids in  $\text{PG}(3, q)$  are the elliptic quadrics, which exist for all  $q$ , and the Suzuki-Tits ovoids, which exist for  $q = 2^m$ , where  $m \geq 3$  is odd. It is well known that every ovoid of  $\text{PG}(3, q)$ , with  $q$  odd, is an elliptic quadric (see for example [9]). On the other hand, the classification problem for  $q$  even has been resolved only for  $q \leq 64$ , see [15, 16]. For this reason, our main interest is the case when  $q$  is even.

Usually ovoids are studied in the standard presentation of the projective space  $\text{PG}(3, q)$ . In this paper, we study ovoids in another model of  $\text{PG}(3, q)$  described in [5], following the approach by Jamison [12] (see also [4]). The point set of this model is identified with a cyclic subgroup of the multiplicative group  $\mathbb{F}_{q^4} \setminus \{0\}$ . We will refer to this model as *the cyclic presentation* of  $\text{PG}(3, q)$ .

In the first part of the paper, we consider the set  $\mathcal{O}$  of  $(q^2 + 1)$ -th roots of unity in the finite field  $\mathbb{F}_{q^4}$ . In [2], it was shown that the set  $\mathcal{O}$  consists of zeros of a quadratic form and it determines an elliptic quadric in  $\text{PG}(3, q)$ . However, a presentation of  $\text{PG}(3, q)$  to which the elliptic quadric  $\mathcal{O}$  belongs was not provided. In Theorem 3.2, we will show that the set  $\mathcal{O}$  is an elliptic quadric in the cyclic presentation of  $\text{PG}(3, q)$ . We highlight the recent work on the zeros of the projective polynomial  $X^{q^2+1} + X + a$  by Kim et al. in [13, 14], which provides necessary tools to obtain our result.

In the remainder of the paper, we study Suzuki-Tits ovoids. These ovoids bear this name because they were first described by Tits in [20] and are stabilized by the Suzuki groups  $Sz(q)$ . These groups  $Sz(q)$ , also known as the twisted Chevalley groups of type  ${}^2B_2(q)$ , were found by Suzuki [21]. A comprehensive treatment of this construction by Tits can be found in the book by Taylor [17]. We also refer the reader to a recent survey [19] and references therein for more on the historical development and impact of the discovery of Suzuki-Tits ovoids.

We mention briefly the general construction of Suzuki-Tits ovoids in  $\text{PG}(3, q)$ ; for the definitions of the terminology, we refer the reader to Subsection 2.3. Associated with  $\text{PG}(3, q)$  is a generalized quadrangle  $W(q)$ . For  $q = 2^m$ , where  $m \geq 3$  is odd, the set of absolute points of a polarity of  $W(q)$  forms a Suzuki-Tits ovoid. The standard way to construct a Suzuki-Tits ovoid is to introduce a symplectic basis of the vector space  $\mathbb{F}_q^4$  over  $\mathbb{F}_q$  and describe related objects with respect to this basis.

An alternative approach to the above, which we will use in this paper, is to consider the cyclic presentation of  $\text{PG}(3, q)$  and the associated description of  $W(q)$  described in [5] and [6].

At the conference Combinatorics 2024, Tao Feng brought our attention to the work of Glauberman [11] in 1996 on Suzuki groups. In [11], Glauberman already considered the

cyclic presentation of  $W(q)$  implicitly and provided suitable polarities for his investigations on the outer automorphisms of  $S_6$ . He did not provide a description of the Suzuki-Tits ovoids, however. Perhaps the reason was that the cyclic presentations of  $\text{PG}(3, q)$  and  $W(q)$  were not described explicitly until 1999 [5] and 2004 [6], respectively.

Following the work of Glauberman, we determine the absolute points of the polarities given in [11] and thereby describe the Suzuki-Tits ovoids in the cyclic presentation of  $\text{PG}(3, q)$ . As a consequence, we obtain a description of these ovoids as a set of zeroes of a polynomial over  $\mathbb{F}_{q^4}$ . This is the third description of Suzuki-Tits ovoids, after the original description by Tits [20] in 1962 and the constructions by Wilson [22, 23].

The content of the paper is organized as follows. In Section 2, we recall preliminary results from finite geometry and projective polynomials. Elliptic quadrics in the cyclic presentation of  $\text{PG}(3, q)$  are considered in Section 3. New results on Suzuki-Tits ovoids are reported in Section 4.

## 2 Preliminaries

Let  $q = 2^m$ . Let  $E \supset K \supset F$  be a chain of finite fields,  $|E| = q^4$ ,  $|K| = q^2$ ,  $|F| = q$ . In the sequel, we define the following sets:

$$\mathcal{P} := \{x \in E \mid x^{q^3+q^2+q+1} = 1\},$$

$$\mathcal{O} := \{x \in E \mid x^{q^2+1} = 1\},$$

$$S := \{x \in K \mid x^{q+1} = 1\}.$$

Each element  $x \in \mathcal{P}$  has a unique decomposition  $x = \lambda u$ , where  $\lambda \in S$  and  $u \in \mathcal{O}$ .

