FORBIDDEN SUBGRAPHS IN THE NORM GRAPH

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ABSTRACT. We show that the norm graph with \( n \) vertices about \( \frac{1}{2} n^{2-1/t} \) edges, which contains no copy of the complete bipartite graph \( K_{t,(t-1)!+1} \), does not contain a copy of \( K_{t+1,(t-1)!-1} \).

1. INTRODUCTION

Let \( H \) be a fixed graph. The Turán number of \( H \), denoted \( ex(n, H) \), is the maximum number of edges a graph with \( n \) vertices can have, which contains no copy of \( H \). The Erdős-Stone theorem from [7] gives an asymptotic formula for the Turán number of any non-bipartite graph, and this formula depends on the chromatic number of the graph \( H \).

When \( H \) is a complete bipartite graph, determining the Turán number is related to the “Zarankiewicz problem” (see [3], Chap. VI, Sect. 2, and [9] for more details and references). In many cases even the question of determining the right order of magnitude for \( ex(n, H) \) is not known.

Let \( K_{t,s} \) denote the complete bipartite graph with \( t \) vertices in one class and \( s \) vertices in the other. The probabilistic lower bound for \( K_{t,s} \)

\[
ex(n, K_{t,s}) \geq cn^{2-(s+t-2)/(st-1)}
\]

is due to Erdős and Spencer [6]. Kővari, Sós and Turán [15] proved that for \( s \geq t \)

\[
ex(n, K_{t,s}) \leq \frac{1}{2} (s - 1)^{1/t} n^{2-1/t} + \frac{1}{2} (t - 1) n.
\]

The norm graph \( \Gamma(t) \), which we will define the next section, has \( n \) vertices and about \( \frac{1}{2} n^{2-1/t} \) edges. In [1] (based on results from [14]) it was proven that the graph \( \Gamma(t) \) contains no copy of \( K_{t,(t-1)!+1} \), thus proving that for \( s \geq (t-1)!+1 \),

\[
ex(n, K_{t,s}) > cn^{2-1/t}
\]

for some constant \( c \).

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In [2], it was shown that $\Gamma(4)$ contains no copy of $K_{5,5}$, which improves on the probabilistic lower bound of Erdős and Spencer [6] for $ex(n,K_{5,5})$. In this article, we will generalise this result and prove that $\Gamma(t)$ contains no copy of $K_{t+1,(t-1)!-1}$. For $t \geq 5$, this does not improve the probabilistic lower bound of Erdős and Spencer, but, as far as we are aware, it is however the deterministic construction of a graph with $n$ vertices containing no $K_{t+1,(t-1)!-1}$ with the most edges.

2. The norm graph

Suppose that $q = p^h$, where $p$ is a prime, and denote by $\mathbb{F}_q$ the finite field with $q$ elements. We will use the following properties of finite fields. For any $a, b \in \mathbb{F}_q$, $(a+b)^q = a^q + b^q$, for any $i \in \mathbb{N}$. For all $a \in \mathbb{F}_q^\ast$, $a^q = a$ if and only if $a \in \mathbb{F}_q$. Finally $N(a) = a^{1+q+\cdots+q^{k-1}} \in \mathbb{F}_q$, for all $a \in \mathbb{F}_q^\ast$, since $N(a)^q = N(a)$.

Let $\mathbb{F}$ denote an arbitrary field. We denote by $\mathbb{P}_n(\mathbb{F})$ the projective space arising from the $(n+1)$-dimensional vector space over $\mathbb{F}$. Throughout dim will refer to projective dimension. A point of $\mathbb{P}_n(\mathbb{F})$ (which is a one-dimensional subspace of the vector space) will often be written as $\langle u \rangle$, where $u$ is a vector in the $(n+1)$-dimensional vector space over $\mathbb{F}$.

Let $\Gamma(t)$ be the graph with vertices $(a, \alpha) \in \mathbb{F}_{q^{t-1}} \times \mathbb{F}_q$, $\alpha \neq 0$, where $(a, \alpha)$ is joined to $(a', \alpha')$ if and only if $N(a+a') = \alpha \alpha'$. The graph $\Gamma(t)$ was constructed in [14], where it was shown to contain no copy of $K_{t,t+1}$. In [1] Alon, Rónyai and Szabó proved that $\Gamma(t)$ contains no copy of $K_{t,(t-1)!+1}$. Our aim here is to show that it also contains no $K_{t+1,(t-1)!-1}$, generalizing the same result for $t = 5$ presented in [2].

