17th "Lluís Santaló" Research School

Enriching Abstract Algebra with GA

RSME-UIMP

22-26 August, 2016

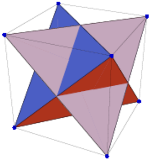
Enriching Abstract Algebra with GA

- **Introduction**. Goodman's book. Artin's book.
- **Classification of the $\mathcal{G}_{r,s}$**. Even algebra isomorphisms. The basic ingredients. A corner of the Clifford chessboard. Induction formulas. The full chessboard. Periodicity mod 8. The classification theorem. The complex case.
- **Pin and Spin representations**. Behaviour of $\mathbf{i}_{r,s}$. Basic notions. Irreducible representations of $\mathbf{K}(n)$. Pinor representations. Pinor synopsis. Spinor representations. Spinor synopsis. Tables for $0 \leqslant n \leqslant 7$.
- **References**. Porteous-1995 [1]. Figueroa-2006 [2]. Garling-2011 [3].

Introduction

ALGEBRA
ABSTRACT AND CONCRETE
EDITION 2.6

FREDERICK M. GOODMAN



SemiSimple Press
Iowa City, IA

Last revised on May 1, 2015.

Bookmarks

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Chapter 4. Symmetries of Polyhedra
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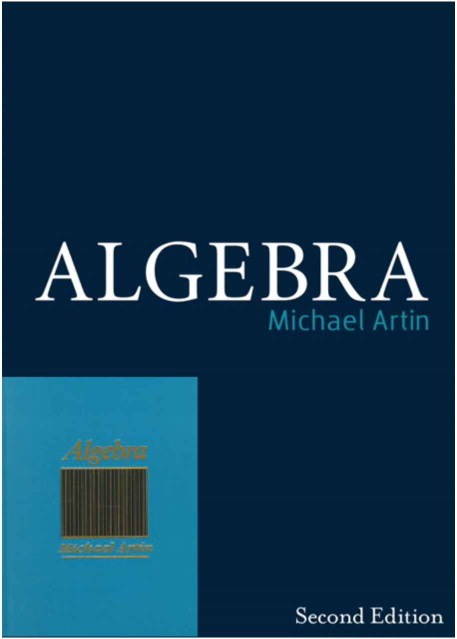
Download from the author's page: F. M. Goodman

"For further study of group theory, my own preference is for the theory of representations and applications. I recommend

- W. Fulton and J. Harris, *Representation Theory, A First Course*, Springer-Verlag, 1991.
- B. Simon, *Representations of Finite and Compact Groups*, American Mathematical Society, 1996.
- S. Sternberg, *Group Theory and Physics*, Cambridge University Press, 1994.

These books are quite challenging, but they are accessible with a knowledge of this course, linear algebra, and undergraduate analysis."


ALGEBRA
Michael Artin
Algebra
Michael Artin
Second Edition

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APPENDIX

Background Material
A.1 About Proofs
A.2 The Integers
A.3 Zorn's Lemma
A.4 The Implicit Function Theorem

"In writing this book, I tried to follow these principles:

1. The basic examples should precede the abstract definitions.
2. Technical points should be presented only if they are used elsewhere in the book.
3. All topics should be important for the average mathematician.

Although these principles may sound like motherhood and the flag, I found it useful to have them statete explicitly. They are, of course, violated here and there."

One may ask:

What about topics not covered that may be important [for the average mathematician] *and very relevant for an Abstract Algebra Course?*

Classification of the $\mathcal{G}_{r,s}$

Remark

In all cases, we set $X_n = X_{n,0}$, $\bar{X}_n = X_{\bar{n}} = X_{0,n}$ ($X_n(\mathbf{C})$ in the complex case), where $X_{r,s}$ stands for any of the symbols defined in prevoious lectures.

$$\mathrm{O},\ \mathrm{SO},\ \mathrm{SO}^{+},\ \mathcal{G},\ \mathcal{G}^{\times},\ \Gamma = V,\ \Gamma^{+} = V^{+},\ \mathrm{Pin},\ \mathrm{Spin},\ \mathrm{Spin}^{+}.$$

Note X_n and $X_{\bar{n}}$ point to difference structures, as for example $\mathcal{G}_n$ and $\mathcal{G}_{\bar{n}}$. The exceptions are O and SO, for it is plain that $\mathrm{O}_n = \mathrm{O}_{\bar{n}}$ and $\mathrm{SO}_n = \mathrm{SO}_{\bar{n}}$.

◊ For any signature (r,s), $C_{r,s} \simeq C^+_{r,s+1} \simeq C^+_{s+1,r}$.

□ Take a standard basis of $C_{r,s+1}$ of the form γ_j $(j = 1,\ldots,r)$, $\bar\gamma_k$ $(k = 1,\ldots,s+1)$ and write $\bar\gamma = \bar\gamma_{s+1}$. Now consider the elements $\Gamma_j = \bar\gamma\gamma_j$ $(j = 1,\ldots,r)$ and $\bar\Gamma_k = \bar\gamma\bar\gamma_k$ $(k = 1,\ldots,s)$. These elements belong to $C^+_{r,s+1}$, are linearly independent, anticommute and satisfy the standard relations for the signature (r,s): $\Gamma_j^2 = 1$ $(j = 1,\ldots,r)$ and $\bar\Gamma_k^2 = -1$ $(k = 1,\ldots,s)$. This implies that $C_{r,s} \simeq C^+_{r,s+1}$.

