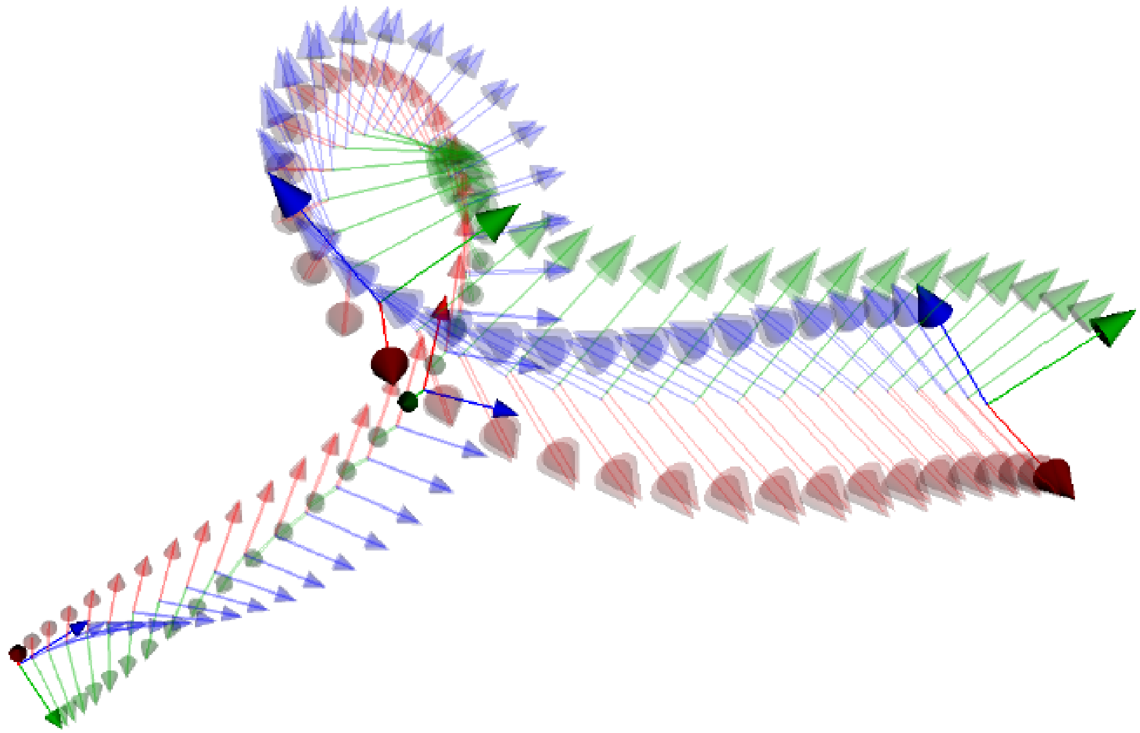
The Devil of Rotations is Afoot!

(James Watt in 1781)



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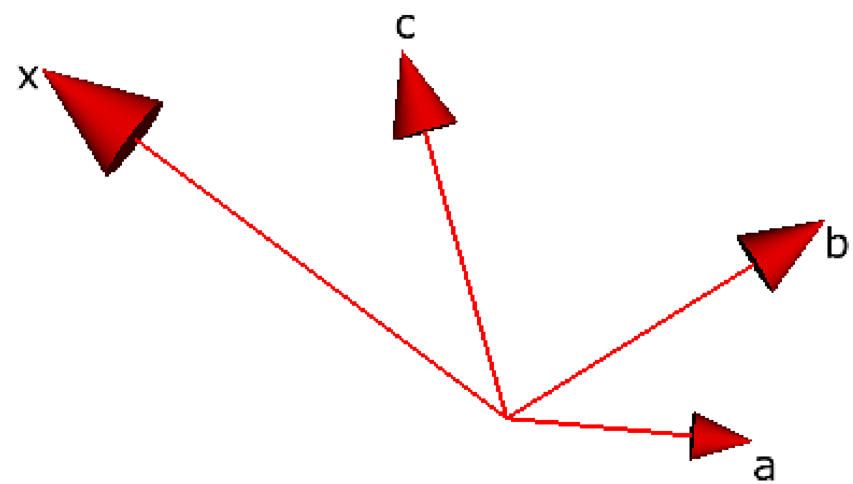
1 The ratio of vectors is an operator in 2D

Given **a** and **b**, find a vector **x**
that is to **c**
what **b** is to **a**?
So, solve **x** from:

$$\mathbf{x} \ : \ \mathbf{c} = \mathbf{b} \ : \ \mathbf{a}.$$



The answer is, by geometric product:

$$\begin{aligned} \mathbf{x} &= (\mathbf{b}/\mathbf{a})\,\mathbf{c} \\ &= \frac{\|\mathbf{b}\|}{\|\mathbf{a}\|}\Big(\cos(\phi) - \mathbf{I}\sin(\phi)\Big)\ \mathbf{c} \\ &= \rho\, e^{-\mathbf{I}\phi}\,\mathbf{c}, \qquad \text{an operator on } \mathbf{c}! \end{aligned}$$

Here $\mathbf{I}$ is the unit-2-blade of the plane 'from **a** to **b**' (so $\mathbf{I}^2 = -1$), ρ is the ratio of their norms, and ϕ is the angle between them. (Actually, it is better to think of $\mathbf{I}\phi$ as the angle.)

