17th "Lluís Santaló" Research School

Algebra and geometry in current curricula

S. Xambó

RSME-UIMP

22-26 August, 2016

Algebra and Geometry in current curricula

- **Ll. Santaló**. Pictures. Timeline. Other recipients of the PAP.
- **Two perspectives: Weyl & Dirac**. Physical insights of a mathematician. Mathematical insights of a theoretical physicist.
- **Parallel universes?** Some Nobel laureates in Theoretical Physics. Some Fields medalists with impact in Physics. An IAS experience.
- **Interlude: A 4-point manifesto**.
- **Some background notions**. Groups. Subgroups. Group homomorphisms. Vectors and linear operators. Algebras. Ideals. Examples of algebras. Metrics (quadratic forms). Orthogonal basis and signatures. Special signatures. Why GA? Orthogonal groups.
- **Algebra and Geometry curricula**. Spanish Conference of Deans Whitebook (2004).
- **Non-GA yardstick texts**.
- **Closing remarks**. Two souls? A royal road to research?
- **References**

Who was Lluís Santaló?

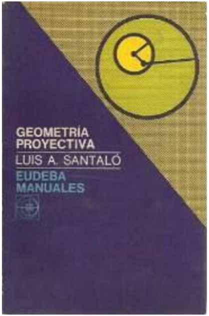
Julio Rey Pastor (1888-1962)

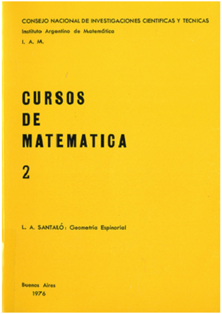


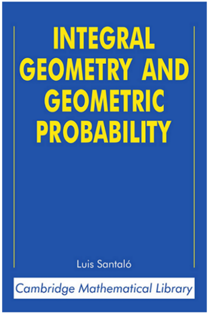
Lluís Santaló (1911-2001)

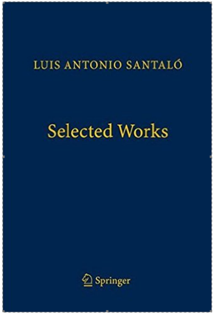


Santalo-2009 [1] (200 r, 20 b, 70 g, 40 e).
Santalo-1966 [2] (Projective Geometry)
Santalo-1976-masterpiece [3].
Santalo-1976-spinors [4] (after Morand-1973 [5]).
12 PhD students.









1911: Born in Girona (9 october). 4th of 7 siblings. Marcel, the second, was also a mathematician.

1927-1934: Univeristy of Madrid. Begins Studies in Civil Engineering. Degree in Mathematics. Teachers: **Rey Pastor**, **Esteve Terradas** (1883-1950), ... Both with part-time positions in Argentinian Universities.

1934-1936: Grant to work in Hamburg University (promoted by Julio Rey Pastor) with **Wilhelm Blaschke** (1885-1962). Colleague of S. S. Chern. PhD, presented at the University of Madrid, with Pedro Pineda Gutiérrez acting as rapporteur: *New applications of the concept of kinematical measure in the plane and in the space*.

1936-1939: Fights in the Spanish Civil War (1936-1939) in the republican air force. Attains Captain rank. Exile.

1939-1947: France (Elie Cartan). Argentina (Rey Pastor). Universidad Nacional del Litoral (Rosario). Beppo Levi. In 1945 marries Hilda Rossi.

1948-1949: **Guggenheim Foundation Prize**. **Institute for Advanced Study** and Chicago.

1949-1956: Universidad Nacional de La Plata, Argentina.

1957: Becomes Full Professor at the University of Buenos Aires.

1977: Doctor Honoris Causa, Universistat Politècnica de Catalunya. "Mathematics is **art**, in that it is creative and uses fantasy; is **science**, as through it we achieve a better knowledge of things, of their composition and causes; and it is a **technique**, because it provides methods and means to solve problems and to act on nature and its phenomena".

1983: Premio "Príncipe de Asturias" for Scientific and Technological Research.

1984: Narcís Monturiol Medal of the Catalan Government.

1986: Doctor Honoris Causa, Universistat Autònoma de Barcelona.

1990: Doctor Honoris Causa, Universisdad de Sevilla.

1994: Sant Jordi Cross of the Catalan Government.

2000: Honorary member of the Catalan Mathematical Society and of the Spanish Mathematical Society.

2001: 17 October, multitudinous tribute by the "Sociedad Argentina de Educación Matemática". November 22: dies in Buenos Aires (aged 90).

2011: Inauguration of the **Instituto de Investigaciones Matemáticas "Luis A. Santaló"** (IMAS-UBA).

Prince Felipe (Now Felipe VI) awarding the "Príncipe de Asturias Prize".

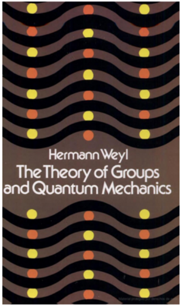
1983: Lluís Santaló (Integral geometry).