### 2.1 The roots of the polynomial $P_a(X) = X^{q+1} + X + a$

For  $a \in E$ , let

$$P_a(X) = X^{q+1} + X + a.$$

We note that the more general polynomial forms  $X^{q+1} + rX^q + sX + t$  with  $s \neq r^q$  and  $t \neq rs$  can be transformed into this form by the substitution  $X = (s - r^q)^{1/q}X_1 - r$ . It is clear that  $P_a(X)$  has no multiple roots. A comprehensive method for solving the equation  $P_a(X) = 0$  was recently developed in [13, 14]. Adopting the notation in [13], we define

$$A_4(X) = 1 + X^q + X^{q^2}.$$

From [13], we have the following characterization of  $P_a(X)$  when it has  $q + 1$  roots.

**Lemma 2.1** ([13]). *Let  $a \in E$ . Then the polynomial  $P_a(X) = X^{q+1} + X + a$  has  $q + 1$  zeros in  $E$  if and only if  $A_4(a) = 0$ . In this case, there exists  $u \in E \setminus K$  such that  $a = \frac{(u + u^q)^{q^2+1}}{(u + u^{q^2})^{q+1}}$ . Let  $c = (u + u^q)^{q-1}$ . Then the  $q + 1$  zeros in  $E$  of  $P_a(X)$  are  $x_0 = \frac{1}{1 + c}$  and  $x_\gamma = x_0(u + \gamma)^{q^2-q}$  for  $\gamma \in F$ .*

## 2.2 The cyclic presentation of $\text{PG}(3, q)$

We still assume that  $q = 2^m$ . In this section we recall the cyclic presentation of  $\text{PG}(3, q)$  first described explicitly in [5, Section 5.1]. The point set  $\mathcal{P}$  consists of  $(q^3 + q^2 + q + 1)$ -th roots of unity in  $E$ . Planes are given by the zeros of equations of the form

$$\sigma^{q^2+q+1}x^{q^2+q+1} + \sigma^{q+1}x^{q+1} + \sigma x + 1 = 0,$$

where  $\sigma \in \mathcal{P}$ . Lines are given by the zeros of polynomials of the form

$$L_{\alpha\beta}(x) := x^{q+1} + \alpha x + \beta,$$

where  $\beta \in \mathcal{P}$  and  $\alpha \in E$  such that

$$\alpha^{q+1} = \beta^q + \beta^{q^2+q+1}. \quad (1)$$

We first note the following.

**Lemma 2.2.** *Let  $\beta \in \mathcal{P}$  and  $\alpha \in E$  satisfy condition (1). If  $\alpha \neq 0$ , then  $\alpha^{q^2-1}\beta^{q+1} = 1$ .*

*Proof.* We have

$$\begin{aligned} (\alpha^{q+1})^{q-1}\beta^{q+1} &= (\beta^q + \beta^{q^2+q+1})^{q-1}\beta^{q+1} = \beta^{q(q-1)}(1 + \beta^{q^2+1})^{q-1}\beta^{q+1} \\ &= \beta^{q^2+1} \frac{1 + \beta^{q^3+q}}{1 + \beta^{q^2+1}} = \frac{\beta^{q^2+1} + \beta^{q^3+q^2+q+1}}{1 + \beta^{q^2+1}} = 1, \end{aligned}$$

and the proof follows.  $\square$

It was shown in [5] how the lines  $L_{\alpha\beta}(x)$  are obtained. In view of recent results from [13], an alternative way to see that these lines contain points from  $\mathcal{P}$  is as follows.

**Lemma 2.3.** *Let  $\beta \in \mathcal{P}$  and  $\alpha \in E$  satisfy condition (1). Then the equation*

$$X^{q+1} + \alpha X + \beta = 0 \quad (2)$$

*has  $q + 1$  roots in  $\mathcal{P}$ .*

*Proof.* 1. We note that  $q$  is even. If  $\alpha = 0$ , then  $\beta^{q^2+1} = 1$ . In particular, there exists  $e \in E$  such that  $e^{q^2-1} = \beta$ . Then the set of solutions of (2) is  $\{se^{q-1} \mid s \in S\}$ , and each element  $se^{q-1}$  is in  $\mathcal{P}$ .

2. Consider the case  $\alpha \neq 0$ . By substituting  $X = \alpha^{q^3}Y$ , the equation (2) can be transformed into

$$Y^{q+1} + Y + a = 0, \quad (3)$$

where  $a := \beta\alpha^{-(q^3+1)}$ . From Lemma 2.2, we have  $\alpha^{q^2-1} = \beta^{q^3+q^2}$ , so that

$$\alpha^{q^3+1} = \alpha^{q^3-q}\alpha^{q+1} = \beta^{q^3+1}\alpha^{q+1} = \beta^{q^3+1}(\beta^q + \beta^{q^2+q+1}) = \beta^{q^3+q+1}(1 + \beta^{q^2+1}).$$

Then

$$a = \frac{\beta}{\alpha^{q^3+1}} = \frac{\beta}{\beta^{q^3+q+1}(1 + \beta^{q^2+1})} = \frac{1}{\beta^{q^3+q}(1 + \beta^{q^2+1})} = \frac{\beta^{q^2+1}}{1 + \beta^{q^2+1}}.$$

And so  $a \in K$ .

Following Subsection 2.1, we let  $A_4(a) = a^{q^2} + a^q + 1$ . It follows readily that  $A_4(a) = 0$ . By Lemma 2.1, there exists  $u \in E \setminus K$  such that  $a = \frac{(u + u^q)^{q^2+1}}{(u + u^{q^2})^{q+1}}$ . Let  $c = (u + u^q)^{q-1} \in \mathcal{P}$ .

The roots of the equation (3) are given by  $y_0 = \frac{1}{1+c}$ , and  $y_0(u + \gamma)^{q^2-q}$ , for each  $\gamma \in F$ .

