

# Multiple blocking sets and arcs in finite planes \*

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## Abstract

This paper contains two main results relating to the size of a multiple blocking set in  $PG(2, q)$ . The first gives a very general lower bound, the second a much better lower bound for prime planes. The latter is used to consider maximum sizes of  $(k, n)$ -arcs in  $PG(2, 11)$  and  $PG(2, 13)$ , some of which are determined. In addition, a summary is given of the value of  $m_n(2, q)$  for  $q \leq 13$ .

## 1. Introduction

A  $t$ -fold blocking set  $B$  in a projective or affine plane, is a set of points such that each line contains at least  $t$  points of  $B$  and some line contains exactly  $t$  points of  $B$ . A 1-fold blocking set is called a blocking set. A blocking set in a projective plane is defined with the extra condition it should contain no line. A 2-fold blocking set is called a *double* blocking set and a 3-fold blocking set is called a *triple* blocking set. A  $(k, n)$ -arc in a projective plane  $\Pi$ , is a set of  $k$  points such that some  $n$ , but no  $n + 1$  of them, are collinear. A  $(k, 2)$ -arc is called a  $k$ -arc. Note that  $(k, n)$ -arcs and multiple blocking sets are in fact just complements of each other in a projective plane, with  $n + t = q + 1$ . Define  $m_n(2, q)$  to be the maximum size of a  $(k, n)$ -arc in  $PG(2, q)$ . An  $i$ -secant to a set  $K$  of points is a line meeting  $K$  in exactly  $i$  points. Let  $\tau_i$  be the number of  $i$ -secants of a set  $K$ . For basic properties relating to projective planes see [21].

In 1947 Bose [11] proved that  $m_2(2, q) = q + 1$  for  $q$  odd, and  $m_2(2, q) = q + 2$  for  $q$  even. In the mid 1950's Segre proved that for  $q$  odd every  $(q + 1)$ -arc is a conic, [29, 30], for  $q=2$ ,  $q = 4$ ,  $q = 8$  every  $(q + 2)$ -arc is a conic plus its nucleus [31], and for  $q = 16$ ,  $q = 32$ ,  $q = 2^h$  ( $h \geq 7$ ), there exists a  $(q + 2)$ -arc other than the conic plus its nucleus.

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\*Mathematics Subject Classification 51E21

In 1956 Barlotti [4] proved the first of many results in the attempt to determine the value of  $m_n(2, q)$ . This has proved to be far from simple. Early results by Barlotti bounded  $m_n(2, q)$  with  $m_n(2, q) \leq (n - 1)q + n$  and for  $(n, q) = 1$  and  $n > 2$  he proved  $m_n(2, q) \leq (n - 1)q + n - 2$ . Lunelli and Sce [24] improved these bounds in 1964, proving that  $m_n(2, q) \leq (n - 1)q + n - 3$  for  $4 \leq n \leq q$ , with  $(n, q) = 1$  and  $m_n(2, q) \leq (n - 1)q + n - 4$  for  $9 \leq n < q$ , with  $(n, q) = 1$ . They also conjectured that  $m_n(2, q) \leq (n - 1)q + 1$  for  $(n, q) = 1$  but this was shown to be false by Hill and Mason [19] in 1981. However, this conjecture was shown to hold in the affine case by Blokhuis [10]; if a  $(k, n)$ -arc has an external line and  $(n, q) = 1$  then  $k \leq (n - 1)q + 1$ . It will be shown, in this paper, to hold in prime planes when  $n \leq (q + 3)/2$ .

The value of  $m_n(2, q)$  is known for  $2 \leq n < q \leq 9$ . A full table of these values is given in the final section. Few values of  $m_n(2, q)$  are known in general; the following are those that have been determined.

1.  $m_2(2, q) = q + 1$  for  $q$  odd.
2.  $m_n(2, q) = (n - 1)q + n$  for  $q = 2^h$  and  $n = 2^e$ .
3.  $m_n(2, q) = (n - 1)q + 1$  for  $q$  odd prime,  $n = (q + 1)/2$ ,  $n = (q + 3)/2$ .
4.  $m_n(2, q) = (n - 1)q + q - 2\sqrt{q} - 1$ , for  $q > 4$  square,  $n = q - 1$ .
5.  $m_n(2, q) = (n - 1)q + q - 3\sqrt{q} - 2$ , for  $q > 121$  an odd square,  $n = q - 2$ .

1. Bose [11], 2. Denniston [18], 3. Barlotti [5] and Theorem 4.2,
4. Ball and Blokhuis [1], 5. Ball [2]

Blocking sets were first studied in detail by di Paola [26], who determined the minimum size of a non-trivial blocking set in the Desarguesian projective planes of order 4, 5, 7, 8 and 9. They were originally defined as a game theory problem in [28]. Richardson defines a finite projective game by taking as players the points of the plane and specifying the lines as the minimal winning coalitions. A survey of blocking sets in Desarguesian planes can be found in [8].

As mentioned before, multiple blocking sets ( $t > 1$ ) are simply complements of  $(k, n)$ -arcs in projective planes, with  $n + t = q + 1$ . In general we will talk about  $(k, n)$ -arcs when  $n$  is small and multiple blocking sets when  $t$  is small.

The two main results of this paper are the following.

**Theorem 1.1** *Let  $B$  be a  $t$ -fold blocking set in  $PG(2, q)$ . If  $B$  contains no line then it has at least  $tq + \sqrt{tq} + 1$  points.*

**Theorem 1.2** *Let  $B$  be a  $t$ -fold blocking set in  $PG(2, p)$ ,  $p > 3$  prime.*

1. *If  $t < p/2$  then  $|B| \geq (t + \frac{1}{2})(p + 1)$ .*
2. *If  $t > p/2$  then  $|B| \geq (t + 1)p$ .*

The bound in Theorem 1.1 can be much improved for  $t = 2$  and 3. The following two theorems come from [1] and [2]. These theorems also give the bound in Theorem 1.2 although improvements were made for small planes.