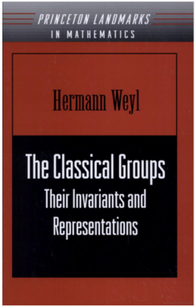
1993: Amable Liñán (Combustion).

2006: Juan I. Cirac (Quantum information).

2013: Peter Higgs, François Englert, and CERN.

Two perspectives: Weyl & Dirac



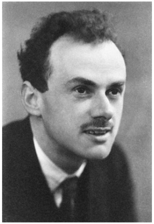


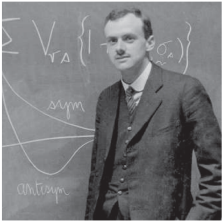
Herrmann Weyl (1885-1955)

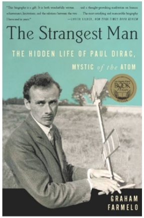
1918. *Raum, Zeit, Materie*.

1928. *Gruppentheorie und Quantenmechanik* (English: 1931)

1938: *The Classical Groups* (key aspects of the theory were developed as early as 1925).







1926: PhD (Cambridge, supervised by Ralph Fowler, "first thesis on quantum mechanics to be submitted anywhere". Postdoc at Copenhagen (Bohr) and Göttingen (Hilbert, Weyl).

1928: Dirac's equation: $i\gamma \cdot \partial\psi = m\psi$ (*Quantum elecdrodynamics)*.

1930: *Principles of Quantum Mechanics*.

1932: Lucasian Professor of Mathematics.

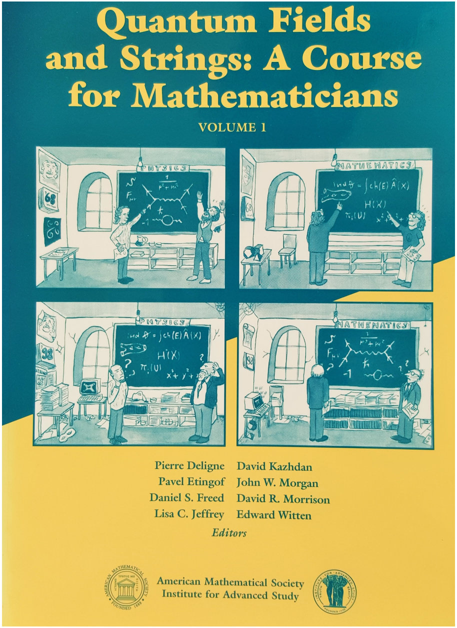
1933: Nobel Prize (shared with Erwin Schrödinger).

Paul A. M. Dirac (1902-1984) Dirac-1931 [6]:

[...] the modern physical developments have **required a mathematics that continually shifts its foundations and gets more abstract**. Non-euclidean geometry and non-commutative algebra, [...] have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation. [...] The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities.

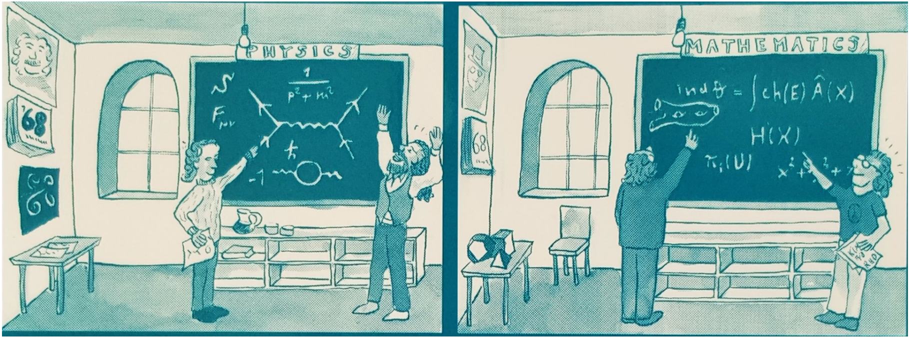
1918 M. Planck (1858-1947). **1921** Albert Einstein (1879-1955). **1922** Niels Bohr (1885-1962). **1932** Werner Heissenberg (1901-1976). **1933** Erwin Schrödinger (1887-1961) & Dirac. **1945** Wolfgang Pauli (1900-1958). **1949** Hideyuki Yukawa (1907-1981). 1963 Eugen Wigner (1902-1995). **1965** Richard Feynman (1918-1988), Julian Schwinger (1918-1994), Sin-Itiro Tomonaga (1906-1979). **1969** Murray Gell-Mann (1929). **1979** S. Glashow (1932), A. Salam (1926-1996), S. Weinberg (1933). **1999** G. T'Hooft (1946), M. Veltman (1931).

