17th "Lluís Santaló" Research School

A view of F. Klein's Erlangen Program through GA

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A view of F. Klein's Erlangen Program through GA

- **What is Geometry?**. Erlangen Program (EP). EP/Introduction. EP (§1, §3 & §4, §5, §6, §7 & §8, §9 & §10, Notes). 1974: Views after one century. A current view. A score card.
- **Lie groups with GA**. Summary of notations from SX2. The Lipschitz approach to versors. Primacy of the rotor group. A criterion for rotors for $n \leqslant 5$. Examples. Plane rotors. Topology of the rotor group.
- **Lie algebras with GA** Infinitesimal rotors. Infinitesimal rotors are bivectors. Identification of bivectors with skew-symmetric endomorphisms. The exponential map from bivectors to rotors. Any bivector is an infinitesimal rotor. Infinitesimal rotors in euclidean and lorentzian spaces.
- **Outshoots of the EP**. Cartan's spaces. Lie theory in Physics. Gauge theories. Invariant theory.
- **References**

What is Geometry

Vergleichende Betrachtungen

über

neuere geometrische Forschungen

von

Dr. Felix Klein,
o. ö. Professor der Mathematik an der Universität Erlangen.

Erlangen.
Verlag von Andreas Deichert.
1872.

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Programm

zum Eintritt in die philosophische Facultät und den Senat
der k. Friedrich-Alexanders-Universität
zu Erlangen.

Erlangen.
Verlag von Andreas Deichert.
1872.

Klein-1872 [1]

Introduction

- Primacy of projective geometry: "the projective method now embraces the whole of geometry."
- On metrical properties: "relations to the circle at infinity common to all spheres."
- Need of general principle: "beside the elementary and the projective geometry, [...] geometry of reciprocal radii vectors, the geometry of rational transformations, ...".
- Stress on the unity in geometry: "geometry, [...] one in substance, [...] broken up [...] into a series of almost distinct theories." "[...] distinction between modern synthetic and modern analytic geometry must no longer be regarded as essential."

- On Lie: "Our respective investigations, [...] have led to the same generalized conception here presented."
- Exceedingly incomplete: "projective geometry leaves untouched [...] the theory of the curvature of surfaces."
- Mathematical physics: bemoans that "the mathematical physicist disregards the advantages afforded him in many cases by only a moderate cultivation of the projective view."
- On n-folds: Useful as possible parameter spaces of geometric figures, like for lines in 3D (which make up a 4-fold, *Klein's quadric*).

§*1*

- Most essential idea: "*group* of space-transformations." (emphasis in original). Examples: group of rigid motions, group of collineations. It is assumed that the group is not finite (or even discrete).

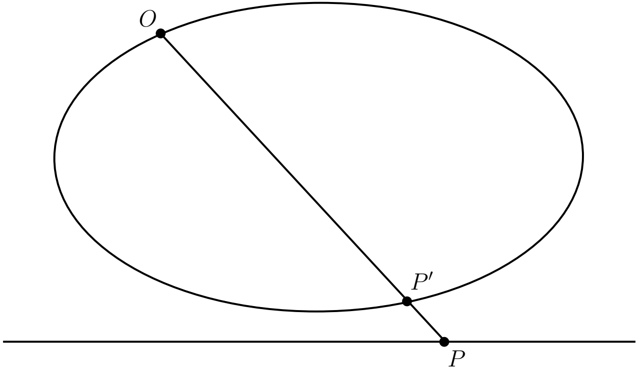
  Motto: Geometry is the study of invariants under a group of transformations.

  Actions. An *action* of a group G on a space X is a map $G \times X \to X$, $(g, x) \mapsto g \cdot x$, such that $1_G \cdot x = x$ for all $x \in X$ and $g' \cdot (g \cdot x) = (g'g) \cdot x$ for all $x \in X$ and all $g, g' \in G$. It is easy to see that $T_g : X \to X$, $x \mapsto g \cdot x$, is a *transformation* of X (one-to-one and onto), and that $T_g^{-1} = T_{g^{-1}}$.

As expressed in the title, this view allows to compare geometries for which the group of one is contained in the other. For example, the euclidean group is a subgroup of the affine group and this in turn is a subgroup of the projective group.

§3: *Projective geometry*. "Metrical properties are to be considered as projective relations to a fundamental configuration, the circle at infinity."

§4: *Transfer of properties by representations*.



§5: *On the arbitrariness of the choice of the space-element. Hesse's principle of transference. Line geometry*. "[...] as long as we base our geometrical investigation upon the same group of transformations, the substance of the geometry remains unchanged."

"The essential thing is, then, the group of transformations; the number of dimensions to be assigned to a sapce appears of secondary importance."

Example: Klein's quadric.

§6: *The geometry of reciprocal radii. Interpretation of $x + iy$*. "[...] the processes here involved have not yet, like projective geometry, been united into a special geometry, whose fundamental group would be the totality of the transformations resulting from a combination of the principal group with geometric inversion."

"In the geometry of reciprocal radii the elementary ideas are the *point*, *circle*, and *sphere*. The line and the plane are special cases of the latter, characterized by the property that they contain a point which, however, has no further special significance in the theory, namely, the *point at infinity*. *If we regard this point as fixed, elementary geometry is the result*."

The geometry of reciprocal radii in the plane and the projective geometry on a quadric surface are one and the same; and, similarly: *The geometry of reciprocal radii in space is equivalent to the projective treatment of a space represented by a quadratic equation between five homogeneous variables*.

§7: *Extension of the preceding considerations. Lie's sphere geometry.*
"[...] we have the following correspondence:

- *the space geometry whose element is the plane and whose group is formed of the linear transformations converting a sphere into itself*, and
- *the plane geometry whose element is the circle and whose group is the group of geometric inversion*.

§8: *Enumeration of other methods based on a group of point-transformations*.

8.1 The Group of Rational Transformations.

8.2 Analysis situs.

8.3 The Group of all Point-transformations.

§9. *On the group of all contact-transformations*. Here the basic element is a flag point-plane (five dimensions) and later also point-line-plane (six dimensions).

§10. *On spaces of any number of dimensions*.

10.1 The Projective Method or Modern Algebra (Theory of Invariants).

10.2 The spaces of constant curvature.

10.3 Flat spaces.

§Notes

V. On the so-called non-euclidean geometry.

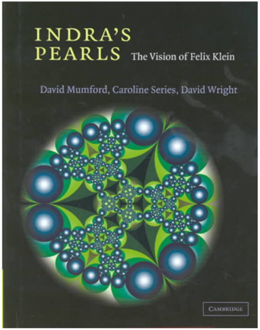
... the axiom of parallels is not a mathematical consequence of the other axioms usually assumed, but the expression of an essentially new principle of space-perception [...]

... an important mathematical idea, –the idea of a space of constant curvature.

