On sets defining few ordinary planes

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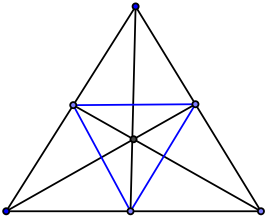
Ordinary lines

Let S be a set of n points in the real plane.

An *ordinary line* is a line incident with precisely 2 points of S.

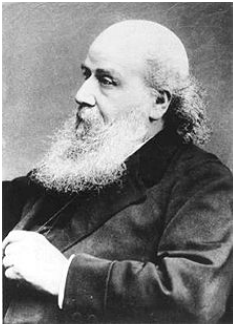
Example the "broken" Fano plane



A set of seven points spanning three ordinary blue lines.

[Sylvester 1893]
Is it true that a set of n points, not all collinear, spans at least one ordinary line?

[Gallai 1944]
Yes.

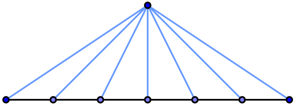


At worst a set of n points will span $\frac{1}{2}n(n-1)$ ordinary lines.

Is it possible to do significantly better?

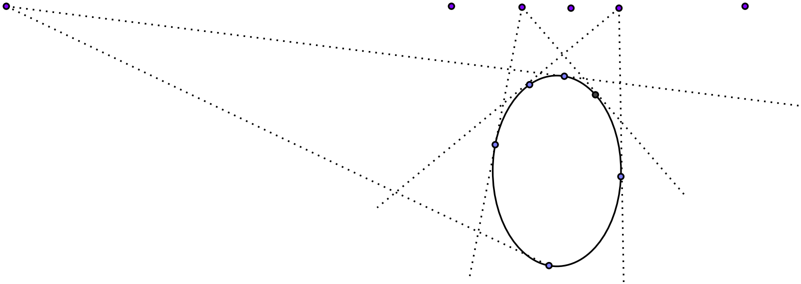
Are there sets of points spanning $< Kn$ ordinary lines, for some $K = o(n)$?

Example



A set of n points spanning $n-1$ ordinary lines.

Example X_n - the regular $\frac{1}{2}n$-gon with $\frac{1}{2}n$ collinear points.



A set of 12 points spanning 6 ordinary lines.

Example

$n = 3$ modulo 4.

Removing a point from X_{n+1} gives a set of n points spanning $\frac{3}{4}n - \frac{9}{4}$ ordinary lines.

$n = 1$ modulo 4.

Adding the centre point of the polygon to X_{n-1} gives a set of n points spanning $\frac{3}{4}n - \frac{3}{4}$ ordinary lines.

Example

Let γ be the set of non-singular points of a cubic curve.

Then there is a binary operation $\oplus$ such that $(\gamma, \oplus)$ is a group with the property that

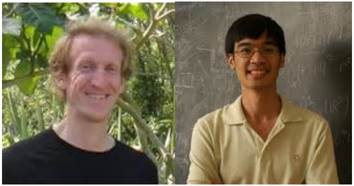
$$x \oplus y \oplus z = 0,$$

if and only if x, y and z are collinear.

If the cubic curve is elliptic or acnodal then this group has finite subgroups.

Any coset of a finite subgroup of size n spans $n-1$ or $n-3$ ordinary lines.

Green-Tao 2013



Let S be a set of n points in the real plane spanning at most Kn ordinary lines, where $K < c(\log\log n)^c$ for some constant c.

Then for n sufficiently large, either

(i) $n - O(K)$ points of S are collinear.

(ii) S differs from a regular polygon example by $O(K)$ points.

(iii) S differs from a cubic curve example by $O(K)$ points.

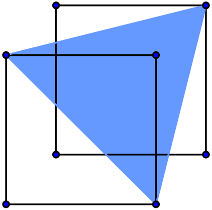
[Are there other examples spanning $o(n^2)$ ordinary lines?]

Ordinary planes

Let S be a set of n points in real 3-space.

