An alternative way to generalise the pentagon

joint work with John Bamberg, Alice Devillers and Klara Stokes

Let $\mathcal{P}$ be a set whose elements are called points.

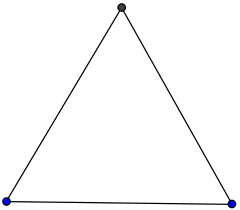
Let $\mathcal{L}$ be a set of subsets of $\mathcal{P}$, whose elements are called lines.

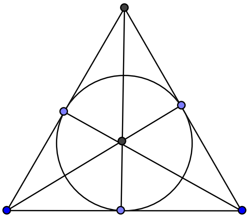
$\Gamma = (\mathcal{P}, \mathcal{L})$ is a partial linear space if

1. any two points are contained in at most one line,
2. it is regular if every point is incident with exactly r lines,
3. and uniform if every line contains s points.

(also called a configuration.)

These two partial linear spaces have the additional property that any two lines meet and any two points are collinear.





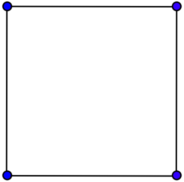
A partial linear space with these additional properties is called a *generalised triangle* or a *projective plane*.

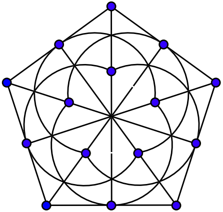
In a generalised triangle $s = r$ (the number of points incident with a line is the same as the number of lines incident with a point) and they are known to exist whenever $s - 1$ is the power of a prime.

[Bruck-Ryser 1949]
If $s = 2$ or 3 modulo 4 and $s - 1$ is not the sum of two squares then there is no generalised triangle of order s.

The prime power conjecture asserts that $s - 1$ is a prime power.

These two partial linear spaces have the additional property that for any point x and any line ℓ with $x \notin \ell$, there is a unique line m with the property that $x \in m$ and ℓ and m meet.





A partial linear space with these additional properties is called a *generalised quadrangle*.

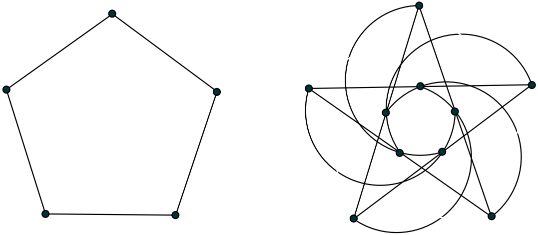
The *incidence graph* of a partial linear space is a bipartite graph with vertices $\mathcal{P} \cup \mathcal{L}$ where (x, ℓ) is an edge iff $x \in \ell$.

[Tits]
A *generalised n-gon* is a regular partial linear space whose incidence graph has diameter n and girth $2n$.

[Feit-Higman 1964]
A finite generalised n-gon exists iff $n = 3, 4, 6$ or 8.

Feit-Higman also give various conditions that r and s must satisfy.

These two partial linear spaces have the additional property that for any point x, the points not collinear with x form a line.

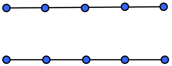


[Ball-Bamberg-Devillers-Stokes]
A regular uniform partial linear space with this additional property is called a *pentagonal geometry*.

A *pentagonal geometry* is a regular uniform partial linear space in which for each point x the points not collinear with x form a line.

Denote this line by x^{opp} and call it the opposite line of x.

This regular partial linear space trivially satisfies the axiom.

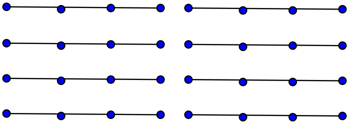


Moreover, this structure can occur as an induced substructure in a larger pentagonal geometry.

We call such a pair of lines an *opposite line pair*.

[Blokhuis] An example containing 4 opposite line pairs.

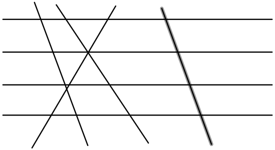
Take 4 horizontal lines of $\mathrm{AG}(2,8)$ and break them in half.



The induced non-horizontal lines of $\mathrm{AG}(2,8)$ and the 4 opposite line pairs form a pentagonal geometry of order $(4,9)$.

This is a special case of the following construction...

Take s copies of a pentagonal geometry Γ of order (s, r) and put the points of each copy on a horizontal "line".



The lines of a transversal design together with the lines of the s copies of Γ form a pentagonal geometry of order $(s, rs + s + 1)$.

This can be done iff there are $s - 2$ mutually orthogonal latin square of order $(s - 1)r + s + 1$.

However, ...

In these examples the deficiency graph (i.e. the non-collinearity graph) is disconnected, so they are kind of degenerate.

What we would like (and need) are pentagonal geometries with connected deficiency graph (or at least with no opposite line pairs).

