17th "Lluís Santaló" Research School

On axiom systems for GA

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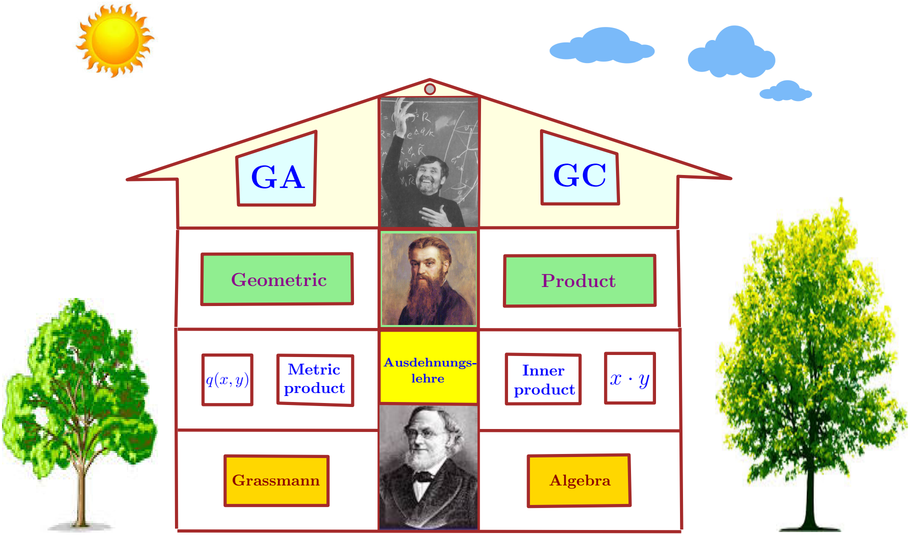
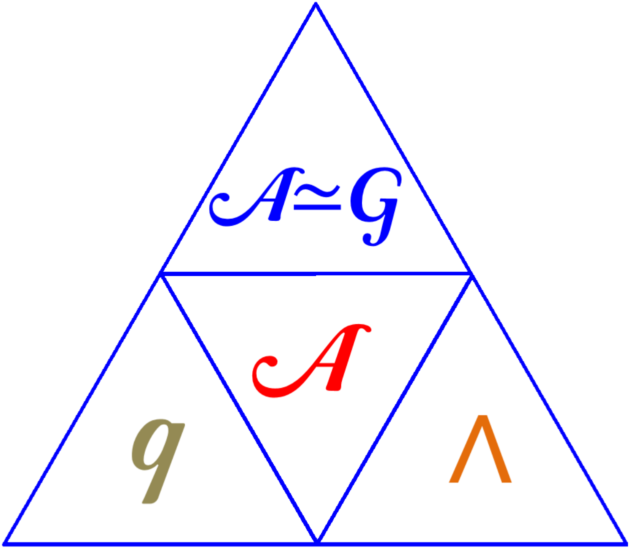
RSME-UIMP

22-26 August, 2016



A constructive perspective of GA



A design perspective of GA

Major references. Many references have influenced my understanding of both perspectives, particularly (in cronological order) Chevalley-1946 [1], Riesz-1958 [2], Hestenes-1966 [3], Porteous-1969 [4], Casanova-1976 [5], Hestenes-Sobczyk-1984 [6], Hestenes-1986 [7], Hestenes-1999 [8], Hitzer-2003-ax [9], Doran-Lasenby-2003 [10], Dorst-Fontijne-Mann-2007 [11]...

What follows is structured as a combination of both perspectives.

The caption: A set of mathematical footnotes to the GA part of the excellent paper Hestenes-Li-Rockwood-2001 [12].

"[...] tools and methods to enrich cassical geometry by *integrating* it more fully into the whole system."

In next **Index**, points that may have some novelty are highlighted.

Axiomatics

- **Grassmann algebra**. Grading and exterior product. Multivectors. Blades. Functorialities. Involutions of $\wedge E$.
- **Metric Grassmann algebra**. The Gram rule. Inner product. Involutions of the inner product. Laplace rule. Inner product in terms of the metric.
- **The design approach**. Basic axioms. The metric. Linear generators. Uniqueness results. Existence of full GAs (Clifford algebra). Existence and structure of folded GAs. The grading, outer and inner products. Involutions. The fundamental formula. Riesz' formulas. Grades of a geometric product. Alternative form of the metric (agrees with Hestenes' *natural scalar product*). A metric adjuntion formula.

- **Geometry with GA**. Introduction. Axial symmetries and reflections. Versors. Pinors and spinors. Rotors. Geometric covariance theorem. A quote from Feynman (1963). An archetypal example. On a theorem of Riesz.
- **Duality**. Pseudoscalars. Properties of a pseudoscalar. Hodge dual.
- **Appendices**. A: Proof of Laplace's formula. B: Existence and structure of folded geometric algebras.
- **Exercises**.
- **References**

Grassmann algebra

The *exterior algebra* associated to E, $(\bigwedge E, \wedge)$ is the direct sum of the *exterior powers* $\bigwedge^k E$ of E with the *exterior* (or *outer*) product multiplication, $\wedge$. Since $\bigwedge^k E = 0$ for $k > n$, this is a finite sum:

$$\bigwedge E = \bigoplus_{k=0}^{n} \bigwedge^k E = \mathbf{R} \oplus E \oplus \bigwedge^2 E \oplus \cdots \oplus \bigwedge^n E$$

It is a *graded algebra*, which means that $x \wedge y \in \bigwedge^{r+s} E$ when $x \in \bigwedge^r E$ and $y \in \bigwedge^s E$.

The exterior product is *skewcommutative* (or *supercommutative*): for $x \in \bigwedge^r E$ and $y \in \bigwedge^s E$,

$$x \wedge y = (-1)^{rs} y \wedge x.$$

◊ If $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ is a basis of E, the $\binom{n}{k}$ products $\widehat{\boldsymbol{e}}_J = \boldsymbol{e}_{j_1} \wedge \cdots \wedge \boldsymbol{e}_{j_r}$ $(1 \leqslant j_1 < \ldots < j_r \leqslant n)$ form a basis of $\bigwedge^k E$. In particular, $\dim \bigwedge^k E = \binom{n}{k}$. Hence $\dim \bigwedge E = 2^n$. □

In general, the elements of $\wedge E$ are called *multivectors*.

If $x \in \wedge E$, $x = \sum_J \lambda_J \widehat{\boldsymbol{e}}_J$, we write $x_r \in \wedge^r E$ to denote the component of x of degree r, $x_r = \sum_{|J|=r} \lambda_J \widehat{\boldsymbol{e}}_J$. So $x = x_0 + x_1 + \cdots + x_n$, and this decomposition is unique.

Remark. Many authors, including our invited speakers, write $\langle x \rangle_r$ instead of x_r, and simply $\langle x \rangle$ for $\langle x \rangle_0$.

The multivectors of $\wedge^r E$ are called *r-vectores*, or *homogeneous multivectors of grade r*.

For $r = 0, 1, 2, n-1, n$ the r-vectors receive particular names: *scalars*, *vectors*, *bivectors*, *pseudovectors* and *pseudoscalars*, respectively.

Since $\wedge^0 E = \mathbf{R}$, the *scalars are real numbers*. Similarly, the vectors are the elements of $\wedge^1 E = E$. Scalars and vectors will be denoted as explained in SX1/32: Greek and bold italic letters, respectively.

r	**Name**	$\dim \Lambda^r$
0	*scalar*	1
1	*vector*	n
2	*bivector*	$\binom{n}{2}$
3	*trivector*	$\binom{n}{3}$
$n-1$	*pseudovector*	n
n	*pseudoscalar*	1

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r$ be vectors and set $X = \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_r$, which is an r-vector. A fundamental property of the exterior algebra is that *$X \neq 0$ if and only if the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r$ are linearly independent*, and in this case we say that X is an *r-blade*.

A general r-vector is not an r-blade (see the remark on page 12) and **E1**, page 85.

If X is an r-blade, let $[X]$ denote the class of X with respect to the *proportionality relation*: $[X] = [X']$ if and only there exists a scalar λ such that $X' = \lambda X$.

$\Diamond$ There is natural bijection between the set $S_r E$ of vector subspaces F of E of dimension r and the set B_r of classes of r-blades.

$\Box$ The r-blades $X = \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_r$ and $X' = \boldsymbol{x}'_1 \wedge \cdots \wedge \boldsymbol{x}'_r$ corresponding to two bases $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r$ and $\boldsymbol{x}'_1, \cdots, \boldsymbol{x}'_r$ of F are proportional, because $X' = dX$, where d is the determinant of the second basis with respect to the first. In other words, $[X] = [X']$, and this shows that the map $S_r \to B_r$, $F \mapsto [X]$ is well defined. This map is clearly onto (or *surjective*) and it is one-to-one (or *injective*), because

$$F = \{\boldsymbol{x} \in E \,|\, \boldsymbol{x} \wedge X = 0\}. \qquad \Box$$

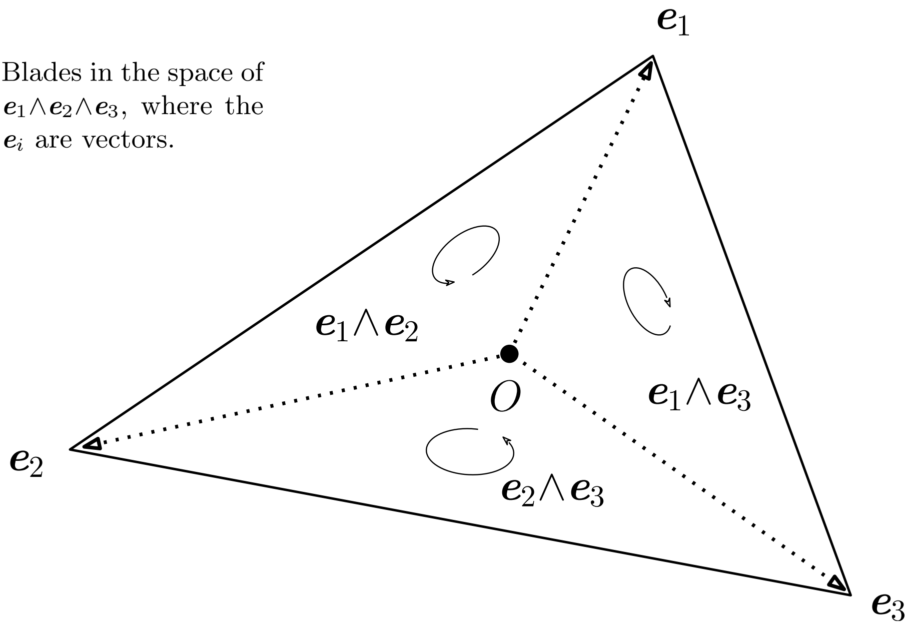
It is therefore natural to identify a subspace F of E of dimension r with the class $[X]$ of the r-blade X formed with any basis of F.

In doing so, we are allowed to write $\boldsymbol{x} \in [X]$ as equivalent to $\boldsymbol{x} \in F$. Note that $\boldsymbol{x} \in [X] \Leftrightarrow \boldsymbol{x} \wedge X = 0$.

Each blade X such that $F = [X]$ represents an *amount of r-volume of F*. Any two such quantities are proportional, and we say that they have the *same* (*opposite*) *orientation* if the proportionality factor is *positive* (*negative*).

Remark. If X is an r-blade, $[X]$ is a point in $S_1(\wedge^r E)$. So $S_r E \simeq B_r E$ yields $S_r E \to S_1(\wedge^r E)$, which turns out to be 1-to-1 (*Plücker embedding*). Moreover, the image is a smooth submanifold of $S_1(\wedge^r E)$ of dimension $(n-r)r$. Since $S_1(\wedge^r E)$ has dimension $\binom{n}{r} - 1$, the B_r is a set of measure 0 in the set of multivectors except for $r = 1$ (every 1-vector is a 1-blade) and $r = n-1$ (every $(n-1)$-vector is an $(n-1)$-blade). For $n = 4$, $r = 2$, the dimensions of $S_1(\wedge^2 E_4)$ and $S_2 E_3$ are 5 and 4, respectively.

Blades in the space of $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3$, where the $\boldsymbol{e}_i$ are vectors.



Let $f \in \mathrm{End}(E)$. Then

- There is a unique linear map $f^{\otimes k} : T^k E \to T^k E$ such that $f^{\otimes k}(\boldsymbol{e}_1 \otimes \cdots \otimes \boldsymbol{e}_k) = f(\boldsymbol{e}_1) \otimes \cdots \otimes f(\boldsymbol{e}_k)$. Adding up for all k we get a linear map $f^{\otimes} : TE \to TE$ that is an algebra endomorphism.
- There is a unique linear map $f^{\wedge k} : \bigwedge^k E \to \bigwedge^k E$ such that $f^{\wedge k}(\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_k) = f(\boldsymbol{e}_1) \wedge \cdots \wedge f(\boldsymbol{e}_k)$. Adding up for all k we get a linear map $f^{\wedge} : \bigwedge E \to \bigwedge E$ which in fact is an algebra endomorphism.