An *ordinary plane* is a plane incident with exactly 3 points of S.

Example the cube

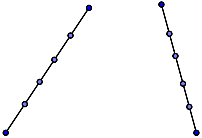


A set of 8 points spanning 8 ordinary planes.

[Extrapolating Sylvester]
Is it true that a set of n points in 3-space, not all co-planar, spans at least one ordinary plane?

[Motzkin 1952]
No.

Example at least four collinear points

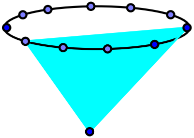


A set of n points in 3-space can span no ordinary planes.

At worst a set of n points will span $\frac{1}{6}n(n-1)(n-2)$ ordinary planes. Is it possible to do significantly better?

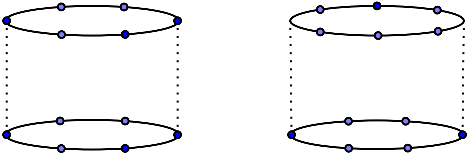
What are the sets of points, no 4 points collinear, spanning $< Kn^2$ ordinary planes, for some $K = o(n)$?

Example cone



A set of n points in 3-space spanning $\frac{1}{2}(n-1)(n-2)$ ordinary planes.

Example P_n - The prism and the skew-prism



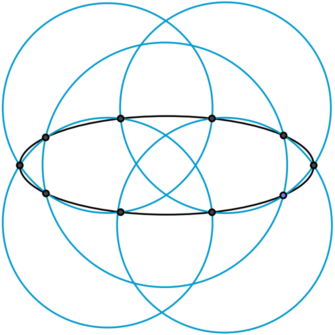
A set of n points in 3-space spanning $\frac{1}{4}n^2 + O(n)$ ordinary planes.

For n odd, remove a point from P_{n+1} gives a set spanning $\frac{3}{8}n^2 + O(n)$ ordinary planes.

Swanepoel-Lin 2016

Consider the subset of the planar ellipse $x^2 + (y/a)^2 = 1$,

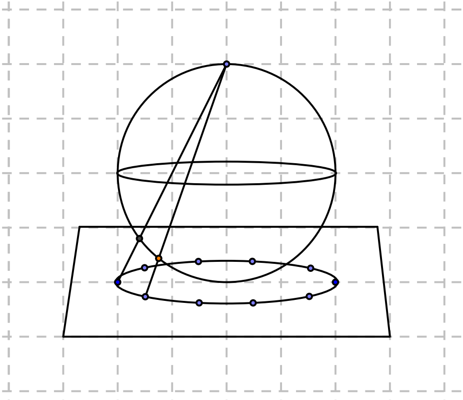
$$E = \{(\cos(2\pi j/n), a\sin(2\pi j/n)) \mid j \in \mathbb{Z}/(n\mathbb{Z})\}.$$



Most circles which pass through three of the points of E, pass through four of them.

There are $\frac{1}{2}n^2 + O(n)$ *ordinary* circles which pass through precisely three points of E.

Lift E to a subset S of the sphere by inversely projecting from the north pole. Circles in the plane lift to plane sections of the sphere.



There are only $\frac{1}{2}n^2 + O(n)$ ordinary circles in the plane.

By stereographic projection …

…there are only $\frac{1}{2}n^2 + O(n)$ ordinary planes.

What are the sets of points, no 4 collinear, spanning $< Kn^2$ ordinary planes, for some $K = o(n)$?

We have seen examples of the cone, the prism, the skew-prism and an example in the intersection of an elliptic cone and a sphere.