The roots of the original equation (2) are then  $x_0 = \frac{\alpha^{q^3}}{1+c}$ , and  $x_0(u + \gamma)^{q^2-q}$ , for each  $\gamma \in F$ . In [1, p.516], it was shown that

$$a = \frac{c^q}{(1+c)^{q+1}},$$

so that

$$a(1+c)^2 = \frac{c^q}{(1+c)^{q-1}} \in \mathcal{P}.$$

We have

$$x_0^2 = \frac{\alpha^{2q^3}}{(1+c)^2} = \frac{a\alpha^{2q^3}}{a(1+c)^2} = \frac{\alpha^{q^3-1}\beta}{a(1+c)^2}.$$

We note that  $\alpha^{q^3-1}\beta$  is a  $(q-1)$ -th power and so it is in  $\mathcal{P}$ . It follows that  $x_0 \in \mathcal{P}$ , and since

$$(u + \gamma)^{q^2-q} = (u^q + \gamma)^{q-1},$$

all the  $q+1$  roots of (2) are also in  $\mathcal{P}$ . □

**Lemma 2.4** ([5]). *The line joining two points  $y$  and  $z$  is given by the zeroes of the polynomial  $X^{q+1} + \alpha X + \beta = 0$ , where*

$$\alpha = \frac{y^{q+1} + z^{q+1}}{y + z},$$

$$\beta = \frac{y^{q+1}z + z^{q+1}y}{y + z}.$$

### 2.3 Known results on the Suzuki-Tits ovoid

We collect some materials from [7, p.41], [6] and [18].

A *correlation* of  $\text{PG}(3, q)$  is a bijection  $\sigma$  of the point set of  $\text{PG}(3, q)$  onto the plane set of  $\text{PG}(3, q)$ , such that the  $q+1$  points of any line  $l$  are mapped onto the  $q+1$  planes containing a line  $l'$ . The line  $l'$  is the image of  $l$  under  $\sigma$ . The point set of any plane  $P$  of  $\text{PG}(3, q)$  is mapped onto the set of all planes containing the point  $\sigma^{-1}(P)$ . A *polarity* of  $\text{PG}(3, q)$  is a correlation of order 2. A polarity  $\pi$  is *symplectic* if and only if every point

$p$  of  $\text{PG}(3, q)$  is absolute, that is  $p \in \pi(P)$ . A line  $l$  is *totally isotropic* (also known as self-polar in [18]) with respect to a correlation  $\sigma$  if  $\sigma(l) = l$ .

Let  $W(q)$  be the point-line geometry formed by all points of  $\text{PG}(3, q)$  and all totally isotropic lines with respect to a symplectic polarity of  $\text{PG}(3, q)$ . The geometry  $W(q)$  is called a tactical configuration in [7] but otherwise is more commonly known as a *generalized quadrangle* of order  $(q, q)$ , see for example [6].

A *correlation* of  $W(q)$  is a bijection  $\alpha$  of  $\mathcal{P} \cup \mathcal{L}$  onto itself such that  $\alpha(\mathcal{P}) = \mathcal{L}$ ,  $\alpha(\mathcal{L}) = \mathcal{P}$ ,  $p \in l$  if and only if  $\alpha(l) \in \alpha(p)$  for  $p \in \mathcal{P}$  and  $l \in \mathcal{L}$ . Such a correlation  $\alpha$  is a *polarity* if  $\alpha^2 = 1$ . A point  $p$  of  $\mathcal{P}$  is *absolute* with respect to  $\alpha$  if  $p \in \alpha(p)$ .

**Theorem 2.5** ([18, 20]). *Suppose that  $q = 2^m$ . The geometry  $W(q)$  admits a polarity  $\alpha$  if and only if  $m$  is odd. In such case, the absolute points of  $\alpha$  form an ovoid  $\mathcal{T}$  of  $\text{PG}(3, q)$ . These ovoids are called *Suzuki-Tits ovoids*. A Suzuki-Tits ovoid is an elliptic quadric if and only if  $q = 2$ .*

### 3 Elliptic quadrics in the cyclic presentation of $\text{PG}(3, q)$

Let  $N_{E/K}$  be the norm map from  $E$  to  $K$ . We recall from [2] the following description of elliptic quadrics in  $\text{PG}(3, q)$ .

**Theorem 3.1.** *Let  $E \supset K \supset F$  be a chain of finite fields,  $|E| = q^4$ ,  $|K| = q^2$ ,  $|F| = q$ ,  $q = 2^m$ . Then*

$$Q(x) = \text{Tr}_{K/F}(N_{E/K}(x))$$

*is a non-degenerate quadratic form on the 4-dimensional vector space  $E$  over  $F$ . Moreover, the set*

$$\mathcal{O} = \{u \in E \mid N_{E/K}(u) = 1\} = \{u \in E \mid u^{q^2+1} = 1\}$$

*determines an elliptic quadric in  $\text{PG}(3, q)$ .*

It is readily checked that  $\mathcal{O} \in \mathcal{P}$ . In [2], a presentation of  $\text{PG}(3, q)$  to which the elliptic quadric  $\mathcal{O}$  belongs was not provided. Here we show that  $\mathcal{O}$  is an ovoid in the cyclic presentation of  $\text{PG}(3, q)$ .