In practice there is no harm in using the same symbol f to denote $f^{\otimes}$ and $f^{\wedge}$, which is just a form of *overloading operators* by using the type of the argument to decide how to evaluate an expression. Thus, for example, $f(\boldsymbol{e} \otimes \boldsymbol{e}') = f(\boldsymbol{e}) \otimes f(\boldsymbol{e}')$ while $f(\boldsymbol{e} \wedge \boldsymbol{e}') = f(\boldsymbol{e}) \wedge f(\boldsymbol{e}')$.

Parity involution. The involutive linear automorphism $E \to E$, $\boldsymbol{x} \mapsto -\boldsymbol{x}$, extends to an automorphism $x \mapsto x^\alpha$ of $\wedge E$. It is an involutive *algebra automorphism*,

$$(x \wedge y)^\alpha = x^\alpha \wedge y^\alpha.$$

It is clear that the restriction of α to $\wedge^r E$ is given by

$$x \mapsto (-1)^r x.$$

The parity involution α is also called *main involution* or *grade involution*.

Let $\wedge^+ E = \{x \in \wedge E \,|\, x^\alpha = x\}$ and $\wedge^- E = \{x \in \wedge E \,|\, x^\alpha = -x\}$. It is clear that $\wedge^+ E = \bigoplus_{j\geqslant 0} \wedge^{2j} E$, $\wedge^- E = \bigoplus_{j\geqslant 0} \wedge^{2j+1} E$, and $\wedge E = \wedge^+ E \oplus \wedge^- E$ as vector subspaces. Furthermore, $\wedge^+ E$ is a subalgebra of $\wedge E$ (the *even subalgebra*).

Reversion. There is a unique linear automorphism $\wedge E \to \wedge E$, $x \mapsto x^{\tau} = \widetilde{x} = x^{\dagger}$, such that

$$(x_1 \wedge \cdots \wedge x_r)^{\tau} = x_r \wedge \cdots \wedge x_1 \;(x_1, \ldots, x_r \in E,\; 0 \leqslant r \leqslant n) \qquad (1)$$

Since $x_r \wedge \cdots \wedge x_1$ is a multilinear alternating function of $x_1, \ldots, x_r$, the claim is a consequence of the universal property of the exterior algebra.

The automorphism τ is clearly an involution and it can be immediately checked that it is an *antiautomorphism* of $\wedge E$:

$$(x \wedge y)^{\tau} = y^{\tau} \wedge x^{\tau}. \qquad (2)$$

If $x \in \wedge^r E$, the alternating character of $\wedge$ implies that

$$x^{\tau} = (-1)^{\binom{r}{2}} x = (-1)^{r//2} x,$$

where $r//2 = \lfloor \frac{r}{2} \rfloor$ is the integer quotient of r by 2.

Clifford conjugation. The composition $\kappa = \alpha\tau = \tau\alpha$ is an antiautomorphim of the exterior product and it is called *Clifford conjugation*. Instead of $\kappa(x)$ we also write x^κ or $\bar{x}$. It is immediate to check that the sign for grade r is $(-1)^{(r+1)//2}$.

	$r \mod 4$			
	0	1	2	3
α	$+$	$-$	$+$	$-$
τ	$+$	$+$	$-$	$-$
κ	$+$	$-$	$-$	$+$

Metric Grassmann algebra

Assume that E is endowed with a metric q (cf. SX1). Then q *induces a metric on* $\wedge E$, which will be denoted with the same symbol q.[1]

With respect to this metric, $\wedge^r E$ and $\wedge^s E$ are orthogonal when $r \neq s$, while for the r-blades $X = \boldsymbol{x}_1 \wedge \ldots \wedge \boldsymbol{x}_r$ and $Y = \boldsymbol{y}_1 \wedge \ldots \wedge \boldsymbol{y}_r$ we have, according to the usual mathematical prescription,

$$q(X, Y) = G(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r; \boldsymbol{y}_1, \ldots, \boldsymbol{y}_r), \tag{3}$$

where $G = G(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r; \boldsymbol{y}_1, \ldots, \boldsymbol{y}_r)$ is the *Gram determinant*

$$G = \begin{vmatrix} q(\boldsymbol{x}_1, \boldsymbol{y}_1) & \cdots & q(\boldsymbol{x}_1, \boldsymbol{y}_r) \\ \vdots & & \vdots \\ q(\boldsymbol{x}_r, \boldsymbol{y}_1) & \cdots & q(\boldsymbol{x}_r, \boldsymbol{y}_r) \end{vmatrix} \tag{4}$$

[1] If we regard q as a linear map $q : E \to E^*$ ($q(\boldsymbol{e})(\boldsymbol{e}') = q(\boldsymbol{e}, \boldsymbol{e}')$) then we have a graded algebra map $q^\wedge : \wedge E \to \wedge(E^*) = \wedge(E)^*$ and hence a metric $q^\wedge(x, y) = q^\wedge(x)(y)$ for $\wedge E$. As we will see (page 51), it agrees with Hestenes' *natural scalar product*.

In particular we have

$$q(X) = G(x_1, \ldots, x_r), \tag{5}$$

where $G(x_1, \ldots, x_r) = G(x_1, \ldots, x_r; x_1, \ldots, x_r)$ takes the form

$$G(x_1, \ldots, x_r) = \begin{vmatrix} q(x_1, x_1) & \cdots & q(x_1, x_r) \\ \vdots & & \vdots \\ q(x_r, x_1) & \cdots & q(x_r, x_r) \end{vmatrix} \tag{6}$$

Example. If q is the Euclidean metric of E_n, then

$$G(x_1, \ldots, x_r) = V(x_1, \ldots, x_r)^2, \tag{7}$$

where $V(x_1, \ldots, x_r)$ is the Euclidean r-volume of the parallelepiped defined by $x_1, \ldots, x_r$ (in **E2**, page 85, you can work out the details). In particular we see that *the induced metric on $\wedge E_n$ is again Euclidean*.

In general, the signature of $\wedge E_{r,s}$ can be determined easily using the formulas (3) and (5). Indeed, if $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is an orthonormal basis of $E_{r,s}$, then the basis $\widehat{\boldsymbol{e}}_J = e_{j_1} \wedge \cdots \wedge \boldsymbol{e}_{j_k}$ is an orthogonal basis of $\wedge E_{r,s}$ and $q(\boldsymbol{e}_J) = q(\boldsymbol{e}_{j_1}) \cdots q(\boldsymbol{e}_{j_k}) = (-1)^{\nu(J)}$, where we set $\nu(J)$ to denote the number of negative terms in the sequence $q(\boldsymbol{e}_{j_1}), \ldots, q(\boldsymbol{e}_{j_k})$, or, in other words, the number of j_l such that $j_l > r$.

◊ If $s > 0$, the signature of $\wedge E_{r,s}$ is $(2^{n-1}, 2^{n-1})$.

□ A positive $\widehat{\boldsymbol{e}}_J$ contains an arbitrary selection of the first r vectors (2^r possibilities) and an arbitrary selection of an even number of the last s vectors, which amounts to $2^s/2 = 2^{s-1}$ possibilities if $s > 0$. So $2^r 2^{s-1} = 2^{n-1}$ is the number of positive terms. □

Remark. The signature of $\wedge^k E_{r,s}$ can be obtained in a similar way. The number of positive and negative $\boldsymbol{e}_J$ of grade k (p and n) are given by:

$$p = \sum_{0 \leqslant 2j \leqslant s} \binom{r}{k-2j}\binom{s}{2j}, \; n = \sum_{0 \leqslant 2j \leqslant s-1} \binom{r}{k-2j-1}\binom{s}{2j+1}.$$

The *inner product* is a bilinear operation in $\wedge E$ that we denote $x \cdot y$. Bilinearity implies that we only need to define $X \cdot Y$ when $x = X$ and $y = Y$ are blades, say $X = \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_r$, $Y = \boldsymbol{y}_1 \wedge \cdots \wedge \boldsymbol{y}_s$.

The basic case is for $r = 1$ ($X = \boldsymbol{x}_1 = \boldsymbol{e} \in E$), and is defined as the (left) *contraction* with $\boldsymbol{e}$:

$$\boldsymbol{e} \cdot Y = \delta_{\boldsymbol{e}}(Y) = \begin{cases} 0 & \text{if } \; s = 0 \\ \sum_{k=1}^{s} (-1)^{k-1} q(\boldsymbol{e}, \boldsymbol{y}_k) Y_k & \text{if } \; s > 0 \end{cases} \tag{8}$$

where $Y_k = \boldsymbol{y}_1 \wedge \cdots \wedge \boldsymbol{y}_{k-1} \wedge \boldsymbol{y}_{k+1} \wedge \cdots \wedge \boldsymbol{y}_s$.

The fundamental property of the operator $\delta_{\boldsymbol{e}}$ (often denoted $i_{\boldsymbol{e}}$) is that it is a *skew-derivation* of grade -1 of the exterior product: if x and y are multivectors, then (*Leibnitz rule*)

$$\delta_{\boldsymbol{e}}(x \wedge y) = \delta_{\boldsymbol{e}}(x) \wedge y + x^{\alpha} \wedge \delta_{\boldsymbol{e}}(y) \tag{9}$$

The case $s = 1$ ($Y = \boldsymbol{y}_1 = \boldsymbol{e}$) is defined in a similar way using the *right contraction* of $\boldsymbol{e}$ with X, *which is equivalent to* $(-1)^{r+1}\boldsymbol{e} \cdot X$.

Note, in particular, that if $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors, then we get, either way, $\boldsymbol{x} \cdot \boldsymbol{y} = q(\boldsymbol{x}, \boldsymbol{y})$.

Thus, except for the case $r = s = 0$, we can assume that $r, s \geqslant 2$, in which case the definition is given by the following recursive rules:

$$(\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_r) \cdot (\boldsymbol{y}_1 \wedge \cdots \wedge \boldsymbol{y}_s) = \begin{cases} (\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_{r-1}) \cdot (\boldsymbol{x}_r \cdot Y) & \text{if } r \leqslant s \\ (X \cdot \boldsymbol{y}_1) \cdot (\boldsymbol{y}_2 \wedge \cdots \wedge \boldsymbol{y}_s) & \text{if } r \geqslant s \end{cases} \tag{10}$$

In fact, it is easy to see, using the definition for $s = 1$ and induction, that the case $r \leqslant s$ is sufficient to evaluate any inner product because

$$X \cdot Y = (-1)^{rs+s} Y \cdot X. \tag{11}$$

when $r \geqslant s$.

In particular we see that *the inner product is symmetric when* $r = s$. More generally: it is symmetric if and only if r and s have the same parity or else when the least of the two grades is even. Otherwise it is skew-symmetric.

Remark. For a vector $\boldsymbol{e}$ and a scalar λ, we have been led to the relation $\boldsymbol{e} \cdot \lambda = 0$ (hence also to $\lambda \cdot \boldsymbol{e} = 0$). By the recursive rules, we also get that $x \cdot \lambda = 0$ (hence also $\lambda \cdot x = 0$) for any r-vector x, $r > 0$. So we have defined all cases except the inner product of two scalars, which boils down to the definition of $1 \cdot 1$. We just take the simplest possibility, namely $1 \cdot 1 = 0$, as this will not ve used in what follows.

Example. Given vectors x_1, x_2, y_1, y_2,

$$(x_1 \wedge x_2) \cdot (y_1 \wedge y_2) = -G(x_1, x_2; y_1, y_2) = -q(x_1 \wedge x_2, y_1 \wedge y_2).$$

The proof is a straightfoward computation:

$$\begin{aligned}
(x_1 \wedge x_2) \cdot (y_1 \wedge y_2) &= x_1 \cdot (x_2 \cdot (y_1 \wedge y_2)) \\
&= x_1 \cdot (q(x_2, y_1)y_2 - q(x_2, y_2)y_1) \\
&= q(x_2, y_1)q(x_1, y_2) - q(x_2, y_2)q(x_1, y_1) \\
&= -((x_1 \cdot y_1)(x_2 \cdot y_2) - (x_1 \cdot y_2)(x_2 \cdot y_1))
\end{aligned}$$

Example. If $X = \boldsymbol{x}_1 \wedge \boldsymbol{x}_2$ and $Y = \boldsymbol{y}_1 \wedge \boldsymbol{y}_2 \wedge \boldsymbol{y}_3$, a similar computation yields

$$\begin{aligned} X \cdot Y &= -G(X, Y_{2,3})\boldsymbol{y}_1 + G(X, Y_{1,3})\boldsymbol{y}_2 - G(X, Y_{1,2})\boldsymbol{y}_3 \\ &= (X \cdot Y_{2,3})\boldsymbol{y}_1 - (X \cdot Y_{1,3})\boldsymbol{y}_2 + (X \cdot Y_{1,2})\boldsymbol{y}_3, \end{aligned}$$

where $Y_{i,j} = \boldsymbol{y}_i \wedge \boldsymbol{y}_j$.

Remark. These two examples are special cases of the formulas that we establish in next slides.

◊1 The involution α is an automorphism of the inner product: $(x \cdot y)^\alpha = x^\alpha \cdot y^\alpha$.