S. Lie (1842-1899)

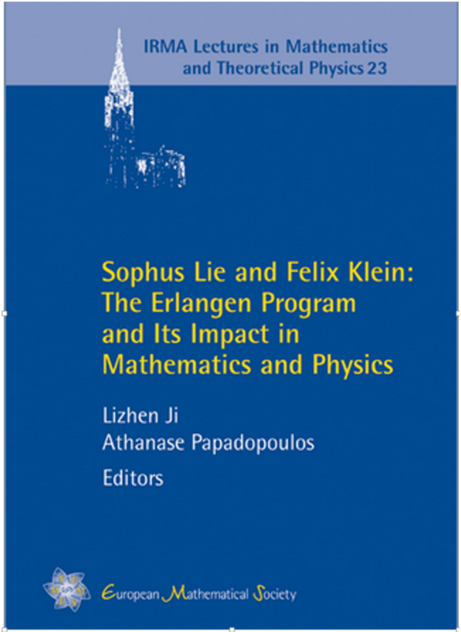
F. Klein (1849-1925)

F. Klein "Erlangen Program" is considered, quite rightly, one of the most important milestones in the history of mathematics in the XIX century. With the hindsight of one century, we can say that it constitutes a kind of "watershed line": it appears as a result of the long and successful development of projective geometry since the beginning of the century, which it sums up, condenses and "explains" thanks to the valorization of the fundamental role played by the concept of group. In doing so, it inaugurated the dominance that the theory of groups will gradually exercise on all of mathematics (not only Geometry), and the amalgamation increasingly close of concepts from algebra, geometry or analysis: tendencies that are among the more characteristic of today's mathematics.



Felix Klein, […] discovered in mathematics an idea prefigured in Buddhist mythology: the heaven of Indra contained a net of pearls, each of which was reflected in its neighbour, so that the whole Universe was mirrored in each pearl. Klein studied infinitely repeated reflections and was led to forms with multiple coexisting symmetries. For a century, these images barely existed outside the imagination of mathematicians. However, in the 1980s, the authors embarked on the first computer exploration of Klein's vision, and in doing so found many further extraordinary images. Join the authors on the path from basic mathematical ideas to the simple algorithms that create the delicate fractal filigrees, most of which have never appeared in print before. Beginners can follow the step-by-step instructions for writing programs that generate the images. Others can see how the images relate to ideas at the forefront of research.

The Erlangen program expresses a ***fundamental point of view on the use of groups and transformation groups in mathematics and physics***. The present volume is the first modern comprehensive book on that program and its *impact in contemporary mathematics and physics*. Klein spelled out the program, and **Lie**, who contributed to its formulation, is the **first** mathematician **who made it effective** in his work. The theories that these two authors developed are also linked to their personal history and to their relations with each other and with other mathematicians, including Hermann **Weyl**, Élie **Cartan**, Henri **Poincaré**, and many others. All these facets of the Erlangen program appear in the present volume.



- 1. S. Lie, a giant in mathematics (*Lizhen Li*)
- 2. Felix Klein: his life and mathematics (*Lizhen Li*)
- 3. Klein and the Erlangen Programme (*Jeremy J. Gray*)
- 4. Klein's "Erlanger Programm": do traces of it exist in physical theories? (*Hubert Goenner*)
- 5. On Klein's *So-called Non-Euclidean geometry* (*Norbert A'Campo*, *Athanase Papadopouloulos*)
- 11. Three-dimensional gravity – an application of Felix Klein's ideas in physics (C. Meusburger)
- 12. Invariaces in physics and group theory (J.-B. Zuber)

- A philosophy about what geometry is about and the role it should play in Mathematics (Russo, Appendix to the French edition of the EP).
- In a geometry, the group becomes primary, the spaces on which it acts secondary. One group may act on many different spaces.
- Spaces that are quite different may be isomorphic geometries, like the conformal geometry of the euclidean 3-space and the hyperbolic geometry of dimension 4.
- Algebraically, the geometric properties and relations are expressed by invariants and covariants, which can be calculated by the symbolic method.
- Has fostered the discovery of many new geometries, and a deepening into those already known.

Onishchik-Sulanke-2006 [2] (emphasis not in original):

"Projective geometry, and the Cayley-Klein geometries embedded into it, are rather ancient topics of geometry, which originated in the 19th century with the work of V. Poncelet, J. Gegonne, Ch. v. Staudt, A.-F. Möbius, **A. Cayley**, **F. Klein**, **S. Lie**, N. I. Lobatschewski, and many others. [...] the most important classical geometries arre systematically developed following the principles founded by A. Cayley and F. Klein, which rely on distinguishing an *absolute* and then *studying the resulting invariants of geometric objects*. These methods, determined by linear algebra and the theory of transformation groups, are just what it is needed in algebraic as well as differential geometry. Furthemore, they may rightly be considered as an integrating factor for the development of analysis, where we mainly have in mind the harmonic or geometric analysis as based on the thery of Lie groups."

Lie groups with GA

- $E = E_{r,s}$: regular space of *signature* (r, s).
- $E^{\times} = E^{\times}_{r,s}$: set of *non-null* (or invertible) vectors.
- $\mathcal{G} = \mathcal{G}_{r,s}$: the *full geometric algebra* of signature (r, s).
- $\mathcal{G}^{+} = \mathcal{G}^{+}_{r,s}$: *even subalgebra*.
- $\mathcal{G}^{\times} = \mathcal{G}^{\times}_{r,s}$ group of *invertible multivectors*.
- $V = V_{r,s}$: *versor group* (subgroup of $\mathcal{G}^{\times}$ generated by $E^{\times}$).
- $V^{+} = V^{+}_{r,s}$: *even versor group*, $V \cap \mathcal{G}^{+}$.
- $P = \mathrm{Pin}_{r,s}$: group of *unit versors*, or *pinors* ($v\widetilde{v} = \pm 1$).
- $S = \mathrm{Spin}_{r,s}$: group of *even unit versors*, or *spinors*.
- $S^{+} = S^{+}_{r,s}$: spinors such that $v\widetilde{v} = 1$ (*rotors*).
- $\mathrm{O} = \mathrm{O}_{r,s}$: *orthogonal group* (endoisometries of E).
- $SO = \mathrm{SO}_{r,s}$: *special O* (proper endoisometries of E).

- $\widetilde{\rho} : V \to \mathrm{O}$, $v \mapsto \underline{v}$, $\underline{v}(\boldsymbol{x}) = v^{\alpha}\boldsymbol{x}v^{-1}$.
  It is onto and $\ker(\widetilde{\rho}) = \mathbf{R}^{\times}$ (SX2/58).
- $\rho : V^{+} \to \mathrm{SO}$, $v \mapsto \underline{v}$, $\underline{v}(\boldsymbol{x}) = v\boldsymbol{x}v^{-1}$.
  It is onto and $\ker(\rho) = \mathbf{R}^{\times}$.
- $\widetilde{\rho} : P \to \mathrm{O}$, onto and $\ker(\widetilde{\rho}) = \{\pm 1\}$ (SX2/60).
- $\rho : S \to \mathrm{SO}$, onto and $\ker(\rho) = \{\pm 1\}$.