For the other isomorphism, take a standard basis of $C_{s+1,r}$ of the form γ_k $(k = 1,\ldots,s+1)$, $\bar\gamma_j$ $(j = 1,\ldots,r)$ and write $\gamma = \gamma_{r+1}$. Now consider the elements $\Gamma_j = \gamma\bar\gamma_j$ $(j = 1,\ldots,r)$ and $\bar\Gamma_k = \gamma\gamma_k$ $(k = 1,\ldots,s)$. These elements belong to $C^+_{s+1,r}$, are linearly independent, anticommute and satisfy the standard relations for the signature (r,s): $\Gamma_j^2 = 1$ $(j = 1,\ldots,r)$ and $\bar\Gamma_k^2 = -1$ $(k = 1,\ldots,s)$. This implies the isomorphism $C_{r,s} \simeq C^+_{s+1,r}$. □

◊ If $s > 0$, then $C^+_{r,s} \simeq C_{r,s-1}$.

◊ If $r > 0$, then $C^+_{r,s} \simeq C_{s,r-1}$.

◊ If $n > 0$, then $C^+_n \simeq \bar{C}_{n-1}$ and $\bar{C}_n \simeq C_{n-1}$. □

Notations. $\mathbf{K}$ will denote one of the fields $\mathbf{R}$ (real field), $\mathbf{C}$ (complex field) and $\mathbf{H}$ (quaternion field). For any integer $n \geq 2$, $\mathbf{K}(n)$ will denote the ring of $n \times n$ matrices with coefficients in $\mathbf{K}$. Since $\mathbf{K}(n) = \mathbf{K} \otimes \mathbf{R}(n)$, its real dimension is $d_{\mathbf{K}} n^2$, where $d_{\mathbf{K}} = \dim_{\mathbf{R}} \mathbf{K} = 1, 2, 4$, respectively. *Note*: $\mathbf{K}(m) \otimes \mathbf{R}(n) \simeq \mathbf{K}(mn)$.

◊1 $\mathbf{C} \otimes \mathbf{C} \simeq \mathbf{C} \oplus \mathbf{C}$

◊2 $\mathbf{C} \otimes \mathbf{H} \simeq \mathbf{C}(2)$

◊3 $\mathbf{H} \otimes \mathbf{H} \simeq \mathbf{R}(4)$

□1 Since $(i \otimes i)^2 = 1 \otimes 1$, the elements $e_\pm = \frac{1}{2}(1 \otimes 1 \pm i \otimes i)$ are idempotents with $e_+ + e_- = 1 \otimes 1$ and $e_+ e_- = e_- e_+ = 0 \otimes 0$. Then the map $\mathbf{C} \oplus \mathbf{C} \to \mathbf{C} \otimes \mathbf{C}$, $(x, y) \mapsto xe_+ + ye_-$, satisfies $(xe_+ + ye_-)(x'e_+ + y'e_-) = xx'e_+ + yy'e_-$ and with this it is easy to prove that it is an isomorphism.

□2 If z is a complex number and q a quaternion, let $f_{z,q} : \mathbf{H} \to \mathbf{H}$ be defined by $f_{z,q}(h) = zh\bar{q}$. Then $f_{z,q}$ is $\mathbf{C}$-linear, so that we have a map $\mathbf{C} \times \mathbf{H} \to \mathrm{End}_{\mathbf{C}}(\mathbf{H})$, $(x, q) \mapsto f_{z,q}$. The map is clearly bilinear and hence induces a linear map $\mathbf{C} \otimes \mathbf{H} \to \mathrm{End}_{\mathbf{C}}(\mathbf{H})$. This map is an algebra homomorphism, for

$$z_2 z_1 h \bar{q}_1 \bar{q}_2 = (z_1 z_2) h \, \overline{q_2 q_1}. \tag{1}$$

It can be checked that this map sends the basis $\{1, i\} \otimes \{1, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}\}$ into linearly independent endomorphisms, and hence the map is an isomorphism, for both sides have dimension 8. Finally note that $\mathrm{End}_{\mathbf{C}}(\mathbf{H}) \simeq \mathrm{End}_{\mathbf{C}}(\mathbf{C}^2) \simeq \mathbf{C}(2)$.

□3 If $q_1, q_2 \in \mathbf{H}$, define $f_{q_1,q_2} : \mathbf{H} \to \mathbf{H}$ by $f_{q_1,q_2}(h) = q_1 h \bar{q}_2$. In this way we get, as in 2), an algebra homomorphism $\mathbf{H} \otimes \mathbf{H} \to \mathrm{End}(\mathbf{H})$ which can be shown to be an isomorphism (both sides have dimension 16). Finally $\mathrm{End}(\mathbf{H}) \simeq \mathrm{End}(\mathbf{R}^4) \simeq \mathbf{R}(4)$. □

We are aiming at giving isomorphic descriptions of $C_{r,s}$ and $C^+_{r,s}$ in terms of basic algebra forms. It will turn out that it is enough to achieve this for $0 \leq r, s \leq 7$. So we will first look at how to fill in the slots in this 8×8 *chessboard*.

The main tools will be the explicit description of $C_{r,s}$ for slots close to the corner $(0, 0)$, which contains $C_{0,0} = \mathbf{R}$, and three *inductive formulas*.

Let us begin with the slots near $(0, 0)$:

$r \backslash s$	0	1	2
0	$\mathbf{R}$	$\mathbf{C}$	$\mathbf{H}$
1	$\mathbf{R} \oplus \mathbf{R}$	$\mathbf{R}(2)$	
2	$\mathbf{R}(2)$		

For row 0, $C_{0,s} = \bar{C}_s$, and we have:

$\bar{C}_1 \simeq \mathbf{C}$. In fact in this case the pseudoscalar $\boldsymbol{i}$ is a vector, $\bar{C}_1 = \langle 1, \boldsymbol{i} \rangle$ and $\boldsymbol{i}^2 = -1$.