1950 L. Schwartz (1915-2002) **1954** J.-P. Serre (1926) **1966** M. Atiyah (1929) **1974** D. Mumford (1937) **1982** A. Connes (1947) **1986** S.-T. Yau (1949) **1986** S. Donaldson (1957) **1990** V. Jones (1952) **1990** E. Witten (1951) **1994** J. Bourgain (1954) **1998** M. Kontsevich (1964) **2014** A. Avila (1979)


Quantum Fields and Strings: A Course for Mathematicians
VOLUME 1
PHYSICS
MATHEMATICS
PHYSICS
MATHEMATICS
Pierre Deligne
Pavel Etingof
Daniel S. Freed
Lisa C. Jeffrey
David Kazhdan
John W. Morgan
David R. Morrison
Edward Witten
Editors
American Mathematical Society
Institute for Advanced Study

1968

Physics office and Mathematics office



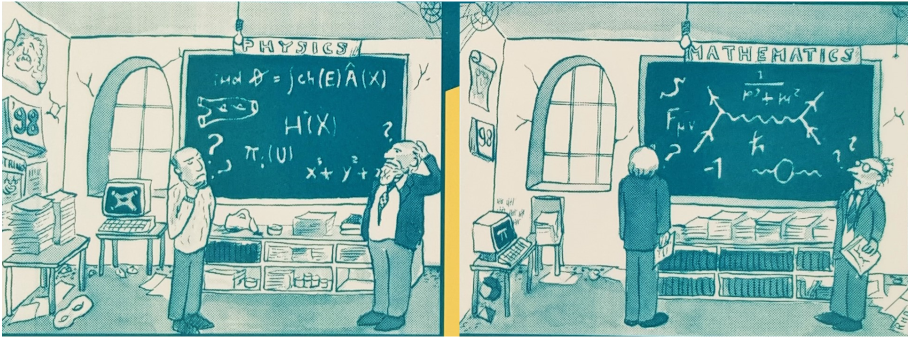
Feynman-Schwinger-Tomonaga (Nobel Prize 1965)
Atiyah-Singer index theorem (1962), Fields Medal 1966 (Atiyah), Abel Prize 2004.

Tomonaga-Schwinger-Feynman

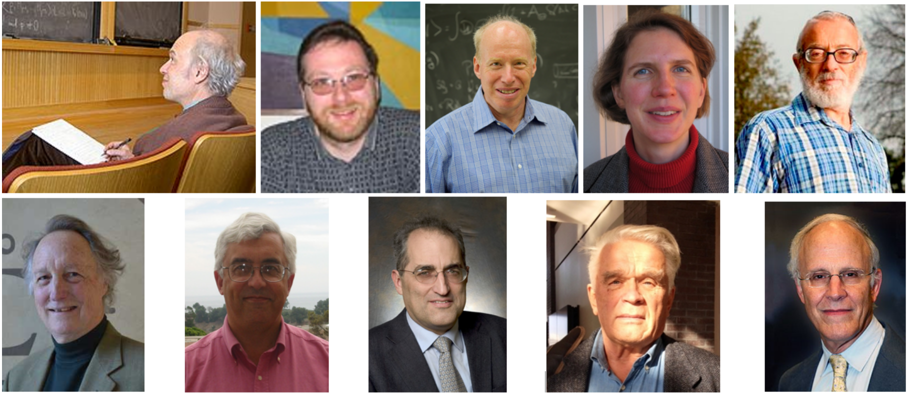
Atiyah-Singer

1998

Physics office and Mathematics office



That metamorphosis happened in Enumerative Geometry: Mirror symmetry (ca. 1990), Manin-Kontsevich (1994), ... (cf. Xambo-2014-sas [7]). Also in low dimensional topology (Solution of the Poincaré conjecture, ca. 2003)



1 Pierre Delighe (IAS). **2** Pavel Etingof (MIT). **3** Daniel S. Freed (UT Austin). **4** Lisa C. Jeffrey (Toronto). **5** David Kazhdan (HUJ, Harvard). **6** John W. Morgan (Simons Center for Geometry and Physics). **7** David Morrison (UCSB). **8** Ed Witten (IAS). **9** Ludwig Fadeev (S. Petersburg). **10** David Gross (UCSB; Nobel 2004, shared with H. D. Plotizer and F. Wilczek –asymptotic freedom of s. f.).

Volume 1

Part 1: Classical fields and supersymmetry

Part 2: Formal aspects of QFT

Volume 2

Part 3: Conformal Field Theory and Strings

Part 4: Dynamical Aspects of QFT

Interlude:
A 4-point manifesto

- Constructive dialogs between Mathematics and Physics are possible and beneficial for both sciences.
- To that purpose, and in terms of the ratio (scope) / (learning effort), the most productive language is Geometric Algebra.
- State of the art engineering applications are being created with GA as the main conceptual frame.
- This frame has the potential to transform the mathematics / physics / computer science / engineering curricula into programs that overcome the fragmented presentation of knowledge in small units that have little connection to the others, and which are altogether quite insufficient for the development of adequate professional competences.

I am proud to also declare that our invited speakers will show us how best to flesh and enliven those points in the lectures ahead.

Background on some mathematical notions

- Groups. Subgroups. Group homomorphisms.
- Vectors and linear operators.
- Algebras. Ideals. Examples of algebras.
- Metrics (quadratic forms). Orthogonal basis and signature of a metric. Special signatures.
- Orthogonal groups.