Remark. Let SO^{+} be the image of the map $\rho : S^{+} \to SO$. We will see that SO^{+} is the connected component of the identity of SO except for $(r, s) = (1, 1)$. Since for euclidean and anti-euclidean spaces we have $S = S^{+}$, we get that SO_n is connected for all n.

Remark. ρ and $\widetilde{\rho}$ are often called the *adjoint* (Ad) and *twisted adjoint* ($\widetilde{\mathrm{Ad}}$) *representations*.

The *Lipschitz group* of $E_{r,s}$, $\Gamma = \Gamma_{r,s}$, is the subgroup of $\mathcal{G}^\times$ formed by the invertible elements $x \in \mathcal{G}^+ \cup \mathcal{G}^-$ such that $xEx^{-1} = E$. We clearly have that the versor group $V = V_{r,s}$ is a subgroup of Γ.

◊ $\Gamma = V$.

□ Take $x \in \Gamma$ and define $\underline{x} : E \to E$, $\boldsymbol{e} \mapsto x^\alpha \boldsymbol{e} x^{-1}$, which is clearly linear. In fact, $\underline{x} \in \mathrm{O}_{r,s}$ (**1**). So we have a map $\rho^\sharp : \Gamma \to O_{r,s}$, $x \mapsto \underline{x}$. This map is an onto homomorphism (**2**). Finally, the kernel of $\rho^\sharp : \Gamma \to \mathrm{O}_{r,s}$ is $\mathbf{R}^\times$ (**3**). Now if $x \in \Gamma$ and $v \in V$ is such that $\rho^\sharp(x) = \rho^\sharp(v)$, then $\lambda = xv^{-1} \in \ker \rho^\sharp = \mathbf{R}^\times$ and $x = \lambda v \in V$. □

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1. $q(\underline{x}\boldsymbol{e}) = (\underline{x}\boldsymbol{e})^2 = x^\alpha \boldsymbol{e} x^{-1} x^\alpha \boldsymbol{e} x^{-1}$. But $x^\alpha = \pm x$, hence
$q(\underline{x}\boldsymbol{e}) = x\boldsymbol{e}x^{-1}x\boldsymbol{e}x^{-1} = \boldsymbol{e}^2 = q(\boldsymbol{e})$.

2. It is onto because its restriction to $V_{r,s}$ is $\widetilde{\rho}$, which is onto. And for $x, y \in \Gamma$, $\underline{xy}(\boldsymbol{e}) = (xy)^\alpha \boldsymbol{e}(xy)^{-1} = x^\alpha y^\alpha \boldsymbol{e} y^{-1} x^{-1} = \underline{x}(\underline{y}\boldsymbol{e}) = (\underline{x}\,\underline{y})(\boldsymbol{e})$.

3. If $\rho^\sharp(x) = 1$, then $x^\alpha \boldsymbol{e} = \boldsymbol{e} x$ for all vectors $\boldsymbol{e}$. Hence $\boldsymbol{e} \cdot x = 0$ for all vectors $\boldsymbol{e}$ and this implies that x is a scalar (vide SX2/59).

Remark. The equality $\Gamma = V$ also gives $\rho^\sharp = \widetilde{\rho}$.

◊ Any versor is the product of at most n (non-null) vectors.

□ Let v be a versor. Then $\underline{v}$ is the product of at most n reflections: $\underline{v} = m_{\boldsymbol{u}_k} \cdots m_{\boldsymbol{u}_1}$ $(\boldsymbol{u}_j \in E^\times)$. Since $m_{\boldsymbol{u}_j} = \widetilde{\rho}(\boldsymbol{u}_j)$, we get $\widetilde{\rho}(v) = \underline{v} = \widetilde{\rho}(u)$, where $u = \boldsymbol{u}_k \cdots \boldsymbol{u}_1$. Therefore $vu^{-1} = \lambda \in \ker(\widetilde{\rho}) = \mathbf{R}^\times$ and hence $v = u\lambda$ is the product of $\leqslant n$ vectors. □

Remark. Since SO and O are isomorphic to matrix groups, they are automatically Lie groups (Hall-2003 [3]). Then their double covers S and P are also Lie groups. On the other hand, we have an onto homomorphism $\mathbf{R}^\times \times S \to V$, $(\lambda, v) \mapsto \lambda v$ with kernel $\{(1,1),(-1,-1)\}$ and form this it follows that V is a Lie group.

$\Diamond$ If $(r,s)=(n,0)$ or $(r,s)=(0,n)$, then $S=S^{+}$ and $P=S^{+}\sqcup \boldsymbol{u}S^{+}$ for any unit vector $\boldsymbol{u}$.

$\square$ Let $v=\boldsymbol{u}_1\cdots\boldsymbol{u}_k\in P$, where the $\boldsymbol{u}_j$ are unit vectors. If k is even, then $v\in S$ and $v\tilde{v}=\boldsymbol{u}_1^2\cdots\boldsymbol{u}_k^2=1$ (in the antieuclidean case the product is $(-1)^k$, which is 1 because k is even). So $v\in S^{+}$. If k is odd, then $v=\boldsymbol{u}(\pm\boldsymbol{u}v)$, and $\pm\boldsymbol{u}v\in S^{+}$. $\square$

If $(r,s)\geqslant(1,1)$, let $\boldsymbol{u}$ and $\bar{\boldsymbol{u}}$ be any unit vectors such that $\boldsymbol{u}^2=1$ and $\bar{\boldsymbol{u}}^2=-1$. Notice that $(\boldsymbol{u}\bar{\boldsymbol{u}})(\boldsymbol{u}\bar{\boldsymbol{u}})^{\tau}=-1$. Then we have:

$\Diamond$ $S=S^{+}\sqcup\boldsymbol{u}\bar{\boldsymbol{u}}S^{+}$ and $P=S^{+}\sqcup\boldsymbol{u}S^{+}\sqcup\bar{\boldsymbol{u}}S^{+}\sqcup\boldsymbol{u}\bar{\boldsymbol{u}}S^{+}$.

$\square$ Plainly, $S=S^{+}\sqcup S^{-}$, where $v\in S^{\pm}\Leftrightarrow v\tilde{v}=\pm1$, and it is easy to check that $w\in S^{-}\Leftrightarrow v=\boldsymbol{u}\bar{\boldsymbol{u}}w\in S^{+}$. Similarly, $\boldsymbol{u}S^{+}$ and $\bar{\boldsymbol{u}}S^{+}$ are the odd pinors v such that $v\tilde{v}=+1$ and $v\tilde{v}=-1$, respectively. $\square$

◊ If $n \leqslant 5$, $\mathrm{Spin}^+ = \{R \in \mathcal{G}^+ \mid R\widetilde{R} = 1\}$ (Lundholm-Svensson-2009 [4], Proposition 6.20).