$\bar{C}_2 \simeq \mathbf{H}$. If $\boldsymbol{e}_1, \boldsymbol{e}_2$ is an orthonormal basis, $\bar{C}_2 = \langle 1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_1\boldsymbol{e}_2 \rangle$ and the linear isomorphism $1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_1\boldsymbol{e}_2 \mapsto 1, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}$ is an algebra isomorphism.

$C_{1,0} = C_1 \simeq \mathbf{R} \oplus \mathbf{R}$. If $\boldsymbol{e}$ is a unit vector, $C_1 = \langle 1, \boldsymbol{e} \rangle$ with $\boldsymbol{e}^2 = 1$. The elements $e_+ = (1+e)/2$ and $e_- = (1-e)/2$ satisfy $e_+^2 = e_+$, $e_-^2 = e_-$ and $e_+e_- = 0$. It follows that the map $\mathbf{R} \oplus \mathbf{R} \to C_1$, $(\alpha, \beta) \mapsto \alpha e_+ + \beta e_-$ is an isomorphism.

$C_{1,1} \simeq \mathbf{R}(2)$. Let $\boldsymbol{e}_1, \boldsymbol{e}_2$ be an orthonormal basis. Then $C_{1,1} = \langle 1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_1\epsilon_2\rangle$. Consider the linear map $C_{1,1} \to \mathbf{R}(2)$ given by $1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_1\boldsymbol{e}_2 \mapsto I_2, E_1, E_2, E_3$, where

$$E_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, E_2 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, E_3 = \begin{pmatrix} & -1 \\ -1 & \end{pmatrix}.$$

This map is a linear isomorphism, because I_2, E_1, E_2, E_3 are linearly independent, and since $E_1^2 = I_2$, $E_2^2 = -I_2$ and $E_3 = E_1 E_2$, it is also an algebra isomorphism.

$C_{2,0} = C_2 \simeq \mathbf{R}(2)$. Like $C_{1,1}$, but using

$$E_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, E_2 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, E_3 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

◊1 $C_{r+2} \simeq \bar{C}_r \otimes C_2 \simeq \bar{C}_r \otimes \mathbf{R}(2)$.

◊2 $\bar{C}_{r+2} \simeq C_r \otimes \bar{C}_2 \simeq C_r \otimes \mathbf{H}$

◊3 $C_{r+1,s+1} \simeq C_{r,s} \otimes \mathbf{R}(2)$.

□1 Let $\bar{\gamma}_1, \ldots, \bar{\gamma}_r$ be standard generators of $\bar{C}_r$, so $\bar{\gamma}_k^2 = -1$, and γ_1, γ_2 standard generators of C_2, so $\gamma_1^2 = \gamma_2^2 = 1$. Let $\boldsymbol{i}_2 = \gamma_1\gamma_2$, so that $\boldsymbol{i}_2^2 = -1$.

Consider the elements $\Gamma_k \in \bar{C}_r \otimes C_2$ defined by $\Gamma_k = \bar{\gamma}_k \otimes \boldsymbol{i}_2$ $(k = 1, \ldots, r)$, and $\Gamma_{r+\ell} = 1 \otimes \gamma_\ell$ $(\ell = 1, 2)$.

The Γ_j $(j = 1, \ldots, r+2)$ are linearly independent and satisfy the relations of a standard basis of C_{r+2}.

So we have an injective homomorphism $C_{r+2} \to \bar{C}_r \otimes C_2$, which must be an isomorphism because both algebras have dimension 2^{r+2}.

□2 Let $\gamma_1, \ldots, \gamma_r$ be standard generators of C_r, so $\gamma_k^2 = 1$, and $\bar{\gamma}_1, \bar{\gamma}_2$ standard generators of $\bar{C}_2$, so $\bar{\gamma}_1^2 = \bar{\gamma}_2^2 = -1$. Let $\boldsymbol{i}_2 = \bar{\gamma}_1\bar{\gamma}_2$, so that $\boldsymbol{i}_2^2 = -1$.

Consider the elements $\bar{\Gamma}_k \in C_r \otimes \bar{C}_2$ defined by $\bar{\Gamma}_k = \gamma_k \otimes \boldsymbol{i}_2$ $(k = 1, \ldots, r)$, and $\bar{\Gamma}_{r+\ell} = 1 \otimes \bar{\gamma}_\ell$ $(\ell = 1, 2)$.

The $\bar{\Gamma}_j$ $(j = 1, \ldots, r+2)$ are linearly independent and satisfy the relations of a standard basis of $\bar{C}_{r+2}$.

So we have an injective homomorphism, $\bar{C}_{r+2} \to C_r \otimes \bar{C}_2$, which must be an isomorphism because both algebras have dimension 2^{r+2}.

□3 Let $\gamma_1, \ldots, \gamma_r, \bar{\gamma}_1, \ldots, \bar{\gamma}_s$ be standard generators of $C_{r,s}$: $\gamma_j^2 = 1$ ($j = 1, \ldots, r$) and $\bar{\gamma}_k^2 = -1$ ($k = 1, \ldots, s$). Let $\gamma, \bar{\gamma}$ be standard generators of $C_{1,1}$ ($\gamma^2 = 1$, $\bar{\gamma}^2 = -1$) and let $\boldsymbol{i}_2 = \gamma\bar{\gamma}$, so that $\boldsymbol{i}_2^2 = 1$.

Consider the elements Γ_j and $\bar{\Gamma}_k$ of $C_{r,s} \otimes C_{1,1}$, $j = 1, \ldots, r+1$, $k = 1, \ldots, s+1$, defined as $\Gamma_j = \gamma_j \otimes \boldsymbol{i}_2$ ($j = 1, \ldots, r$), $\Gamma_{r+1} = 1 \otimes \gamma$, $\bar{\Gamma}_k = \bar{\gamma}_k \otimes \boldsymbol{i}_2$ ($k = 1, \ldots, r$) and $\bar{\Gamma}_{s+1} = 1 \otimes \bar{\gamma}$.