◊2 The involution τ is an antiautomorphism of the inner product: $(x \cdot y)^\tau = y^\tau \cdot x^\tau$.

□ We can assume that x and y are homogeneous multivectors, say of grades r and s, and then we can conclude by grade accounting.

The first is reduced to check that $|r-s|$ and $r+s$ have the same parity (which is obvious).

The second is reduced to check that $s//2 + r//2 + rs + \min(r,s)$ and $|r-s|//2$ also have the same parity, which may be left as an exercise. □

If $r \neq s$, the formula (11) tells us that it suffices to consider the case $r \leqslant s$ in order to find an expression for the inner product $X \cdot Y$ of two blades of grades r and s.

The result is what can be called *Laplace rule*:

$$\diamondsuit\ X \cdot Y = \sum_J (-1)^{t(J,J')}(X \cdot Y_J)Y_{J'} = \sum_J (-1)^{t(J,J')} q(\widetilde{X}, Y_J)Y_{J'} \quad (12)$$

where the sum is extended to all multiindices $J \subseteq \{1, \ldots, s\}$ of grade r, $J' = \{1, \ldots, s\} - J$ and Y_L is the exterior product of the factors of Y with index in L. **E3**, page 85, gives a condition for $X \cdot Y = 0$.

We have seen the case $r = 2$ and $s = 3$ on page 26.

☐ For $r = 1$, the formula agrees with (8) and for $r > 1$ we can use the recursive rule (10) and induction. The details are interesting, but a bit tedious, and are collected in the Appendix A, page 77. ☐

Notice that in the case $r = s$, the inner product can be expressed with the following *metric formula*:

$$x \cdot y = q(\widetilde{x}, y) = (-1)^{r//2} q(x, y). \tag{13}$$

This follows from the Laplace rule (for r-blades) and bilinearity. In particular we get the following formula for the *metric norm* $q(x)$ of an r-vector x in terms of the inner product:

$$q(x) = \widetilde{x} \cdot x = (-1)^{r//2} x \cdot x. \tag{14}$$

Example. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n \in E$ be an orthogonal basis and J, K two multiindices of grade r and s, respectively. Assume that $r \leqslant s$. Then the Laplace rule gives

$$\widehat{\boldsymbol{e}}_J \cdot \widehat{\boldsymbol{e}}_K = \begin{cases} 0 & \text{if } J \not\subseteq K \\ (-1)^{t(J,K)} q(\widehat{\boldsymbol{e}}_J) \widehat{\boldsymbol{e}}_{K-J} & \text{otherwise.} \end{cases}$$

Thus it has grade $s - r$ or is 0.

The design approach to GA

A *geometric algebra* is a structure with the ingredients described in **A0** and satisfying the properties **A1** and **A2** below. We will also assume the non-degeneration condition **A3**.

A0. *Structure*: An algebra $\mathcal{A}$ with a distinguished subspace $E \subseteq \mathcal{A}$ not containing $1 = 1_{\mathcal{A}}$. This structure is denoted $(\mathcal{A}, E)$. The elements of $\mathbf{R} \subseteq \mathcal{A}$ are called *scalars* and those of E and $\mathcal{A}$, *vectors*.

A1. *Contraction rule*: $\boldsymbol{x}^2 \in \mathbf{R}$ for any vector $\boldsymbol{x}$ (**1**).

A2. $\mathcal{A}$ is generated by E as a $\mathbf{R}$-algebra (**2**).

Notation. If $a = a_1, \ldots, a_r$ is a sequence of elements of $\mathcal{A}$ and $J = j_1, \ldots, j_s$ is a sequence of integers in $\{1, \ldots, r\}$, then we will write a_J to denote the product $a_{j_1} \cdots a_{j_s}$.

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1 The *magnitude* $|\boldsymbol{x}| \geqslant 0$ of $\boldsymbol{x}$ can defined by $|\boldsymbol{x}|^2 = \epsilon_{\boldsymbol{x}} \boldsymbol{x}^2$, where $\epsilon_{\boldsymbol{x}}$ is the sign of $\boldsymbol{x}^2$ (and called *signature* of $\boldsymbol{x}$). **2** If $E' \subseteq E$ is a vector subspace and $\mathcal{A}' \subseteq \mathcal{A}$ the subalgebra generated by E', then $(\mathcal{A}', E')$ is a GA.

◊ If $x, y \in E$, let $q(x,y) = \frac{1}{2}(xy + yx)$. Then $q(x,y) \in \mathbf{R}$ and since it is symmetric and bilinear, it is a metric for E (*Clifford metric* or just *metric*).

□ The algebra allows us to write

$$(x+y)^2 = x^2 + xy + yx + y^2.$$

Since $x^2, y^2, (x+y)^2 \in \mathbf{R}$, it follows that

$$xy + yx = 2q(x,y) \in \mathbf{R}. \qquad \square$$

This is *Clifford's relation*. Setting $y = x$, we get $q(x) = x^2$, which means that the contraction rule and the Clifford relation are equivalent.

Two vectors are ortogonal if and only if anticommute.

A3. Henceforth we will assume that q is non-degenerate. Its signature will be denoted (r,s).

Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n \in E$ be an orthonormal basis and set $B = \{\boldsymbol{e}_J\}$, $J \subseteq N = \{1, \ldots, n\}$ a multiindex.

◊ B generates $\mathcal{A}$ as a vector space. Therefore $\dim \mathcal{A} \leqslant 2^n$.

☐ The elements of the form $\boldsymbol{e}_K = \boldsymbol{e}_{k_1} \cdots \boldsymbol{e}_{k_l}$, $K = k_1, \ldots, k_l \in N$, generate $\mathcal{A}$ as a vector space (**1**). The $\boldsymbol{e}_K$ with $k_1 \leqslant \cdots \leqslant k_l$ also generate $\mathcal{A}$ as a vector space (**2**). Now any repeated factors appear together and can be symplified with the contraction rule. The result will be a scalar multiple of some $\boldsymbol{e}_J \in B$. ☐

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1. Use **A2** and the bilinearity of the product.

2. Since $\boldsymbol{e}_k \boldsymbol{e}_j = -\boldsymbol{e}_j \boldsymbol{e}_k$, the product $\boldsymbol{e}_K$ is equal to $(-1)^{t(K)} \boldsymbol{e}_{\widetilde{K}}$ where $\widetilde{K}$ is the result of sorting K in non-decreasing order.

Example. If we follow the procedures explained in the proof above to evaluate $\boldsymbol{e}_I \boldsymbol{e}_J$, I and J multiindices, we get *Artin's rule*:

$$\boldsymbol{e}_I \boldsymbol{e}_J = (-1)^{t(I,J)} q(I \cap J) \boldsymbol{e}_{I \triangle J}, \tag{15}$$

where $I \triangle J$ is the (sorted) symmetric difference of I and J and $q(K) = q(\boldsymbol{e}_{k_1}) \cdots q(\boldsymbol{e}_{k_r})$ for any multiindex K.

In particular, $\boldsymbol{e}_J^2 = (-1)^{r//2} q(J)$, $r = |J|$. Hence any $\boldsymbol{e}_J \in B$ is invertible and $\boldsymbol{e}_J^{-1} = (-1)^{r//2} q(J) \boldsymbol{e}_J$. If K is another multiindex, $\boldsymbol{e}_K \boldsymbol{e}_J^{-1}$ is, up to a sign, an element of B.

Example (A commutation formula). $\boldsymbol{e}_J \boldsymbol{e}_I = (-1)^c (-1)^{rs} \boldsymbol{e}_I \boldsymbol{e}_J$, where $r = |I|$, $s = |J|$, $c = |I \cap J|$. Indeed, there are rs pairs (i_k, j_l) $(k = 1, \ldots, r, j = 1, \ldots, s)$. The number of pairs with $i_k > i_l$ is $t(I, J)$, the number of pairs with $i_k < i_l$ is $t(J, I)$, and there are c pairs such that $i_k = j_l$ (coincidences). Thus $rs = t(I, J) + t(J, I) + c$ and $t(J, I) \equiv rs + c + t(I, J) \mod 2$. Now the claim is immediate, for $J \cap I = I \cap J$ and $J \triangle I = I \triangle J$.

$\lozenge$1 If n is even, the set B is linearly independent, and hence $\dim \mathcal{A} = 2^n$.

$\square$ Suppose we have a linear relation $\sum_J \lambda_J \boldsymbol{e}_J = 0$. To prove that all λ_J must vanish, it is sufficient to show that the coefficient $\lambda_\emptyset$ of $1 = \boldsymbol{e}_\emptyset$ must vanish. Indeed, multiplying the original relation by $\boldsymbol{e}_J^{-1}$ we get a similar relation in which the coefficient of 1 is λ_J. Now for any index k, the original relation implies $\sum_J \lambda_J \boldsymbol{e}_k \boldsymbol{e}_J \boldsymbol{e}_k^{-1} = 0$. Since $\boldsymbol{e}_k$ either commutes or anticommutes with $\boldsymbol{e}_J$, we derive the relation $\sum_J \lambda_J \boldsymbol{e}_J = 0$ where the sum only involves the $\boldsymbol{e}_J$ that commute with all $\boldsymbol{e}_k$. Since $\boldsymbol{e}_J$ anticommutes with any of its factors when $|J|$ is even and non-zero, and anticommutes with any $\boldsymbol{e}_k$ with $k \notin J$ when J is odd (such k exist because n is even), it turns out that the relation implies $\lambda_\emptyset = 0$. $\square$

Let B^+ be the set of the $\boldsymbol{e}_J$ such that $|J|$ is even.

$\Diamond$2 The set B^+ is linearly independent.

$\square$ By $\Diamond$1 we may assume that n is odd, and then it is immediate to adapt the above argument to this case, for $\boldsymbol{e}_N \notin B^+$. $\square$

$\Diamond$3 If n is odd, say $n = 2m + 1$, and the set B is linearly **dependent**, then $n//2 + s = m + s$ is even, $\boldsymbol{e}_N = \pm 1$ and $\dim \mathcal{A} = 2^{n-1}$.

$\square$ In this case the argument used in the proof of $\Diamond$1 works just as well: starting with a non-trivial linear relation $\sum_J \lambda_J \boldsymbol{e}_J = 0$, we can get a similar relation in which $\lambda_\emptyset \neq 0$ (multiply by any $\boldsymbol{e}_J^{-1}$ for which $\lambda_J \neq 0$), but in this case *we cannot get rid of N*, because all $\boldsymbol{e}_k$ commute with $\boldsymbol{e}_N$. So we obtain a non-trivial relation of the form $\lambda_\emptyset + \lambda_N \boldsymbol{e}_N = 0$. This implies that $\boldsymbol{e}_N \in \mathbf{R}$.

Let us work out the consequences of this.

First, $\boldsymbol{e}_N^2$ is a positive, but we also have $\boldsymbol{e}_N^2 = (-1)^m(-1)^s$, and hence $m+s$ must be even and $\boldsymbol{e}_N = \pm 1$. So for any J such that $|J|$ is odd, $\boldsymbol{e}_J = \pm \boldsymbol{e}_{J'}$, with $J' = N - J$. Since $\boldsymbol{e}_{J'} \in B^+$, we conclude that B^+ generates $\mathcal{A}$ linearly. To finish, use ◊2. □

Thus we have $\dim \mathcal{A} = 2^n$ unless n is odd (say $2m+1$) and $m+s$ is even, in which case $\dim \mathcal{A} = 2^n$ if $\boldsymbol{e}_N^2 \neq \pm 1$ and $\dim \mathcal{A} = 2^{n-1}$ otherwise. Signatures with n odd and $m+s$ even will be called *special*, and *regular* otherwise. Here is a table of special signatures up to $n = 9$:

n	3		5			7				9				
r	2	0	5	3	1	6	4	2	0	9	7	5	3	1
s	1	3	0	2	4	1	3	5	7	0	2	4	6	8

Note that the STA signature $(1,3)$ and the CGA signature $(4,1)$ are regular. The Euclidean signatures $(n,0)$ and Lorentzian signatures $(1, n-1)$ are regular unless $n = 1 + 4m$, $m \geqslant 1$.

A geometric algebra $\mathcal{A}$ will be said to be *full* if $\dim\mathcal{A} = 2^n$ and *folded* if $\dim\mathcal{A} = 2^{n-1}$ (or equivalently if the signature is special and $\boldsymbol{e}_N = \pm 1$ for any orthonormal basis).

◊ Let $\mathcal{A}$ and $\mathcal{A}'$ be geometric algebras with the same signature and E and E' the corresponding vector spaces. Let $f : E \to E'$ be an isometry. If $\mathcal{A}$ is full, then there is a unique algebra homomorphism $f^\sharp : \mathcal{A} \to \mathcal{A}'$ that agrees with f on E, and $f^\sharp$ is onto.

If $\mathcal{A}'$ is also full, then $f^\sharp$ is an isomorphism.