□ By definition, Spin^+ is contained in $\{R \in \mathcal{G}^+ \mid R\widetilde{R} = 1\}$.

Let then R be an even multivector such that $R\widetilde{R} = 1$. To prove that $R \in \mathrm{Spin}^+$, by the theorem on page 24 it is enough to see that $\boldsymbol{y} = R\boldsymbol{x}\widetilde{R}$ is a vector when $\boldsymbol{x}$ a vector.

Now $\widetilde{\boldsymbol{y}} = \boldsymbol{y}$, and for $n \leqslant 5$ we must have, since $\boldsymbol{y}$ is odd, $\boldsymbol{y} = \boldsymbol{y}_1 + \boldsymbol{y}_5$. So we will be done if we show that $\boldsymbol{y}_5 = 0$.

Since this is obviously true if $n < 5$, we can assume that $n = 5$. In this case $\boldsymbol{y} = \boldsymbol{y}_1 + \lambda \boldsymbol{i}$, where $\boldsymbol{i}$ is the pseudoscalar. Then we have, using that $\boldsymbol{i}$ is a central element,

$$\lambda = \langle R\boldsymbol{x}\widetilde{R}\boldsymbol{i}^{-1}\rangle = \langle R\boldsymbol{x}\boldsymbol{i}^{-1}\widetilde{R}\rangle = \langle \boldsymbol{x}\boldsymbol{i}^{-1}\widetilde{R}R\rangle = \langle \boldsymbol{x}\boldsymbol{i}^{-1}\rangle = 0. \qquad \square$$

Remark. The statement above is not true for $n \geqslant 6$.

Since the possible signatures are $1 \sim (1,0)$ and $\bar{1} \sim (0,1)$, we have

$$S_1 = S_1^+ = S_{\bar{1}} = S_{\bar{1}}^+.$$

In this case $S_1^+ = \{\pm 1\}$ and hence

$$S_1 = \{\pm 1\} \simeq \mathbf{Z}_2.$$

On the other hand

$$P_1 = P_{\bar{1}} = \{\pm 1, \pm \boldsymbol{e}\}.$$

This group is $\simeq \mathbf{Z}_4$ if $\boldsymbol{e}^2 = -1$ and $\simeq \mathbf{Z}_2 \times \mathbf{Z}_2$ if $\boldsymbol{e}^2 = 1$.

Finally $\mathrm{O}_1 = \{\pm \mathrm{Id}\}$ and $\mathrm{SO}_1 = \{\mathrm{Id}\}$, with $\widetilde{\rho}(\pm 1) = \mathrm{Id}$ and $\widetilde{\rho}(\pm e) = -\mathrm{Id}$.

We note that S_1 is not connected.

The cases E_2 and $E_{\bar{2}}$ are similar. If $\boldsymbol{i}$ is the unit area, $\mathcal{G}^{+} = \{\alpha + \beta\boldsymbol{i} \,|\, \alpha, \beta \in \mathbf{R}\}$. Since $\boldsymbol{i}^2 = -1$ and $\boldsymbol{i}^{\tau} = -\boldsymbol{i}$, $(\alpha + \beta\boldsymbol{i})(\alpha + \beta\boldsymbol{i})^{\tau} = \alpha^2 + \beta^2$ and

$$S_2 = S_{\bar{2}} = \{\alpha + \beta\boldsymbol{i} \,|\, \alpha^2 + \beta^2\} = \mathrm{U}_1.$$

This is the circle group, which is connected but not simply connected (going once around cannot be shrunk to 1; in fact, $\pi_1(\mathrm{U}_1) \simeq \mathbf{Z}$).

If $\boldsymbol{e}_1$ is any unit vector, then

$$P_2 = P_{\bar{2}} = S^{+} \sqcup S^{-} = \mathrm{U}_1 \sqcup \boldsymbol{e}_1\mathrm{U}_1,$$

and $\widetilde{\rho}(\boldsymbol{e}_1 e^{\boldsymbol{i}\theta})$ is the symmetry along $\boldsymbol{v} = \boldsymbol{e}_1 e^{\boldsymbol{i}\theta}$: $-\boldsymbol{v}\boldsymbol{v}\boldsymbol{v}^{-1} = -\boldsymbol{v}$.

It remains, for dimension 2, to analise the lorentzian $E_{1,1}$. Let $\boldsymbol{e}_0, \boldsymbol{e}_1$ by an orthonormal basis and $\boldsymbol{i} = \boldsymbol{e}_1\boldsymbol{e}_0$ (we follow the convention of placing the temporal vector on the vertical axis). We still have $\mathcal{G}^+ = \{\alpha + \beta\boldsymbol{i} \,|\, \alpha, \beta \in \mathbf{R}\}$ and $(\alpha + \beta\boldsymbol{i})^\tau = \alpha - \beta\boldsymbol{i}$, but

$$(\alpha + \beta\boldsymbol{i})(\alpha + \beta\boldsymbol{i})^\tau = \alpha^2 - \beta^2$$

because $\boldsymbol{i}^2 = 1$. Thus

$$S^+_{1,1} = \{\alpha + \beta\boldsymbol{i} \,|\, \alpha^2 - \beta^2 = 1\}$$

and so $S^+_{1,1}$ has two connected components (the two branches of a hyperbola in $\mathcal{G}^+$, both simply connected because $\simeq \mathbf{R}$, and distinguished by the sign of α). The two branches are parameterized by $\alpha = \epsilon\,\mathrm{ch}\,\lambda$, $\beta = \mathrm{sh}\,\lambda$ ($\epsilon = \pm 1$, $\lambda \in \mathbf{R}$).

The action of $R = R_{\epsilon,\lambda} = \epsilon\,\mathrm{ch}\,\lambda + \boldsymbol{i}\,\mathrm{sh}\,\lambda = \epsilon e^{\epsilon t\boldsymbol{i}}$ on $\boldsymbol{e}_0$ and $\boldsymbol{e}_1$ can be calculated straightforwardly, and we get:

$$\underline{R}(\boldsymbol{e}_0) = \boldsymbol{e}_0 \,\mathrm{ch}\, 2\epsilon\lambda + \boldsymbol{e}_1 \,\mathrm{sh}\, 2\epsilon\lambda,$$

$$\underline{R}(\boldsymbol{e}_1) = \boldsymbol{e}_0 \,\mathrm{sh}\, 2\epsilon\lambda + \boldsymbol{e}_1 \,\mathrm{ch}\, 2\epsilon\lambda \;\; (\mathbf{1}).$$

So $R_{\epsilon,\lambda}$ and $-R_{\epsilon,\lambda} = R_{-\epsilon,-\lambda}$ give the same rotation, as we know they should, and $\mathrm{SO}^+_{1,1}$ is isomorphic to $\mathbf{R}$ via the map (in matrix form)

$$t \mapsto H_t = \begin{pmatrix} \mathrm{ch}\, 2t & \mathrm{sh}\, 2t \\ \mathrm{sh}\, 2t & \mathrm{ch}\, 2t \end{pmatrix},$$

for $H_t H_{t'} = H_{t+t'}$.