A *group* is a set G endowed with an operation $G \times G \to G$, $(x, y) \mapsto xy$ such that

- $(xy)z = x(yz)$ for all $x, y, z \in G$ (*associativity*).
- There is $1_G \in G$ such that $1_G x = x 1_G = x$ for all $x \in G$ (*unit* or *neutral* element, also denoted $e = e_G$; it is unique).
- For each $x \in G$ there exists $x' \in G$ such that $xx' = x'x = 1_G$. Given x, this element x' is unique, is denoted x^{-1} and is called the *inverse* of x. Since $(xy)(y^{-1}x^{-1}) = 1_G$, we have $(xy)^{-1} = y^{-1}x^{-1}$.

For specific groups, the operation, the unit and the inverse are denoted in various ways.

For example, if $xy = yx$ for all $x, y \in G$, in which case we say that G is *commutative* or *abelian*, the operation is often denoted as $x + y$, the unit as 0 (*zero*) and the inverse as $-x$ (*opposite*).

Example. The set S_n of *permutations* $[j_1, \ldots, j_n]$ of $\{1, \ldots, n\}$, with the composition operation $s \circ t$, is a group for all integers $n \geqslant 0$. Recall that $(s \circ t)(j) = s(t(j))$ for all $j \in \{1, \ldots, n\}$. For $n \geqslant 3$, S_n is not abelian. The unit is the identity permutation and the inverse of s is the *reciprocal* permutation s^{-1} (it maps $k \mapsto j$ if s maps $j \to k$).

$(\mathbf{R}, +)$ is a basic example of abelian group.

Also $(\mathbf{R}^{\times}, \cdot)$, where $\mathbf{R}^{\times} = \mathbf{R} - \{0\}$, but in this case the unit is 1 and $\alpha^{-1} = 1/\alpha$ for any non-zero real number α.

A subset H of a group G is said to be a *subgroup* if $1_G \in H$, $xy \in H$ for any $x, y \in H$ (H is closed with the group operation), and $x^{-1} \in H$ for any $x \in H$.

Examples: $(\mathbf{Z}, +)$ and $(\mathbf{Q}, +)$ are subgroups of $(\mathbf{R}, +)$. And $(\mathbf{Q}^{\times}, \cdot)$ is a subgroup of $(\mathbf{R}^{\times}, \cdot)$, where $\mathbf{Q}^{\times} = \mathbf{Q} - \{0\}$. With $\mathbf{Z}^{\times} = \{\pm 1\}$, $(\mathbf{Z}^{\times}, \cdot)$ is also a subgroup of $(\mathbf{R}^{\times}, \cdot)$.

Example. For a permutation $s \in S_n$, let $i(s)$ be the number of its *inversions*, that is, the number of pairs $j < k$ such that $s(j) > s(k)$ (*inverted pairs for s*). We say that s is *odd* (*even*) if $i(s)$ is odd (even). Since $i(s \circ t) = i(s) + i(t) \mod 2$, the set A_n of even permutations is a subgroup of S_n. For instance,

$$A_3 = \{[1,2,3], [2,3,1], [3,1,2]\} \text{ (\textit{cyclic permutations})},$$

while $[2,1,3], [1,3,2], [3,2,1]$ are the odd permutations (in this case they are *transpositions*: $[2,1,3] = (1,2)$, $[1,3,2] = (2,3)$, $[3,2,1] = (1,3)$, where (j,k) denotes the transposition of j and k.

Given groups G and G', and a map $f : G \to G'$, we say that f is a (group) *homomorphism* if $f(xy) = f(x)f(y)$ for all $x, y \in G$. The relations $f(1_G) = 1_{G'}$ and $f(x^{-1}) = f(x)^{-1}$ are a consequence of the definition.

The *kernel* of f, denoted $\ker(f)$, is the set $\{x \in G \mid f(x) = 1_{G'}\}$. It is a subgroup of G.

This subgroup is *normal*, or *invariant*, because for any $x \in \ker(f)$ and any $y \in G$, we have $yxy^{-1} \in \ker(f)$:

$$f(yxy^{-1}) = f(y)f(x)f(y^{-1}) = f(y)1_{G'}f(y)^{-1} = 1_{G'}.$$

The *quotient* of a group G by a normal subgroup H, denoted G/H, is a group endowed with a surjective homomorphism $\pi : G \to G/H$ (called *canonical projection*) such that $\ker(\pi) = H$ and with the following *universal property*: for any homomorphism $f : G \to G'$ such that $H \subseteq \ker(f)$ there is a unique homomorphism $\bar{f} : G/H \to G'$ such that $f = \bar{f} \circ \pi$ (which means that $\bar{f}(\pi(x)) = f(x)$ for all $x \in G$).

R will denote the field of *real numbers*. Its elements will be called *scalars* and will be denoted by Greek letters $(\alpha, \lambda, \rho, \omega, \dots)$.

By a *vector space* we mean an **R**-vector space E of finite dimension n (unless said otherwise explicitly). Its elements will be called *vectors* and will be denoted by latin boldface italic letters $(\boldsymbol{e}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}, \boldsymbol{y} \dots)$.