**Theorem 1.3** *Let  $B$  be a double blocking set in  $PG(2, q)$ .*

1. *If  $q < 9$  then  $B$  has at least  $3q$  points.*
2. *If  $q = 11, 13, 17$  or  $19$  then  $|B| \geq (5q + 7)/2$ .*
3. *If  $19 < q = p^{2d+1}$  then  $|B| \geq 2q + p^d[(p^{d+1} + 1)/(p^d + 1)] + 2$ .*
4. *If  $4 < q$  is a square then  $|B| \geq 2q + 2\sqrt{q} + 2$ .*

**Theorem 1.4** *Let  $B$  be a triple blocking set in  $PG(2, q)$ .*

1. *If  $q = 5, 7$  or  $9$  then  $B$  has at least  $4q$  points and if  $q = 8$  then  $B$  has at least  $31$  points.*
2. *If  $q = 11, 13$  or  $17$  then  $|B| \geq (7q + 9)/2$ .*
3. *If  $17 < q = p^{2d+1}$  then  $|B| \geq 3q + p^d[(p^{d+1} + 1)/(p^d + 1)] + 3$ .*
4. *If  $4 < q$  is an even square or  $q = 25, 49, 81$  or  $121$  then  $|B| \geq 3q + 2\sqrt{q} + 3$ .*
5. *If  $121 < q$  is an odd square then  $|B| \geq 3q + 3\sqrt{q} + 3$ .*

The following section is included for completeness since Theorem 1.1 restricts the set to containing no line. If  $B$ , a  $t$ -fold blocking set, contains a line  $l$  then  $B \setminus l$  is a  $(t - 1)$ -fold blocking set in  $AG(2, q) = PG(2, q) \setminus l$ .

## 2. Multiple blocking sets in $AG(2, q)$

The problem of finding the size of the smallest blocking set in  $AG(2, q)$  was effectively finished by Jamison in [23] and independently by Brouwer and Schrijver [13] who proved respectively Theorem 2.1 and Theorem 2.2.

**Theorem 2.1** *Let  $V$  be a vector space of dimension  $n$  over  $GF(q)$ . If  $0 < k < n$ , then any covering of  $V^*$  (the non-zero elements in  $V$ ) with non-zero  $k$ -flats contains at least*

$$q^{n-k} - 1 + k(q - 1)$$

*$k$ -flats. Furthermore, a covering with this number of  $k$ -flats is always possible.*

**Theorem 2.2** *The blocking number of the affine space  $AG(s, q)$  is  $s(q-1)+1$ .*

The blocking number being the number of points required to form a blocking set. The bound in the  $AG(2, q)$  case is attained by, for example, the union of points on any two intersecting lines. The following generalisation was proven by Bruen [14], for  $t = 2$  and  $s = 2$ , and then in general by Bruen in [15].

**Theorem 2.3** *A  $t$ -fold blocking set in  $AG(s, q)$  has at least  $(t+s-1)(q-1)+1$  points.*

This result was improved by Blokhuis when  $(t, q) = 1$  and  $s = 2$ . He proves the following theorem in [10].

**Theorem 2.4** *Let  $B$  be a  $t$ -fold blocking set in  $AG(2, q)$ , where  $(t, q) = 1$ . Then  $B$  has at least  $(t + 1)q - 1$  points.*

If  $(t, q) > 1$  and  $t > q/2$  then the lower bound  $t(q + 1)$ , obtained by counting the points of the  $t$ -fold blocking set on lines through a point not in the set, is in fact a better bound than that proved by Bruen. This is trivial but by combining this comment with the previous two theorems the following theorem is implied.

**Theorem 2.5** *Let  $B$  be a  $t$ -fold blocking set in  $PG(2, q)$  that contains a line.*

1. *If  $(t - 1, q) = 1$ , then  $|B| \geq q(t + 1)$ .*
2. *If  $(t - 1, q) > 1$  and  $t \leq \frac{q}{2} + 1$ , then  $|B| \geq tq + q - t + 2$ .*
3. *If  $(t - 1, q) > 1$  and  $t \geq \frac{q}{2} + 1$ , then  $|B| \geq t(q + 1)$ .*

To be able to prove the bound relating to the prime planes a theorem on lacunary polynomials is needed. This theorem was also used in [1], [2] and is a very useful tool.

### 3. Lacunary polynomials

A polynomial in  $GF(q)[x]$  is called *fully reducible* if it factors completely into linear factors over  $GF(q)$ . If in the sequence of coefficients of a polynomial a long run of zeros occurs we call this polynomial *lacunary*. In [27] Rédei studied properties of lacunary polynomials that are fully reducible. The following theorem copied together with the proof from [8] is really just a slight generalisation of Theorem 24' in [27]. In the following  $q = p^h$ , where  $p$  is prime.

**Theorem 3.1** *Let  $f \in GF(q)[x]$  be fully reducible, and suppose that  $f(x) = x^q v(x) + w(x)$ , where  $v$  and  $w$  have no common factor. Let  $m < q$  be the maximum of the degrees of  $v$  and  $w$ . Let  $e$  be maximal such that  $f$  (and hence  $v$  and  $w$ ) is a  $p^e$ -th power. Then one of the following holds:*

1.  $e = h$  and  $m = 0$ ;
2.  $e \geq h/2$  and  $m \geq p^e$ ;
3.  $e < h/2$  and  $m \geq p^e \lceil (p^{h-e} + 1)/(p^e + 1) \rceil$ ;
4.  $e = 0$ ,  $m = 1$  and  $f(x) = a(x^q - x)$ .

Note that in particular when  $q$  is prime and  $m > 1$ , then  $m \geq (q + 1)/2$ .