**Theorem 3.2.** *Let  $\beta \in \mathcal{P}$  and  $\alpha \in E$  satisfy condition (1). Let  $\lambda \in S$ . Then the system (I) of equations*

$$\begin{cases} X^{q+1} + \alpha X + \beta = 0 & (4) \\ X^{q^2+1} = \lambda & (5) \end{cases}$$

*has at most two solutions in  $\mathcal{P}$ .*

*Proof.* We follow the notation in the proof of Lemma 2.3. The  $q+1$  roots of (4) are in  $\mathcal{P}$  and are given by  $x_0 = \frac{\alpha^{q^3}}{1+c}$  and  $x_0(u^q + \gamma)^{q-1}$  for each  $\gamma \in F$ .

1. Suppose that there exists  $\gamma \in F$  such that  $x_0(u^q + \gamma)^{q-1}$  is a solution of (5), that is,

$$x_0^{q^2+1}(u^q + \gamma)^{(q-1)(q^2+1)} = \lambda.$$

Let

$$s := \frac{\lambda}{x_0^{q^2+1}} = (u^q + \gamma)^{(q-1)(q^2+1)} = \frac{(u^{q^2} + \gamma)^{q^2+1}}{(u^q + \gamma)^{q^2+1}}.$$

Since  $u \in E \setminus K$  and  $\gamma \in F$ , we have that  $(u^q + \gamma)^{q^2+1} \neq 0$ . Then

$$\begin{aligned} (u^q + \gamma)^{q^2+1}s &= (u^{q^2} + \gamma)^{q^2+1} \\ \iff (u^{q^3} + \gamma)(u^q + \gamma)s &= (u^{q^2} + \gamma)(u + \gamma) \\ \iff A\gamma^2 + B\gamma + C &= 0, \end{aligned}$$

where

$$\begin{aligned} A &= s + 1, \\ B &= s(u^{q^3} + u^q) + u^{q^2} + u, \\ C &= su^{q^3+q} + u^{q^2+1}. \end{aligned}$$

2. The calculation above shows that  $x_0(u^q + \gamma)^{q-1}$  is a solution of (I) if and only if  $\gamma$  is a solution of the equation

$$AX^2 + BX + C = 0. \tag{6}$$

If  $s = 1$ , then  $A = 0$  and (6) has a unique solution  $\gamma_0 \in F$ . Then (I) has two solutions, which are  $x_0$  and  $x_0(u^q + \gamma_0)^{q-1}$ .

If  $s \neq 1$ , then  $A \neq 0$  and (6) has at most two solutions in  $K$  (and consequently in  $F$ ). Furthermore, since  $s = \lambda/x_0^{q^2+1}$ , we see that  $x_0$  is not a solution of (5). It follows that (I) has at most two solutions.  $\square$

## 4 Suzuki-Tits ovoids

### 4.1 Polarities in the cyclic presentation of $\text{PG}(3, q)$

Let  $q = 2^m$ , where  $m \geq 3$  is odd. We still consider the cyclic presentation of  $\text{PG}(3, q)$  and follow the terminology introduced in Subsection 2.3. For every point  $\lambda$  of  $\mathcal{P}$ , let

$$\omega_\lambda(x) := (\lambda^{q^2}x)^{q^2+q+1} + (\lambda^{q^2}x)^{q+1} + (\lambda^{q^2}x) + 1$$

be a polynomial over  $\mathbb{F}_{q^4}$  whose zeros correspond to the points of a plane. Then a symplectic polarity  $\omega$  of  $\text{PG}(3, q)$  was given in [5] as  $\omega : \lambda \mapsto \omega_\lambda(x) = 0$ .

For  $\beta \in \mathcal{P}$ , let  $L(\beta)$  be the line defined by the equation

$$X^{q+1} + (\beta^{(q^2+q+2)/2} + \beta^{(q+1)/2})X + \beta = 0.$$

The totally isotropic lines with respect to  $\omega$  are the lines  $L(\beta)$ , where  $\beta \in \mathcal{P}$ . This was proved in [6]. Let  $\mathcal{L}$  be the set of all such totally isotropic lines. Let  $W(q) = (\mathcal{P}, \mathcal{L})$  be the corresponding generalized quadrangle.

This presentation of the generalized quadrangle  $W(q)$  was already considered by Glauberman in [11], where it was referred to as the ‘‘symplectic geometry  $\mathcal{G}$  on  $E$ ’’. However, the totally isotropic lines were not described explicitly by Glauberman.

In [11, p. 64], a correlation  $\delta$  of  $W(q)$  was defined with respect to a symplectic basis of  $E$  over  $F$ . Furthermore, it was proved (also in [11, p. 64]) that this map  $\delta$  can be defined equivalently by the following.

**Definition 4.1** (The map  $\delta$ ). Let  $\delta : W(q) \rightarrow W(q)$  be defined as follows:

1.  $\delta(x) = L(x^{2q})$  for every point  $x \in \mathcal{P}$ ,
2.  $\delta(L(x)) = x$  for every line  $L(x) \in \mathcal{L}$ .

**Remark 1.** It was pointed out in [6, p.6] that the point  $x$  lies on  $L(\beta)$  if and only if  $\beta$  lies on  $L(x^{2q})$ . For a proof, we refer the reader to Lemma 4.5. This provides an alternative way to show that  $\delta$  is a correlation of  $W(q)$ .

**Definition 4.2** (The maps  $\sigma_i$  and  $c(\sigma)$ ). For each nonnegative integer  $i$ , let  $\sigma_i$  be the automorphism  $x \mapsto x^{2^i}$  of  $E$ . For each element  $\sigma$  of the Galois group  $\text{Gal}(E)$ , let  $c(\sigma)$  denote the collineation of  $W(q)$  given by

$$W^{c(\sigma)} = \{\sigma(w) \mid w \in W\},$$

for every point or line  $W$ .