A typical basis of E will be denoted $\mathbf{e} = \boldsymbol{e}_1, \dots, \boldsymbol{e}_n$.

If f is an *endomorphism* of E (also called a *linear operator*), its matrix with respect to $\mathbf{e}$ is the matrix M defined by the relation

$$f(\mathbf{e}) = \mathbf{e}M. \tag{1}$$

The group of linear automorphisms of E will be denoted $\mathrm{GL}(E)$. If f is a linear automorphism, then its matrix M is invertible ($\Leftrightarrow \det(M) \neq 0$), so that $M \in \mathrm{GL}_n$. The map $GL(E) \to \mathrm{GL}_n$, $f \mapsto M$, is a group isomorphism.

In these lectures, by an *algebra* we understand a non-zero vector space A (that may have infinite dimension) endowed with a bilinear map $A \times A \to A$, $(x, y) \mapsto x * y$.

Unless we say otherwise explicitely, we will also assume that it is *associative* ($(x * y) * z = x * (y * z)$) and *unital* (there is $1_A \in A$ such that $1_A \neq 0_A$ and $1_A * x = x * 1_A = x$ for all x).

Via the map $\mathbf{R} \to A$, $\rho \mapsto \rho 1_A$, we identify $\mathbf{R}$ with its image in A, which means that we regard ρ and $\rho 1_A$ as equal.

If A and B are algebras, a map $f : A \to B$ is an *homomorphism of algebras* if it is linear, $f(aa') = f(a)f(a')$ for all $a, a' \in A$ and $f(1_A) = 1_B$.

A subset B of an algebra A is said to be a *subalgebra* if it is a vector subspace such that $B * B \subseteq B$ and $1_A \in B$.

The group of invertible elements of an algebra A will be denoted $A^{\times}$.

A subset I of an algebra A is a *left ideal* if is an additive subgroup and $A * I \subseteq I$.

If $I * A \subseteq I$ we say that it is a *right ideal*.

A left and right ideal is said to be a *bilateral ideal* (or simply an *ideal*).

The *kernel* of a homomomorphism f, $\ker(f) = \{a \in A \,|\, f(a) = 0\}$ is an ideal of A.

With an ideal I of A we can form the *quotient algebra* A/I. There is a *canonical projection* $\pi : A \to A/I$, wich is surjective, $\ker(\pi) = I$, and satisfying the following *universal property*: If $f : A \to B$ is a homomorphism of algebras such that $f(I) = \{0\}$, then there exists unique homomorphism $\bar{f} : A/I \to B$ such that $f = \bar{f}\pi$. Thus $\bar{f}(\bar{a}) = f(a)$ for all $a \in A$.

♢ End(E). The vector space of *endomorphisms* of E has an algebra structure given by the composition of endomorphisms.

The unit is the *identity* endomorphism I (or Id).

It has dimension n^2 if E has dimension n. A basis of E provides an isomorphism of End(E) with $\mathbf{R}(n)$, the algebra of square matrices with the usual matrix product and with unit the identity matrix I_n: it is the map $f \mapsto A$ as in formula (1), page 31.

♢ *TE*. The *tensor algebra* associated to E, $(TE, \otimes)$, is the direct sum of the *tensor powers* T^kE of E $(k \geqslant 0)$,

$$TE = \bigoplus_{k \geqslant 0} T^kE = \mathbf{R} \oplus E \oplus T^2E \oplus \cdots$$

with the tensor product multiplication, $\otimes$. It is *graded algebra*, which means that $x \otimes x' \in T^{k+k'}E$ when $x \in T^kE$ and $x' \in T^{k'}E$

If $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ is a basis of E, the n^r products $\boldsymbol{e}_{j_1} \otimes \cdots \otimes \boldsymbol{e}_{j_r}$ $(j_1, \ldots, j_r \in \{1, \ldots, n\})$ form a basis of T^kE. In particular, $\dim T^kE = n^k$. Hence $\dim TE = \infty$.

♢ *Universal property*. If $f : E \to E'$ is a linear map, there is a unique algebra homomorphism $f^{\otimes} : TE \to TE'$ such that $f^{\otimes}(\boldsymbol{e}) = f(\boldsymbol{e})$ for all $\boldsymbol{e} \in E$.

◊ $\bigwedge E$. The *exterior algebra* associated to E, $(\bigwedge E, \wedge)$, or *Grassmann algebra*, is the direct sum of the *exterior powers* $\bigwedge^k E$ of E $(0 \leqslant k \leqslant n)$,

$$\bigwedge E = \oplus_{k=0}^{n} \bigwedge^k E = \mathbf{R} \oplus E \oplus \bigwedge^2 E \oplus \cdots \oplus \bigwedge^n E$$

with the exterior product multiplication, $\wedge$. It is *graded algebra*, which means that $x \wedge x' \in \bigwedge^{k+k'} E$ when $x \in \bigwedge^k E$ and $x' \in \bigwedge^{k'} E$, with the convention that $\bigwedge^r E = \{0\}$ for $r > n$. The exterior product is *skewcommutative* (or *supercommutative*): for $x \in \bigwedge^k E$ and $x' \in \bigwedge^{k'} E$,

$$x \wedge x' = (-1)^{kk'} x' \wedge x.$$

If $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ is a basis of E, the $\binom{n}{k}$ products $\boldsymbol{e}_{j_1} \wedge \cdots \wedge \boldsymbol{e}_{j_r}$ $(1 \leqslant j_1 < \ldots < j_r \leqslant n)$ form a basis of $\bigwedge^k E$. In particular, $\dim \bigwedge^k E = \binom{n}{k}$. Hence $\dim \bigwedge E = 2^n$.