**Proof :** Assume  $e < h/2$ , for the first two possibilities are easily checked. Write  $E = p^e$ . Let  $f(x) = f_1(x)^E$  and define  $v_1$  and  $w_1$  similarly. Then extracting  $E$ -th roots we get

$$f_1 = x^{q/E} v_1 + w_1.$$

Now write  $f_1(x) = \alpha(x)\beta(x)$  where  $\alpha(x)$  contains all different linear factors of  $f_1$  exactly once, and  $\beta(x)$  the rest. Since  $\alpha \mid x^q - x$  and  $\alpha \mid f = x^q v + w$  these imply

$$\alpha \mid xv + w.$$

Since  $\beta \mid f'_1$  and  $\beta \mid f_1$  these imply  $\beta \mid f'_1 v_1 - v'_1 f_1$  or

$$\beta \mid w'_1 v_1 - v'_1 w_1.$$

Note that  $(v, w) = 1$  and  $v_1$  and  $w_1$  are not both  $p$ -th powers, so the right hand side does not vanish. Combining these two divisibility relations we get

$$f_1 (= \alpha\beta) \mid (xv + w)(w'_1 v_1 - v'_1 w_1).$$

Now if  $xv + w = 0$  then  $m = 1$  since  $(v, w) = 1$ , and  $f$  has the desired form. Otherwise the degree of the left hand side is at most equal to that of the right hand side. First consider the case that  $\deg v = m = Em_1$ . In this case

$$q/E + m_1 \leq 1 + Em_1 + 2m_1 - 2.$$

Hence

$$m_1 \geq \frac{q/E + 1}{E + 1}.$$

The other case ( $\deg v < \deg w = m$ ) is similar and gives the same conclusion.  $\square$

#### 4. Two main results

The object of this section is to obtain good lower bounds for the size of multiple blocking sets in  $PG(2, q)$ . Theorem 4.1 is a very general bound. Although there are no known examples which attain it when  $t > 1$ , and it is unlikely to be sharp, it is an improvement on previous results. Theorem 4.2 uses the theorem on lacunary polynomials. It is a significant improvement to Theorem 4.1 for  $q$  prime, and is sharp for some values of  $t$ .

**Theorem 4.1** *Let  $B$  be a  $t$ -fold blocking set in  $PG(2, q)$ . If  $B$  contains no line then it has at least  $tq + \sqrt{tq} + 1$  points.*

**Proof :** Let  $B$  be a  $t$ -fold blocking set in  $PG(2, q)$  of size  $tq + m$  with  $l$ , the line at infinity, being a  $t$ -secant. Let  $S = B \setminus l$ ; so  $S$  is contained in the affine plane  $AG(2, q) = PG(2, q) \setminus l$ , and has size  $tq + m - t$ .

Assume  $B$  has a secant of length at least  $\lceil \sqrt{tq} + 1 \rceil$ . Take a point  $P$  on this secant but not in  $B$ . Then all other lines through  $P$  must contain  $t$  points of  $B$ ; so  $B$  has at least  $tq + \lceil \sqrt{tq} \rceil + 1 \geq tq + \sqrt{tq} + 1$  points unless the secant was a  $(q + 1)$ -secant, in which case no such  $P$  would exist.

So assume  $B$  has secants of length at most  $\lceil \sqrt{tq} \rceil$ . Every two points of  $S$  determine a direction, the slope of the line joining them, and so  $S$  determines  $\binom{tq + m - t}{2}$  directions. Let  $P$  be a point of  $B$  on the line at infinity, let  $d_P$  be the direction of the lines meeting at  $P$  and let  $m_P$  be the number of the directions determined by  $S$  that are equal to  $d_P$ . Since  $P$  is a point of  $B$  any secant to  $S$  through  $P$  has at most  $\lceil \sqrt{tq} \rceil - 1 < \sqrt{tq}$  points. The parameter  $m_P$  is maximised if as many as possible of the secants through  $P$  are of maximum length. In particular,  $m_P$  will be less than the value obtained by assuming all secants to  $S$  are either  $\sqrt{tq}$ -secants or  $(t - 1)$ -secants. Let  $x$

be the number of  $\sqrt{tq}$ -secants (secants with respect to  $S$ ) and  $y$  the number of  $(t-1)$ -secants. Then the following equations hold:

$$(\sqrt{tq})x + (t-1)y = |S| = tq + m - t, \quad (1)$$

$$x + y = q. \quad (2)$$

Solving these equations gives  $x = (m + q - t)/(\sqrt{tq} - t + 1)$  and  $y = (q\sqrt{tq} + t - m - tq)/(\sqrt{tq} - t + 1)$ . A secant of length, say  $i$ , determines  $\binom{i}{2}$  of the directions of  $S$ ; hence

$$m_P \leq x \binom{\sqrt{tq}}{2} + y \binom{t-1}{2}.$$

Since there are  $t$  such points  $P$  of  $B$  lying on the line at infinity,

$$\sum_{P \in l \cap B} m_P \leq \frac{t(m + q - t)}{\sqrt{tq} - t + 1} \binom{\sqrt{tq}}{2} + \frac{t(q\sqrt{tq} + t - m - tq)}{\sqrt{tq} - t + 1} \binom{t-1}{2} = g_{t,q}(m).$$

This leaves at least

$$\binom{tq + m - t}{2} - g_{t,q}(m)$$

directions for the rest of the  $m_Q$  where  $Q$  is a point of  $l$  not in  $B$ . There are  $q + 1 - t$  such points; so there must be one point, say  $Q$ , such that

$$m_Q \geq \frac{1}{q + 1 - t} \binom{tq + m - t}{2} - \frac{g_{t,q}(m)}{q + 1 - t}.$$

But  $m_Q$  is maximised when  $Q$  lies on an  $m$ -secant, in which case

$$m_Q \leq \binom{m}{2} + (q-1) \binom{t}{2}.$$

To combine these inequalities, define  $f_{t,q}(m)$  to be

$$\begin{aligned} f_{t,q}(m) = & 2 \left( \binom{tq + m - t}{2} - \frac{t(m + q - t)}{\sqrt{tq} + 1 - t} \binom{\sqrt{tq}}{2} \right. \\ & \left. - \frac{t(q\sqrt{tq} + t - m - tq)}{\sqrt{tq} + 1 - t} \binom{t-1}{2} \right) \\ & - (q + 1 - t) \left( \binom{m}{2} - (q + 1 - t)(q - 1) \binom{t}{2} \right) (\sqrt{tq} + 1 - t). \end{aligned}$$