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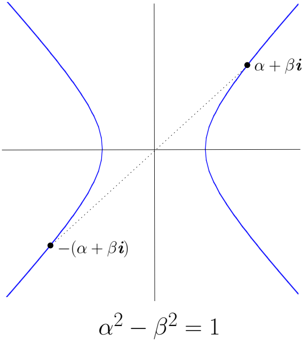
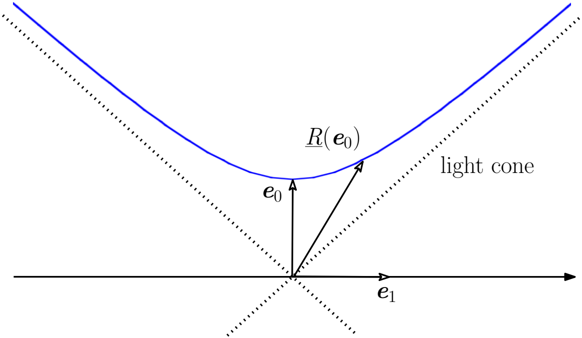
1 Use that $\boldsymbol{i}$ anticommutes with $\boldsymbol{e}_0$ and $\boldsymbol{e}_1$ and properties of ch, sh:

$$\begin{aligned} \underline{R}(\boldsymbol{e}_0) &= (\epsilon \,\mathrm{ch}\, \lambda + \boldsymbol{i} \,\mathrm{sh}\, \lambda)\boldsymbol{e}_0(\epsilon \,\mathrm{ch}\, \lambda - \boldsymbol{i} \,\mathrm{sh}\, \lambda) \\ &= \boldsymbol{e}_0(\epsilon \,\mathrm{ch}\, \lambda - \boldsymbol{i} \,\mathrm{sh}\, \lambda)(\epsilon \,\mathrm{ch}\, \lambda - \boldsymbol{i} \,\mathrm{sh}\, \lambda) \\ &= \boldsymbol{e}_0(\mathrm{ch}^2\lambda + \mathrm{sh}^2\lambda - 2\epsilon\boldsymbol{i} \,\mathrm{sh}\, \lambda \,\mathrm{ch}\, \lambda) \\ &= \boldsymbol{e}_0(\mathrm{ch}\, 2\lambda - \epsilon\boldsymbol{i} \,\mathrm{sh}\, 2\lambda) \\ &= \boldsymbol{e}_0 \,\mathrm{ch}\, 2\epsilon\lambda + \boldsymbol{e}_1\boldsymbol{i} \,\mathrm{sh}\, 2\epsilon\lambda. \end{aligned}$$

The other equality is proved in a similar way.





Since $S^+_{1,1}$ has two components, $P_{1,1}$ has eight components. If we set $S^{+\epsilon} = \{R_{\epsilon,\lambda}\}$ (we can call *right* and *left* rotors according to whether $\epsilon = +$ or $\epsilon = -$), then the components are

$$S^{++}, S^{+-}, \boldsymbol{e}_0 S^{++}, \boldsymbol{e}_0 S^{+-}, \boldsymbol{e}_1 S^{++}, \boldsymbol{e}_1 S^{+-}, \boldsymbol{i} S^{++}, \boldsymbol{i} S^{+-}.$$

From this we get that $\mathrm{O}_{1,1}$ has four components:

$$\mathrm{SO}^+_{1,1}, m_{\boldsymbol{e}_1}\mathrm{SO}^+_{1,1}, m_{\boldsymbol{e}_0}\mathrm{SO}^+_{1,1}, \text{ and } m_{\boldsymbol{e}_1} m_{\boldsymbol{e}_0}\mathrm{SO}^+_{1,1} = -\mathrm{SO}^+_{1,1}.$$

$S_3 = S_{\bar{3}} = S_3^+ = S_{\bar{3}}^+ = \{\alpha + \boldsymbol{x}\boldsymbol{i}_3 \mid \alpha^2 + |\boldsymbol{x}|^2 = 1\} = \mathrm{SU}_2$ (unit quaternions). Since SU_2 is the 3-sphere, it is connected and simply connected.

The 2:1 covering $S_3 \to \mathrm{SO}_3$ gives the classical 2:1 covering $\mathrm{SU}_2 \to \mathrm{SO}_3$ in which a unit quaternion $\boldsymbol{u}$ acts on E_3 as $\boldsymbol{x} \mapsto \boldsymbol{u}\boldsymbol{x}\widetilde{\boldsymbol{u}}$.

If we take for $\boldsymbol{u}$ the quaternion units $\boldsymbol{i}_1 = \boldsymbol{e}_2\boldsymbol{e}_3$, $\boldsymbol{i}_2 = \boldsymbol{e}_3\boldsymbol{e}_1$, $\boldsymbol{i}_3 = \boldsymbol{e}_1\boldsymbol{e}_e$, the corresponding isometries are the axial symmetries with respect to $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, respectively.

The two components of P_3 are S_3 and $\boldsymbol{u}S_3$, where is any fixed unit vector, and the two components of O_3 are SO_3 and $m_{\boldsymbol{u}}\mathrm{SO}_3$. If f is a rotation, and $\boldsymbol{v} = f^{-1}(\boldsymbol{u})$, or $f(\boldsymbol{v}) = \boldsymbol{u}$, then $m_{\boldsymbol{u}}f$ is the symmetry along $\boldsymbol{v}$ (or across $\boldsymbol{v}^{\perp}$).

Given two unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, $R = \boldsymbol{vu}$ is a spinor. We will say that it is the *plane spinor* defined by $\boldsymbol{u}$ and $\boldsymbol{v}$. Since $R\widetilde{R} = \boldsymbol{u}^2\boldsymbol{v}^2$, we see that R is a rotor (in which case we will say that it is a *plane rotor*) if and only if $\boldsymbol{u}^2\boldsymbol{v}^2 = 1$, that is, if either $\boldsymbol{u}^2 = \boldsymbol{v}^2 = 1$ or $\boldsymbol{u}^2 = \boldsymbol{v}^2 = -1$.

$\Diamond$ If $R = \boldsymbol{vu}$ is a plane rotor, then there exists $\beta \in \mathbf{R}$, $\beta \geqslant 0$, and $\epsilon \in \{\pm 1\}$ such that $R = \epsilon e^{-\beta \boldsymbol{u}\wedge\boldsymbol{v}}$.