Our space E will be endowed with a bilinear map $q : E \times E \to \mathbf{R}$ (the *metric*).

Instead of $q(\boldsymbol{e}, \boldsymbol{e})$ we will simply write $q(\boldsymbol{e})$. The function $q(\boldsymbol{e})$ is the *quadratic form* associated to q. It determines q by the *polarization relation*

$$2q(\boldsymbol{e}, \boldsymbol{e}') = q(\boldsymbol{e} + \boldsymbol{e}') - q(\boldsymbol{e}) - q(\boldsymbol{e}'). \quad (2)$$

Two vectors $\boldsymbol{e}$ and $\boldsymbol{e}'$ are said to be *q-orthogonal* when $q(\boldsymbol{e}, \boldsymbol{e}') = 0$.

Two sets of vectors F and F' are said to be q-orthogonal if $q(\boldsymbol{e}, \boldsymbol{e}') = 0$ for all $\boldsymbol{e} \in F$ and $\boldsymbol{e}' \in F'$.

If F is a set of vectors, $F^{\perp}$ denotes the set of vectors that are q-orthogonal to F. The bilinearity of q implies that $F^{\perp}$ is vector subspace of E.

Henceforth the metric will be assumed to be *non-degenerate*, which means that $E^{\perp} = \{0\}$. In this case $\dim F^{\perp} = n - \dim F$ for all F.

A basis $\mathbf{e} = \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is said to be *q-orthogonal* if $q(\boldsymbol{e}_j, \boldsymbol{e}_k) = 0$ for all $j \neq k$.

◊1 There exist q-orthogonal bases.

Indeed, if all vectors are *isotropic* (that is, $q(\boldsymbol{e}) = 0$ for all $\boldsymbol{e} \in E$), then $q \equiv 0$, by (2), and any basis is orthogonal. Otherwise, pick a non-isotropic $\boldsymbol{e}_1 \in E$ and use induction on $\boldsymbol{e}_1^{\perp}$.

◊2 If $\mathbf{e}$ is orthogonal, and r (s) denotes the number of j such that $q(\boldsymbol{e}_j) > 0$ ($q(\boldsymbol{e}_j) < 0$), we say that (r, s) is the *signature* of q. Notice that $r + s = n$, as there is no j such that $q(e_j) = 0$.

The signature does not depend on the orthogonal basis used to compute it (*Sylvester's law of inertia*).

If $\mathbf{e}$ is an orthogonal basis, we say that it is *orthonormal* if $q(\boldsymbol{e}_j) = 1$ for $j = 1, \ldots, r$ and $q(\boldsymbol{e}_j) = -1$ for $j = r + 1, \ldots, r + s = n$.

We will write $E_{r,s}$ to indicate that the metric q has signature r, s.

Instead of $E_{n,0}$ we will simply write E_n (*Euclidean space*). In this case the metric is *positive definite* ($q(\boldsymbol{e}) > 0$ for all non-zero $\boldsymbol{e}$) and we will use the standard notions:

- $|\boldsymbol{e}| = +\sqrt{q(\boldsymbol{e})}$ (*norm* or *length* of $\boldsymbol{e}$).
- $\theta = \theta(\boldsymbol{e}, \boldsymbol{e}') = \operatorname{acos}\frac{q(\boldsymbol{e},\boldsymbol{e}')}{|\boldsymbol{e}||\boldsymbol{e}'|}$. This is the *angle* formed by $\boldsymbol{e}$ and $\boldsymbol{e}'$ ($\theta \in [0, \pi]$). We have $\theta = 0$ ($\theta = \pi$) if and only if $\boldsymbol{e}' = \lambda\boldsymbol{e}$ with $\lambda > 0$ ($\lambda < 0$).

The cases $n = 2$ and $n = 3$ are used in plane and space Euclidean geometry. The space $E_{1,3}$ is the ground structure for special relativity (*Minkowski space*) and $E_{4,1}$ for the conformal group of the 3d space.

The spectacular success of the investigation of such structures with GA (including *electromagnetism* and the *Dirac equation* in Minkowski space), is one of the most compelling reasons to embrace this formalism. Let us just mention two crowning jewels: *Space-Time Alebra* (STA) and *conformal geometric algebra* (CGA).