Writing  $f_{t,q}(m)$  as a quadratic expression in  $m$  this simplifies to

$$f_{t,q}(m) = [-(q-t)(\sqrt{tq} - t + 1)]m^2 + [(2t+1)q\sqrt{tq} + (-3t^2 + t + 1)q +$$

$$(-2t)\sqrt{tq} - (t^3 - t)]m + tq^2\sqrt{tq} + (-2t^2 + t)q^2 + (-2t)q\sqrt{tq} + (2t^3 - t)q + (-t^3 + 2t^2)\sqrt{tq} + (-t^3 + t^2).$$

The inequality from which  $f_{t,q}(m)$  was formed implies that  $f_{t,q}(m) \leq 0$  for all valid values of  $m$ . Now,  $f_{t,q}(m)$  is a quadratic function of  $m$ . If the upper root of  $f_{t,q}(m)$  is say  $m_0$  then  $B$  must have at least  $tq + m_0$  points. It will now be shown that  $f_{t,q}(\sqrt{tq} + 1) \geq 0$  and hence that  $m_0 \geq \sqrt{tq} + 1$ .

$$\begin{aligned} f_{t,q}(\sqrt{tq} + 1) &= -(q - t)(\sqrt{tq} + 1 - t)(tq + 2\sqrt{tq} + 1) + (2t + 1)q\sqrt{tq}(\sqrt{tq} + 1) + \\ &(-3t^2 + t + 1)q(\sqrt{tq} + 1) + (-2t)\sqrt{tq}(\sqrt{tq} + 1) + (t^3 - t)(\sqrt{tq} + 1) + tq^2\sqrt{tq} + \\ &(-2t^2 + t)q^2 + (-2t)q\sqrt{tq} + (2t^3 - t)q + (-t^3 + 2t^2)\sqrt{tq} + (-t^3 + t^2). \end{aligned}$$

Writing this as an expression for  $q$  gives

$$f_{t,q}(\sqrt{tq} + 1) = (t^2 - t)q^2 + (-2t^2 + 3t - 1)q\sqrt{tq} + t(t - 1)^2q,$$

which simplifies to

$$f_{t,q}(\sqrt{tq} + 1) = (t - 1)q\left(\left(\sqrt{tq} + \frac{1}{2} - t\right)^2 - \frac{1}{4}\right) \geq 0.$$

Hence  $B$  has at least  $tq + \sqrt{tq} + 1$  points unless  $B$  contains a line.  $\square$

In the case that  $q$  is a prime the following theorem obtains a useful lower bound for general multiple blocking sets by generalising a theorem of Blokhuis in [9].

**Theorem 4.2** *Let  $B$  be a  $t$ -fold blocking set in  $PG(2, p)$ ,  $p > 3$  prime.*

1. *If  $t < p/2$  then  $|B| \geq (t + \frac{1}{2})(p + 1)$ .*
2. *If  $t > p/2$  then  $|B| \geq (t + 1)p$ .*

**Proof :** Let  $l$ , the line  $z = 0$ , be a  $t$ -secant to a  $t$ -fold blocking set  $B$ . Assume  $(1, 0, 0) \in B$  and that  $B$  has  $tp + m + t$  points. Let  $S = B \setminus l$  so that  $S$  has  $tp + m$  points. Assume  $t + m < p$  since otherwise  $|B| \geq (t + 1)p$ . In coordinates

$$S = \{(a_i, b_i) | i = 1, \dots, tp + m\} \subset AG(2, p) = PG(2, p) \setminus l.$$

Let  $d_1, \dots, d_{t-1}$  represent the directions determined by the points in  $B \cap (l \setminus (1, 0, 0))$ . Indeed, for each point  $P_j \in B \cap l$ , let  $d_j$  be the negative reciprocal

of the slope of the lines meeting in  $P_j$ . Each line  $x + uy + s = 0$  has  $t$  solutions in  $S$  whenever  $u \notin \{d_j | j = 1, \dots, t-1\}$ . So it follows that

$$F(s, u) = \prod_{j=1}^{t-1} (u - d_j) \prod_{i=1}^{tp+m} (s + a_i + b_i u)$$

has zeros of degree  $t$  for all  $s$  and  $u$ . This implies that  $F(s, u)$  lies in the ideal generated by  $((s^p - s)^r (u^p - u)^{t-r})$ , for  $r = 0, \dots, t$ . This implies there exist polynomials  $G_r(s, u)$  such that

$$F(s, u) = \sum_{r=0}^t (s^p - s)^r (u^p - u)^{t-r} G_r(s, u).$$

Let  $F^*(s, u)$  and  $G_r^*(s, u)$  be the parts of  $F(s, u)$  and  $G_r(s, u)$  that are homogeneous of total degree  $tp + m + t - 1$  and  $m + t - 1$  respectively. Then

$$F^*(s, u) = \sum_{r=0}^t s^{rp} u^{(t-r)p} G_r^*(s, u) = u^{t-1} \prod_{i=1}^{tp+m} (s + b_i u).$$

Since the equation is homogeneous the variable  $u$  can be fixed; so define  $f(s) = F^*(s, 1)$ ,  $g_r(s) = G_r^*(s, 1)$ . Then

$$f(s) = \sum_{r=0}^t s^{rp} g_r(s) = \prod_{i=1}^{tp+m} (s + b_i).$$

This implies that the degree of  $g_t$  is  $m$  and the degrees of the other  $g_r$  are at most  $m + t - 1$ .