Let

$$V = \{(1, a) \otimes (1, a^q) \otimes \cdots \otimes (1, a^{q^{h-2}}) \mid a \in \mathbb{F}_{q^{t-1}}\} \subset \mathbb{P}_{2^{t-1}-1}(\mathbb{F}_{q^{t-1}}).$$

The set $V$ is the affine part of an algebraic variety that is in turn a subvariety of the Segre variety

$$\Sigma = \mathbb{P}_1 \times \mathbb{P}_1 \times \cdots \times \mathbb{P}_1,$$

where $\mathbb{P}_1 = \mathbb{P}_1(\mathbb{F}_q)$. We briefly recall that a Segre variety is the image of the Segre embedding:

$$\sigma : (v_1, v_2, \ldots, v_k) \in \mathbb{P}_{n_1-1} \times \mathbb{P}_{n_2-1} \times \cdots \times \mathbb{P}_{n_k-1} \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_k \in \mathbb{P}_{n_1n_2\cdots n_k-1},$$

i.e. it is the set of points corresponding to the simple tensors. For the reader that is not familiar to tensor products we remark that, up to a suitable choice of coordinates, if $v_i = (x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{n_i-1})$, then $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ is the vector of all possible products of type $x_{j_1}^{(1)} x_{j_2}^{(2)} \cdots x_{j_k}^{(k)}$ (see [12] for an easy overview on Segre varieties over finite fields).
Then, the affine point \( P_a = (1, a) \otimes (1, a^q) \otimes \cdots \otimes (1, a^{q^{t-2}}) \) has coordinates indexed by the subsets of \( T := \{0, 1, \ldots, t - 1\} \), where the \( S \)-coordinate is
\[
(\prod_{i \in S} a^q),
\]
for any non-empty subset \( S \) of \( T \) and
\[
\prod_{i \in S} a^q = 1
\]
when \( S = \emptyset \) (see [16]).

Let \( n = 2^{t-1} - 1 \).

We order the coordinates of \( \mathbb{P}_n(\mathbb{F}_{q^{t-1}}) \) so that if the \( i \)-th coordinate corresponds to the subset \( S \), then the \( (n - i) \)-th coordinate corresponds to the subset \( T \setminus S \).

Embed the \( \mathbb{P}_n(\mathbb{F}_{q^{t-1}}) \) containing \( V \) as a hyperplane section of \( \mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}}) \) defined by the equation \( x_{n+1} = 0 \).

Let \( b \) be the symmetric bilinear form on the \( (n+2) \)-dimensional vector space over \( \mathbb{F}_{q^{t-1}} \) defined by
\[
b(u, v) = \sum_{i=0}^{n} u_i v_{n-i} - u_{n+1} v_{n+1}.
\]

Let \( \perp \) be defined in the usual way, so that given a subspace \( \Pi \) of \( \mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}}) \), \( \Pi \perp \) is the subspace of \( \mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}}) \) defined by
\[
\Pi \perp = \{ v \mid b(u, v) = 0, \text{ for all } u \in \Pi \}.
\]

We wish to define the same graph \( \Gamma(t) \), so that adjacency is given by the bilinear form. Let \( P = (0, 0, 0, \ldots, 1) \). Let \( \Gamma' \) be a graph with vertex set the set of points on the lines joining the aff points of \( V \) to \( P \) obtained using only scalars in \( \mathbb{F}_q \), distinct from \( P \) and not contained in the hyperplane \( x_{n+1} = 0 \). Join two vertices \( \langle u \rangle \) and \( \langle u' \rangle \) in \( \Gamma' \) if and only if \( b(u, u') = 0 \). It is a simple matter to verify that the graph \( \Gamma' \) is isomorphic to the graph \( \Gamma(t) \) by the map \( P_a + \alpha P \mapsto (a, \alpha) \) since
\[
N(a + b) - \alpha \beta = \sum_{\emptyset \subseteq T \subseteq S} \prod_{i \in S, j \in T \setminus S} a^q b^q - \alpha \beta = b(u, v),
\]
where
\[
u = (1, a) \otimes (1, a^q) \otimes \cdots \otimes (1, a^{q^{t-2}}) + \alpha P,
\]
and
\[
v = (1, b) \otimes (1, b^q) \otimes \cdots \otimes (1, b^{q^{t-2}}) + \beta P.
\]