$\square$ Let us start with $R = \boldsymbol{u}\cdot\boldsymbol{v} - \boldsymbol{u}\wedge\boldsymbol{v}$. Replacing R by $-R = (-\boldsymbol{u})\boldsymbol{v}$ if necessary, we may assume that $\boldsymbol{u}\cdot\boldsymbol{v} \geqslant 0$. Since $(\boldsymbol{u}\wedge\boldsymbol{v})^2 = (\boldsymbol{u}\wedge\boldsymbol{v})\cdot(\boldsymbol{u}\wedge\boldsymbol{v}) = (\boldsymbol{u}\cdot\boldsymbol{v})^2 - \boldsymbol{u}^2\boldsymbol{v}^2 = (\boldsymbol{u}\cdot\boldsymbol{v})^2 - 1$, let us distinguish the cases (I) $(\boldsymbol{u}\cdot\boldsymbol{v})^2 < 1$ and (II) $(\boldsymbol{u}\cdot\boldsymbol{v})^2 - 1 > 0$ (we exclude $\boldsymbol{u}\cdot\boldsymbol{v} = 1$ as it it implies $R = 1$).

In case (I), there is $\alpha \in (0, \pi/2]$ such that $\boldsymbol{u} \cdot \boldsymbol{v} = \cos\alpha$. This implies that $(\boldsymbol{u} \wedge \boldsymbol{v})^2 = -\sin^2\alpha$ and hence $R = \cos\alpha - U\sin\alpha$, with $U^2 = -1$ ($U = \boldsymbol{u} \wedge \boldsymbol{v}/\sin\alpha$). Thus, finally, $R = e^{-\alpha U} = e^{-\beta \boldsymbol{u}\wedge\boldsymbol{v}}$, $\beta = \alpha/\sin\alpha$.

In case (II), there is $\alpha \in \mathbf{R}$, $\alpha > 0$, such that $\boldsymbol{u} \cdot \boldsymbol{v} = \operatorname{ch}\alpha$, which implies that $(\boldsymbol{u} \wedge \boldsymbol{v})^2 = \operatorname{sh}^2\alpha$ and hence $R = \operatorname{ch}\alpha - U\operatorname{sh}\alpha$, with $U^2 = 1$ ($U = \boldsymbol{u} \wedge \boldsymbol{v}/\operatorname{sh}\alpha$). Thus $R = e^{-\alpha U} = e^{-\beta \boldsymbol{u}\wedge\boldsymbol{v}}$, $\beta = \alpha/\operatorname{sh}\alpha$. $\square$

Remark. Letting $\alpha \to 0$, we see that a plane rotor is connected to 1 or to -1. In other words, either R or $-R$ is connected to 1.

$\Diamond$ Any rotor $R \in S^+$ is path connected to 1 or to -1. If $r \geqslant 2$ or $s \geqslant 2$, then any rotor is connected to 1 and, as a consequence, S^+ is connected.

$\square$ Let $R \in S^+$, say $R = \boldsymbol{u}_1 \cdots \boldsymbol{u}_{2k}$, with the $\boldsymbol{u}_j$ unitary. It is easy to see that we can reexpress R as a product of unit vectors in such a way that all the negative ones appear after the positive ones **(1)**. Since both the number of positive terms and the number of negative terms are even, R can be expressed as the product of k plane rotors. Each of these rotors is path connected to 1 or to -1 and so the same is true for R. In case $r \geqslant 2$ or $s \geqslant 2$, we can pick two unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ of the same sign, so that $(\boldsymbol{e}_1\boldsymbol{e}_2)^2 = -1$, and then the path $t \mapsto e^{t\boldsymbol{e}_1\boldsymbol{e}_2}$ connects 1 $(t = 0)$ to -1 $(t = \pi)$. $\square$

(1) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two unit vectors, then

$$\boldsymbol{v}\boldsymbol{u} = \boldsymbol{v}\boldsymbol{u}\boldsymbol{v}^{-1}\boldsymbol{v} = -\underline{\boldsymbol{v}}(\boldsymbol{u})\boldsymbol{v} = \boldsymbol{u}'\boldsymbol{v},$$

and $\boldsymbol{u}' = -\underline{\boldsymbol{v}}(\boldsymbol{u})$ is a unit vector because $\underline{\boldsymbol{v}}$ is an isometry.

◊ The 2 to 1 surjection $\text{Spin}_{r,s} \to \text{SO}_{r,s}$ is non-trivial if $r \geqslant 2$ or $s \geqslant 2$.

□ It will be enough to construct a path on $\text{Spin}_{r,s}$ connecting 1 and -1. To that end, let $\boldsymbol{u}_1, \boldsymbol{u}_2$ be an orthonormal pair of positive ($\epsilon = 1$) or negative ($\epsilon = -1$) vectors. Now define $s(t) \in \text{Spin}_{r,s}$, $t \in [0, \pi/2]$, as follows:

$$\begin{aligned} s(t) &= (\boldsymbol{u}_1 \cos(t) + \boldsymbol{u}_2 \sin(t))(\boldsymbol{u}_1 \cos(t) - \boldsymbol{u}_2 \sin(t)) \\ &= \epsilon \cos^2(t) - \epsilon \sin^2(t) - \boldsymbol{u}_1\boldsymbol{u}_2 \sin(t)\cos(t) + \boldsymbol{u}_2\boldsymbol{u}_1 \sin(t)\cos(t) \\ &= \epsilon \cos(2t) - \boldsymbol{u}_1\boldsymbol{u}_2 \sin(2t). \end{aligned}$$

Now it is clear that $s(0) = \epsilon$ and $s(\pi/2) = -\epsilon$. □

We define an *infinitesimal rotor* as a tangent vector to the rotor groups S^+ at 1.

◊ *Any infinitesimal rotor is a bivector*.

□ By definition, any infinitesimal rotor has the form $R'(0)$, where $R(t) \in S^+$ is defined and is differentiable for t in an open interval around 0 and $R(0) = 1$. We want to prove that with these conditions $R'(0)$ is a bivector.

Taking the derivative of $R(t)\widetilde{R(t)}$ with respect to t at 0, we get $\widetilde{R'(0)} = -R'(0)$. Since $R'(0)$ is an even multivector, this condition shows that the possible grades $r = 2j$ of $R'(0)$ are such that $r//2 = k$ must be odd, or $r = 4k + 2$, $k \geqslant 0$. So we find that $R'(0) = b + z$, where b is a bivector and the least grade of z is $\geqslant 6$.