The  $b_i$ 's must take each value in  $GF(q)$  at least  $t - 1$  times since each line  $y = b_i$  contains at least  $t - 1$  points of  $S$ . Hence  $(s^p - s)^{t-1} | f(s)$ . The divisibility condition implies that  $s^{t-1} | g_0$  and so define  $\gamma(s)$  to be such that  $g_0 = (-1)^{t-1} s^{t-1} \gamma(s)$ . Since  $(s^p - s)^{t-1} | f(s)$  define  $h(s)$  to be such that  $f(s) = (s^p - s)^{t-1} (s^p g_t(s) + h(s) + \gamma(s))$ . The degree of  $h$  is at most  $m + t - 1 < p - 1$ . The divisibility  $s^p | (f(s) - g_0(s))$  implies that  $s^{p-t+1} | h(s)$ . As  $f(s)$  has no terms with exponent equal to  $-1 \pmod{p}$  it follows that neither does the function  $(s^p - s)^{t-1} h(s)$ . Write

$$(s^p - s)^{t-1} h(s) = \sum_{r=0}^{t-1} \binom{t-1}{r} (-1)^r s^{(t-1-r)p+r} h(s).$$

The binomial coefficients  $\binom{t-1}{r}$ ,  $r = 0, \dots, t-1$ , are non-zero so  $h(t)$  has no terms of order greater than  $p - t$ . But, as mentioned before,  $s^{p-t+1} | h(s)$  and so  $h \equiv 0$ .

Divide  $s^p g_t(s) + \gamma(s)$  by the greatest common divisor of  $g_t(s)$  and  $\gamma(s)$ , this possibly reduces  $m$ . Now apply Theorem 3.1.

If  $m = 0$  then we are in Case 1 of Theorem 3.1 and  $s^p g_t(s) + \gamma(s)$  contains a factor of the form  $s^p - a = (s - a)^p$ . In addition to the  $(s - a)^{t-1}$  present in the  $(s^p - s)^{t-1}$ , this gives too many factors  $s - a$  in  $f(s)$ . If  $m = 1$  then we are in Case 4 of Theorem 3.1 but the degree of  $g_t$  is not less than the degree of  $\gamma$  so this case cannot occur.

So  $m > 1$  and as mentioned at the end of Theorem 3.1 this implies that  $m \geq (p + 1)/2$ .  $\square$

In 1965 Barlotti [5] provided examples of  $t$ -fold blocking sets with  $(t + 1)q$  points when  $q$  is odd and  $t = (q - 1)/2$  and  $t = (q + 1)/2$ . So if  $q = p > 3$  is a prime then the bound in Theorem 4.2 is attained. It is also attained when  $t = p - 1$  by the complement of a conic.

The theorem above will be used in the next two sections to determine exact values and improved bounds on the maximum sizes of  $(k, n)$ -arcs in the Desarguesian projective planes of order eleven and thirteen.

## 5. The value of $m_n(2, 11)$

In this section examples of large  $(k, n)$ -arcs in  $PG(2, 11)$  are given. Improvements on the upper bounds of  $m_n(2, 11)$  that come from Theorem 4.2 are made.

Examples 5.1 and 5.2 were constructed by taking random subsets of the internal points of a conic.

**Example 5.1** *The set*

{ (0, 1, 1)(0, 1, 4)(0, 1, 5)(0, 1, 9)(1, 2, 3)(1, 2, 5)(1, 2, 8)(1, 3, 5)(1, 3, 8)  
 (1, 3, 9)(1, 4, 1)(1, 4, 4)(1, 5, 1)(1, 5, 3)(1, 5, 7)(1, 6, 4)(1, 6, 8)(1, 6, 9)(1, 6, 10)  
 (1, 7, 5)(1, 7, 7)(1, 7, 10)(1, 8, 2)(1, 8, 3)(1, 9, 3)(1, 9, 6)(1, 9, 8)(1, 9, 9)(1, 10, 6)  
 (1, 10, 7)(1, 10, 9) }

*forms a (31, 4)-arc in  $PG(2, 11)$  with secant distribution  $\tau_0 = 16$ ,  $\tau_1 = 9$ ,  $\tau_2 = 24$ ,  $\tau_3 = 21$  and  $\tau_4 = 63$ .*

**Example 5.2** *The set*

{ (0, 1, 1)(0, 1, 4)(1, 1, 2)(1, 1, 4)(1, 1, 5)(1, 1, 6)(1, 1, 10)(1, 2, 1)(1, 2, 2)  
 (1, 2, 3)(1, 2, 5)(1, 2, 8)(1, 3, 2)(1, 3, 5)(1, 4, 1)(1, 4, 4)(1, 4, 7)(1, 4, 8)(1, 5, 1)  
 (1, 5, 2)(1, 5, 7)(1, 5, 10)(1, 6, 4)(1, 6, 8)(1, 6, 9)(1, 6, 10)(1, 7, 3)(1, 7, 4)  
 (1, 7, 5)(1, 7, 7)(1, 8, 2)(1, 8, 3)(1, 8, 4)(1, 8, 6)(1, 8, 9)(1, 9, 3)(1, 9, 8)(1, 9, 9)  
 (1, 9, 10)(1, 10, 1)(1, 10, 5)(1, 10, 7)(1, 10, 9) }

$A$	$(0, 1, 1)$	$F$	$(1, 0, 10)$	$K$	$(1, 2, 1)$
$B$	$(0, 1, 2)$	$G$	$(1, 1, 0)$	$L$	$(1, 2, 2)$
$C$	$(0, 1, 7)$	$H$	$(1, 1, 3)$	$M$	$(1, 3, 3)$
$D$	$(1, 0, 1)$	$I$	$(1, 1, 7)$	$N$	$(1, 6, 0)$
$E$	$(1, 0, 3)$	$J$	$(1, 2, 0)$	$O$	$(1, 8, 1)$

Table 1. The fifteen  $c_2$  points in Example 5.3.