We shall refer to \( \Gamma' \) as \( \Gamma(t) \) from now on.

We recall some known properties of \( \Sigma \) and its subvariety
\[
\mathcal{V} = \{ (a, b) \otimes (a^q, b^q) \otimes \cdots (a^{q^{t-2}}, b^{q^{t-2}}) \mid (a, b) \in \mathbb{P}_1(\mathbb{F}_{q^{t-1}}) \}.
\]
and prove a new one in Theorem 2.5.

Let \( \mathbb{F}_q \) denote the algebraic closure of \( \mathbb{F}_q \) and consider \( \Sigma \) as the Segre variety over \( \mathbb{F}_q \).

**Theorem 2.1.** \( \Sigma \) is a smooth irreducible variety.

**Theorem 2.2.** The dimension of \( \Sigma \) (as algebraic variety) is \( t - 1 \) and its degree is \( (t - 1)! \).

**Proof.** The (Segre) product \( X \times Y \) of two varieties \( X \) and \( Y \) of dimension \( d \) and \( e \) has dimension \( d + e \), see, for example [13], page 138. The Hilbert polynomial of \( X \times Y \) is the product of the Hilbert polynomials of \( X \) and \( Y \) (see [13, Chapter 18]). The Hilbert polynomial \( h(m) \) of \( \mathbb{P}_1 \) is \( m + 1 \), hence the Hilbert polynomial of \( \Sigma = \mathbb{P}_1 \times \mathbb{P}_1 \times \cdots \times \mathbb{P}_1 \) \( t-1 \) times is \( h_\Sigma(m) = (m + 1)^{t-1} \). Since the leading term of \( h_\Sigma \) is 1 and the dimension of \( \Sigma \) is \( t - 1 \), we have that the degree of \( \Sigma \) is \( (t - 1)! \).

**Theorem 2.3.** [16] Any \( t \) points of \( \mathcal{V} \) are in general position.

**Theorem 2.4.** [11] If \( t + 1 \) points span a \( (t - 1) \)-dimensional projective space, then that space contains \( q + 1 \) points of \( \mathcal{V} \).

**Theorem 2.5.** If a subspace of codimension \( t \) contains a finite number of points of \( \Sigma \) then it contains at most \( (t - 1)! - 2 \) points of \( \Sigma \).

**Proof.** By Theorem 2.1, \( \Sigma \) is smooth, so it is regular at each of its points, i.e., if \( T_P \Sigma \) is the tangent space of \( \Sigma \) at a point \( P \in \Sigma \), then \( \dim T_P \Sigma = t - 1 \).

Let \( \Pi \) be a subspace of codimension \( t \) containing a finite number of points of \( \Sigma \). Let \( P \in \Pi \cap \Sigma \). Then \( \dim \langle T_P \Sigma, \Pi \rangle \leq n - 1 \). Therefore, there is a hyperplane \( H \) containing \( \langle T_P \Sigma, \Pi \rangle \).

Suppose that \( H \) contains another tangent space \( T_R \Sigma \), with \( R \in \Pi \cap \Sigma \). The algebraic variety \( H \cap \Sigma \) has dimension \( t - 2 \) (since \( \Sigma \) is irreducible) and it has two singular points, \( P \) and \( R \). Since \( \dim H \cap \Sigma = t - 2 \) as an algebraic variety, there must be a linear subspace \( \Pi_1 \) of codimension \( t - 2 \) in \( H \) containing \( \Pi \) and such that \( \Pi_1 \cap H \cap \Sigma \) consists of \( \deg(H \cap \Sigma) \leq (t - 1)! \) points of \( \Sigma \) counted with their multiplicity. Since \( \Pi_1 \) contains \( P \) and \( R \), which are singular points and so with multiplicity at least 2, we have that

\[ |\Pi \cap \Sigma| \leq |\Pi_1 \cap \Sigma| \leq (t - 1)! - 2. \]