The $\Gamma_1, \ldots, \Gamma_{r+1}, \bar{\Gamma}_1, \ldots, \bar{\Gamma}_{s+1}$ are linearly independent and satisfy the relations of a standard basis of $C_{r+1,s+1}$.

Now argue as in the previous cases. □

Remark. The C_r and $\bar{C}_r$, $r = 0, \ldots, 7$, fill the chessboard 0-th column and 0-th row, respectively, and ◊1 and ◊2, page 18, say that if for either one we know the values up to r, then we can know the values of the other up to $r + 2$. Since we know the values up to $r = 2$ for both of them, the determination of the other values can be carried out, for example, as follows:

$C_3 \simeq \bar{C}_1 \otimes \mathbf{R}(2) \simeq \mathbf{C} \otimes \mathbf{R}(2) \simeq \mathbf{C}(2)$; $C_4 \simeq \bar{C}_2 \otimes \mathbf{R}(2) \simeq \mathbf{H}(2)$;

$\bar{C}_3 \simeq C_1 \otimes \mathbf{H} \simeq \mathbf{H} \oplus \mathbf{H}$; $\bar{C}_4 \simeq C_2 \otimes \mathbf{H} \simeq \mathbf{H}(2)$;

$\bar{C}_5 \simeq C_3 \otimes \mathbf{H} \simeq \mathbf{C}(2) \otimes \mathbf{H} \simeq \mathbf{C}(4)$ (use the ◊s on page 13);

$\bar{C}_6 \simeq C_4 \otimes \mathbf{H} \simeq \mathbf{H}(2) \otimes \mathbf{H} \simeq \mathbf{R}(8)$ (again by the ◊s on page 13);

$C_5 \simeq \bar{C}_3 \otimes \mathbf{R}(2) \simeq \mathbf{H}(2) \oplus \mathbf{H}(2)$; $C_6 \simeq \bar{C}_4 \otimes \mathbf{R}(2) \simeq \mathbf{H}(4)$;

$C_7 \simeq \bar{C}_5 \otimes \mathbf{R}(2) \simeq \mathbf{C}(8)$; $\bar{C}_7 \simeq C_5 \otimes \mathbf{H} \simeq \mathbf{R}(8) \oplus \mathbf{R}(8)$.

Now use the recursive formulas on page 18 to fill in the rest:

$r \backslash s$	0	1	2	3
0	$\mathbf{R}$	$\mathbf{C}$	$\mathbf{H}$	$\mathbf{H} \oplus \mathbf{H}$
1	$\mathbf{R} \oplus \mathbf{R}$	$\mathbf{R}(2)$	$\mathbf{C}(2)$	$\mathbf{H}(2)$
2	$\mathbf{R}(2)$	$\mathbf{R}(2) \oplus \mathbf{R}(2)$	$\mathbf{R}(4)$	$\mathbf{C}(4)$
3	$\mathbf{C}(2)$	$\mathbf{R}(4)$	$\mathbf{R}(4) \oplus \mathbf{R}(4)$	$\mathbf{R}(8)$
4	$\mathbf{H}(2)$	$\mathbf{C}(4)$	$\mathbf{R}(8)$	$\mathbf{R}(8) \oplus \mathbf{R}(8)$
5	$\mathbf{H}(2) \oplus \mathbf{H}(2)$	$\mathbf{H}(4)$	$\mathbf{C}(8)$	$\mathbf{R}(16)$
6	$\mathbf{H}(4)$	$\mathbf{H}(4) \oplus \mathbf{H}(4)$	$\mathbf{H}(8)$	$\mathbf{C}(16)$
7	$\mathbf{C}(8)$	$\mathbf{H}(8)$	$\mathbf{H}(8) \oplus \mathbf{H}(8)$	$\mathbf{H}(16)$

$r \backslash s$	4	5	6	7
0	$\mathbf{H}(2)$	$\mathbf{C}(4)$	$\mathbf{R}(8)$	$\mathbf{R}(8) \oplus \mathbf{R}(8)$
1	$\mathbf{H}(2) \oplus \mathbf{H}(2)$	$\mathbf{H}(4)$	$\mathbf{C}(8)$	$\mathbf{R}(16)$
2	$\mathbf{H}(4)$	$\mathbf{H}(4) \oplus \mathbf{H}(4)$	$\mathbf{H}(8)$	$\mathbf{C}(16)$
3	$\mathbf{C}(8)$	$\mathbf{H}(8)$	$\mathbf{H}(8) \oplus \mathbf{H}(8)$	$\mathbf{H}(16)$
4	$\mathbf{R}(16)$	$\mathbf{C}(16)$	$\mathbf{H}(16)$	$\mathbf{H}(16) \oplus \mathbf{H}(16)$
5	$\mathbf{R}(16) \oplus \mathbf{R}(16)$	$\mathbf{R}(32)$	$\mathbf{C}(32)$	$\mathbf{H}(32)$
6	$\mathbf{R}(32)$	$\mathbf{R}(32) \oplus \mathbf{R}(32)$	$\mathbf{R}(64)$	$\mathbf{C}(64)$
7	$\mathbf{C}(32)$	$\mathbf{R}(64)$	$\mathbf{R}(64) \oplus \mathbf{R}(64)$	$\mathbf{R}(128)$

♢1 $C_{n+8} \simeq C_n \otimes \mathbf{R}(16)$.

♢2 $\bar{C}_{n+8} \simeq \bar{C}_n \otimes \mathbf{R}(16)$.

♢3 $C_{r+4,s+4} \simeq C_{r,s} \otimes \mathbf{R}(16)$.