Now use that $\boldsymbol{x}(t) = R(t)\boldsymbol{e}\widetilde{R(t)} \in E$ for any fixed $\boldsymbol{e} \in E$. Taking the derivative at 0 we get

$$\begin{aligned} \boldsymbol{x}'(0) &= R'(0)\boldsymbol{e} + \boldsymbol{e}\widetilde{R'(0)} \\ &= (b+z)\boldsymbol{e} - \boldsymbol{e}(b+z) \\ &= b\boldsymbol{e} - \boldsymbol{e}b + z\boldsymbol{e} - \boldsymbol{e}z \\ &= 2b \cdot \boldsymbol{e} + 2z \cdot \boldsymbol{e}. \end{aligned}$$

Since $\boldsymbol{x}'(0)$ and $b \cdot \boldsymbol{e}$ are vectors and the least grade of $z \cdot \boldsymbol{e}$ is $\geqslant 5$, we get that $z \cdot \boldsymbol{e} = 0$. Since $\boldsymbol{e}$ is any vector, this implies that $z = 0$ (SX2/58). $\square$

To prove that any bivector is an infinitesimal rotor, we have to work a bit more.

Let $B \in \mathcal{G}^2$ be a bivector, and define the linear map $\mathrm{ad}_B : \mathcal{G} \to \mathcal{G}$ by the formula $\mathrm{ad}_B(x) = Bx - xB = [B, x]$.

$\Diamond$ ad_B is a grade preserving derivation of the geometric product.

$\square$ It is clear that ad_B vanishes on scalars and in the previous lecture we saw that ad_B maps vectors to vectors. If now $x \in \mathcal{G}^r$, $r \geqslant 2$, then $Bx = B \cdot x + \langle Bx \rangle_r + F \wedge x$ and $xB = x \cdot B + \langle xB \rangle_r + x \wedge B$. But in this case $x \cdot B = B \cdot x$ and $x \wedge B = B \wedge x$, and hence $[B, x] = \langle Bx \rangle_r - \langle xB \rangle_r$, which is an r-vector.

And ad_B is derivation:
$\mathrm{ad}_B(xy) = Bxy - xyB = Bxy - xBy + xBy - xyB = \mathrm{ad}_B(x)y + x\mathrm{ad}_B(y)$. $\square$

◊ Let $\mathbf{so}(E) \subseteq \mathrm{End}(E)$ be the linear subspace of skew-symmetric endomorphisms. Then $\mathrm{ad}_B \in \mathbf{so}(E)$ and the map $B \mapsto \mathrm{ad}_B$ provides a natural isomorphism $\mathcal{G}^2 \simeq \mathbf{so}(E)$.

□ Since $\mathrm{ad}_B(\boldsymbol{e}) = 2B \cdot \boldsymbol{e}$, we have

$$\mathrm{ad}_B(\boldsymbol{e}) \cdot \boldsymbol{e}' = 2(B \cdot \boldsymbol{e}) \cdot \boldsymbol{e}' = 2B \cdot (\boldsymbol{e} \wedge \boldsymbol{e}').$$

Similarly,

$$\begin{aligned}&\boldsymbol{e} \cdot \mathrm{ad}_B(\boldsymbol{e}') = 2\boldsymbol{e} \cdot (B \cdot \boldsymbol{e}') = -2\boldsymbol{e} \cdot (\boldsymbol{e}' \cdot B)\\ &= -2(\boldsymbol{e} \wedge \boldsymbol{e}') \cdot B = -2B \cdot (\boldsymbol{e} \wedge \boldsymbol{e}').\end{aligned}$$

Therefore $\mathrm{ad}_B(\boldsymbol{e}) \cdot \boldsymbol{e}' = \boldsymbol{e} \cdot (-\mathrm{ad}_B(\boldsymbol{e}'))$, or $\mathrm{ad}_B^\dagger = -\mathrm{ad}_B$.

Since $\mathrm{ad}_B(\boldsymbol{e}) = -2\boldsymbol{e} \cdot B$, the kernel of the map $\mathcal{G}^2 \to \mathbf{so}(E)$ consists of the bivectors B such that $\boldsymbol{e} \cdot B = 0$ for all vectors $\boldsymbol{e}$, and we know that this implies $B = 0$ (SX2/58). Therefore, the map is injective. Now the image is $\mathbf{so}(E)$ because $\dim \mathbf{so}(E) = \binom{n}{2} = \dim \mathcal{G}^2$. □

Example. If $B = \boldsymbol{x} \wedge \boldsymbol{y}$, $\mathrm{ad}_B(\boldsymbol{e}) = 2((\boldsymbol{e} \cdot \boldsymbol{y})\boldsymbol{x} - (\boldsymbol{e} \cdot \boldsymbol{x})\boldsymbol{y})$.

Example. If $f \in \mathbf{so}$, then $f = \mathrm{ad}_B$, where $B = \frac{1}{4}\sum_k f(\boldsymbol{e}_k) \wedge \boldsymbol{e}^k$, where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n \in E$ is a basis and $\boldsymbol{e}^1, \ldots, \boldsymbol{e}^n$ is the reciprocal basis (defined by the relations $\boldsymbol{e}^j \cdot \boldsymbol{e}_k = \delta^j_k$).

Let $f(\boldsymbol{e}_j) = \sum_l a_{jl}\boldsymbol{e}_l$. We will show that $\mathrm{ad}_B(\boldsymbol{e}_j) = f(\boldsymbol{e}_j)$ for $j = 1, \ldots, n$. Indeed,

$$\mathrm{ad}_B(\boldsymbol{e}_j) = \tfrac{1}{2}\textstyle\sum_k(\boldsymbol{e}^k \cdot \boldsymbol{e}_j)f(\boldsymbol{e}_k) - \tfrac{1}{2}\textstyle\sum_k(f(\boldsymbol{e}_k) \cdot \boldsymbol{e}_j)\boldsymbol{e}^k.$$

It is clear that the first term is $\frac{1}{2}f(\boldsymbol{e}_j)$. As for the second, it is equal to $\frac{1}{2}\sum_k(\boldsymbol{e}_k \cdot f(\boldsymbol{e}_j))\boldsymbol{e}^k$ (for f is skew-symmetric) and

$$\textstyle\sum_k(\boldsymbol{e}_k \cdot f(\boldsymbol{e}_j))\boldsymbol{e}^k = \sum_k(\sum_l a_{jl}g_{kl})\boldsymbol{e}^k = \sum_l a_{jl}\sum_k g_{kl}\boldsymbol{e}^k = f(\boldsymbol{e}_j)$$

where $g_{kl} = \boldsymbol{e}_k \cdot \boldsymbol{e}_l$ and hence $\sum_k g_{kl}\boldsymbol{e}^k = \boldsymbol{e}_l$.

◊ For any bivector $B \in \mathcal{G}^2$, $\pm e^B \in S^+$ ([4], Th. 6.17).