□ Let $\mathbf{e} = \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be an orthonormal basis of E. Let $\mathbf{e}' = \boldsymbol{e}'_1, \ldots, \boldsymbol{e}'_n$, with $\boldsymbol{e}'_k = f(\boldsymbol{e}_k)$. Since f is an isometry, $\mathbf{e}'$ is an orthonormal basis of E'. Then $B = \{\boldsymbol{e}_J\}$ is linear basis of $\mathcal{A}$ and $B' = \{\boldsymbol{e}'_J\}$ is a linearly generating set for $\mathcal{A}'$. If $f^\sharp$ exists, $f^\sharp(\boldsymbol{e}_J) = \boldsymbol{e}'_J$, and hence $f^\sharp$ is uniquely determined as a linear map, and is onto.

To see that $f^\sharp$ is an algebra homomorphism, it is enough to show that $f(\boldsymbol{e}_J\boldsymbol{e}_K) = \boldsymbol{e}'_J\boldsymbol{e}'_K$ for any multiindices J and K.

But this is an immediate consequence of Artin's rule, for if $L = J \triangle K$, then $\boldsymbol{e}_J\boldsymbol{e}_K = \epsilon\boldsymbol{e}_L$ and $\boldsymbol{e}'_J\boldsymbol{e}'_K = \epsilon\boldsymbol{e}'_L$ (the same sign ϵ).

Finally, in case $\mathcal{A}'$ is also full, B' is a linear basis of $\mathcal{A}'$ and so $f^\sharp$ is a linear isomorphism, hence an algebra isomorphism. □

Let (E, q) be a regular metric vector space of finite dimension n. Let C_qE be the quotient algebra TE/I_qE, where I_qE is the ideal generated by the tensors of the form $\boldsymbol{e} \otimes \boldsymbol{e} - q(\boldsymbol{e})$ (C_qE is called the *Clifford algebra* of (E, q)). Since non-zero vectors cannot belong to I_qE, the restriction of the quotient map to $E = T^1E$ is one-to-one, and hence we will identify E to its image in C_qE. It is also immediate that $1 \notin E$.

◊ (C_qE, E) is a full geometric algebra with metric q.

□ We just checked the conditions **A0**.

The fact that $\boldsymbol{e} \otimes \boldsymbol{e} - q(\boldsymbol{e})$ maps to 0 in C_qE shows that $\boldsymbol{e}^2 = q(\boldsymbol{e})$ in C_qE, which is the contraction rule **A1**. This also shows that the metric of E defined by C_qE is q and in particular that **A3** is satisfied as well.

Since TE is generated by E as an $\mathbf{R}$-algebra, C_qE has the same property. This shows that **A2** is also satisfied.

To end the proof, we have to show that C_qE is full. Since this is true for n even, we can assume that n is odd.

For this we will use the parity involution $\alpha : TE \to TE$, uniquely determined by the rule

$$\boldsymbol{e}_1 \otimes \cdots \otimes \boldsymbol{e}_r \mapsto (-\boldsymbol{e}_1) \otimes \cdots \otimes (-\boldsymbol{e}_r) = (-1)^k \boldsymbol{e}_1 \otimes \cdots \otimes \boldsymbol{e}_r.$$

Observe that I_qE is invariant by α, for

$$(\boldsymbol{e} \otimes \boldsymbol{e} - q(\boldsymbol{e}))^\alpha = \boldsymbol{e} \otimes \boldsymbol{e} - q(\boldsymbol{e}),$$

and hence α induces an involutory algebra automorphism of C_qE uniquely determined by the rule

$$\boldsymbol{e}_1 \cdots \boldsymbol{e}_r \mapsto (-1)^k \boldsymbol{e}_1 \cdots \boldsymbol{e}_r.$$

Now if $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is any basis of E, $(\boldsymbol{e}_1 \ldots \boldsymbol{e}_n)^\alpha = -\boldsymbol{e}_1 \ldots \boldsymbol{e}_n$ (because n is odd). Hence $\boldsymbol{e}_1 \ldots \boldsymbol{e}_n \notin \mathbf{R}$ and we know that this implies that C_qE is full. $\square$

Although it is not clear what is their role in the realm of GA, for completeness we include a proof (appendix B, page 80) that *folded GAs exist*, and that there is *only one*, up to a canonical isomorphism, for each special signature. Moreover, we show how to contruct this algebra as an explicit quotient of the full algebra for the same signature.

Here lets return to the mainstream.

For each signature (r, s) there is a unique *full* geometric algebra, up to a canonical isomorphism. We will denote it $\mathcal{G} = \mathcal{G}_{r,s}$. Its product will be called *geometric product*.

Recall that we let $B = \{\boldsymbol{e}_J\}$ denote the linear basis of $\mathcal{G}$ associated to an orthonormal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r$ of E. We will also write $B^r = \{\boldsymbol{e}_J\}_{|J|=r}$.

Consider the map alt : $E^r \to \mathcal{G}$ given by

$$\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r \mapsto \mathrm{alt}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r) = \tfrac{1}{r!} \sum_J (-1)^{t(J)} \boldsymbol{x}_{j_1} \cdots \boldsymbol{x}_{j_r},$$

where the sum is extended to all permutations $J = [j_1, \ldots, j_r]$ of $\{1, \ldots, r\}$. This map is multilinear and alternating, and hence there is a unique linear map alt : $\bigwedge^r E \to \mathcal{G}$ such that

$$\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_r \mapsto \mathrm{alt}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r).$$

◊ The map $\mathrm{alt} : \wedge^r E \to \mathcal{G}$ is one-to-one and its image is the space $\mathcal{G}^r$ spanned by B^r. In particular, $\mathcal{G}^r$ is independent of the orthonormal basis $\mathbf{e}$ and $\dim \mathcal{G}^r = \binom{n}{r}$.

□ If the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r$ are pair-wise orthogonal, they anticommute, and this implies that $\mathrm{alt}(\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_r) = \boldsymbol{x}_1 \cdots \boldsymbol{x}_r$. In particular $\mathrm{alt}(\widehat{\boldsymbol{e}}_J) = \boldsymbol{e}_J$ for any J such that $|J| = r$. □

So we have a *canonical linear grading* $\mathcal{G} = \mathcal{G}^0 \oplus \mathcal{G}^1 \oplus \cdots \oplus \mathcal{G}^n$ and a *canonical graded linear isomorphism* $\mathrm{alt} : \wedge E \to \mathcal{G}$. This isomorphism allows us to define the exterior (outer) product and the interior product in $\mathcal{G}$ by grafting the exterior and interior products of $\wedge E$ via this map. Thus, by definition, $\mathrm{alt}(x) \wedge \mathrm{alt}(y) = \mathrm{alt}(x \wedge y)$ and $\mathrm{alt}(x) \cdot \mathrm{alt}(y) = \mathrm{alt}(x \cdot y)$.

The notions of *multivector*, *r-vector* and *r-blade* are also transferred to $\mathcal{G}$: they are respectively the elements of G, of $\mathcal{G}^r$, and the non-zero r-vectors of the form $\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_r$, $\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_r \in E$.

◊ If x is an r-vector and y an s-vector,

$$x \wedge y = (xy)_{r+s} \text{ and } x \cdot y = (xy)_{|r-s|}. \tag{16}$$

□ It is enough to check these relations for $x = \boldsymbol{e}_J$, $y = \boldsymbol{e}_K$.

From the definitions, it follows that $\boldsymbol{e}_J \wedge \boldsymbol{e}_K = \mathrm{alt}(\widehat{\boldsymbol{e}}_J \wedge \widehat{\boldsymbol{e}}_K)$ and $\boldsymbol{e}_J \cdot \boldsymbol{e}_K = \mathrm{alt}(\widehat{\boldsymbol{e}}_J \cdot \widehat{\boldsymbol{e}}_K)$.

Thus $\boldsymbol{e}_J \wedge \boldsymbol{e}_K$ is zero if $J \cap K \neq \emptyset$ and is $\boldsymbol{e}_J\boldsymbol{e}_K$ otherwise, which agrees with $(\boldsymbol{e}_J\boldsymbol{e}_K)_{r+s}$ in both cases.

For the interior product, assume $r \leqslant s$. In this case $\boldsymbol{e}_J\boldsymbol{e}_K = (-1)^{t(J,K)}q(L)\boldsymbol{e}_L$, with $L = J \triangle K$ (the sorted symmetric difference), has grade $r + s - 2|L| \geqslant r + s - 2r = s - r$, with equality if an only if $J \subseteq K$. Thus $(\boldsymbol{e}_J\boldsymbol{e}_K)_{s-r} = 0$ if $J \not\subseteq K$ and $= (-1)^{t(J,K)}q(J)\boldsymbol{e}_{K-L}$ otherwise. And these values agree with $\boldsymbol{e}_J \cdot \boldsymbol{e}_K$ (use the example on page 29). The case $r \geqslant s$ is analyzed in a similar way and is left as an exercise. □

The involutions α and τ can also be transported to $\mathcal{G}$ via alt.

The *main involution* α of $\wedge E$ becomes the involution α of $\mathcal{G}$ defined on page 41, for on r-vectors both agree with multiplication by the sign $(-1)^r$. It is immediate to check that it is an involutive automorphism of the geometric product:

$$(xy)^\alpha = x^\alpha y^\alpha.$$

The *reversion* on r-vectors is the multiplication by the sign $(-1)^{\binom{r}{2}} = (-1)^{r//2}$. Since for products of orthogonal vectors this is the sign produced by reversing the order of the factors, this holds in general: $(\boldsymbol{e}_1 \cdots \boldsymbol{e}_r)^\tau = \boldsymbol{e}_r \cdots \boldsymbol{e}_1$. From this relation it follows that τ is also an involutive anti-automorphism of the geometric product:

$$(xy)^\tau = y^\tau x^\tau.$$

The *Clifford involution* $\kappa = \tau\alpha = \alpha\tau$ is also an involutive anti-automorphism of the geometric product: $(xy)^\kappa = y^\kappa x^\kappa$.

◊ Let $\boldsymbol{e} \in E$ an $x \in \mathcal{G}$. Then

$$\boldsymbol{e}x = \boldsymbol{e} \cdot x + \boldsymbol{e} \wedge x. \tag{17}$$

□ Since both sides are bilinear expressions of $\boldsymbol{e}$ and x, it is enough to check the relation for $\boldsymbol{e} = \boldsymbol{e}_k$ and $x = \boldsymbol{e}_J$, $k \in N$ and J a multiindex. If $k \notin J$, $\boldsymbol{e}_k \cdot \boldsymbol{e}_J = 0$ and $\boldsymbol{e}_k\boldsymbol{e}_J = \boldsymbol{e}_k \wedge \boldsymbol{e}_J$. If $k \in J$, then $\boldsymbol{e}_k \wedge \boldsymbol{e}_J = 0$ and $\boldsymbol{e}_k\boldsymbol{e}_J = (-1)^{t(k,J)}q(\boldsymbol{e}_k)\boldsymbol{e}_{J-\{k\}} = \boldsymbol{e}_k \cdot \boldsymbol{e}_J$. □

We also have the formula

$$x\boldsymbol{e} = x \cdot \boldsymbol{e} + x \wedge \boldsymbol{e}. \tag{18}$$

□ Instead of proceeding as in the proof above, we can apply (17) to x^τ and then apply τ to the result:

$$x\boldsymbol{e} = (\boldsymbol{e}x^\tau)^\tau = (\boldsymbol{e} \cdot x^\tau + \boldsymbol{e} \wedge x^\tau)^\tau = x \cdot \boldsymbol{e} + x \wedge \boldsymbol{e}.$$ □

◊ (*Riesz formulas*). Taking into account that $x \cdot \boldsymbol{e} = (-1)^{r+1}\boldsymbol{e} \cdot x$ and $x \wedge \boldsymbol{e} = (-1)^r \boldsymbol{e} \wedge x$, we can write

$$x\boldsymbol{e} = (-1)^r(-\boldsymbol{e} \cdot x + \boldsymbol{e} \wedge x). \tag{19}$$

Together with (17), it is immediate to get the expressions

$$2\boldsymbol{e} \wedge x = \boldsymbol{e}x + (-1)^r x\boldsymbol{e}, \qquad 2\boldsymbol{e} \cdot x = \boldsymbol{e}x - (-1)^r x\boldsymbol{e}. \tag{20}$$

◊ For any vector $\boldsymbol{e}$, the operator $\delta_{\boldsymbol{e}}$ is an antiderivation of the geometric product: $\delta_{\boldsymbol{e}}(xy) = (\delta_{\boldsymbol{e}}x)y + x^{\alpha}(\delta_{\boldsymbol{e}}y)$. **E4**, page 85. □

Given a vector $\boldsymbol{e}$, we define $\mu_{\boldsymbol{e}} : \mathcal{G} \to \mathcal{G}$ by $\mu_{\boldsymbol{e}}(x) = \boldsymbol{e} \wedge x$. Then formula (17) can be written as

$$\boldsymbol{e}x = (\delta_{\boldsymbol{e}} + \mu_{\boldsymbol{e}})(x). \tag{21}$$

♢ Let $x \in \mathcal{G}^r$, $y \in \mathcal{G}^s$. If $k \in \{0, 1, \ldots, n\}$ and $(xy)_k \neq 0$, then $k = |r - s| + 2i$ with $i \geqslant 0$ and $k \leqslant r + s$. Moreover, if $r, s > 0$, then

$$(xy)_{r+s} = x \wedge y, \quad (xy)_{|r-s|} = x \cdot y. \tag{22}$$

□ Since $(xy)_k$ depends linearly of x, we do not loose generality if we assume that x is a non-zero r-blade, say $X = \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_r$. Moreover, we may assume that $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_r$ is an orthogonal basis of $[X]$, in which case $X = \boldsymbol{x}_1 \cdots \boldsymbol{x}_r$ and, using (21),

$$xy = (\mu_{\boldsymbol{x}_1} + \delta_{\boldsymbol{x}_1}) \cdots (\mu_{\boldsymbol{x}_r} + \delta_{\boldsymbol{x}_r})(y).$$

If we choose i times the summand μ, and hence $r - i$ times δ, we get a homogeneous multivector of grade $s + i - (r - i) = s - r + 2i$. The maximum grade we can form in this way is $r + s$ (with $i = r$), and the corresponding term is $x \wedge y$ (in agreement with (16)). Now if $s \geqslant r$, the minimum grade we can get is $s - r$ (with $i = 0$), and we know that the corresponding term is $x \cdot y$ (by (16)).