□ ♢1 and ♢2 on page 18 allow us to write:

$$\begin{aligned} C_{n+8} &\simeq \bar{C}_{n+6} \otimes C_2 \simeq C_{n+4} \otimes \bar{C}_2 \otimes C_2 \\ &\simeq \bar{C}_{n+2} \otimes C_2 \otimes \bar{C}_2 \otimes C_2 \\ &\simeq C_n \otimes \bar{C}_2 \otimes C_2 \otimes \bar{C}_2 \otimes C_2 \end{aligned}$$

Now we have, using the chessboard and ♢3 on page 13,

$$\begin{aligned} \bar{C}_2 \otimes C_2 \otimes \bar{C}_2 \otimes C_2 &\simeq \mathbf{H} \otimes \mathbf{R}(2) \otimes \mathbf{H} \otimes \mathbf{R}(2) \\ &\simeq \mathbf{H} \otimes \mathbf{H} \otimes \mathbf{R}(4) \\ &\simeq \mathbf{R}(4) \otimes \mathbf{R}(4) \simeq \mathbf{R}(16). \end{aligned}$$

With this we conclude the proof of ♢1.

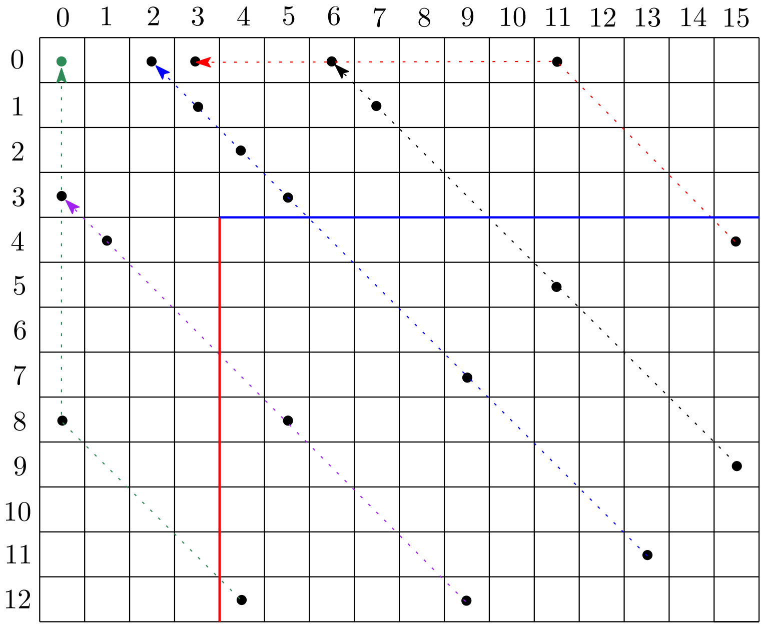
The proof of $\lozenge 2$ follows the same pattern as the proof for $\lozenge 1$:

$$\begin{aligned}\bar{C}_{n+8} \simeq C_{n+6} \otimes \bar{C}_2 \simeq \bar{C}_{n+4} \otimes C_2 \otimes \bar{C}_2 \\ \simeq C_{n+2} \otimes \bar{C}_2 \otimes C_2 \otimes \bar{C}_2 \\ \simeq \bar{C}_n \otimes C_2 \otimes \bar{C}_2 \otimes C_2 \otimes \bar{C}_2\end{aligned}$$

and clearly $C_2 \otimes \bar{C}_2 \otimes C_2 \otimes \bar{C}_2 \simeq \mathbf{R}(16)$.

The proof of $\lozenge 3$ is simpler: it suffices to apply the rule $\lozenge 3$ on page 18 four successive times to conlude that

$$C_{r+4,s+4} \simeq C_{r,s} \otimes \mathbf{R}(2)^{\otimes 4} \simeq C_{r,s} \otimes \mathbf{R}(16).$$


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
0 1 2 3 4 5 6 7 8 9 10 11 12

Reduction to the chessboard. Given r, s, let $m = \min(r, s)$ and k the greatest non-negative integer such that $4k \leq m$. Let $r' = r - 4k$, $s' = s - 4k$ and $m' = m - 4k = \min(r', s')$. Then ◊3 on page 23 tells us that $C_{r,s} \simeq C_{r',s'} \otimes \mathbf{R}(16^k)$ and by ◊3 on page 18, that $C_{r',s'} \simeq C_{r'',s''} \otimes \mathbf{R}(2^{m'})$, with $r'' = r' - m'$, $s'' = s' - m'$, or $C_{r,s} \simeq C_{r'',s''} \otimes \mathbf{R}(2^{m'}16^k)$. Since either $s'' = 0$ (when $s \leqslant r$) or $r'' = 0$ (when $r \leqslant s$), we see that $C_{r,s} \simeq C_{r''}$ (when $s \leqslant r$) or $C_{r,s} \simeq \bar{C}_{s''}$ (when $r \leqslant s$).

The integer $\nu = r - s \mod 8$ is clearly invariant in the reduction process. It follows that $C_{r,s} \simeq C_\nu \otimes \mathbf{R}(d)$ if $r \geqslant s$ and $C_{r,s} \simeq \bar{C}_{8-\nu} \otimes \mathbf{R}(d')$ if $r < s$, where d and d' are positive integers. Now in the 15 algebras C_ν $(\nu = 0, \ldots, 7)$ and $\bar{C}_{8-\nu}$ $(\nu = 1, \ldots, 7)$ there appear exactly 5 forms (up to tensoring by $\mathbf{R}(2^m)$, for some m):

ν	0, 2	1	3, 7	4, 6	5
Form	$\mathbf{R}$	$\mathbf{R} \oplus \mathbf{R}$	$\mathbf{C}$	$\mathbf{H}$	$\mathbf{H} \oplus \mathbf{H}$

Algorithm

While $r, s \geqslant 4$, jump to $r-4, s-4$ and update the matrix factor by $\mathbf{R}(16)$. So we may assume $\min(r,s) \leqslant 3$.