$x + 2y + 10z = 0$	$B D H$	$x + 3y + 4z = 0$	$B K M$
$x + 4y + z = 0$	$C F L$	$x + 5y + 3z = 0$	$B J O$
$x + 5y + 7z = 0$	$E I J$	$x + 6y + 10z = 0$	$D I L$
$x + 9y + 4z = 0$	$H N O$	$x + 9y + 7z = 0$	$E L N$
$x + 10y + z = 0$	$A F G K$	$x + 10y + 6z = 0$	$B G L$
$x + 10y + 7z = 0$	$E G O$	$x + 10y + 8z = 0$	$C G M$

Table 2. The eleven 3-secants and the one 4-secant in Example 5.3 together with the points they pass through.

forms a  $(43, 5)$ -arc in  $PG(2, 11)$  with secant distribution  $\tau_0 = 13$ ,  $\tau_2 = 12$ ,  $\tau_3 = 3$ ,  $\tau_4 = 42$  and  $\tau_5 = 63$ .

**Example 5.3** (*This technique for constructing arcs was suggested in [20].*) Consider a 5-arc made up of the points  $\{ (1, 0, 0)(0, 1, 0)(0, 0, 1)(1, 1, 1)(1, 2, 3) \}$ . This arc has fifteen  $c_2$  points ( that is points lying on two bisecants of the arc ). Take the dual of this set to form  $S$ , a set of 50 points. Every line of the plane must go through at least 4 points of this set except the 15 points that are dual to the  $c_2$  points. These 15 points form a  $(15, 4)$ -arc with one 4-secant and eleven 3-secants. They are the points which have been labelled with the letters  $A$  to  $O$ .

Suppose  $T$  is a subset of these lines such that each of the 15 points lies on at least one of the lines of  $T$ . Then the set  $S \cup T^*$ , where  $T^*$  is the dual of  $T$ , is a 4-fold blocking set in  $PG(2, 11)$  and hence its complement is a  $(k, 8)$ -arc.

Suppose there exists a set of five of these lines with such a property. The subset must contain the 4-secant since it is the only one containing the point  $A$ . The two lines containing  $C$  both meet the 4-secant, in  $F$  and  $G$  respectively; so all other lines must not meet in the 15 points. So for the lines through  $B$  the subset must contain either  $x + 2y + 10z = 0$  or  $x + 5y + 3z = 0$ . The former line implies that a line through  $O$  always intersects one of the chosen lines and the latter line implies that lines through  $H$  are no good.

There are various subsets of six lines with the desired property, for example,  $\{x+5y+3z=0, x+6y+10z=0, x+9y+4z=0, x+9y+7z=0, x+10y+z=0, x+10y+8z=0\}$ . Taking the dual of this set and adding these points to  $S$ , the set

$\{ (0, 0, 1)(0, 1, 0)(0, 1, 1)(0, 1, 2)(0, 1, 3)(0, 1, 4)(0, 1, 5)(0, 1, 6)(0, 1, 7)$   
 $(0, 1, 8)(0, 1, 9)(0, 1, 10)(1, 0, 0)(1, 0, 1)(1, 0, 2)(1, 0, 3)(1, 0, 4)(1, 0, 5)(1, 0, 6)$   
 $(1, 0, 7)(1, 0, 8)(1, 0, 9)(1, 0, 10)(1, 1, 0)(1, 1, 9)(1, 1, 10)(1, 2, 0)(1, 2, 2)(1, 2, 8)$   
 $(1, 3, 0)(1, 3, 5)(1, 3, 7)(1, 4, 0)(1, 4, 6)(1, 4, 8)(1, 5, 0)(1, 5, 5)(1, 6, 0)(1, 6, 3)$   
 $(1, 6, 4)(1, 7, 0)(1, 7, 3)(1, 7, 6)(1, 8, 0)(1, 8, 2)(1, 8, 9)(1, 9, 0)(1, 9, 1)(1, 10, 0)$   
 $(1, 10, 4)(1, 5, 3)(1, 6, 10)(1, 9, 4)(1, 9, 7)(1, 10, 1)(1, 10, 8) \}$

is a 56 point 4-fold blocking set in  $PG(2, 11)$ . The complement of which is a  $(77, 8)$ -arc with secant distribution  $\tau_0 = 5, \tau_4 = 1, \tau_5 = 6, \tau_6 = 18, \tau_7 = 42$  and  $\tau_8 = 61$ .

The fact that no five line subset of the one 4-secant and eleven 3-secants existed with the desired property implies that no  $(78, 8)$ -arc can be constructed from this 5-arc. There are two projectively distinct 5-arcs in  $PG(2, 11)$  and in a similar way to this it can be shown that the other one cannot yield a  $(78, 8)$ -arc either. If three external lines meet outside a  $(k, 8)$ -arc then  $k$  is at most 72, by counting. Hence there exists no  $(78, 8)$ -arc in  $PG(2, 11)$  with more than four external lines.

**Lemma 5.4** *There exists no  $(79, 8)$ -arc in  $PG(2, 11)$ .*

**Proof :** Finding a maximum  $(k, 8)$ -arc is equivalent to finding the minimum 4-fold blocking set by considering complements. Theorem 4.2 says that since 11 is prime a 4-fold blocking set must have at least 54 points. If there were a 4-fold blocking set with exactly 54 points then the degree of  $\alpha$  in the proof of Theorem 3.1 must be exactly equal to  $m + 1$  which is equal to 7 in this case. This implies that each point of the 4-fold blocking set is on exactly five 4-secants since the degree of  $\alpha$  is equal to the number of non 4-secants through a point. The total number of 4-secants must therefore be  $5 \cdot 54 / 4$  which is not an integer. Hence a 54 point 4-fold blocking set does not exist and equivalently a  $(79, 8)$ -arc does not exist.  $\square$

## 6. The value of $m_n(2, 13)$

In this section examples of large  $(k, n)$ -arcs in  $PG(2, 13)$  are given. Improvements to the upper bounds of  $m_n(2, 13)$  that come from Theorem 4.2 are made.

Example 6.1 was constructed by random search. Examples 6.2, 6.3 and 6.4 are subsets of the internal points of a conic.