Suppose now that \( H \) does not contain any other tangent space \( T_R \Sigma \) with \( R \in \Pi \cap \Sigma \), \( R \neq P \). Then take \( R \in \Pi \cap \Sigma \) and consider a hyperplane \( H' \neq H \) containing \( \langle T_R \Sigma, \Pi \rangle \). Then the tangent spaces of \( P \) and \( R \) with respect to \( H \cap H' \cap \Sigma \) are \( T_P \Sigma \cap H' \) and \( T_R \Sigma \cap H \), and they both have dimension \( t - 2 \) (as linear spaces).

If \( \dim H \cap H' \cap \Sigma = t - 3 \) as an algebraic variety, then \( P \) and \( R \) are two singular points of \( H \cap H' \cap \Sigma \) and we can find, as before, a linear subspace \( \Pi_1 \) of codimension \( t - 3 \) in \( H \cap H' \) such that it contains \( \Pi \) and intersects \( H \cap H' \cap \Sigma \) in \( \deg(H \cap H' \cap \Sigma) \leq (t - 1)! \)
points, counted with their multiplicity. Since $P$ and $R$ have multiplicity at least 2, we have

$$|\Pi \cap \Sigma| \leq |\Pi_1 \cap \Sigma| \leq (t - 1)! - 2.$$  

If $\dim H \cap H' \cap \Sigma = t - 2$ as an algebraic variety, then $H \cap \Sigma$ is reducible. Hence, we have

$$H \cap \Sigma = V_1 \cup V_2 \cup \cdots \cup V_r,$$

where $V_i$ is an irreducible variety of dimension $t - 2$, for all $i = 1, \ldots, r$. So we have

$$H \cap H' \cap \Sigma = V_1 \cup V_2 \cup \cdots \cup V_s \cup W_{s+1} \cup W_{s+2} \cup \cdots \cup W_r,$$

where $W_i$ is a hyperplane section of $V_i$, for all $i = s+1, \ldots, r$. We observe that also $H' \cap \Sigma$ has to be reducible and, since the decomposition in irreducible components is unique, we have

$$H' \cap \Sigma = V_1 \cup V_2 \cup \cdots \cup V_{s'} \cup V_{s'+1} \cup V_{s'+2} \cup \cdots \cup V_r,$$

where $V_i$ and $V'_j$ are irreducible varieties of dimension $t - 2$.

We have, by hypothesis, that $T_P \Sigma \subset H$ and $P \in \Pi$. So either $P \in V_i$ and it is singular for $V_i$, for some $i \in \{1, 2, \ldots, r\}$, or it is not singular for $V_i$, for any $\ell \in \{1, 2, \ldots, r\}$.

Suppose we are in the first case. We know that $P \in \Pi \subset H'$. If $V_i \subset H'$, then $P$ is singular for an irreducible component of $H' \cap \Sigma$ and so $T_P \Sigma \subset H'$, contradicting our hypothesis, so $V_i$ is not contained in $H'$ and $H' \cap V_i = W_i$. We have that $\dim T_P \Sigma \cap H' = t - 2$ (as linear subspace) and $\dim W_i = t - 3$ (as algebraic variety), so $P$ is singular for $W_i$.

Suppose now that $P$ is not singular for any $V_i$, so the dimension of $T_P V_i$, as a subspace, is $t - 2$. If $P \notin V_j$, for any $i \neq j$, then

$$T_P (H \cap \Sigma) = T_P (V_i) = T_P (\Sigma),$$

a contradiction since the dimension of $T_P (\Sigma)$ is $t - 1$. Hence $P \in V_i \cap V_j$, and so $P$ is contained in the intersection of two components of $H' \cap \Sigma$, so it is again a singular (or multiple) point. The same is true for the point $R$ such that $T_R \Sigma \subset H'$, so in