While $r, s \geqslant 1$, jump to $r-1, s-1$ and update the matrix factor by $\mathbf{R}(2)$. After at most three steps, we are going to hit the C_n boundary or the $\bar{C}_n$.

While $n \geqslant 8$, jump to the slot $n-8$ along the boundary and update the matrix factor by $\mathbf{R}(16)$. So we may assume that we have landed on C_n or $\bar{C}_n$ with $0 \leqslant n \leqslant 7$.

Output: Let $\nu = r - s \mod 8$ and define $d_k = 2^{(n-k)/2}$, for $k = 0, \ldots, 4$ (in each usage below, d_k is an integer). Then return the algebra $F(d)$ indicated by the following table:

ν	$0,2$	1	$3,7$	$4,6$	5
F	$\mathbf{R}$	$\mathbf{R}\oplus\mathbf{R}$	$\mathbf{C}$	$\mathbf{H}$	$\mathbf{H}\oplus\mathbf{H}$
d	d_0	d_1	d_1	d_2	d_3

Now the isomorphisms $C^+_{r,s} \simeq C_{r,s-1}$ if $s > 0$ and $C^+_n \simeq \bar{C}_{n-1}$ imply that the form F^+_ν of $C^+_{r,s}$ is $F_{\nu+1}$ (in all cases), and so:

ν	$1,7$	0	$2,6$	$3,5$	4
F^+	$\mathbf{R}$	$\mathbf{R}\oplus\mathbf{R}$	$\mathbf{C}$	$\mathbf{H}$	$\mathbf{H}\oplus\mathbf{H}$
d^+	d_1	d_2	d_2	d_3	d_4

That the d's are as claimed follows by counting dimensions. The dimension of $C_{r,s}$ is 2^n, and the dimensions of the five forms are

Form	$\mathbf{R}(m)$	$\mathbf{R}(m) \oplus \mathbf{R}(m)$	$\mathbf{C}(m)$	$\mathbf{H}(m)$	$\mathbf{H}(m) \oplus \mathbf{H}(m)$
$d(m)$	m^2	$2m^2$	$2m^2$	$4m^2$	$8m^2$

Solving for m in the equation $2^n = d(m)$ we get the claimed expressions. For example, if $2^n = 8m^2$, then $m^2 = 2^{n-3}$ and hence $m = 2^{(n-3)/2} = d_3$.

$n \mod 2$	C_n	C_n^+
0	$\mathbf{C}(d_0)$	$\mathbf{C}(d_2) \oplus \mathbf{C}(d_2)$
1	$\mathbf{C}(d_1) \oplus \mathbf{C}(d_1)$	$\mathbf{C}(d_3)$

Pin and Spin representations

Let $n = r + s$ and $\nu = r - s \mod 8$.

We defined $d_k = 2^{(n-k)/2}$ (it will be used for $k = 0, 1, \ldots, 4$ and in cases that will guarantee that $(n-k)/2$ is an integer).

Let $\mathbf{i} = \mathbf{i}_{r,s}$ be the pseudoscalar (volume element) of $C_{r,s}$.

◊1 $\mathbf{i}^2 = (-1)^{s+n//2} = (-1)^{(r-s)//2} = (-1)^{\nu//2}$. Thus

$$\mathbf{i}^2 = 1 \text{ if } \nu \equiv 0, 1 \mod 4$$

$$\mathbf{i}^2 = -1 \text{ if } \nu \equiv 2, 3 \mod 4$$

◊2 For any vector $\boldsymbol{e}$, $\boldsymbol{e}\mathbf{i} = (-1)^{n-1}\mathbf{i}\boldsymbol{e}$. Therefore, $\mathbf{i}$ is central if n is odd and anticommutes with vectors if n is even (so it anticommutes with odd multivectors and commutes with even multivectors). Since $n \equiv \nu \mod 2$, we can use ν instead of n.

Let $\mathbf{K}$ be one of the fields $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$.

A $\mathbf{K}$-representation of a *real* algebra A is an $\mathbf{R}$-linear homomorphism $\rho : A \to \mathrm{End}_{\mathbf{K}}(E)$ for some $\mathbf{K}$-vector space E.

Equivalent $\mathbf{K}$-representations are defined as usual: isomorphic under a $\mathbf{K}$-linear isomorphism. Note that ρ defines an A-module structure on E.

A representation ρ is irreducible if the only there are no non-trivial submodules.

Similar definitions can be phrased for groups instead of algebras.

Facts

(1) Every irreducible $\mathbf{R}$-representation of the real algebra $\mathbf{R}(n)$ is isomorphic to $\mathbf{R}^n$

(2) Every irreducible $\mathbf{H}$-representation of the real algebra $\mathbf{H}(n)$ is isomorphic to $\mathbf{H}^n$ (as a right $\mathbf{H}$-vector space).

(3) Every irreducible $\mathbf{C}$-representation of the real algebra $\mathbf{C}(n)$ is isomorphic either to $\mathbf{C}^n$ or to $\bar{\mathbf{C}}^n$.

A *pinor representation* of $\mathrm{Pin}_{r,s}$ is the restriction to $\mathrm{Pin}_{r,s}$ of an *irreducible* representation of $C_{r,s}$.

♢ The type of the pinor representations depends only on ν.

ν **even**. Unique pinor respresentation $P_{s,t}$.

$\nu = 0, 2$: real of dimension d_0 (*Majorana*, *M*): $\mathbf{R}^{d_0}$.