Result not fully dependent on **a** and **b**, so better parametrize by ρ and $\mathbf{I}\phi$.

```
GAViewer:  a = e1, label(a), b = e1+e2, label(b), c = -e1+2 e2, dynamic{x = (b/a) c,}
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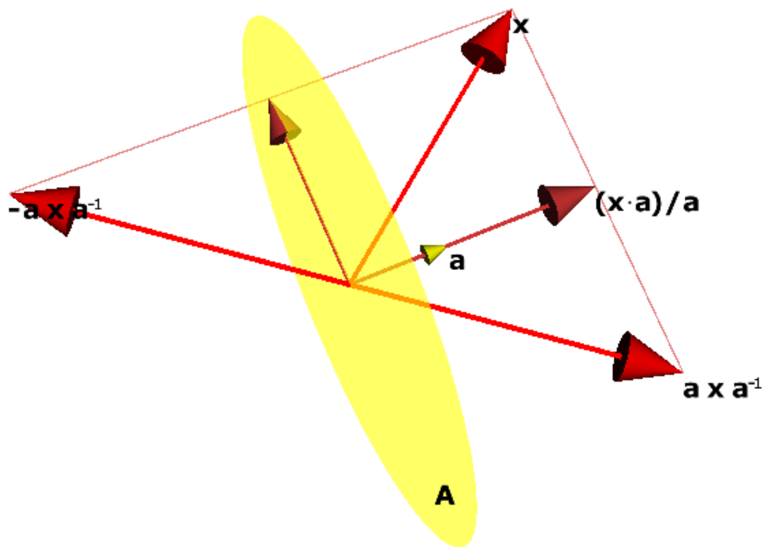
2 Another idea: rotation as multiple reflection



FIG(7,1)

Reflection in an origin plane with unit normal $\mathbf{a}$

$$\mathbf{x} \;\mapsto\; \mathbf{x} - 2(\mathbf{x}\cdot\mathbf{a})\,\mathbf{a}/\|\mathbf{a}\|^2 \quad \textit{(classic LA)}.$$

Now consider the dot product as the symmetric part of a more fundamental geometric product:

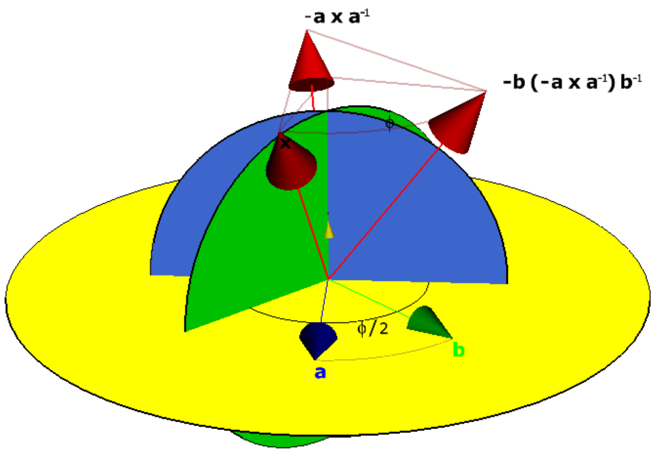
$$\mathbf{x}\cdot\mathbf{a} = \tfrac{1}{2}(\mathbf{x}\,\mathbf{a} + \mathbf{a}\,\mathbf{x}).$$

Then rewrite (with linearity, associativity):

$$\mathbf{x} \;\mapsto\; \mathbf{x} - (\mathbf{x}\,\mathbf{a} + \mathbf{a}\,\mathbf{x})\,\mathbf{a}/\|\mathbf{a}\|^2 \quad \textit{(GA product)}$$
$$= \boxed{-\mathbf{a}\,\mathbf{x}\,\mathbf{a}^{-1}}$$

with *the geometric inverse of a vector*: $\mathbf{a}^{-1} = \mathbf{a}/\|\mathbf{a}\|^2$.

3 Orthogonal Transformations as Products of Unit Vectors



FIG(7,2)

A reflection in two successive origin planes $\mathbf{a}$ and $\mathbf{b}$:

$$\begin{aligned}\mathbf{x} &\mapsto -\mathbf{b}\,(-\mathbf{a}\,\mathbf{x}\,\mathbf{a}^{-1})\,\mathbf{b}^{-1} \\ &= (\mathbf{b}\,\mathbf{a})\,\mathbf{x}\,(\mathbf{b}\,\mathbf{a})^{-1}\end{aligned}$$

So a rotation is represented by the geometric product of two vectors $\mathbf{b}\,\mathbf{a}$, also an element of the algebra.

(Actually, in 3D these are quaternions.)

Multiple reflections are the fundamental representation for operators in GA:

> *The geometric product of (invertible) vectors is called a versor.*
> *It acts as an orthogonal transformation by sandwiching.*

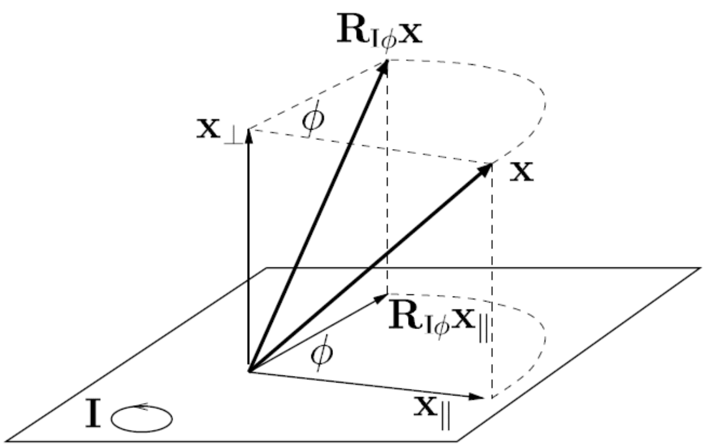
As we will see, versors perform structure-preserving actions on all elements.

It is common to use normalized versors and call those rotors.

4 Sandwiching Rotors can do Rotations in Space ($\mathbb{R}^3$, even $\mathbb{R}^n$)