Let  $\theta_0 = \sigma_{\frac{m+1}{2}}$  and  $\theta_1 = \sigma_{\frac{m+1}{2}+2m}$ . Then  $\theta_0(x) = x^{\sqrt{2q}}$  and  $\theta_1(x) = x^{q^2\sqrt{2q}}$ , for all  $x \in E$ . In [11, Proposition 3.6], Glauberman proved the following.

**Lemma 4.1.** *The maps  $\pi_i := c(\theta_i)^{-1}\delta$  are polarities of  $W(q)$ .*

**Remark 2.** We can describe the maps  $\pi_0$  and  $\pi_1$  explicitly. For  $x \in \mathcal{P}$ , we have that

$$\pi_0(x) = c(\theta_0)^{-1}\delta(x) = c(\theta_0)^{-1}(L(x^{2q})) = L(x^{\sqrt{2q}}).$$

Using the relation  $x^{1/q^2} = x^{q^2}$  for  $x \in E$ , the map  $\pi_0 : W(q) \rightarrow W(q)$  can be defined as follows:

1.  $\pi_0(x) = L(x^{\sqrt{2q}})$  for every point  $x \in \mathcal{P}$ ,
2.  $\pi_0(L(x)) = x^{1/\sqrt{2q}}$  for every line  $L(x) \in \mathcal{L}$ .

Similarly, the map  $\pi_1 : W(q) \rightarrow W(q)$  is defined as follows:

1.  $\pi_1(x) = L(x^{q^2\sqrt{2q}})$  for every point  $x \in \mathcal{P}$ ,
2.  $\pi_1(L(x)) = x^{q^2/\sqrt{2q}}$  for every line  $L(x) \in \mathcal{L}$ .

## 4.2 Absolute points of polarities

Recall that  $q = 2^m$ , where  $m \geq 3$  is odd. Let  $s := q - \sqrt{2q} + 1$ ,  $t := q + \sqrt{2q} + 1$ . We define  $\mathcal{O}_s := \{x \in E \mid x^s = 1\}$ , and  $\mathcal{O}_t := \{x \in E \mid x^t = 1\}$ . Then  $q^3 + q^2 + q + 1 = (q+1)(q^2+1) = (q+1)st$ ,  $\mathcal{O} = \mathcal{O}_s\mathcal{O}_t$ , and every element  $x \in \mathcal{P}$  can be decomposed uniquely as  $x = \lambda uv$ ,  $\lambda \in S$ ,  $u \in \mathcal{O}_s$ ,  $v \in \mathcal{O}_t$ .

**Lemma 4.2.** *Let  $x \in \mathcal{O}_s \cup \mathcal{O}_t$  such that  $x \neq 1$ . Let  $\lambda = \left(x^{q-1} + \frac{1}{x^{q-1}}\right)^{q-1}$ . Then*

$$x^{q+1} + \sqrt{(\lambda\sqrt{2q} + 1)\lambda x + \lambda^{\sqrt{2q}/2}} = 0.$$

*Proof.* We first note that  $\lambda \in S$ . By Lemma 2.4, the equation that describes the line going through  $x$  and  $1/x$  is given by  $X^{q+1} + \alpha X + \beta = 0$ , where  $\beta = \left(x + \frac{1}{x}\right)^{q-1}$  and

$$\alpha = \frac{x^{q+1} + 1/x^{q+1}}{x + 1/x}.$$

Since  $x \in \mathcal{O}_s \cup \mathcal{O}_t \subset \mathcal{O}$ , we have that  $x^{q-1} = x^{q^2+q} = x^{q(q+1)}$ . Hence  $x^{q-1} = x^{q\sqrt{2q}}$  or  $x^{q-1} = x^{-q\sqrt{2q}}$ . This implies that

$$\lambda = \left(x^{q-1} + \frac{1}{x^{q-1}}\right)^{q-1} = \left(x^{q\sqrt{2q}} + \frac{1}{x^{q\sqrt{2q}}}\right)^{q-1} = \left(x + \frac{1}{x}\right)^{(q-1)q\sqrt{2q}} = \beta^{q\sqrt{2q}}.$$

In particular,  $\beta = \lambda^{\sqrt{2q}/2}$ . It remains to prove that  $\alpha^2 = (\lambda\sqrt{2q} + 1)\lambda$ . For convenience, we denote  $x^n + 1/x^n$  by  $D_n$ . We have

$$\begin{aligned} D_{q+1}^{q+1} &= D_{q+1}^q D_{q+1} = D_{q^2+q} D_{q+1} = D_{q-1} D_{q+1} = D_1^{2q} + D_1^2, \\ (\beta + \beta^q) D_1^{q+1} &= (D_1^{q-1} + D_1^{q^2-q}) D_1^{q+1} = D_1^{2q} + D_1^2. \end{aligned}$$

This implies that  $D_{q+1}^{q+1} = (\beta + \beta^q) D_1^{q+1}$ , so that

$$\alpha^{q+1} = \left(\frac{x^{q+1} + 1/x^{q+1}}{x + 1/x}\right)^{q+1} = \left(\frac{D_{q+1}}{D_1}\right)^{q+1} = \beta + \beta^q. \quad (7)$$