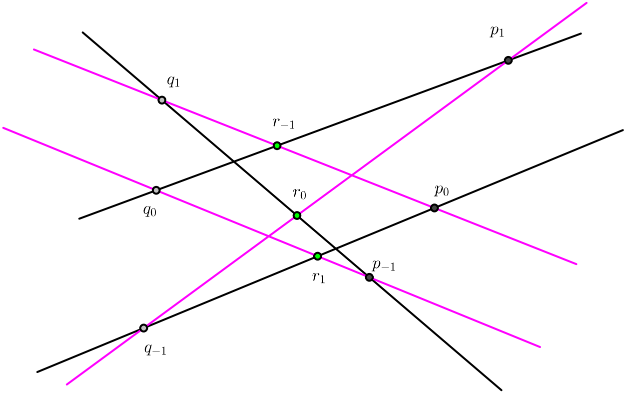
All of these examples have $cn^2 + O(n)$ ordinary planes.

Our aim is to prove structural results on sets spanning few ordinary planes.

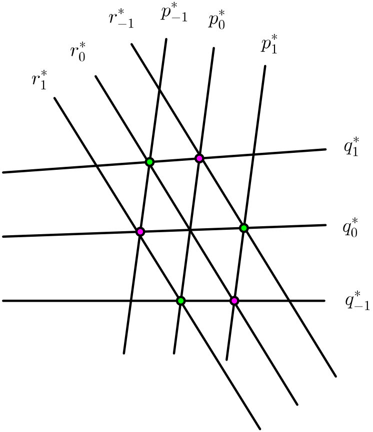
Cayley-Bacharach theorem

Cayley (1889) and Bacharach (1886)

Consider two sets of three lines and the nine intersection points. Any cubic curve which passes through 8 of the points passes through the 9-th.

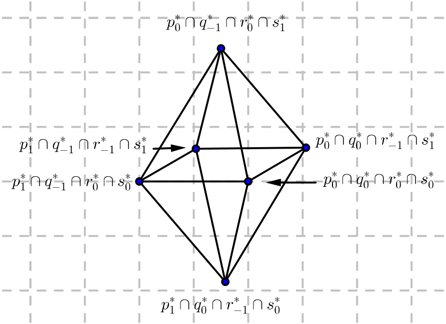


In the dual projective plane the nine points dualise to nine lines cutting out a hexagon.

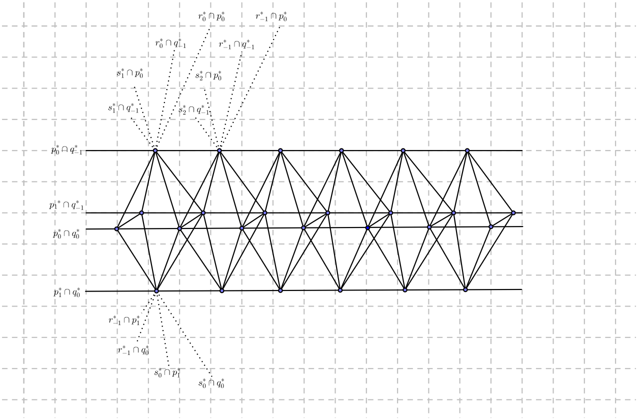


Consider three sets of two planes defining eight intersection points. Any quadric which passes through 7 of the points passes through the 8-th.

In dual 3-space the eight points dualise to eight planes cutting out a diamond.



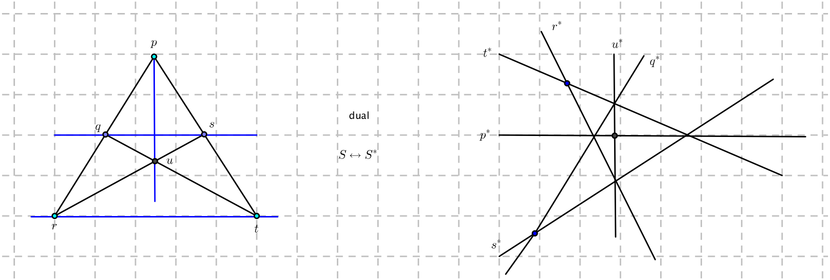
The points dual to the planes cutting out a string of diamonds are in the intersection of two quadrics.