If $r \geqslant s$, the minimum grade in xy is the minimum grade appearing in $(xy)^\tau = y^\tau x^\tau$, namely $r-s$, and the corresponding term is $((y^\tau x^\tau)_{r-s})^\tau = (y^\tau \cdot x^\tau)^\tau = x \cdot y$. □

Examples. Let $\boldsymbol{e} \in E_{r,s}$ and $b \in \mathcal{G}^2_{r,s}$. Then $\boldsymbol{e}b - b\boldsymbol{e} = 2\boldsymbol{e} \cdot b$, which is a vector. This property also happens for $b = \boldsymbol{xy}$, $\boldsymbol{x}, \boldsymbol{y} \in E$:

$$\boldsymbol{exy} - \boldsymbol{xye} = 2\boldsymbol{e} \cdot (\boldsymbol{xy}) = 2(\boldsymbol{e} \cdot \boldsymbol{x})\boldsymbol{y} - 2(\boldsymbol{ey})\boldsymbol{x}.$$

With the same notations, we have $\boldsymbol{e}b + b\boldsymbol{e} = 2\boldsymbol{e} \wedge b$, a trivector. But for $b = \boldsymbol{xy}$, $\boldsymbol{exy} + \boldsymbol{xye} = 2(\boldsymbol{x} \cdot \boldsymbol{y})\boldsymbol{e} + 2\boldsymbol{e} \wedge \boldsymbol{x} \wedge \boldsymbol{y}$.

Example. For any r-blade X, $X^2 \in \mathbf{R}$. Indeed, we may assume that X is the product of orthogonal vectors, $X = \boldsymbol{x}_1 \cdots \boldsymbol{x}_r$, and $X^2 = (-1)^{r//2} X\widetilde{X} = (-1)^{r//2} q(X) \in \mathbf{R}$. In particular we see that if X is *non-nul* (meaning $X^2 \neq 0$), then X is invertible, with $X^{-1} = (-1)^{r//2} X/q(X)$.

Exercises: **E5**, page 86 and **E6**, page 86.

◊ $q(x,y) = (x^\tau y)_0$. In particular, $q(x) = (x^\tau x)_0$.

□ Since both expressions are bilinear, we may assume that x and y are homogeneous, say of grade r and s, respectively. Then (16) tells us that $(x^\tau y)_0 = 0$ if $r \neq s$, which agrees with $q(x,y)$ as this also vanishes. So we may assume that $r = s$, and in this case (16) again tells us that $(x^\tau y)_0 = x^\tau \cdot y$ and then formula (13) allows us to conclude that $x^\tau \cdot y = q(x,y)$. □

Remark. This form of the metric is called *natural scalar product* in Hestenes-Li-Rockwook-2001.

◊ Let $x \in \mathcal{G}^r$ and $y \in G^s$ and $z \in \mathcal{G}^m$, where $m = |s - r|$ (the grade of $x \cdot y$). Then

$$q(x \cdot y, z) = \begin{cases} q(y, x \wedge z) & \text{if} \quad r \leqslant s \\ q(x, z \wedge y) & \text{if} \quad r \geqslant s \end{cases}$$

□ If $r \geqslant s$, $x \cdot y = (-1)^{(r-s)s} y \cdot x$, while

$$q(x, z \wedge y) = (-1)^{ms} q(x, y \wedge z) = (-1)^{(r-s)s} q(x, y \wedge z).$$

This shows that the second case is reduced to the first and so we may assume that $r \leqslant s$.

Since the two sides of the claimed equality are linear in x, y, and z, it suffices to prove it for three basis elements: $x = \boldsymbol{e}_J \in \mathcal{G}^r$, $y = \boldsymbol{e}_K \in \mathcal{G}^s$, $z = \boldsymbol{e}_L \in \mathcal{G}^{s-r}$ ($m = s - r$ in this case). The value of $\boldsymbol{e}_J \cdot \boldsymbol{e}_K$ follows directly from the Laplace formula (12): $(-1)^{t(J,K-J)} \widehat{\boldsymbol{e}}_{K-J}$ if $J \subseteq K$ and 0 otherwise.

Therefore $q(\boldsymbol{e}_J \cdot \boldsymbol{e}_K, \boldsymbol{e}_L) = (-1)^{t(J,K-J)} q_{K-J}$ if $J \subseteq K$ and $L = K - J$, 0 in any other case. On the other hand $q(\boldsymbol{e}_K, \boldsymbol{e}_J \wedge \boldsymbol{e}_L)$ can only be non-zero if L and J are disjoint and $K = J \cup L$, which is the same thing as saying that $J \subseteq K$ and $L = K - J$, and in this case the result is also, taking into account the reordering of $\boldsymbol{e}_J \wedge \boldsymbol{e}_L = \boldsymbol{e}_J \wedge \boldsymbol{e}_{K-J}$, $(-1)^{t(J,K-J)} q_{K-J}$. □

Geometry with GA

Let $\mathcal{G} = \mathcal{G}_{r,s}$ denote the full geometric algebra of signature (r, s), which is endowed with the exterior, interior and geometric products. The *even subalgebra* is denoted $\mathcal{G}^+$.

The group of multivectors that are *invertible with respect to the geometric* product will be denoted $\mathcal{G}^\times$. Note that $\mathbf{R}^\times = \mathbf{R} - \{0\}$ is a subgroup of $\mathcal{G}^\times$ and that it contains the set $E_{r,s}^\times$ of *invertible vectors* (are the *non-null*, or *non-isotropic* vectors). The exercise **E5**, page 86, gives sufficient (and necessary) conditions for a blade to be invertible.

As we see it, one of the fundamental reasons to study $\mathcal{G}$ is that it *provides an effective general way to work with the group $O_{r,s}$ of isometries of $E_{r,s}$, and to investigate the problems* (geometrical or physical) *in which such groups are essential*. Important instances: the isometries of the Euclidean space E_n (*orthogonal group* O_n), of the Minkowski space $E_{1,3}$ (*Lorentz group*) and of $E_{4,1}$ (*conformal group* of E_3).

◊ If $\boldsymbol{u}$ is a non-isotropic vector, then the map $s_{\boldsymbol{u}} : E_{r,s} \to E_{r,s}$, $\boldsymbol{x} \mapsto \boldsymbol{u}\boldsymbol{x}\boldsymbol{u}^{-1}$ is the *axial symmetry* with respect to the line (*axis*) $\langle \boldsymbol{u} \rangle$.

□ Since $\boldsymbol{u}\boldsymbol{u}^{-1} = 1$, $s_{\boldsymbol{u}}(\boldsymbol{u}) = \boldsymbol{u}$. If $\boldsymbol{x} \in \boldsymbol{u}^{\perp}$, then $\boldsymbol{u}$ and $\boldsymbol{x}$ anticommute and hence $s_{\boldsymbol{u}}(x) = \boldsymbol{u}\boldsymbol{x}\boldsymbol{u}^{-1} = -\boldsymbol{x}\boldsymbol{u}\boldsymbol{u}^{-1} = -\boldsymbol{x}$. Thus $s_{\boldsymbol{u}}$ is indeed the linear map that leaves $\boldsymbol{u}$ fixed and is $-\mathrm{Id}$ on $\boldsymbol{u}^{\perp}$, in agreement with the definition of axial symmetry. □

Corollary. If $\boldsymbol{u}$ is a non-isotropic vector, then the map $m_{\boldsymbol{u}} : E_{r,s} \to E_{r,s}$, $\boldsymbol{x} \mapsto -\boldsymbol{u}\boldsymbol{x}\boldsymbol{u}^{-1}$ (thus $m_{\boldsymbol{u}} = -s_{\boldsymbol{u}}$) is the (mirror) *reflection* across the hyperplane $\boldsymbol{u}^{\perp}$.

□ Indeed, $m_{\boldsymbol{u}}$ is the identity on $\boldsymbol{u}^{\perp}$ and maps $\boldsymbol{u}$ to $-\boldsymbol{u}$. □

Remark. For non-zero λ, $\boldsymbol{u}$ and $\lambda\boldsymbol{u}$ define the same axial symmetry (reflection). Therefore we can always assume that the vector $\boldsymbol{u}$ used to specify an axial symmetry (reflexion) is a *unit vector* (that is, $q(\boldsymbol{u}) = \pm 1$).

A *versor* is an element $v \in \mathcal{G}$ that can be expressed as a product of non-null vectors: $v = \boldsymbol{u}_k \cdots \boldsymbol{u}_1$. The set of versors $V_{r,s} = V(E_{r,s})$ forms a subgroup of $\mathcal{G}^{\times}$ (**1**).

◊ Given a versor v, the map $\underline{v}(\boldsymbol{x}) = v^{\alpha}\boldsymbol{x}v^{-1}$ is an isometry of $E_{r,s}$.

□ Indeed, we have

$$\begin{aligned} v^{\alpha}\boldsymbol{x}v^{-1} &= (-1)^k \boldsymbol{u}_k \cdots \boldsymbol{u}_1 \boldsymbol{x} \boldsymbol{u}_1^{-1} \cdots \boldsymbol{u}_k^{-1} \\ &- \boldsymbol{u}_k(\cdots(-\boldsymbol{u}_1 \boldsymbol{x} \boldsymbol{u}_1)\cdots)\boldsymbol{u}_k^{-1} \\ &= m_{\boldsymbol{u}_k}(\cdots(m_{\boldsymbol{u}_1}(\boldsymbol{x}))\cdots) = (m_{\boldsymbol{u}_k}\cdots m_{\boldsymbol{u}_1})(\boldsymbol{x}) \end{aligned}$$

and hence $\underline{v} = m_{\boldsymbol{u}_k}\cdots m_{\boldsymbol{u}_1}$, which is an isometry. □

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1. It is clear that the product of two versors is a versor, that 1 is a versor (actually any non-zero scalar λ is a versor, as $\lambda = (\lambda\boldsymbol{u})\boldsymbol{u}^{-1}$ for any invertible vector $\boldsymbol{u}$) and that the inverse of versor v is $v^{-1} = \boldsymbol{u}_1^{-1}\cdots\boldsymbol{u}_k^{-1}$. Since $v\tilde{v} = q(v) = \boldsymbol{u}_1^2\cdots\boldsymbol{u}_k^2 \neq 0$, we can also write $v^{-1} = \tilde{v}/q(v)$.

Let $V_{r,s} = V(E_{r,s})$ be the group of versors and $\mathrm{O}_{r,s} = \mathrm{O}(E_{r,s})$ the group of isometries of $E_{r,s}$ (*orthogonal group*).

$\Diamond$ The map $\rho : V_{r,s} \to \mathrm{O}_{r,s}$ given by $v \mapsto \underline{v}$ (*adjoint map*) is an onto homomorphism and its kernel is $\mathbf{R}^{\times}$ (the multiplicative group of non-zero real numbers).

$\square$ It is a homomorphism because if v and w are versors, then $\underline{wv}(\boldsymbol{x}) = (wv)^{\alpha}\boldsymbol{x}(wv)^{-1} = w^{\alpha}v^{\alpha}\boldsymbol{x}v^{-1}w^{-1} = \underline{w}(\underline{v}(\boldsymbol{x}))$, which shows that $\underline{wv} = \underline{w}\,\underline{v}$.

That it is onto is a direct consequence of the Cartan-Dieudonné theorem, which asserts that any isometry is a product of at most n reflections.

Since $\underline{\lambda}(\boldsymbol{x}) = \lambda\boldsymbol{x}\lambda^{-1} = \boldsymbol{x}$ for $\lambda \in \mathbf{R}^{\times}$, it is clear that $\mathbf{R}^{\times}$ is contained in the kernel of $V_{r,s} \to \mathrm{O}_{r,s}$. So it remains to prove that any element of the kernel is in fact a scalar.