$\nu = 4, 6$: quaternionic of dimension d_2 (*symplectic M*, *sM*): $\mathbf{H}^{d_2}$.

ν **odd**. Two pinor representations.

$\nu = 1, 5$, so $\mathbf{i}^2 = 1$. There are two pinor representations $P^{\pm}_{r,s}$, distinguished by the action ($+1$ or -1) of $\mathbf{i}$.

$\nu = 1$: real of dimension d_1 (*M*): $\mathbf{R}^{d_1}, \bar{\mathbf{R}}^{d_1}$

$\nu = 5$: quaternionic of dimension d_3 (*sM*): $\mathbf{H}^{d_3}, \bar{\mathbf{H}}^{d_3}$.

$\nu = 3, 7$, so $\mathbf{i}^2 = -1$: complex $P_{r,s}$ and $\bar{P}_{r,s}$ of complex dimension d_1, distinguished by the action ($+\mathbf{i}$ or $-\mathbf{i}$) of $\mathbf{i}$ (*Dirac*, *D*): $\mathbf{C}^{d_1}, \bar{\mathbf{C}}^{d_1}$.

$$\nu \begin{cases} \text{even} \begin{cases} 0,2 \to \mathbf{R}^{d_0} \ (M) \\ 4,6 \to \mathbf{H}^{d_2} \ (sM) \end{cases} \\ \\ \text{odd} \begin{cases} 1,5 \ (\mathbf{i}^2 = 1) \begin{cases} 1 \to \mathbf{R}^{d_1}, \bar{\mathbf{R}}^{d_1} \ (M) \\ 5 \to \mathbf{H}^{d_3}, \bar{\mathbf{H}}^{d_3} \ (sM) \end{cases} \\ 3,7 \ (\mathbf{i}^2 = -1) \to \mathbf{C}^{d_1}, \bar{\mathbf{C}}^{d_1} \ (D) \end{cases} \end{cases}$$

Remark. Note that the k appearing in the d_k has the same parity as ν. Thus for ν even (odd), only d_0 and d_2 (d_1 and d_3) appear.

A *spinor representation* of $\mathrm{Spin}_{r,s}$ is the restriction to $\mathrm{Spin}_{r,s}$ of an *irreducible* representation of $C^+_{r,s}$.

◊ The type of the spinor representations depends only on ν.

ν odd. There is a unique spinor representation $S_{r,s}$.

$\nu = 1, 7$: real of dimension d_1: $\mathbf{R}^{d_1}$ (*M*).

$\nu = 3, 5$: quaternionic of dimension d_3: $\mathbf{H}^{d_3}$ (*sM*).

ν even. Two representations (*Weyl spinors*, *W*).

$\nu = 2, 6$ ($\mathbf{i}^2 = -1$): S and $\bar{S}$ of complex dimension d_2, distinguished by the action of $\mathbf{i}$ (i and $-i$): $\mathbf{C}^{d_2}, \bar{\mathbf{C}}^{d_2}$.

$\nu = 0, 4$ ($\mathbf{i}^2 = 1$): $S^{\pm}$, distinguished by the action of $\mathbf{i}$ ($+1$ and -1):

- $\nu = 0$: real, dimension d_2: $\mathbf{R}^{d_2}, \bar{\mathbf{R}}^{d_2}$ (*MW*).
- $\nu = 4$: quaternionic, dimension d_4: $\mathbf{H}^{d_4}, \bar{\mathbf{H}}^{d_4}$ (*sMW*).

$$
\nu \begin{cases} \text{odd} \to \begin{cases} 1,7 \to \mathbf{R}^{d_1}\ (M) \\ 3,5 \to \mathbf{H}^{d_3}\ (sM) \end{cases} \\ \\ \text{even} \begin{cases} 2,6\ (\mathbf{i}^2=-1) \to \mathbf{C}^{d_2}, \bar{\mathbf{C}}^{d_2}\ (DW) \\ \\ 0,4\ (\mathbf{i}^2=1) \to \begin{cases} 0 \to \mathbf{R}^{d_2}, \bar{\mathbf{R}}^{d_2}\ (MW) \\ 4 \to \mathbf{H}^{d_4}, \bar{\mathbf{H}}^{d_4}\ (sMW) \end{cases} \end{cases} \end{cases}
$$

Remark. The forms corresponding to $\nu = 5, 6, 7$ are the same as those for $\nu = 3, 2, 1$. This means that the row of the 8 forms indexed by $\nu = 0, \dots, 7$ is symmetric with respect to $\nu = 4$.

Note also that the k appearing in the d_k has the same parity as ν. Thus for ν odd (even), only d_1 and d_3 (d_2 and d_4) appear.

For a given n ($1 \leqslant n \leqslant 7$), there are $n+1$ signatures: $(r, n-r)$, $0 \leqslant r \leqslant n$. The corresponding $\nu = 2r - n$ decrease from $\nu = n$ to $\nu = -n$ in steps of -2, but in case $n \geqslant 3$ it is only necessary to find the forms for the first four values of ν because the remaining $n-3$ cases repeat the beginning of the sequence, as

$$\nu(r,s) = \nu(r+4, s-4) \mod 8.$$

In the tables that follow, we first specify the dimension n and the relevant d_k. Then the first row contains the $n+1$ signatures, the second the corresponding ν's, while the third and forth specify the key data of the corresponding pinor and spinor representations. If the representation is unique (up to isomorphism), it is denoted F^d, with $F = \mathbf{R}, \mathbf{C}, \mathbf{H}$ and d the F-dimension of the representation. If there are to 'conjugate' representations of dimension d, they are denoted F^d and $\bar{F}^d$. The latter is like the former, but with the action of the $\mathbf{i}$ (multiplication by i for $F = \mathbf{C}$ and by 1 for $F = \mathbf{R}$ or $F = \mathbf{H}$) reversed in sign.