$$V_1 \cup V_2 \cup \cdots \cup V_s \cup W_{s+1} \cup W_{s+2} \cup \cdots \cup W_r$$

there are at least two multiple points and when we sum up all the degrees, we count at least two points twice, hence, by

$$\sum_{i=1}^s \deg V_i + \sum_{j=s+1}^r \deg W_j \leq (t - 1)!,$$

we get that the number of points in

$$\Pi \cap (V_1 \cup V_2 \cup \cdots \cup V_s \cup W_{s+1} \cup W_{s+2} \cup \cdots \cup W_r),$$

is at most $(t - 1)! - 2$. \qed
Remark One could wonder whether one could try with one more hyperplane $H''$ such that $T_Q\Sigma \subset H''$, $T_Q\Sigma \notin H$, $T_Q\Sigma \notin H'$ and $Q \in \Pi$. However, it can happen that $H \cap H' \cap H'' = H \cap H'$, so $\dim T_Q\Sigma \cap H \cap H' \cap H'' = t - 2$ (as a linear space) and $\dim H \cap H' \cap H'' \cap \Sigma = t - 2$, so $Q$ would not be a singular point of $H \cap H' \cap H'' \cap \Sigma$.

The locus of hyperplanes containing a tangent space to a variety $X$ of $\mathbb{P}^n$ is a variety $X^*$ of the dual space $(\mathbb{P}_n)^*$ (see, e.g., [13, Chapter 15]). Let $\Sigma^*$ be the dual variety of $\Sigma$. From [17], we know that $\Sigma^*$ is a hypersurface, hence, if $d$ is the degree of $\Sigma^*$, then the number of points of $\Sigma^*$ on a general line of $(\mathbb{P}_n)^*$ is $d$. Suppose that the line of $(\mathbb{P}_n)^*$ defined by $H \cap H'$ is general, hence if $|\Pi \cap \Sigma| > d$, then we could find a point $Q \in \Pi \cap \Sigma$ such that $T_Q\Sigma \subset H''$ and $H''$ is a hyperplane not containing $H \cap H'$. If $d > (t - 1)! - 2$ then we would not be able to get a better bound than the bound in Theorem 2.5. The degree of $\Sigma^*$ is found in [10], where it is given by $N_{t-1}$, where $N_r$ is defined by the generating function

$$\sum_{r \geq 0} N_r z^r = \frac{e^{-2z}}{(1 - z)^2}.$$  

Hence $d = \deg \Sigma^*$, is the evaluation of

$$\left(\frac{e^{-2z}}{(1 - z)^2}\right)^{(t-1)}$$

at $z = 0$, where we denote by $f^{(n)}$ the $n$–th derivative of the function $f$.

Let $F = fg$, where $f$ and $g$ are two functions, then

$$F^{(n)} = \sum_{i=0}^{n} \binom{n}{i} f^{(i)} g^{(n-i)}.$$  

Let

$$f = e^{-2z} \text{ and } g = (1 - z)^{-2}.$$  

It is easy to see that

$$f^{(i)} = (-2)^i f \text{ and } g^{(i)} = (i + 1)!(1 - z)^{-(i+2)}.$$  

Since $f(0) = 1$, we have that $F^{(n)}$, evaluated at $z = 0$, is

$$\sum_{i=0}^{n} \binom{n}{i} (-2)^i (n + 1 - i)!.$$  

When $n = t - 1$ and we have

$$d = N_{t-1} = \sum_{i=0}^{t-1} \binom{t-1}{i} (-2)^i (t - i)!.$$
Now
\[ \sum_{i=0}^{t-1} \binom{t-1}{i} (-2)^i (t-i)! = (t-1)! \sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i). \]

Note that
\[ \sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i) = 1 \]

for \( t = 5 \) and
\[ \sum_{i=0}^{t} \frac{(-2)^i}{i!} (t+1-i) - \sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i) = \sum_{i=0}^{t} \frac{(-2)^i}{i!}. \]

Since \( \sum_{i=0}^{5} \frac{(-2)^i}{i!} = \frac{1}{15} \) and
\[ \frac{(-2)^{n-1}}{(n-1)!} - \frac{(-2)^n}{n!} = \frac{2^{n-1}(n-2)}{n!} > 0 \]

when \( n \geq 3 \) is odd,
\[ \sum_{i=0}^{t} \frac{(-2)^i}{i!} > 0 \]

for all \( t \geq 4 \) and so
\[ \sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i) \]

is an increasing function. Thus, for \( t \geq 5 \),
\[ \sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i) \geq 1, \]

and so
\[ (t-1)! \sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i) \geq (t-1)! \]

and hence \( d = N_{i-1} > (t-1)! - 2 \).