To show the inclusion $\ker(\rho) \subseteq \mathbf{R}^\times$, suppose that $v \in V_{r,s}$ is an element of $\ker(\rho)$. Then $v^\alpha \boldsymbol{e} v^{-1} = \boldsymbol{e}$, or $v^\alpha \boldsymbol{e} = \boldsymbol{e} v$, for all $\boldsymbol{e} \in E_{r,s}$. But the Riesz formulas tell us that this relation is equivalent to say that $\boldsymbol{e} \cdot v = (\boldsymbol{e} v - v^\alpha \boldsymbol{e})/2 = 0$ for all $\boldsymbol{e} \in E_{r,s}$ **(1)**, and this implies that v must be a scalar **(2)**. □

---

(1) $v^\alpha \boldsymbol{e} = (-1)^r v\boldsymbol{e} = (-1)^r v \cdot \boldsymbol{e} + (-1)^r v \wedge \boldsymbol{e} = -\boldsymbol{e} \cdot v + \boldsymbol{e} \wedge v$, $\boldsymbol{e} v = \boldsymbol{e} \cdot v + \boldsymbol{e} \wedge v$, so $\boldsymbol{e} v - v^\alpha \boldsymbol{e} = 2\boldsymbol{e} \cdot v$.

(2) For any grade r, $\boldsymbol{e} \cdot x_r = 0$ for all $\boldsymbol{e}$. So it is enough to see that $\boldsymbol{e} \cdot x_r = 0$ for all vectors $\boldsymbol{e}$ and $r > 0$ imply $x_r = 0$. Use an orthogonal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ and write $x_r = \sum_{|J|=r} \lambda_J \boldsymbol{e}_J$. Since $\boldsymbol{e}_1 \cdot x_r = 0$, and $\boldsymbol{e}_1 \cdot \boldsymbol{e}_J = q(\boldsymbol{e}_1)\boldsymbol{e}_{J-\{1\}}$ if $1 \in J$ and 0 otherwise, we get $\lambda_J = 0$ if $1 \in J$. So $\boldsymbol{e}_1$ does not appear in the above expansion. Arguing in a similar way using $\boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$, we get that no $\boldsymbol{e}_k$ appears in the expansion, and so $x_r = 0$.

Let $\mathrm{Pin}_{r,s}$ be the subgroup of $V_{r,s}$ of unit versors, that is, versors v such that $q(v) = \pm 1$ or, equivalently, $v\widetilde{v} = \pm 1$. The elements of $\mathrm{Pin}_{r,s}$ are called *pinors*.

◊ The group $\mathrm{Pin}_{r,s}$ coincides with the subgroup of $V_{r,s}$ whose elements are products of unit vectors.

□ Since it is clear that a product of unit vectors is a pinor, what remains is to see that any pinor is a product of unit vectors. Let then $v = \boldsymbol{u}_k \cdots \boldsymbol{u}_1$ be a pinor. Since $q(v) = v\widetilde{v} = \pm 1$, we have that $q(\boldsymbol{u}_k) \cdots q(\boldsymbol{u}_1) = \pm 1$. Let $\varepsilon_j = \pm 1$ and $\lambda_j > 0$ be such that $q(\boldsymbol{u}_j) = \varepsilon_j \lambda_j^2$. Then

$$\pm 1 = \varepsilon_1 \cdots \varepsilon_k \lambda_1^2 \cdots \lambda_k^2,$$

which implies that $\lambda_1 \cdots \lambda_k = 1$. Therefore $v = \boldsymbol{u}'_k \cdots \boldsymbol{u}'_1$, with $\boldsymbol{u}'_j = \boldsymbol{u}_j / \lambda_j$, and $\boldsymbol{u}'_j$ is a unit vector, for $q(\boldsymbol{u}'_j) = q(\boldsymbol{u}_j)/\lambda_j^2 = \varepsilon_j$. □

◊ The homomorphism $\mathrm{Pin}_{r,s} \to \mathrm{O}_{r,s}$, $v \mapsto \underline{v}$, is onto and its kernel is $\{\pm 1\}$.

☐ It is surjective because any reflexion has the form $m_{\boldsymbol{u}}$ with $\boldsymbol{u}$ a unit vector. The kernel consists of scalars λ such that $q(\lambda) = \lambda^2 = 1$. ☐

Consider the subgroup $V_{r,s}^{+}$ of $V_{r,s}$ formed by the even elements of $V_{r,s}$. For any $v \in V^{+}$, $\underline{v}$ is the product of an even number of reflections and hence it belongs to $\mathrm{SO}_{r,s}$ (*special orthogonal group*). The map $V_{r,s}^{+} \to \mathrm{SO}_{r,s}$ is onto (again by the Cartan-Dieudonné theorem) and its kernel is $\mathbf{R}^{\times}$.

The group $\mathrm{Spin}_{r,s}$ is the subgroup of even elements of $\mathrm{Pin}_{r,s}$, that is, pinors that are the product of an even number of unit vectors (*spinors*). The same reasoning as in the previous paragraph shows that we have an onto map $\mathrm{Spin}_{r,s} \to \mathrm{SO}_{r,s}$ and that its kernel is $\{\pm 1\}$.

Since spinors R are unit versors, in general we have $R\widetilde{R} = \pm 1$. If $R\widetilde{R} = 1$, the spinor is called a *rotor*. In the Euclidean case, all spinors are rotors, but this is not so in general.

Rotors form a normal subgroup, which we will denote $\mathrm{Spin}^+_{r,s}$, of $\mathrm{Spin}_{r,s}$. In fact, the map $\mathrm{Spin}_{r,s} \to \{\pm 1\}$, $S \mapsto S\widetilde{S}$, is a homomorphism and its kernel is the rotor group.

As we will see in next lecture (with the exception of $(r,s) = (1,1)$) the group $\mathrm{Spin}^+_{r,s}$ is path connected to 1 and its image by the adjoin map is the $\mathrm{SO}^+_{r,s}$, the connected component of 1 of $\mathrm{SO}_{r,s}$ (*rotation group*). This gives a 2:1 cover $\mathrm{Spin}^+_{r,s} \to \mathrm{SO}^+_{r,s}$ which is the *universal cover*.

◊ *Let $v \in V^{+}_{r,s}$. Then $\underline{v} : \mathcal{G} \to \mathcal{G}$ is an automorphism of the geometric algebra* (that is, a linear automorphism that preserves grades and which is an automorphism of the geometric, exterior and interior products).

□ Indeed, $\underline{v}$ is linear and is a homomorphism of the geometric product. Since it maps vectors to vectors, it follows that it preserves grades. The fact that it is also an homomorphism of the exterior and interior products follows from the preservation of grades and the characterization of those operations given by the formulas (16). □

"The most remarkable formula in mathematics is:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This is our **jewel**. We may *relate the geometry to the algebra* by representing complex numbers in a plane

$$x + iy = re^{i\theta}$$

This is the *unification of algebra and geometry*."

R. Feynman, *Lecture Notes in Physics*, Volume I, Section 22-6.

Comment. Emphasis not in the original. We also note, from the introduction of chapter 22: "So, ultimately, in order to understand nature it may be necessary to have a deeper understanding of mathematical relationships".

Let us see what GA has to say about that jewel!

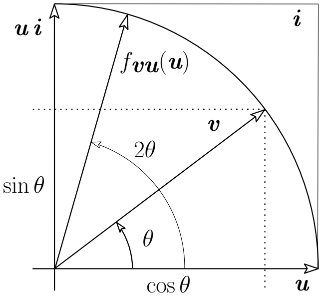
So far in this section we have shown the value of versors (or of pinors) for representing isometries and of even versors (or of spinors) for representing proper isometries. But this value is more theoretical than practical, because it hardly gives any clue about the detailed properties of an isometry in terms of the versor producing it.

Example. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be linearly independent unit vectors in the Euclidean space E_n and consider the rotor $R = \boldsymbol{vu}$. This generates the rotation $\underline{R}(\boldsymbol{x}) = R\boldsymbol{x}R^{-1}$. Fine, but *what is its axis and amplitude in terms of* $\boldsymbol{u}$ *and* $\boldsymbol{v}$*?*

To find out, let $\theta \in (0,\pi)$ be the Euclidean angle between $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$\cos\theta = \boldsymbol{u}\cdot\boldsymbol{v}.$$

Let $\boldsymbol{i}$ be the unit area in the oriented plane $P = \langle \boldsymbol{u},\boldsymbol{v}\rangle$. So $\boldsymbol{i} = \boldsymbol{u}_1\boldsymbol{u}_2$ for any positive orthonormal basis $\boldsymbol{u}_1,\boldsymbol{u}_2$ of P. We have $\boldsymbol{i}^2 = -1$. Note also that $\boldsymbol{x}\mapsto\boldsymbol{x}\boldsymbol{i}$ is the anticlockwise rotation by $\pi/2$, for $\boldsymbol{u}_1\boldsymbol{i} = \boldsymbol{u}_2$ and $\boldsymbol{u}_2\boldsymbol{i} = -\boldsymbol{u}_1$). In particular $\boldsymbol{u}$ and $\boldsymbol{u}\boldsymbol{i}$ is a positive orthonormal basis of P and hence $\boldsymbol{v} = \boldsymbol{u}\cos\theta + \boldsymbol{u}\boldsymbol{i}\sin\theta$.



It follows that $\boldsymbol{u} \wedge \boldsymbol{v} = \boldsymbol{u} \wedge \boldsymbol{u}\boldsymbol{i} \sin\theta = \boldsymbol{u}\boldsymbol{u}\boldsymbol{i} \sin\theta = \boldsymbol{i} \sin\theta$. Therefore

$$R = \boldsymbol{v}\boldsymbol{u} = \boldsymbol{v} \cdot \boldsymbol{u} + \boldsymbol{v} \wedge \boldsymbol{u} = \cos\theta - \boldsymbol{i} \sin\theta = e^{-\boldsymbol{i}\theta}. \tag{23}$$

Thus we have what may be called *Euler's spinor formula*:

◊ $$\underline{R}(\boldsymbol{x}) = e^{-\boldsymbol{i}\theta} \boldsymbol{x} e^{\boldsymbol{i}\theta}.$$

And this formula allows us to read directly the geometric elements of the rotation:

◊ *The rotation is in the plane P and its amplitude is 2θ.*

□ If $\boldsymbol{x}$ is orthogonal to P, it anticommutes with $\boldsymbol{u}$ and $\boldsymbol{v}$, hence it commutes with $\boldsymbol{i}$, and $e^{-\boldsymbol{i}\theta} \boldsymbol{x} e^{\boldsymbol{i}\theta} = \boldsymbol{x}$. If $\boldsymbol{x}$ lies in P, it anticommutes with $\boldsymbol{i}$ and hence $e^{-\boldsymbol{i}\theta} \boldsymbol{x} e^{\boldsymbol{i}\theta} = \boldsymbol{x} e^{2\boldsymbol{i}\theta}$, which is the rotation of $\boldsymbol{x}$ by 2θ in the positive direction of P. □

Example. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two linearly independent vectors of the Euclidean space such that $\boldsymbol{u}^2 = \boldsymbol{v}^2$. Let $R = \boldsymbol{v}(\boldsymbol{u}+\boldsymbol{v}) = (\boldsymbol{u}+\boldsymbol{v})\boldsymbol{u}$. Then $\underline{R}$ maps $\boldsymbol{u}$ to $\boldsymbol{v}$. Indeed,

$$\underline{R}(\boldsymbol{u}) = R\boldsymbol{u}R^{-1} = \boldsymbol{v}(\boldsymbol{u}+\boldsymbol{v})\boldsymbol{u}\boldsymbol{u}^{-1}(\boldsymbol{u}+\boldsymbol{v})^{-1} = \boldsymbol{v}.$$

Example. Suppose that $n = 3$. If $\boldsymbol{n}$ is the unit normal vector of the oriented plane P, the unit volume of E_3 is $\mathbf{i} = \boldsymbol{i}\boldsymbol{n}$. So $\boldsymbol{i} = \mathbf{i}\boldsymbol{n}$ and the rotation $f_{\boldsymbol{n},\alpha}$ about the axis $\boldsymbol{n}$ of amplitude α is given by the formula

$$f_{\boldsymbol{n},\alpha}(\boldsymbol{x}) = e^{-\mathbf{i}\boldsymbol{n}\alpha/2}\boldsymbol{x}e^{\mathbf{i}\boldsymbol{n}\alpha/2}.$$

Note that

$$\mathbf{i}^2 = \boldsymbol{i}\boldsymbol{n}\boldsymbol{i}\boldsymbol{n} = \boldsymbol{i}^2 = -1,$$

for $\boldsymbol{n}$ commutes with $\boldsymbol{i}$.

It is a good moment to take a bit of homework: **E7**, page 87.

We can use Euler's spinor formula as many times as we want to produce rotations of the Euclidean space E_n. In parcular we can choose area units $\boldsymbol{i}_1, \ldots, \boldsymbol{i}_k$ in pairwise orthogonal planes, and angles $\alpha_1, \ldots, \alpha_k \in [0, 2\pi)$ (not necessarily distinct), and construct the rotation f_R with

$$R = \exp(-\boldsymbol{i}_k\alpha_k/2) \cdots \exp(-\boldsymbol{i}_1\alpha_1/2).$$

Then the basic classification of Euclidean isometries insures that any element of $\mathrm{SO}(E_n)$ can be obtained in this way.