On the other hand, since  $x \in \mathcal{O}_s \cup \mathcal{O}_t$ , either  $x^{q+1} = x^{\sqrt{2q}}$  or  $x^{q+1} = x^{-\sqrt{2q}}$ , and so

$$\alpha^{q-1} = \left(\frac{x^{q+1} + 1/x^{q+1}}{x + 1/x}\right)^{q-1} = \left(\frac{D_1^{\sqrt{2q}}}{D_1}\right)^{q-1} = \beta^{\sqrt{2q}-1}. \quad (8)$$

From (7) and (8), it follows that  $\alpha^2 = \alpha^{q+1}/\alpha^{q-1} = (\beta + \beta^q)\beta^{1-\sqrt{2q}}$ . From the relations  $\lambda = \beta^{q\sqrt{2q}}$  and  $\beta = \lambda^{\sqrt{2q}/2}$ , we obtain that

$$\alpha^2 = (\beta + \beta^q)\beta^{1-\sqrt{2q}} = (\lambda^{\sqrt{2q}/2} + \lambda^{q\sqrt{2q}/2})\lambda^{\sqrt{2q}/2}\lambda = (\lambda^{\sqrt{2q}} + 1)\lambda.$$

The proof then follows since  $x$  fulfills  $X^{q+1} + \alpha X + \beta = 0$  with  $\alpha = \sqrt{(\lambda\sqrt{2q} + 1)\lambda}$  and  $\beta = \lambda^{\sqrt{2q}/2}$ .  $\square$

We now introduce two sets. Let

$$\mathcal{T}_0 := \mathcal{O}_s \cup \left\{ \left( v^{q-1} + \frac{1}{v^{q-1}} \right)^{q-1} uv \mid u \in \mathcal{O}_s, v \in \mathcal{O}_t \setminus \{1\} \right\},$$

and

$$\mathcal{T}_1 := \mathcal{O}_t \cup \left\{ \left( u^{q-1} + \frac{1}{u^{q-1}} \right)^{q-1} uv \mid u \in \mathcal{O}_s \setminus \{1\}, v \in \mathcal{O}_t \right\}.$$

**Theorem 4.3.** *For  $i \in \{0, 1\}$ , the set  $\mathcal{T}_i$  is the set of absolute points of the polarity  $\pi_i$ . In particular,  $\mathcal{T}_i$  is a Suzuki-Tits ovoid.*

*Proof.* We will prove the theorem for the case  $i = 0$ , as the case  $i = 1$  is similar. Since  $|\mathcal{T}_0| = q^2 + 1$ , it is sufficient to show that every point of  $\mathcal{T}_0$  is an absolute point of  $\pi_0$ , that is  $x \in L(x^{\sqrt{2q}})$  for every  $x \in \mathcal{T}_0$ .

Let  $x \in \mathcal{T}_0$  be of the form  $x = \lambda uv$ , where  $\lambda \in S, u \in \mathcal{O}_s, v \in \mathcal{O}_t$ . Let

$$A := (x^{\sqrt{2q}})^{(q^2+q+2)/2} + (x^{\sqrt{2q}})^{(q+1)/2} = \left( x^{q+1}(x^{q^2+1} + 1) \right)^{\sqrt{2q}/2} = \left( \frac{u}{v} \right)^q (\lambda + 1)^{\sqrt{2q}}.$$

Then

$$\begin{aligned} x \in L(x^{\sqrt{2q}}) &\iff x^{q+1} + Ax + x^{\sqrt{2q}} = 0 \\ &\iff (uv)^{q+1} + \left( \frac{u}{v} \right)^q (\lambda + 1)^{\sqrt{2q}} (\lambda uv) + (\lambda uv)^{\sqrt{2q}} = 0 \\ &\iff v^{2(q+1)} + (\lambda^{\sqrt{2q}} + 1) \lambda v^2 + \lambda^{\sqrt{2q}} = 0 \\ &\iff v^{q+1} + \sqrt{(\lambda^{\sqrt{2q}} + 1) \lambda v} + \lambda^{\sqrt{2q}/2} = 0. \end{aligned}$$

We finally note that if  $x \in \mathcal{O}_s$ , then  $\lambda = 1$  and  $v = 1$ . The last line is then clearly true for  $x \in \mathcal{O}_s$  and is also true for  $x \in \mathcal{T}_0 \setminus \mathcal{O}_s$ , by Lemma 4.2. Therefore,  $x \in L(x^{\sqrt{2q}})$  for every  $x \in \mathcal{T}_0$ . This implies that  $\mathcal{T}_0$  is the set of absolute points of the polarity  $\pi_0$  and also a Suzuki-Tits ovoid, by Theorem 2.5.  $\square$

### 4.3 The sets $\mathcal{T}_i$ as sets of zeroes of polynomials

We now show that  $\mathcal{T}_0$  is the set of zeroes of a short polynomial over  $\mathbb{F}_{q^4}$ .