Theorem 2.6. For \( q \geq (t-1)! + 1 \) the graph \( \Gamma(t) \) contains no \( K_{t+1,(t-1)!-1} \).

Proof. Let \( X = \{x_1, x_2, \ldots, x_{t+1} \} \) be \( t + 1 \) distinct vertices of \( \Gamma(t) \). The set of common neighbours of the elements of \( X \) is \( \Pi^\perp \cap \Gamma(t) \), where \( \Pi \) is the subspace spanned by \( X \). If any two elements of \( X \) project from \( P \) onto the same point of \( V \), then \( P \in \Pi \) and hence \( \Pi^\perp \subset P^\perp \). Since \( P^\perp \) is the hyperplane \( x_{n+1} = 0 \), \( \Pi^\perp \cap \Gamma(t) = \emptyset \), and the elements of \( X \) have no common neighbour.

Therefore, we assume now that all the points in \( X \) project from \( P \) onto distinct points of \( V \). Then, by Theorem 2.3, \( \dim \Pi \geq t - 1 \).
If \( \text{dim} \, \Pi = t - 1 \), then by Theorem 2.4, the projection of \( \Pi \) onto \( V \) contains at least \( q \) points of \( V \) (we recall that \( V \) is the affine part of \( \mathcal{V} \) and the hyperplane section we removed contains just one point of \( \mathcal{V} \)). Therefore, there are at least \( q \) points \( Y \) of \( \Pi \) on the lines joining \( P \) to the points of \( V \). We wish to prove that the points of \( Y \) are vertices of the graph \( \Gamma(t) \). To do this, we have to show that the points of \( Y \), which are of the form \( \langle (v, \lambda) \rangle \), where \( v \in V \) and \( \lambda \in \mathbb{F}_q \), are of the form \( \langle (u, \mu) \rangle \), where \( u \in V \), \( u \neq -v \) and \( \mu \in \mathbb{F}_q \), is a common neighbour of the elements of \( X \) of the form \( \langle (u, \mu) \rangle \), where \( u \in V \), \( u \neq -v \) and \( \mu \in \mathbb{F}_q \), is a common neighbour of the elements of \( X \). Then \( \langle (u, \mu) \rangle \) is in \( \Pi^\perp \) and since \( Y \subset \Pi \),

\[
N(u + v) = \lambda \mu.
\]

Since \( N(u + v) \in \mathbb{F}_q \) and \( \mu \in \mathbb{F}_q \), we have that \( \lambda \in \mathbb{F}_q \) and so the points of \( Y \) are vertices of the graph \( \Gamma(t) \). Therefore, the vertices of \( X \) have at least \( q \) common neighbours. Since \( \Gamma \) contains no \( K_{t,(t-1)!+1} \), if \( q \geq (t - 1)! + 1 \), then this case cannot occur.

If \( \text{dim} \, \Pi = t \) then \( \text{dim} \, \Pi^\perp = n - t \). Let \( Y \) be the points of \( \Pi^\perp \) which project from \( P \) onto \( V \). Arguing as in the previous paragraph, the points \( Y \) are vertices of the graph \( \Gamma(t) \). Since the vertices of \( X \) have at most \( (t - 1)! \) common neighbours, there are a finite number of points in \( Y \) and so a finite number of points in the projection of \( \Pi^\perp \) onto \( V \). By Theorem 2.5, this projection contains at most \( (t - 1)! - 2 \) points of \( V \), so there are at most \( (t - 1)! - 2 \) points in \( Y \). Therefore, the vertices in \( X \) have at most \( (t - 1)! - 2 \) common neighbours. \( \square \)

References


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