**Example 6.1** *The set*

$\{ (0, 0, 1)(0, 1, 0)(0, 1, 1)(1, 0, 0)(1, 0, 1)(1, 2, 2)(1, 3, 5)(1, 4, 7)(1, 4, 10)$   
 $(1, 5, 1)(1, 5, 4)(1, 6, 3)(1, 6, 8)(1, 7, 8)(1, 8, 7)(1, 8, 12)(1, 9, 12)(1, 10, 3)(1, 10, 5)$   
 $(1, 11, 2)(1, 11, 9)(1, 12, 9)(1, 12, 10) \}$

*forms a (23, 3)-arc in  $PG(2, 13)$ . This set has secant distribution  $\tau_0 = 41$ ,  $\tau_1 = 35$ ,  $\tau_2 = 34$  and  $\tau_3 = 73$ .*

**Example 6.2** *The set*

$\{ (0, 1, 1)(0, 1, 5)(0, 1, 11)(1, 2, 4)(1, 2, 8)(1, 2, 10)(1, 2, 11)(1, 3, 1)(1, 3, 2)$   
 $(1, 3, 4)(1, 3, 11)(1, 4, 5)(1, 4, 8)(1, 5, 5)(1, 5, 9)(1, 5, 11)(1, 6, 8)(1, 6, 10)(1, 8, 2)$   
 $(1, 8, 4)(1, 8, 6)(1, 8, 9)(1, 9, 1)(1, 9, 8)(1, 9, 9)(1, 9, 11)(1, 10, 2)(1, 10, 6)$   
 $(1, 10, 10)(1, 11, 5)(1, 11, 9)(1, 11, 10)(1, 12, 5)(1, 12, 6) \}$

*forms a (34, 4)-arc in  $PG(2, 13)$  with secant distribution  $\tau_0 = 24$ ,  $\tau_1 = 21$ ,  $\tau_2 = 24$ ,  $\tau_3 = 49$ ,  $\tau_4 = 65$ .*

**Example 6.3** *The set*

$\{ (0, 1, 1)(0, 1, 5)(0, 1, 6)(0, 1, 7)(0, 1, 8)(1, 1, 7)(1, 1, 8)(1, 1, 9)(1, 1, 12)(1, 2, 8)$   
 $(1, 3, 3)(1, 3, 4)(1, 3, 11)(1, 4, 3)(1, 4, 5)(1, 4, 8)(1, 4, 12)(1, 5, 4)(1, 5, 5)(1, 5, 7)$   
 $(1, 6, 2)(1, 6, 7)(1, 6, 8)(1, 6, 10)(1, 6, 12)(1, 7, 5)(1, 7, 6)(1, 7, 11)(1, 7, 12)$   
 $(1, 8, 1)(1, 8, 2)(1, 8, 4)(1, 8, 6)(1, 8, 8)(1, 9, 1)(1, 9, 5)(1, 9, 11)(1, 10, 2)$   
 $(1, 10, 10)(1, 10, 12)(1, 11, 2)(1, 11, 7)(1, 11, 10)(1, 12, 4)(1, 12, 6)(1, 12, 7) \}$

*forms a (46, 5)-arc in  $PG(2, 13)$  and has secant distribution  $\tau_0 = 15$ ,  $\tau_1 = 7$ ,  $\tau_2 = 22$ ,  $\tau_3 = 34$ ,  $\tau_4 = 36$ ,  $\tau_5 = 69$ .*

**Example 6.4** *The set*

$\{ (0, 1, 1)(0, 1, 5)(0, 1, 6)(0, 1, 7)(0, 1, 8)(0, 1, 11)(1, 1, 3)(1, 1, 6)(1, 1, 7)$   
 $(1, 1, 8)(1, 1, 9)(1, 1, 12)(1, 2, 3)(1, 2, 4)(1, 2, 6)(1, 2, 8)(1, 2, 10)(1, 2, 11)(1, 3, 1)$   
 $(1, 3, 2)(1, 3, 3)(1, 3, 4)(1, 3, 7)(1, 3, 11)(1, 4, 2)(1, 4, 3)(1, 4, 4)(1, 4, 8)(1, 4, 12)$   
 $(1, 5, 4)(1, 5, 5)(1, 5, 7)(1, 5, 9)(1, 5, 11)(1, 5, 12)(1, 6, 1)(1, 6, 7)(1, 6, 12)(1, 7, 1)$   
 $(1, 7, 3)(1, 7, 5)(1, 7, 12)(1, 8, 8)(1, 9, 1)(1, 9, 5)(1, 9, 8)(1, 9, 9)(1, 9, 10)(1, 9, 11)$   
 $(1, 10, 2)(1, 10, 6)(1, 10, 9)(1, 10, 11)(1, 10, 12)(1, 11, 2)(1, 11, 5)(1, 11, 7)$   
 $(1, 12, 4)(1, 12, 5)(1, 12, 6)(1, 12, 10) \}$

*forms a (61, 6)-arc in  $PG(2, 13)$  and has secant distribution  $\tau_0 = 12$ ,  $\tau_1 = 5$ ,  $\tau_2 = 1$ ,  $\tau_3 = 15$ ,  $\tau_4 = 24$ ,  $\tau_5 = 50$ ,  $\tau_6 = 76$ .*

**Example 6.5** *Let  $K$  be the 7-arc consisting of the points  $\{ (0, 0, 1)(0, 1, 0)$   
 $(1, 0, 0)(1, 1, 1)(1, 2, 3)(1, 5, 11)(1, 6, 4) \}$ . Let  $S$  be the points lying on the 7  
lines that are dual to  $K$ . The geometry of  $K$  is such that of all the points  
in the plane there are just seven lying on three of its bisecants, the rest lying  
on fewer. These seven points all lie on the line  $x + 12y + 4z = 0$ . So, by  
forming the union of  $S$  with  $(1, 12, 4)$ , a 78 point 5-fold blocking set is formed  
in  $PG(2, 13)$ ; the complement of which is a (105, 9)-arc.*

**Example 6.6** Consider the dual of a 5-arc in  $PG(2, 13)$ . Every line passes through at least four points of this dual set except the fifteen lines dual to the  $c_2$  points (points lying on two bisecants to the arc). These fifteen points must have a 3-secant which must be an external line to the arc. There exist six other external lines which cover the remaining 12 points since each  $c_2$  point is on 11 external lines to the 5-arc. The dual set formed from these seven lines and the 5-arc is a 4-fold blocking set in  $PG(2, 13)$  with 67 points. The complement of this set is a  $(116, 10)$ -arc.