**Theorem 4.4.** *The set  $\mathcal{T}_0$  is the set of solutions of the equation  $Q_0(x) = 0$ , where*

$$Q_0(x) = x^{q^2+1} + x^{s(\sqrt{2q}+1)} + x^s + 1.$$

*Proof.* Let  $x = \lambda uv \in \mathcal{P}$ , where  $\lambda \in S, u \in \mathcal{O}_s, v \in \mathcal{O}_t$ . Then  $x^{q^2+1} = \lambda^2$ ,  $x^{s(\sqrt{2q}+1)} = \lambda^{s+2}(v^s)^{\sqrt{2q}+1}$  and  $x^s = \lambda^s v^s$ . This means that

$$\begin{aligned} Q_0(x) = 0 &\iff \lambda^{s+2}(v^s)^{\sqrt{2q}+1} + \lambda^s v^s + \lambda^2 + 1 = 0 \\ &\iff v^{s\sqrt{2q}} + (\lambda^2 + 1) \lambda^{(s+2)q} v^{-s} + \lambda^{2q} = 0 \\ &\iff v^{2\sqrt{2q}(q+1)} + (\lambda^2 + 1) \lambda^{(2-\sqrt{2q})q} v^{2\sqrt{2q}} + \lambda^{2q} = 0 \\ &\iff v^{2\sqrt{2q}(q+1)} + (1 + \lambda^{2q}) \lambda^{\sqrt{2q}} v^{2\sqrt{2q}} + \lambda^{2q} = 0. \end{aligned}$$

Here we used the relations  $v^{q+1} = 1/v^{\sqrt{2q}}$ ,  $v^s = v^{2(q+1)} = 1/v^{2\sqrt{2q}}$ ,  $\lambda^{\sqrt{2q}} = 1/\lambda^s$ . Taking power  $1/\sqrt{2q}$  on both sides of the last line, one obtains

$$v^{2(q+1)} + (\lambda^{\sqrt{2q}} + 1)\lambda v^2 + \lambda^{\sqrt{2q}} = 0.$$

From the proof of Theorem 4.3, we see that each element of  $\mathcal{T}_0$  is a solution of  $Q_0(x) = 0$ . We finally note that  $\deg Q_0(x) = |\mathcal{T}_0| = q^2 + 1$  and this proves the theorem.  $\square$

**Lemma 4.5.** *For points  $x$  and  $\beta$  we have*

$$x \in L(\beta) \iff \beta \in L(x^{2q}).$$

*Proof.* Recall that  $x \in L(\beta)$  if and only if

$$x^{q+1} + (\beta^{(q^2+q+2)/2} + \beta^{(q+1)/2})x + \beta = 0. \quad (9)$$

Taking the  $q$ -th power and multiplying by  $\beta$ ,

$$\beta x^{q^2+q} + (\beta^{(q+1)/2} + \beta^{(q^2+q+2)/2})x^q + \beta^{q+1} = 0. \quad (10)$$

Calculating (10) minus  $x^{q-1}$  times (9) gives,

$$\beta^{q+1} + (x^{q^2+q} + x^{q-1})\beta + x^{2q} = 0.$$

Thus,

$$x \in L(\beta) \iff \beta \in L(x^{2q}). \quad \square$$

We now describe the polynomial of degree  $q^2 + 1$  whose roots are the elements of  $\mathcal{T}_1$ . In the following theorem  $\log$  refers to  $\log_2$ .

**Theorem 4.6.** *The set  $\mathcal{T}_1$  is the set of solutions of the equation  $Q_1(x) = 0$ , where*

$$Q_1(x) = x^{q^2+1} + 1 + x^t \left( \frac{1 + x^{\sqrt{2qt}(\sqrt{q/2}-1)}}{1 + x^{\sqrt{2qt}}} \right) + x^t \sum_{j=0}^{\log \sqrt{q/2}-1} x^{2^j(\sqrt{2q}-2)t} (1 + x^{\sqrt{2qt}})^{2^j-1}.$$

*Proof.* Writing, once again,  $x = \lambda uv$ , where  $\lambda^{q+1} = 1$ ,  $u \in \mathcal{O}_s$  and  $v \in \mathcal{O}_t$ , one obtains, as in the proof of Theorem 4.3,

$$x \in L(x^{q^2\sqrt{2q}}) \iff u^{2(q+1)} + (\lambda^{\sqrt{2q}} + 1)\lambda u^2 + \lambda^{\sqrt{2q}} = 0.$$

Using  $u^t = u^{2\sqrt{2q}}$ ,  $\lambda^{q^2+1} = \lambda^{-2q}$  and  $\lambda^t = \lambda^{\sqrt{2q}}$ ,

$$x^{q^2+tq+t+1} + x^{q^2+t+1} + x^t + 1 = 0 \iff u^{2\sqrt{2q}(q+1)}\lambda^{-2q} + \lambda^{\sqrt{2q}-2q}u^{2\sqrt{2q}} + \lambda^{\sqrt{2q}}u^{2\sqrt{2q}} + 1 = 0,$$

which implies, taking  $1/\sqrt{2q}$  power and multiplying by  $\lambda^{\sqrt{2q}}$ , that

$$x \in L(x^{q^2\sqrt{2q}}) \iff x^{q^2+tq+t+1} + x^{q^2+t+1} + x^t + 1 = 0.$$

Taking the right-hand side equation and multiplying by  $x^{(\sqrt{2q}-2)t}$ , we get

$$x^{2qt} + x^{qt} + x^{(\sqrt{2q}-1)t} + x^{(\sqrt{2q}-2)t} = 0. \quad (11)$$

Consecutively squaring this equation and summing, we get

$$x^{\sqrt{2q}qt} + x^{qt} + \sum_{j=0}^{\log \sqrt{q/2}} (x^{2^j(\sqrt{2q}-1)t} + x^{2^j(\sqrt{2q}-2)t}) = 0. \quad (12)$$