□ It is clear that $R = e^B$ is an even multivector. Moreover, $R \in \mathcal{G}^\times$, for $R\widetilde{R} = e^B e^{\widetilde{B}} = e^B e^{-B} = 1$. By the theorem on page 24, to see that $R \in V$ it is enough to prove that $R\boldsymbol{x}\widetilde{R} \in E$ for any $\boldsymbol{x} \in E$.

Let $W = E^\perp$ be the q-orthogonal of E in $\mathcal{G}$, that is, $W = \mathbf{R} \oplus \mathcal{G}^2 \oplus \cdots \oplus \mathcal{G}^n$. Pick any $\boldsymbol{x} \in E$ and any $y \in W$, and define the map $f : \mathbf{R} \to \mathbf{R}$ by the formula $f(t) = q(e^{tB}\boldsymbol{x}e^{-tB}, y)$. The derivatives of f have the following form:

$$f'(t) = q(Be^{tB}\boldsymbol{x}e^{-tB} - e^{tB}\boldsymbol{x}e^{-tB}B, y) = q([B, e^{tB}\boldsymbol{x}e^{-tB}], y)$$

$$f''(t) = q([B, Be^{tB}\boldsymbol{x}e^{-tB} - e^{tB}\boldsymbol{x}e^{-tB}B], y) = q([B, [B, e^{tB}\boldsymbol{x}e^{-tB}], y])$$

and so on. Setting $t = 0$, we obtain

$f^{(k)}(0) = q(\mathrm{ad}_B^k(\boldsymbol{x}), y) = 0$, because ad_B is an endomorphism of E. Thus $f \equiv 0$, as f is real analytic, and therefore $q(e^{tB}\boldsymbol{x}e^{-tB}, y) = 0$ and $e^{tB}\boldsymbol{x}e^{-tB} \in W^\perp = E$. □

◊ If $B \in \mathcal{G}^2$, then B is an infinitesimal rotor.

□ By the preceding result, $R(t) = e^{tB} \in S^{+}$ for all t and

$$R'(0) = B.$$ □

The commutator $[B, B'] = BB' - B'B$ of two bivectors is a bivector. This follows from the grade decomposition

$$BB' = B \cdot B' + \langle BB' \rangle_2 + B \wedge B'$$

and the fact that $B \cdot B' = B' \cdot B$ and $B \wedge B' = B' \wedge B$. Now it is a straigtfoward exercise to show that $\mathcal{G}^2$ with the commutator $[B, B']$ is a *Lie algebra*.

◊ Regarding $\mathcal{G}^2$ as the tangent space to S^{+} at 1, the commutator is identified with the Lie bracket of infinitesimal rotors. In other words, $\mathcal{G}^2$ is canonically isomorphic to $\mathrm{Lie}(S^{+})$.

◊ In the euclidean and lorentzian cases, every bivector F can be written as a sum of *commuting* 2-blades:

$$F = F_1 + \cdots + F_p \quad (2p \leqslant n)$$

where each F_j is a non-zero 2-blade, $F_j F_k = F_k F_j$ for all j, k.

It follows that

$$e^F = e^{F_1} \cdots e^{F_p}$$

which means that every rotor can be factored as a product of plane commuting rotors.

Remark. For additional details about the Euclidean case, see SX2/69.

Let $E = E_{r,s}$, $n = r + s$, and $L \in \mathrm{SO}^+_{r,s}$ (this is the connected component of the identity of $\mathrm{SO}_{r,s}$).

♢1 If (r, s) is of one of the forms $(n, 0)$, $(0, n)$, $(1, n-1)$ or $(n-1, 1)$, there exists a bivector $B \in \mathcal{G}^2$ such that

$$Lx = e^{-B}\boldsymbol{x}e^{B}. \tag{1}$$

This result is false for any other signature.

♢2 If B' is another bivector such that $e^{-B'}\boldsymbol{x}e^{B'} = e^{-B}\boldsymbol{x}e^{B}$ for all vectors $\boldsymbol{x}$, then $e^{B'} = \pm e^{B}$.

□ See Riesz-1958 [5], §4.12. Here let us just notice that it is not effective in the sense that it does not provide clues about how to relate the specific geometric properties of L to the algebraic properties of B. Note also that these 'specifics' are dealt with in detail in other lectures for signatures such as $(1, 3)$ (STA) and $(4, 1)$ (CGA). □

Remark. The differencial of the map $\exp : \mathcal{G}^2 \to S^+$ at 0 is the identity. Therefore, an open neighborhood of 0 in $\mathcal{G}^2$ is mapped diffeomorphically to an open neighborhood U of 1 in S^+. So any $u \in U$ has the form e^B, B a bivector. The results of the previous page tell us that if $r, s \geqslant 2$, then there are rotors that are not the exponential of a bivector. On the other hand, since the subgroup generated by U is open, it follows that any rotor is the product of expoenentials of bivectors, or a limit of such products, and that at least two exponentials are needed to get any other rotor.

Outshoots of the EP

Recommeded reference: Sharpe-1997 [6] ("[...] a stydy of an aspect of Elie Cartan's contribution to the question "What is Geometry?")

The main contribution of Cartan in that respect (his *espaces généralizés*, now called *Cartan geometries*) was a unification of the Cayley-Klein-Lie geometries (the basic structure is a coset space G/H of Lie groups) with (*Riemannian geometry*, Riemann-1944, the other main generalization of Euclidean geometry).

Yang-1977 [7]: "That non-abelian gauge fields are conceptually identical to ideas of the beautiful theory of fibre bundles developed by mathamaticians without reference to the physical world, was a great marvel to me."

On the Hypotheses which lie at the Bases of Geometry.

Bernhard Riemann

Translated by William Kingdon Clifford

[*Nature*, Vol. VIII. Nos. 183, 184, pp. 14–17, 36, 37.]

Transcribed by D. R. Wilkins

Preliminary Version: December 1998

- WP/Principal bundle "Principal bundles have important applications in topology and differential geometry. They have also found application in physics where they form part of the foundational framework of gauge theories."
- WP/Gauge thery "In physics, a gauge theory is a type of field theory in which the Lagrangian is invariant under a continuous group of local transformations.
  The term gauge refers to redundant degrees of freedom in the Lagrangian. The transformations between possible gauges, called gauge transformations, form a Lie group—referred to as the symmetry group or the gauge group of the theory. Associated with any Lie group is the Lie algebra of group generators.

For each group generator there necessarily arises a corresponding field (usually a vector field) called the gauge field. Gauge fields are included in the Lagrangian to ensure its invariance under the local group transformations (called gauge invariance). When such a theory is quantized, the quanta of the gauge fields are called gauge bosons. If the symmetry group is non-commutative, the gauge theory is referred to as non-abelian, the usual example being the Yang–Mills theory."

Tot està per fer,
tot és possible

Todo está por hacer,
todo es posible

All is to be done,
all is possible

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