Hestenes-1966 [8]

Doran-Lasenby-2003 [9]

Dorst-Fontijne-Mann-2007 [10]

Dorst-JLasenby-2011 [11]

Dorst-Doran-JLasenby-2002 [12]

A linear automorphism f of E is said to be a *q-isometry* if $q(f\boldsymbol{e}, f\boldsymbol{e}') = q(\boldsymbol{e}, \boldsymbol{e}')$.

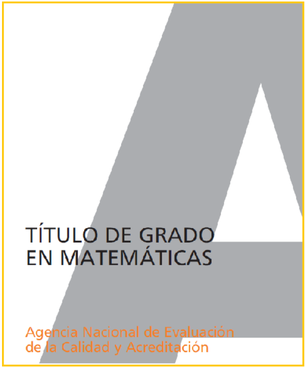
Under composition, the set of q-isometries of E forms a group that is denoted $\mathrm{O}_q(E)$ and is called the *orthogonal group* of q, or of (E, q). It is also denoted $\mathrm{O}_{r,s}$, or just O_n in the case of the Euclidean space.

It is a basic fact that if $f \in \mathrm{O}_q(E)$, then $\det(f) = \pm 1$.

By $\mathrm{SO}_q(E)$ we will understand the subgroup of the isometries f such that $\det(f) = 1$. This is the subgroup of *proper* isometries, and it is called *special orthogonal group*. It is denoted $\mathrm{SO}_{r,s}$, and SO_n in the case of the Euclidean space, when want to specify the signature.

The isometries of $E_{1,3}$ are also called *Lorentz transformations*, and hence $\mathrm{O}_{1,3}$ is the *Lorentz group* and $\mathrm{SO}_{1,3}$ is the *proper* or *special* Lorentz group. The *restricted Lorentz group*, $\mathrm{SO}_{1,3}^{+} \subset \mathrm{SO}_{1,3}$ (proper *orthochronous* Lorentz transformations), will be introduced in later lectures.

Algebra and Geometry curricula



- 240 cr, 8 semesters.
- $\geqslant$60% common topics (144 cr)
- Distributed in 9 blocs.
  LibroBlanco-2004 [13]

DISTRIBUTION OF CREDITS BY BLOCS

Differential and integral calculus. Functions of one complex variable	$34.5\ (32 \leq x_1 \leq 37)$
Linear Algebra and Geometry	$16.5\ (14 \leq x_2 \leq 19)$
Algebraic structures	$13.5\ (\leq x_3 \leq 16)$
Topology and Differential Geometry	$15\ (12.5 \leq x_4 \leq 17.5)$
Probability and Statistics	$15\ (12.5 \leq x_5 \leq 17.5)$
Differential equations	$12\ (9.5 \leq x_6 \leq 14.5)$
Numerical methods and Informatics	$19.5\ (17 \leq x_7 \leq 22)$
Discrete mathematics and Optimization	$12\ (9.5 \leq x_8 \leq 14.5)$
Mathematical models	$6\ (3.5 \leq x_9 \leq 8.5)$
$144 \leq \text{TOTAL} = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 \leq 150$	

- Watertight blocs representing status ‘knowledge areas’ at the time. Relevant areas left out.
- Tendency to a 4/5 rescaling of the previous 5-year degrees.
- In the implementations, local groups were more influential than principles.
- Variable laxity in the definition of contents and how and where they appear in the official programs.

- The ‘bachelor thesis’ was included as compulsory for all.
- Favours plagiarism and even corrupted practices (buying of memoirs, for example).
- Overloads teachers with work that is wasted effort for the most part. The energy would be better spend in caring for motivated students above a suitable qualification threshold. In Spain such ‘elitism’ is only accepted for sports, mainly soccer players.
- Since universities are force to aim at a 100% pass ratio, **in general** students are no longer prepared to endure a serious effort to learn and master subjects.
- The students in this School are of course exceptions!

- Elementary geometry of 2d and 3d.
- Systems of linear equations and matrices.
- Vector spaces and linear maps. Eigenvalues and eigenvectors.
- Bilinear maps and quadratic forms. Diagonalization.
- Affine and Euclidean spaces. Transformations.
- Conics and quadrics.

- Sets, relations, maps.
- Elementary algebraic structures: $\mathbf{Z}$, $\mathbf{Z}_n$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, and Polynomials in one and several variables.
- Groups. Subgroups.
- Rings and ideals. Divisibility and factorization.
- Fields. Resolution of algebraic equations.

Linear algebra

- Vector spaces.
- Matrices, linear systems, determinant.
- Linear maps.
- Diagonalization.
- Euclidean space.

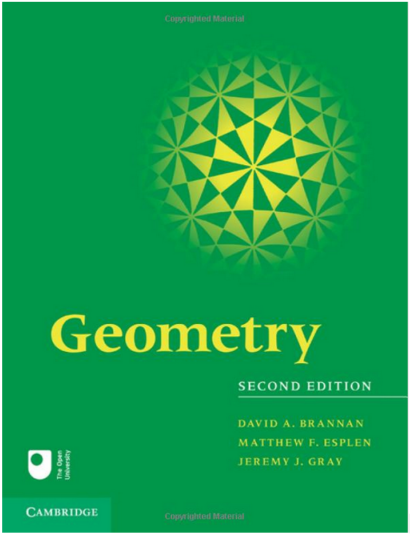
Affine and Euclidean geometry

- Affine space.
- Affine maps.
- Euclidean geometry.
- Isometries (rigid motions).
- Conics and quadrics.