If $\boldsymbol{u}$ a unit vector orthogonal to the unit-area planes, then $-\underline{\boldsymbol{u}R}$ is a reflection (an element of $\mathrm{O}_n - \mathrm{SO}_n$), and all reflections can be obtained in this way.

Remark. Since the area units $\boldsymbol{i}_\ell$ commute, $R = e^{-F}$, where $F = (\boldsymbol{i}_1\alpha_1 + \cdots + \boldsymbol{i}_k\alpha_k)/2 \in \mathcal{G}^2$. This amounts to a simple and effective proof in the Euclidean case of a remarkable theorem or Riesz that we state and comment next.

Let $E = E_{r,s}$, $n = r + s$, and $L \in \mathrm{SO}^+_{r,s}$ (this is the connected component of the identity of $\mathrm{SO}_{r,s}$).

◇1 If (r, s) is of one of the forms $(n, 0)$, $(0, n)$, $(1, n-1)$ or $(n-1, 1)$, there exists a bivector $F \in \mathcal{G}^2$ such that

$$Lx = e^{-F} \boldsymbol{x} e^F. \tag{24}$$

This result is false for any other signature.

◇2 If F' is another bivector such that $e^{-F'} \boldsymbol{x} e^{F'} = e^{-F} \boldsymbol{x} e^F$ for all vectors x, then $e^{F'} = \pm e^F$.

□ See Riesz-1958, §4.12. We will delve into the proof and significance of this result in tomorrow's lecture. Here let us just notice that it is not effective in the sense that it does not provide clues about how to relate the specific geometric properties of L to the algebraic properties of F. Note also that these 'specifics' are dealt with in detail in other lectures for signatures such as $(1, 3)$ (STA) and $(4, 1)$ (CGA). □

Duality

Let $\mathbf{e} = \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be an orthonormal basis of $E_{r,s}$ and define

$$\boldsymbol{i}_{\mathbf{e}} = \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n \in \mathcal{G}^n.$$

We will say that it is the *pseudoscalar* (or also *chiral element*) associated to $\mathbf{e}$.

Note that by the metric formula we have:

$$q(\boldsymbol{i}_{\mathbf{e}}) = q(\boldsymbol{e}_1) \cdots q(\boldsymbol{e}_n) = (-1)^s.$$

If $\mathbf{e}' = \boldsymbol{e}'_1, \ldots, \boldsymbol{e}'_n$ is another orthonormal basis of E, then

$$\boldsymbol{i}_{\mathbf{e}'} = d\,\boldsymbol{i}_{\mathbf{e}},$$

where $d = \det_{\mathbf{e}}(\mathbf{e}')$ is the determinant of the matrix of the vectors $\mathbf{e}'$ with respect to the basis $\mathbf{e}$. Taking the metric norm, we conclude that $d^2 = 1$ and hence $d = \pm 1$. This means that the there is a unique pseudocalar, up to sign. The distinction of one of the pseudoscalars is equivalent to choose an orientation of the space.

Let $\boldsymbol{i} \in \mathcal{G}^n$ be a pseudoscalar. Then we have:

◊1 $\boldsymbol{i} \in \mathcal{G}^\times$, $\boldsymbol{i}^{-1} = (-1)^s \boldsymbol{i}^\tau = (-1)^s(-1)^{n//2}\boldsymbol{i}$, $\boldsymbol{i}^2 = (-1)^{n//2}(-1)^s$.

◊2 (*Hodge duality*) For any $x \in \mathcal{G}^r$, we have $\boldsymbol{i}x, x\boldsymbol{i} \in \mathcal{G}^{n-r}$ and the maps $x \mapsto \boldsymbol{i}x$ and $x \mapsto x\boldsymbol{i}$ are linear isomorphisms $\mathcal{G}^r \to \mathcal{G}^{n-r}$. The inverse maps are $x \mapsto \boldsymbol{i}^{-1}x$ and $x \mapsto x\boldsymbol{i}^{-1}$, respectively.

◊3 If n is odd, $\boldsymbol{i}$ commutes with all elements of $\mathcal{G}$. This is also expressed by saying that $\boldsymbol{i}$ belongs to the *center* of $\mathcal{G}$.

◊4 If n is even, $\boldsymbol{i}$ commutes with even multivectors and anticommutes with odd multivectors.

◊5 If $q(\boldsymbol{i}) = 1$, then the Hodge duality are *isometries*. If $q(\boldsymbol{i}) = -1$, they are *antiisometries*.

If you have not seen it, it is enlightening to work out the case $n = 3$ in detail. See **E10**, page 90.

□1 Since $(-1)^s = q(\boldsymbol{i}) = \boldsymbol{i}^\tau \boldsymbol{i}$, we see that $\boldsymbol{i} \in \mathcal{G}^\times$ and that $\boldsymbol{i}^{-1}$ is given by the stated formula. The value of $\boldsymbol{i}^2$ follows readily from this.

□2 Since $\boldsymbol{i} = \boldsymbol{e}_N$, for any multiindex J of order r we conclude that $\boldsymbol{e}_J \boldsymbol{i}, \boldsymbol{i}\boldsymbol{e}_J \in \mathcal{G}^{n-r}$ using Artin's formula.

□3&4 We can use the formula in the second example on page 34:

$$\boldsymbol{e}_j \boldsymbol{i} = \boldsymbol{e}_j \boldsymbol{e}_N = (-1)^{n+1} \boldsymbol{e}_N \boldsymbol{e}_j = (-1)^{n+1} \boldsymbol{i}\boldsymbol{e}_j,$$

so $\boldsymbol{i}$ commutes (anticommutes) with all vectors for odd n (for n even).

□5 Let us compute $q(x\boldsymbol{i}, y\boldsymbol{i})$, for $x, y \in \mathcal{G}^r$, using the alternative definition of the metric:

$$q(x\boldsymbol{i}, y\boldsymbol{i}) = ((x\boldsymbol{i})(y\boldsymbol{i})^\tau)_0 = (x\boldsymbol{i}\,\boldsymbol{i}^\tau y^\tau)_0 = (xq(\boldsymbol{i})y^\tau)_0 = q(\boldsymbol{i})q(x, y).$$

That $q(\boldsymbol{i}x, \boldsymbol{i}y) = q(\boldsymbol{i})q(x, y)$ is proved similarly, using that

$$(\boldsymbol{i}x)^\tau \boldsymbol{i}y = x^\tau \boldsymbol{i}^\tau \boldsymbol{i}y = x^\tau q(\boldsymbol{i})y = q(\boldsymbol{i})x^\tau y. \qquad \square$$

The following table lists the value of $\boldsymbol{i}^2$ for $1 \leqslant n \leqslant 4$:

n	1		2			3				4				
r	1	0	2	1	0	3	2	1	0	4	3	2	1	0
s	0	1	0	1	2	0	1	2	3	0	1	2	3	4
$\boldsymbol{i}^2$	$+$	$-$	$-$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$-$	$+$

Notice that from the formula giving $\boldsymbol{i}^2$ it follows that its value is $(-1)^s$ if $n \equiv 0, 1 \mod 4$ and $-(-1)^s$ otherwise.

Example. Let $\boldsymbol{i}$ be a pseudoscalar and define, for any multivector x, $x^* = x\boldsymbol{i}$ (*Hodge dual of x*). If X is a non-nul r-blade, then $[X^*] = [X]^{\perp}$ (the orthogonal of $[X]$). Indeed, by taking an orthonormal basis of $[X]$, and completing it to an orthonormal basis of E with the same orientation as $\boldsymbol{i}$, we can assume that $X = \boldsymbol{e}_J$, for some J, and then $X^* = \pm\boldsymbol{e}_{J'}$, $J' = N - J$.

Appendices

The arguments that follow make up a proof of formula (12).

We will proceed by induction with respect to r. Since we already observed that the statement is true for $r = 1$, we can assume that $r > 1$ and that the formula is correct for $r - 1$ (induction hypothesis). Then, setting $X' = \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_{r-1}$ and using the recursive rules, we can write

$$\begin{aligned} X \cdot Y &= X' \cdot (\boldsymbol{x}_r \cdot Y) \\ &= X' \cdot \left(\textstyle\sum_{k=1}^{s} (-1)^{k-1} q(\boldsymbol{x}_r, \boldsymbol{y}_k)\, Y_{k'}\right) \\ &= \textstyle\sum_{k=1}^{s} (-1)^{k-1} q(\boldsymbol{x}_r, \boldsymbol{y}_k)\, X' \cdot Y_{k'}, \end{aligned}$$

with $k' = \{1, \ldots, s\} - \{k\}$. But now we have, by the induction hypothesis,

$$X' \cdot Y_{k'} = \textstyle\sum_L (-1)^{t(L,k'-L)} (X' \cdot (Y_{k'})_L) Y_{k'-L},$$

where L runs over the size $r - 1$ multiindices contained in k' (equivalent to say that L does not contain k).

Now there is a one-to-one correspondence between the set of multiindices L of order $r-1$ not containing k and the set of multiindices J of order r containing k: $L = J - \{k\}$, or $J = (\{k\} \cup L)^{\sim}$ (the reordering of $\{k\} \cup L$ in increasing order). Using this correspondence we have $(Y_{k'})_L = Y_L = Y_{J-\{k\}}$ and $k' - L = J'$ and consequently

$$X \cdot Y = \sum_{k=1}^{s} (-1)^{k-1} q(\boldsymbol{x}_r, \boldsymbol{y}_k) \sum_J (-1)^{t(J-\{k\}, J')} (X' \cdot Y_{J-\{k\}}) Y_{J'}.$$

This sum can be rearranged as follows:

$$\sum_{J \in \mathcal{I}_{r,s}} \left(\sum_{k \in J} (-1)^{k-1} (-1)^{t(J-\{k\}, J')} q(\boldsymbol{x}_r, \boldsymbol{y}_k)(X' \cdot Y_{J-\{k\}}) \right) Y_{J'}. \quad (*)$$

The number of inversions $t(J-\{k\}, J')$ is equal to $t(J, J') - h$, where h is the number of inversions in the sequence (k, J'). If

$$J = j_1 < \cdots < j_{l-1} < k = j_l < j_{l+1} < \cdots < j_r,$$

it is clear that $h = (k-1) - (l-1) = k - l$ and hence that

$$t(J-\{k\},J') = t(J,J') - k + l.$$

So in the expression $(*)$ we can use

$$(-1)^{k-1}(-1)^{t(J-\{k\},J')} = (-1)^{t(J,J')}(-1)^{l-1}$$

and get

$$X\cdot Y = \sum_{J\in\mathcal{J}_{r,s}}(-1)^{t(J,J')}\left(\sum_{l=1}^{r}(-1)^{l-1}q(x_r,y_{j_l})(X'\cdot Y_{J-\{j_l\}})\right)Y_{J'}.$$

Finally,

$$\sum_{l=1}^{r}(-1)^{l-1}q(x_r,y_{j_l})(X'\cdot Y_{J-\{j_l\}}) = X\cdot Y_J,$$

by the recursive rule. $\square$

The problem is the following. If we have a special signature ($n = 2m + 1$, $m + s$ even), we want to find out whether there are folded geometric algebras $\mathcal{A}'$ ($\boldsymbol{e}_N = \epsilon$, $\epsilon = \pm 1$) of that signature and if they exist, how many non-isomorphic can we find.

Suppose $\mathcal{A}'$ is an folded geometric algebra and let $\mathcal{A}$ be the full geometric algebra of the same signature as $\mathcal{A}'$. Then there is a unique homomorphism of algebras $f : \mathcal{A} \to \mathcal{A}'$, which is onto. It follows that $\mathcal{A}' \simeq \mathcal{A}/\mathcal{I}$, where $\mathcal{I} = \ker(f)$. This ideal contains $1 - \epsilon \boldsymbol{e}_N$ (if $\boldsymbol{e}'_N = \epsilon$) and its dimension must be $\dim \mathcal{A}) - \dim(\mathcal{A}') = 2^{n-1}$.

Remark. We can assume that $\epsilon = 1$ if we use that the pseudoscalar is defined up to sign. With $\epsilon = 1$, we have some orientation, and with $\epsilon = -1$, we have the reversed orientation.

◊ Let $\mathcal{A}$ be the full geometric algebra of a special signature (r,s). Let $\boldsymbol{e}_N$ be the pseudoscalar associated to an orthonormal basis. Then the ideal $\mathcal{I}$ of $\mathcal{A}$ generated by $1-\boldsymbol{e}_N$ has dimension 2^{n-1} and the quotient algebra $\mathcal{A}'=\mathcal{A}/\mathcal{I}$ is a restricted geometric algebra of signature (r,s).