[Lemma]
A pentagonal geometry of order (s, r) has $rs - r + s + 1$ points and $(rs - r + s + 1)r/s$ lines, so s divides $r(r - 1)$.

[Lemma]
If there is a pentagonal geometry of order (s, r) with $r > 1$ then $r \geq s$.

[Lemma]
If the pentagonal geometry of order (s, r) contains no opposite line pair then the deficiency graph G is s-regular of girth at least 5.

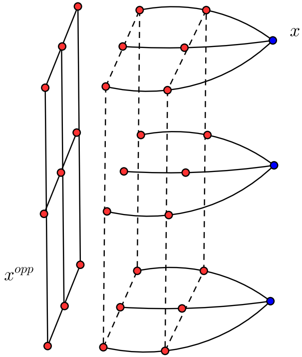
[Theorem]
The deficiency graph of a pentagonal geometry of order (s, s) is a Moore graph, so $s = 3, 7$ or possibly 57.

[Theorem]
A pentagonal geometry of order $(s, s + 1)$ embeds in a pentagonal geometry of order $(s + 1, s + 1)$, so $s = 6$ or possibly 56.

Thus, we have pentagonal geometries with connected deficiency graph of order $(3, 3)$, $(6, 7)$ and $(7, 7)$,

and the only other one we know of has order $(3, 13)$ . . .

Take AG$(3,3)$ and break off a plane and add the three blue points.



Put Desargues configurations in the three horizontal planes.

Break the lines in the two dashed planes into 3 parts and join to the blue points.

Put the blue points and their opposite lines on 3 horizontal lines of AG$(2,4)$ and take the induced non-horizontal lines.

[Proposition]
There is no pentagonal geometry of order $(3, 6)$.

[Proposition]
There is no pentagonal geometry of order $(3, 7)$ without opposite line pairs.

[Proposition]
If the deficiency graph of Γ, a pentagonal geometry of order (s, r) with $r > s + 1$, is distance regular of diameter 3 then either Γ has more than 1025 points or $(s, r) = (7, 28)$ (and the deficiency graph has intersection array $\{7, 6, 6; 1, 1, 2\}$).

I will give 50 euros to the first person to construct a new pentagonal geometry with a connected deficiency graph with at least 4 points on a line.

Let x be a vertex of a graph G and let $N(x)$ be the set of neighbouring vertices.

An *identifying code* of a graph G is a subset S of the vertices with the property that $N(x) \cap S$ is distinct for all x.

An identifying code of the deficiency graph of a partial linear space can be used to identify a malfunction in a sensor network.

We wish to find small identifying codes S. However, ...

Let Γ be a partial linear space with point set $\mathcal{P}$ and deficiency d.

[Proposition]
If S is an identifying code for the deficiency graph of Γ then

$$|S| \geq \left\lceil \frac{2(|P|-1)}{d+1} \right\rceil .$$

If we take Γ to be a pentagonal geometry with no opposite line pairs then $d = s$ and $|\mathcal{P}| = rs - r + s + 1$,

and for the examples of order $(3,3)$, $(6,7)$ and $(3,13)$ we can attain this bound.

Let Γ be a partial linear space.

For any two points x and y of Γ the distance $d(x, y)$ is n, where n is the least number of sequentially concurrent lines needed to connect x and y.

So, if two points x and y are collinear then $d(x, y) = 1$.

An *n-gonal geometry* is a partial linear space with the property that

(G1) for any set S of $m \leq n$ points, no three collinear, there exists an ordering $x_1, \ldots, x_m$ of the elements of S such that

$$d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{m-1}, x_m) + d(x_m, x_1) \leq n,$$

(G2) for every point x there is a point at maximum distance from x,

and if we want a non-degenerate example then we add

(G3) there are $n + 1$ points, no 3 collinear.

[Stokes] There exist n-gonal geometries if and only if $n \leq 5$.

If
Γ is an n-gonal geometry and the dual of Γ is an n-gonal geometry then

[n=3] Γ is a projective plane and $k = r$ and if $k = 2$ or 3 mod 4 then $k - 1$ is the sum of two square. (Bruck-Ryser 1949)

[n=4] Γ is a partial linear space of deficiency one with $k = r$ and k or $k - 2$ is a square. (Bose-Connor 1952)

[n=5] Γ is a pentagonal geometry with $k = r$ and so comes from a Moore graph, $k = 2, 3, 7$ or possible 57. (Hoffman-Singleton 1960).

An *n-gonal geometry of index r* is a partial linear space with the property that

(G1) for any set S of $m \leq r$ points, no three collinear, there exists an ordering $x_1, \ldots, x_m$ of the elements of S such that

$$d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{m-1}, x_m) + d(x_m, x_1) \leq n,$$

(G2) for every point x there is a point at maximum distance from x,

and if we want a non-degenerate example then we add

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