$n = 1$. $d_1 = 1$.

(r,s)	(1,0)	(0,1)
ν	1	7
$P, \bar{P}$	$\mathbf{R}, \bar{\mathbf{R}}$	$\mathbf{C}, \bar{\mathbf{C}}$
S	$\mathbf{R}$	$\mathbf{R}$

$n = 2$. $d_0 = 2, d_2 = 1$.

(r,s)	(2,0)	(1,1)	(0,2)
ν	2	0	6
P	$\mathbf{R}^2$	$\mathbf{R}^2$	$\mathbf{H}$
$S, \bar{S}$	$\mathbf{C}, \bar{\mathbf{C}}$	$\mathbf{R}, \bar{\mathbf{R}}$	$\mathbf{C}, \bar{\mathbf{C}}$

In the lower left corner, $C^+_{2,0} = C^+_2 = \mathbf{C}$ and the representations are the action of $\mathbf{C}$ on itself by multiplication and conjugate multiplication.

$n = 3$. $d_1 = 2, d_3 = 1$.

(r,s)	(3,0)	(2,1)	(1,2)	(0,3)
ν	3	1	7	5
$P, \bar{P}$	$\mathbf{C}^2, \bar{\mathbf{C}}^2$	$\mathbf{R}^2, \bar{\mathbf{R}}^2$	$\mathbf{C}^2, \bar{\mathbf{C}}^2$	$\mathbf{H}, \bar{\mathbf{H}}$
S	$\mathbf{H}$	$\mathbf{R}^2$	$\mathbf{R}^2$	$\mathbf{H}$

$n = 4$. $d_0 = 4, d_2 = 2, d_4 = 1$.

(r,s)	(4,0)	(3,1)	(2,2)	(1,3)	(0,4)
ν	4	2	0	6	4
P	$\mathbf{H}^2$	$\mathbf{R}^4$	$\mathbf{R}^4$	$\mathbf{H}^2$	$\mathbf{H}^2$
$S, \bar{S}$	$\mathbf{H}, \bar{\mathbf{H}}$	$\mathbf{C}^2, \bar{\mathbf{C}}^2$	$\mathbf{R}^2, \bar{\mathbf{R}}^2$	$\mathbf{C}^2, \bar{\mathbf{C}}^2$	$\mathbf{H}, \bar{\mathbf{H}}$

Remark. The space $\mathbf{C}^2$ for the signature $(3,0)$ is the space of *Pauli pinors* and $\mathbf{C}^2 \oplus \bar{\mathbf{C}}^2$ for the signature $(1,3)$, or $(3,1)$, is the space of *Dirac spinors*.

$n = 5.$ $d_1 = 4$, $d_3 = 2.$

(r, s)	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)	(0,5)
ν	5	3	1	7	5	3
$P, \bar{P}$	$\mathbf{H}^2, \bar{\mathbf{H}}^2$	$\mathbf{C}^4, \bar{\mathbf{C}}^4$	$\mathbf{R}^4, \bar{\mathbf{R}}^4$	$\mathbf{C}^4, \bar{\mathbf{C}}^4$	$\mathbf{H}^2, \bar{\mathbf{H}}^2$	$\mathbf{C}^4, \bar{\mathbf{C}}^4$
S	$\mathbf{H}^2$	$\mathbf{H}^2$	$\mathbf{R}^4$	$\mathbf{R}^4$	$\mathbf{H}^2$	$\mathbf{H}^2$

$n = 6.$ $d_0 = 8$, $d_2 = 4$, $d_4 = 2.$

(r, s)	(6,0)	(5,1)	(4,2)	(3,3)	(2,4)	(1,5)	(0,6)
ν	6	4	2	0	6	4	2
P	$\mathbf{H}^4$	$\mathbf{H}^4$	$\mathbf{R}^8$	$\mathbf{R}^8$	$\mathbf{H}^4$	$\mathbf{H}^4$	$\mathbf{R}^8$
$S, \bar{S}$	$\mathbf{C}^4, \bar{\mathbf{C}}^4$	$\mathbf{H}^2, \bar{\mathbf{H}}^2$	$\mathbf{C}^4, \bar{\mathbf{C}}^4$	$\mathbf{R}^4, \bar{\mathbf{R}}^4$	$\mathbf{C}^4, \bar{\mathbf{C}}^4$	$\mathbf{H}^2, \bar{\mathbf{H}}^2$	$\mathbf{C}^4, \bar{\mathbf{C}}^4$

$n = 7.\ d_1 = 8, d_3 = 4.$

(r,s)	(7,0)	(6,1)	(5,2)	(4,3)	(3,4)	(2,5)	(1,6)	(0,7)
ν	7	5	3	1	7	5	3	1
$P, \bar{P}$	$\mathbf{C}^8, \bar{\mathbf{C}}^8$	$\mathbf{H}^4, \bar{\mathbf{H}}^4$	$\mathbf{C}^8, \bar{\mathbf{C}}^8$	$\mathbf{R}^8, \bar{\mathbf{R}}^8$	$\mathbf{C}^8, \bar{\mathbf{C}}^8$	$\mathbf{H}^4, \bar{\mathbf{H}}^4$	$\mathbf{C}^8, \bar{\mathbf{C}}^8$	$\mathbf{R}^8, \bar{\mathbf{R}}^8$
S	$\mathbf{R}^8$	$\mathbf{H}^4$	$\mathbf{H}^4$	$\mathbf{R}^8$	$\mathbf{R}^8$	$\mathbf{H}^4$	$\mathbf{H}^4$	$\mathbf{R}^8$

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