□ Since $\boldsymbol{e}_N$ commutes with any element, the ideal generated by $1-\boldsymbol{e}_N$ is linearly spanned by the elements

$$f_J=\boldsymbol{e}_J-\boldsymbol{e}_J\boldsymbol{e}_N=\boldsymbol{e}_J-(-1)^{t(J,N)}q(J)\boldsymbol{e}_{J'},$$

where $J=j_1,\ldots,j_k$ is any multiindex and $J'=N-J$. There are 2^{n-1} such elements for $k=|J|$ even, and these elements are linearly independent because the corresponding J' are odd.

The above 2^{n-1} elements form a linear basis of $\mathcal{I}$ because, as we will see now, $f_{J'}=\pm f_J$. Indeed, applying the formula to odd J', we get the element

$$\begin{aligned} f_{J'} &= \boldsymbol{e}_{J'} - \boldsymbol{e}_{J'}\boldsymbol{e}_N = \boldsymbol{e}_{J'} - (-1)^{t(J',N)} q(J')\boldsymbol{e}_J \\ &= -(-1)^{t(J',N)} q(J') \left(\boldsymbol{e}_J - (-1)^{t(J',N)} q(J')\boldsymbol{e}_{J'} \right). \end{aligned}$$

Now we will establish the equality of signs

$$(-1)^{t(J',N)} q(J') = (-1)^{t(J,N)} q(J),$$

or, equivalently, that

$$(-1)^{t(J,N)+t(J',N)} q(J)q(J') = 1.$$

But this follows from $q(J)q(J') = q(N) = (-1)^s$,

$$(-1)^{t(J,N)+t(J',N)} = (-1)^{\Sigma J - k + \Sigma J' - (n-k)} = (-1)^{(n+1)//2+n} = (-1)^m$$

and the fact that $m + s$ is even by hypothesis.

◊ The construction above using the orientations $\boldsymbol{e}_N$ and $-\boldsymbol{e}_N$ leads to isomorphic algebras.

□ Indeed, the map defined by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n \mapsto -\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ is an isometry that extends to an automorphism of the full algebra, and this automorphism maps the ideal generated by $1 - \boldsymbol{e}_N$ to the ideal generated by $1 + \boldsymbol{e}_N$, thus yielding an isomorphism of the quotient algebras. □

Exercises

E1. Construct a bivector of E_4 that is not a blade.

E2. The formula (7) is true for $r = 1$, for the 1-volume of x_1 is $|x_1|$ and $|x_1|^2 = q(x_1, x_1)$. Now use induction on r to show that for $r > 1$ the formula is true if x_r is orthogonal to $\langle x_1, \ldots, x_{r-1}\rangle$. Finally show that the formula is true in general by decomposing x_r as a sum $x_r' + x_r''$ with $x_r' \in \langle x_1, \ldots, x_{r-1}\rangle$ and $x_r'' \in \langle x_1, \ldots, x_{r-1}\rangle^\perp$.

E3. Let X be an r-blade and Y and s-blade, $r \leqslant s$. Show that $X \cdot Y = 0$ if one of the factors of X is orthogonal to all the factors of Y.

E4. Given a vector $\boldsymbol{e} \in E = E_{r,s}$, let $\delta_{\boldsymbol{e}}$ be the unique antiderivation of the tensor algebra TE such that $\delta_{\boldsymbol{e}}(\boldsymbol{e}') = q(\boldsymbol{e}, \boldsymbol{e}')$ for any vector $\boldsymbol{e}'$. With the notations used in the proof of the existence of the geometric product, show that $\delta_{\boldsymbol{e}}$ vanishes on the generators of the ideal $I_q E$ and hence that $\delta_{\boldsymbol{e}} I_q E \subseteq I_q E$. Therefore $\delta_{\boldsymbol{e}}$ induces an antiderivation of $C_q E$ and *a fortiori* of the geometric product.

E5. Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r \in E_{r,s}$. Set $U = \boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_r$. Show that if $[U]$ is non-singular (this means that the restriction of q to $[U]$ is non-degenerate) then U is invertible. Is the converse true? *Hint*: First settle the case in which the $\boldsymbol{u}_j$ are pair-wise orthogonal.

E6. Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r \in E_n$ be linearly independent vectors and set $U = \boldsymbol{u}_1 \wedge \ldots \wedge \boldsymbol{u}_r$. Show that for any vector $\boldsymbol{x} \in E_n$ the expressions

$$(\boldsymbol{x} \cdot U)U^{-1} \text{ and } (\boldsymbol{x} \wedge U)U^{-1}$$

yield the *orthogonal projections* of $\boldsymbol{x}$ on $[U]$ and on $[U]^{\perp}$ (the latter is often called the *rejection* of $\boldsymbol{x}$ by $[U]$). *Hints*: Both expressions are linear in $\boldsymbol{x}$. The first vanishes for $\boldsymbol{x} \in [U]^{\perp}$ and coincides with $(\boldsymbol{x}U)U^{-1} = \boldsymbol{x}$ for $\boldsymbol{x} \in [U]$. The second vanishes for $\boldsymbol{x} \in [U]$ and coincides with $(\boldsymbol{x}U)U^{-1} = \boldsymbol{x}$ for $\boldsymbol{x} \in [U]^{\perp}$.

E7. *Olinde Rodrigues formulas*. Let $\boldsymbol{n}, \boldsymbol{n}' \in E_3$ be unit vectors and $\alpha, \alpha' \in \mathbf{R}$. Show that the amplitude α'' and the axis $\boldsymbol{n}''$ of the composition $f_{\boldsymbol{n}',\alpha'} f_{\boldsymbol{n},\alpha}$ is given by the formulas:

$$\cos\tfrac{\alpha''}{2} = \cos\tfrac{\alpha}{2}\cos\tfrac{\alpha'}{2} - (\boldsymbol{n}\cdot\boldsymbol{n}')\sin\tfrac{\alpha}{2}\sin\tfrac{\alpha'}{2}$$

$$\boldsymbol{n}''\sin\tfrac{\alpha''}{2} = \boldsymbol{n}\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha'}{2} + \boldsymbol{n}'\cos\tfrac{\alpha}{2}\sin\tfrac{\alpha'}{2} - (\boldsymbol{n}\times\boldsymbol{n}')\sin\tfrac{\alpha}{2}\sin\tfrac{\alpha'}{2}.$$

E8. Let $\mathcal{G}_2$ be the geometric algebra of the Euclidean plane E_2 and $\bar{\mathcal{G}}_2$ of the *anti-Euclidean* plane $E_{\bar{2}}$ (its metric is $\bar{q} = -q$, q the metric of E_2).

Let $\boldsymbol{e}_1, \boldsymbol{e}_2$ be an *orthonormal basis* E_2. The corresponding linear basis of $\mathcal{G}_2$ (and $\bar{\mathcal{G}}_2$) is $1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_{12} = \boldsymbol{i}$ (the *unit area*). The tables for the geometric product, however, are quite different:

$\mathcal{G}_2$	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{i}$
$\boldsymbol{e}_1$	1	$\boldsymbol{i}$	$\boldsymbol{e}_2$
$\boldsymbol{e}_2$	$-\boldsymbol{i}$	1	$-\boldsymbol{e}_1$
$\boldsymbol{i}$	$-\boldsymbol{e}_2$	$\boldsymbol{e}_1$	-1

$\bar{\mathcal{G}}_2$	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{i}$
$\boldsymbol{e}_1$	-1	$\boldsymbol{i}$	$-\boldsymbol{e}_2$
$\boldsymbol{e}_2$	$-\boldsymbol{i}$	-1	$\boldsymbol{e}_1$
$\boldsymbol{i}$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$	-1

Hint. For the computation of product tables, use the formulas introduced in the examples on page 34.

E9. $\mathcal{G}_3$ is the geometric algebra of the Euclidean space E_3 (*Pauli algebra*). Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ be an *orthonormal basis* and $\boldsymbol{i} = \boldsymbol{e}_{123} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$ (*unit volume*). Note that $\boldsymbol{ie}_1 = \boldsymbol{e}_2\boldsymbol{e}_3$, $\boldsymbol{ie}_2 = \boldsymbol{e}_3\boldsymbol{e}_1$, $\boldsymbol{ie}_3 = \boldsymbol{e}_1\boldsymbol{e}_2$ is a basis of $\mathcal{G}^2$. The multiplication table of the geometric product using this basis is as follows:

$\mathcal{G}_3$	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$	$\boldsymbol{ie}_1$	$\boldsymbol{ie}_2$	$\boldsymbol{ie}_3$	$\boldsymbol{i}$
$\boldsymbol{e}_1$	1	$\boldsymbol{ie}_3$	$-\boldsymbol{ie}_2$	$\boldsymbol{i}$	$-\boldsymbol{e}_3$	$\boldsymbol{e}_2$	$\boldsymbol{ie}_1$
$\boldsymbol{e}_2$	$-\boldsymbol{ie}_3$	1	$\boldsymbol{ie}_1$	$\boldsymbol{e}_3$	$\boldsymbol{i}$	$-\boldsymbol{e}_1$	$\boldsymbol{ie}_2$
e_3	$\boldsymbol{i}e_2$	$-\boldsymbol{i}e_1$	1	$-e_2$	e_1	$\boldsymbol{i}$	$\boldsymbol{i}e_3$
$\boldsymbol{ie}_1$	$\boldsymbol{i}$	$-\boldsymbol{e}_3$	$\boldsymbol{e}_2$	-1	$-\boldsymbol{ie}_3$	$\boldsymbol{ie}_2$	$-\boldsymbol{e}_1$
$\boldsymbol{ie}_2$	$\boldsymbol{e}_3$	$\boldsymbol{i}$	$-\boldsymbol{e}_1$	$\boldsymbol{ie}_3$	-1	$-\boldsymbol{ie}_1$	$-\boldsymbol{e}_2$
$\boldsymbol{ie}_3$	$-\boldsymbol{e}_2$	$\boldsymbol{e}_1$	$\boldsymbol{i}$	$-\boldsymbol{ie}_2$	$\boldsymbol{ie}_1$	-1	$-\boldsymbol{e}_3$
$\boldsymbol{i}$	$\boldsymbol{ie}_1$	$\boldsymbol{ie}_2$	$\boldsymbol{ie}_3$	$-\boldsymbol{e}_1$	$-\boldsymbol{e}_2$	$-\boldsymbol{e}_3$	-1

We see that $\langle 1, \boldsymbol{i}\rangle \simeq \mathbf{C}$ is the center of $\mathcal{G}_3$. We also see that the even subalgebra $\mathcal{G}^+ = \langle 1, \boldsymbol{ie}_1, \boldsymbol{ie}_2, \boldsymbol{ie}_3\rangle$ is isomorphic to the quaternion field $\mathbf{H} = \langle 1, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}\rangle$, via the linear map given by

$$1, \boldsymbol{ie}_1, \boldsymbol{ie}_2, \boldsymbol{ie}_3 \mapsto 1, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}.$$

E10. With the same notations as in **E9**, and using what we have learned in the section *Playing with the pseudoscalar* (page 73), we get that $\boldsymbol{i}^2 = -1$, that $\boldsymbol{i}$ commutes with any element of $\mathcal{G}$, that the map $E_3 = \mathcal{G}^1 \to \mathcal{G}^2$, $\boldsymbol{x} \mapsto \boldsymbol{ix} = \boldsymbol{xi}$, is an isometry. These are particular features of 3D and can be proved directly with no difficulty.

For any vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, show that the vector $-\boldsymbol{i}(\boldsymbol{x} \wedge \boldsymbol{y})$ is equal to the *cross product* $\boldsymbol{x} \times \boldsymbol{y}$: *Hint*: If jkl is a cyclic permutation of 123, then $-\boldsymbol{i}(\boldsymbol{e}_j \wedge \boldsymbol{e}_k) = \boldsymbol{e}_l = \boldsymbol{e}_j \times \boldsymbol{e}_k$.

Show that $\boldsymbol{x} \times \boldsymbol{y} = -(\boldsymbol{ix}) \cdot \boldsymbol{y} = \boldsymbol{y} \cdot (\boldsymbol{ix})$. *Hint*: The second equality is clear and $\boldsymbol{y} \cdot (\boldsymbol{ix}) = (\boldsymbol{y} \cdot \boldsymbol{i})\boldsymbol{x} - \boldsymbol{i}(\boldsymbol{y} \cdot \boldsymbol{x}) = \boldsymbol{iyx} - \boldsymbol{i}(\boldsymbol{y} \cdot \boldsymbol{x}) = \boldsymbol{i}(\boldsymbol{y} \wedge \boldsymbol{x})$, for $\boldsymbol{y} \cdot \boldsymbol{i} = \boldsymbol{yi} - \boldsymbol{y} \wedge \boldsymbol{i} = \boldsymbol{yi} = \boldsymbol{iy}$.

If $\boldsymbol{z}$ is a third vector, $(\boldsymbol{x} \times \boldsymbol{y}) \cdot \boldsymbol{z} = \det(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ (*mixed product*).

$(\boldsymbol{x} \times \boldsymbol{y}) \times \boldsymbol{z} = (\boldsymbol{x} \cdot \boldsymbol{z})\boldsymbol{y} - (\boldsymbol{y} \cdot \boldsymbol{z})\boldsymbol{x}$ (*double cross product*).

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