Euler's formula

[Melchior 1940]

Make a graph embeddable in the projective plane from a set of points S by dualising.



Let τ_i be number of lines incident with i points of S.
In the example, $\tau_2 = 3$ and $\tau_3 = 4$.

Let S be a set of points in the projective plane.

S^* cuts out a (projective) planar graph in the dual space with

$$\text{number of vertices } V = \textstyle\sum_{i \geqslant 2} \tau_i.$$

$$\text{number of edges } E = \textstyle\sum_{i \geqslant 2} i\tau_i = \frac{1}{2} \sum_{i \geqslant 3} i f_i,$$

where f_i is the number of faces with i sides.

Euler's formula implies

$$V - E + F = 1$$

which gives

$$\tau_2 = 3 + \sum_{i \geqslant 4} (i-3)\tau_i + \sum_{i \geqslant 4} (i-3) f_i.$$

$$\tau_2 = 3 + \sum_{i \geqslant 4} (i-3)\tau_i + \sum_{i \geqslant 4} (i-3) f_i.$$

If τ_2 is small, (i.e. there are few ordinary lines) then most vertices should have degree six and most faces should be triangles.

A *good edge* is an edge both of whose vertices have degree six and both faces are triangles.

[Green-Tao 2013] If there are at most Kn ordinary lines then there are at most $16Kn$ bad edges.

Let S be a set of points in projective 3-space.

Let τ_i be number of planes incident with i points of S.

S^* cuts out a graph embedded in projective 3-space.

Considering only faces which are contained in a plane of S^* we get

$$\tau_3 = n + \tfrac{1}{3}\sum_{i \geqslant 5} i(i-4)\tau_i + \tfrac{1}{3}\sum_{i \geqslant 4}(i-3)f_i.$$

A *good edge* is an edge both of whose vertices have degree eight and all four of whose faces are triangles.

[Lemma] If there are at most Kn^2 ordinary planes then there are at most $30Kn^2$ bad edges.

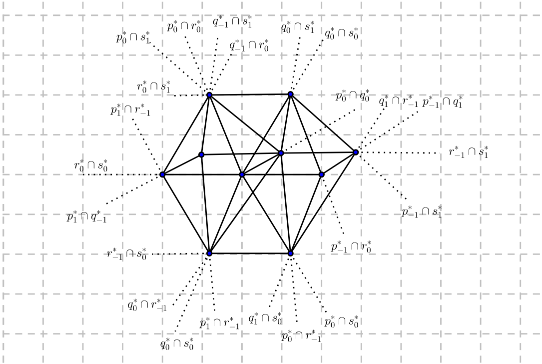
A *rather good edge* is a good edge all of whose neighbouring edges are also good.

[Lemma] If there are at most Kn^2 ordinary planes then there are at most $450Kn^2$ slightly bad (not really good) edges.

[Lemma] The structure of the graph around a rather good edge is a double diamond.

[Lemma] The ten points of S which in the daul space cut out the double diamond, lie in the intersection of two quadrics.

Suppose that the edge joining $p_0^* \cap q_0^* \cap r_{-1}^* \cap s_1^*$ and $p_0^* \cap q_0^* \cap r_0^* \cap s_0^*$ is rather good.



Then the points $\{p_{-1}, p_0, p_1, q_{-1}, q_0, q_1, r_{-1}, r_0, s_0, s_1\}$ lie in the intersection of two quadrics.

[Ball 2016]

Suppose that S is a set of n points in 3-space, no 3 collinear, spanning less than Kn^2 ordinary planes.

Then S is contained in the union of $2700K$ varieties which are the intersection of two quadrics.

This says something about sets spanning $o(n^3)$ ordinary planes.

If we restrict K further. . .

[Ball 2016]

Let S be the set of n points in 3-space spanning at most Kn^2 ordinary planes.

If $K = o(n^{\frac{1}{6}})$ then for n sufficiently large, either all but $O(K)$ points of S lie on the intersection of two quadrics or S contains four collinear points.