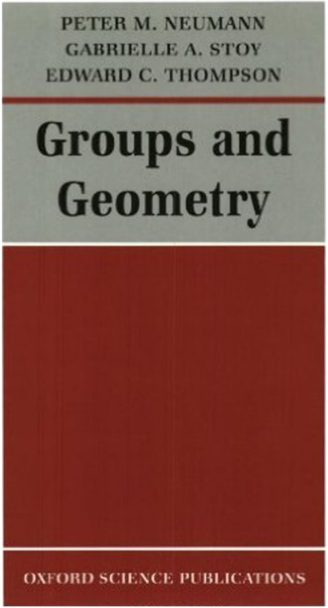
Physics

- Dynamics of one particle. Newton's laws.
- Dynamics of a system of particles. Work and energy. Rigid body.
- Gravitational field.
- Electrostatics.
- Currents. Circuits.
- Magnetostatics.

Non-GA yardstick texts



Brannan-Esplen-Gray-1999 [14]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Groups and Geometry
OXFORD SCIENCE PUBLICATIONS

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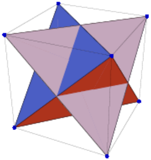

SemiSimple Press
Iowa City, IA

Last revised on May 1, 2015.

Bookmarks

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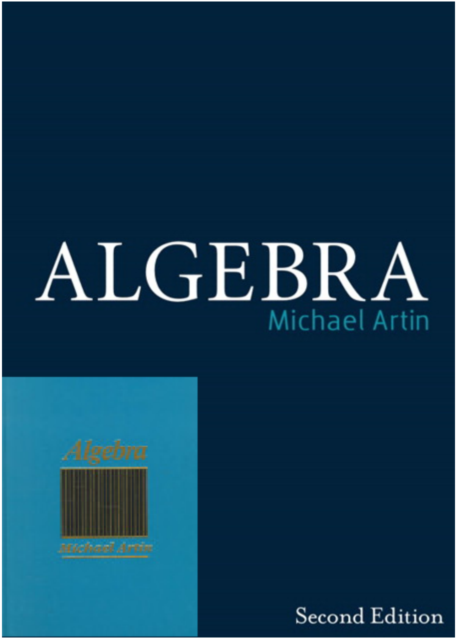
ALGEBRA
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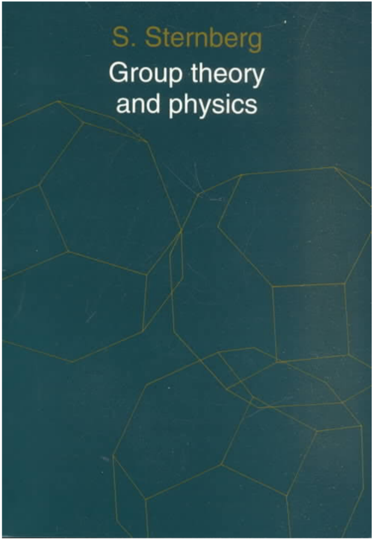

1. Basic definitions and examples ($SL(2,\mathbb{C})$ and the Lorentz group; applications to crystallography; the topology of SU_2 & SO_3; finite subgroups of O_3)

2. Representation theory of finite groups

3. Molecular vibrations and homogeneous vector bundles

4. Compact groups and Lie groups

5. The irreducible representations of SU_n (includes an account of the eight-fold way and the theory of quarks)

Appendix A. The Bravais lattices and the arithmetical crystal classes

Sternberg-1994 [16]

1928: Dirac publishes his equation.

1929: Dirac's hole theory (negative energy solutions), but identifies a hole with a proton.

1930: Dirac proves *charge conjugation* (PQM): negative energy solutions can be transformed into positive energy solutions of opposite charge and the same mass, but continues to identify holes with protons (excess mass accounted for by interactions with filled holes). Oppenheimer points out that lifetime of hydrogen atom would be extremely short. Weyl proved that the theory must have complete symmetry between e and $-e$.

1931: Dirac predicts anti-matter: anti-electron and anti-proton. This is the begining of QFT. In the same paper, he advances the argument that the existence of a single monopole implies quantization of electric charge (first proposal of quantum states with non-trivial topology).

Edward F. Hughes: (Not) How to Write your First Paper

Celebrate:

- **Llull** (1235-1316). “Great influence on Leibniz” (McTutor)
- **Shakespear** (1564-1616)
- **Cervantes** (1547-1616). *Quijote*, 2nd part (1615)
- **Leibniz** (1646-1716). *Dissertatio de arte combinatoria* (1666).
- **Schwarzschild** (1873-1916). First exact solutions of Einstein’s GR equations.
- **Einstein** (1879-1955). GR (1915)
- **Hestenes**. STA (1966)

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With a foreward by Mark Kac.

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