In the following, Lemma 4.5 implies the first equivalence and then taking  $\sqrt{q/2}$  roots in the second equivalence we have

$$x \in L(x^{q^2\sqrt{2q}}) \iff x^{q^2\sqrt{2q}} \in L(x^{2q}) \iff x^{2q^2} \in L(x^{2\sqrt{2q}})$$

which gives

$$x^{2q^2(q+1)} + (x^{(q^2+q+2)\sqrt{2q}} + x^{(q+1)\sqrt{2q}})x^{2q^2} + x^{2\sqrt{2q}} = 0.$$

Multiplying by  $x^{2q+2}$ , implies

$$x^{\sqrt{2q}qt+2t} + x^{(2q-\sqrt{2q}+2)t} + x^{2t} + 1 = 0. \quad (13)$$

Calculating (13) plus  $x^{(-\sqrt{2q}t+2t)}$  times (11) gives

$$x^{\sqrt{2q}qt+2t} + x^{(q-\sqrt{2q}+2)t} + x^{2t} + x^t = 0. \quad (14)$$

Now,  $x^t(12) + x^{-t}(14)$  gives

$$x^{(q-\sqrt{2q}+1)t} + x^t + 1 + x^{(q+1)t} + x^t \sum_{j=0}^{\log \sqrt{q/2}} (x^{2^j(\sqrt{2q}-1)t} + x^{2^j(\sqrt{2q}-2)t}) = 0.$$

This we can rewrite as

$$x^{q^2+1}(1 + x^{\sqrt{2q}t}) + x^t + 1 + x^{\sqrt{2q}t} + x^{(q-\sqrt{2q}+1)t} + x^t \sum_{j=0}^{\log \sqrt{q/2}-1} (x^{2^{j+1}(\sqrt{2q}-1)t} + x^{2^j(\sqrt{2q}-2)t}) = 0$$

and dividing by  $1 + x^{\sqrt{2q}t}$  gives

$$x^{q^2+1} + 1 + x^t \left( \frac{1 + x^{(q-\sqrt{2q})t}}{1 + x^{\sqrt{2q}t}} \right) + x^t \sum_{j=0}^{\log \sqrt{q/2}-1} x^{2^j(\sqrt{2q}-2)t} \left( \frac{x^{2^j(\sqrt{2q})t} + 1}{1 + x^{\sqrt{2q}t}} \right) = 0. \quad \square$$

**Example 1.** We describe some examples obtained from GAP [10]. For  $q = 8$  we have

$$Q_1(x) = x^{65} + x^{39} + x^{13} + 1,$$

and for  $q = 32$  we have

$$Q_1(x) = x^{1025} + x^{861} + x^{697} + x^{533} + x^{369} + x^{287} + x^{41} + 1.$$

## 5 Ovoids fixed by the Frobenius automorphism

Let  $O$  be an ovoid and define a bijective map  $\tau : O \rightarrow O$  by

$$x \in L(\tau(x))$$

for all  $x \in O$ .

We conclude the article with a theorem which allows us to restrict  $\tau$  by assuming that  $x \in O$  implies  $x^2 \in O$  for all  $x \in O$ , i.e. that  $O$  is Frobenius fixed. This is equivalent to assuming that the polynomial of degree  $q^2 + 1$ , whose zeros are the points of  $O$ , is an element of  $\mathbb{F}_2[X]$ .

**Theorem 5.1.** *If the points of  $O$  are Frobenius fixed, then*

$$\tau(\tau(x)^{1/2q}) = x$$

for all  $x \in O$ .

*Proof.* Since  $\tau$  is a bijection, we have that

$$x \in L(\tau(x)) \iff \tau^{-1}(x) \in L(x).$$

By Lemma 4.5, we have

$$x \in L(\tau(x)) \iff \tau(x) \in L(x^{2q}).$$

By taking  $1/2q$  power of the last inclusion we have

$$x \in L(\tau(x)) \iff \tau(x)^{1/2q} \in L(x).$$

Assuming  $O$  to be Frobenius fixed,  $\tau(x) \in O$  implies  $\tau(x)^{1/2q} \in O$ . Since there is a unique point of  $O$  on the line  $L(x)$ , we conclude that

$$\tau(x)^{1/2q} = \tau^{-1}(x)$$

for all  $x \in O$ . □

**Example 2.** The map  $\tau_0(x) = x^{\sqrt{2q}}$  corresponds to the Tits ovoid  $\mathcal{T}_0$ , the map  $\tau_1(x) = x^{q^2\sqrt{2q}}$  corresponds to the Tits ovoid  $\mathcal{T}_1$ . One can see that  $\pi_i(x) = L(\tau_i(x))$ . It is clear that these two examples satisfy  $\tau(\tau(x)^{1/2q}) = x$ .

The map  $\tau(x) = x^{q+1}$  gives the canonical elliptic quadric which are zeros of  $x^{q^2+1} + 1$ . To verify this claim, observe that  $\tau(x)$  is a bijection of  $O$ , since

$$x^{q+1} = y^{q+1} \Rightarrow x^{q^2-1} = y^{q^2-1} \Rightarrow x^2 = y^2 \Rightarrow x = y.$$

Moreover,  $x \in L(\tau(x))$ , since

$$x^{q+1} + (\tau(x))^{(q^2+q+2)/2} + \tau(x)^{(q+1)/2}x + \tau(x) = x^{q+1} + \tau(x) = 0.$$

Finally observe that  $\tau(\tau(x)^{1/2q}) = x^{(q+1)^2/2q} = x$ , for all  $x^{q^2+1} + 1 = 0$ .

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