[Are there other examples spanning $o(n^3)$ ordinary planes?]

[Ball 2016]

Let S be the set of n points in 3-space spanning at most $\frac{1}{2}n^2 - cn$ ordinary planes, for some constant c.

If c is large enough then either

S is a prism or a skew-prism, a prism with a point deleted or a skew-prism with a point deleted, or S contains four collinear points.

[Monserrat 2015]
Let S be the set of n points in 3-space spanning at most $\frac{1}{3}n^2$ ordinary planes, for some constant c. Then either S is a prism or a skew-prism, or S contains four collinear points.

[Long proof]

1. Show that most of S is contained in one intersection of quadric variety.

2. Most points p of the variety have a 1-dimensional tangent space ℓ_p.

3. If a point q projects S onto an irreducible cubic then in most cases the plane $q \oplus \ell_p$ is an ordinary plane.

4. Find a point p such that $q \oplus \ell_p$ is ordinary for many $q \in S$ and p is incident with $O(Kn)$ ordinary planes.

If p projects onto an irreducible cubic then ℓ_p projects to a point incident with many ordinary lines.

Dirac-Motzkin conjecture

Let $e_d(n)$ be the minimum number of ordinary planes that a set of n points S spans, where S is a set of n points in d-space with the property that every d of its points spans a hyperplane.

[Dirac-Motzkin conjecture] $e_2(n) \geqslant \frac{1}{2}n$, whenever $n \neq 7, 13$.

[Green-Tao 2013] For n sufficiently large, the Dirac-Motzkin conjecture is true.

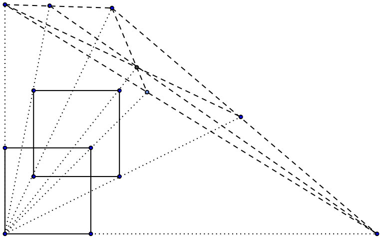
In fact

$$e_2(n) = \tfrac{1}{2}n, \quad n \text{ even},$$
$$e_2(n) = \tfrac{3}{4}n - \tfrac{3}{4}, \quad n = 1 \bmod 4,$$
$$e_2(n) = \tfrac{3}{4}n - \tfrac{9}{4}, \quad n = 3 \bmod 4.$$

[Lemma]

$$e_d(n) \geqslant \frac{n}{d} e_{d-1}(n-1).$$

[Example]



The cube has 8 points and spans 8 ordinary planes. It projects to a "broken" Fano plane of 7 points spanning 3 ordinary lines.

Using the structural theorems, one can prove

[Ball-Monserrat 2016] For n sufficiently large,

$$e_3(n) = \begin{cases} \frac{1}{4}n^2 - n, & n = 0 \mod 4 \\ \frac{3}{8}n^2 - n + \frac{5}{8}, & n = 1 \mod 4, \\ \frac{1}{4}n^2 - \frac{1}{2}n, & n = 2 \mod 4, \\ \frac{3}{8}n^2 - \frac{3}{2}n + \frac{65}{8}, & n = 3 \mod 4, \end{cases}$$

[Conjecture] These are the correct values of $e_3(n)$ for $n \geqslant 8$.

	4	5	6	7	8	9	10	11	12
2	3	4	3	3	4	6	5	6	6
3	.	6	8	11	8	16...22	20	19...37	24
4	.	.	10	20	25...35	18...56	40...84	55...120	57...165
5	.	.	.	15	32	54...70	36...126	88...210	132...330
6	.	.	.	.	21	56	90...126	.	.
7	.	.	.	.	.	28	80	.	.

The value of $e_d(n)$ for small d and n.

Open problems.

1. Prove the Dirac-Motzkin conjecture for $e_2(n)$.
2. Prove the corresponding conjecture for $e_3(n)$.
3. Find sets of size n in the plane spanning $o(n^2)$ ordinary lines.
4. Find sets of size n in 3-space spanning $o(n^3)$ ordinary lines.