**Lemma 6.7** There exists no  $(106, 9)$ -arc in  $PG(2, 13)$ .

**Proof :** Finding a maximum  $(k, 9)$ -arc in  $PG(2, 13)$  is equivalent to finding a minimum 5-fold blocking set. Theorem 4.2 implies, since 13 is prime, that such a set must have at least 77 points. If equality is attained then the degree of  $\alpha$  in the proof of Theorem 3.1 must be equal to  $m + 1$ . Hence each point has exactly 8 lines through it that are not 5-secants and so exactly six lines which are 5-secants. This implies that the total number of 5-secants is  $77 \cdot 6 / 5$  which is not an integer. Since this number must be an integer a 77 point 5-fold blocking set does not exist and so neither does a  $(106, 9)$ -arc.  $\square$

**Lemma 6.8** There exists no  $(120, 10)$ -arc in  $PG(2, 13)$ .

**Proof :** Finding a maximum  $(k, 10)$ -arc in  $PG(2, 13)$  is equivalent to finding a minimum 4-fold blocking set. By Theorem 4.2 such a set must have at least 63 points. If equality is attained then the degree of  $\alpha$  in the proof of Theorem 3.1 is equal to  $m + 1 = 8$ . This implies that each point has exactly 8 lines through it that are not 4-secants and hence exactly 6 lines through it that are 4-secants. The total number of 4-secants is therefore  $6 \cdot 63 / 4$  which is not an integer. Since this number must be an integer such a set cannot exist and hence neither does a  $(120, 10)$ -arc.  $\square$

## 7. The value of $m_n(2, q)$

Table 3 gives a summary of all known values of  $m_n(2, q)$  for  $q \leq 13$ . There are some quite large discrepancies between the lower and upper bounds in some cases for  $PG(2, 11)$  and  $PG(2, 13)$  and further work is needed to deduce the correct values. References are given for each value and bound and Table 4 compares the difference each value makes with  $(n - 1)q + 1$ .

I am most grateful to Dr R. Hill for noticing an error in the original proof of Theorem 4.1 and to Dr A. Blokhuis for improving the original bound of  $(t + \frac{1}{2})p + \frac{3}{2}$  in Theorem 4.2.

$q$	3	4	5	7	8	9	11	13
$n$								
2	4 <sup>a</sup>	6 <sup>a</sup>	6 <sup>a</sup>	8 <sup>a</sup>	10 <sup>a</sup>	10 <sup>a</sup>	12 <sup>a</sup>	14 <sup>a</sup>
3		9 <sup>b</sup>	11 <sup>b</sup>	15 <sup>c</sup>	15 <sup>d</sup>	17 <sup>e</sup>	21 <sup>f</sup> ..22 <sup>g</sup>	23 <sup>h</sup> ..27 <sup>i</sup>
4			16 <sup>b</sup>	22 <sup>b</sup>	28 <sup>j</sup>	28 <sup>b</sup>	31 <sup>k</sup> ..34 <sup>l</sup>	34 <sup>m</sup> ..40 <sup>l</sup>
5				29 <sup>n</sup>	33 <sup>p</sup>	37 <sup>p</sup>	43 <sup>q</sup> ..45 <sup>l</sup>	46 <sup>r</sup> ..53 <sup>l</sup>
6				36 <sup>n</sup>	42 <sup>s</sup>	48 <sup>p</sup>	56 <sup>l</sup>	61 <sup>t</sup> ..66 <sup>l</sup>
7					49 <sup>d</sup>	55 <sup>u</sup>	67 <sup>l</sup>	79 <sup>l</sup>
8						65 <sup>v</sup>	77 <sup>w</sup> ..78 <sup>x</sup>	92 <sup>l</sup>
9							89 <sup>s</sup> ..90 <sup>u</sup>	105 <sup>y</sup>
10							100 <sup>s</sup> ..102 <sup>v</sup>	116 <sup>z</sup> ..119 <sup>α</sup>
11								131 <sup>s</sup> ..133 <sup>u</sup>
12								144 <sup>s</sup> ..147 <sup>v</sup>

Table 3. The value of  $m_n(2, q)$  for  $2 \leq n < q \leq 13$ .

a: Bose [11], b: Lunelli and Sce [24], c: Bramwell [12], d: Bierbrauer [7], e: Barnabei *et al* [6], f: Hirschfeld and Sadeh [22], g: Ball [3], h: Example 6.1, i: Barlotti [4], j: Cossu [17], k: Example 5.1, l: Theorem 4.2, m: Example 6.2, n: Ciecchanowicz [16], p: Mason [25], q: Example 5.2, r: Example 6.3, s: Hill and Mason [19], t: Example 6.4, u: Ball [2], v: Ball and Blokhuis [1], w: Example 5.3, x: Lemma 5.4, y: Example 6.5 and Lemma 6.7, z: Example 6.6,  $\alpha$ : Lemma 6.8.

$q$	3	4	5	7	8	9	11	13
$n$								
2	0	1	0	0	1	0	0	0
3		0	0	0	-2	-2	-2..-1	-4..0
4			0	0	3	0	-3..0	-6..0
5				0	0	0	-2..0	-7..0
6				0	1	2	0	-5..0
7					0	0	0	0
8						1	-1..0	0
9							0..1	0
10							0..2	-2..1
11								0..2
12								0..3

Table 4. The value of  $m_n(2, q)$  minus  $(n - 1)q + 1$ .

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SIMEON BALL: BLOCKING SETS AND ARCS

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