The operator we found is of the form $\mathbf{ba}$. Sandwiching cancels the norms, so we can take both as unit vectors. Then $\mathbf{ba} = e^{-\mathbf{I}\phi'}$.

Now consider vectors parallel or perpendicular to $\mathbf{I}$-plane. They have different (anti-)commutation properties.



geometry: perpendicular/parallel $\leftrightarrow$ algebra: (anti-)commutation

$$\begin{aligned}
e^{-\mathbf{I}\phi'}\,\mathbf{x}\,e^{\mathbf{I}\phi'} &= (\cos\phi' - \mathbf{I}\sin\phi')(\mathbf{x}_\perp + \mathbf{x}_\parallel)(\cos\phi' + \mathbf{I}\sin\phi') \\
&= (\cos^2\phi' + \sin^2\phi')\mathbf{x}_\perp + (\cos\phi' - \mathbf{I}\sin\phi')^2\,\mathbf{x}_\parallel \\
&= \mathbf{x}_\perp + (\cos 2\phi' - \mathbf{I}\sin 2\phi')\,\mathbf{x}_\parallel \\
&= \mathbf{x}_\perp + (\cos\phi - \mathbf{I}\sin\phi)\,\mathbf{x}_\parallel \qquad \text{(clearly, } \phi' = \phi/2 \text{ !!)} \\
&= \mathbf{x}_\perp + e^{-\mathbf{I}\phi}\,\mathbf{x}_\parallel.
\end{aligned}$$

So $\mathbf{x}_\perp$ is unchanged, and $\mathbf{x}_\parallel$ rotates over an angle $\mathbf{I}\phi$. Therefore:

$$\boxed{\mathbf{R}_{\mathbf{I}\phi}\,\mathbf{x} = e^{-\mathbf{I}\phi/2}\,\mathbf{x}\,e^{\mathbf{I}\phi/2}}$$

5 Concatenation of rotations

A rotation of a vector is fully characterized by the rotor (unit spinor) $e^{\mathbf{I}\phi/2}$:

$$\mathbf{R}_{\mathbf{I}\phi}\,\mathbf{x} = e^{-\mathbf{I}\phi/2}\,\mathbf{x}\,e^{\mathbf{I}\phi/2}$$

We can consider the rotor $e^{\mathbf{I}\phi/2}$ as 'representing the rotation', independent of whether we want to use it on a vector or not. (And we will use it on much more!)

Multiplication of rotations on vectors, first over $\mathbf{I}\phi$, then $\mathbf{J}\psi$:

$$\begin{aligned}\mathbf{R}_{\mathbf{J}\psi}\,(\mathbf{R}_{\mathbf{I}\phi}\,\mathbf{x}) &= e^{-\mathbf{J}\psi/2}\left(e^{-\mathbf{I}\phi/2}\,\mathbf{x}\,e^{\mathbf{I}\phi/2}\right)e^{\mathbf{J}\psi/2}\\ &= \left(e^{-\mathbf{J}\phi/2}e^{-\mathbf{I}\psi/2}\right)\mathbf{x}\left(e^{-\mathbf{J}\phi/2}e^{-\mathbf{I}\psi/2}\right)^{-1}\end{aligned}$$

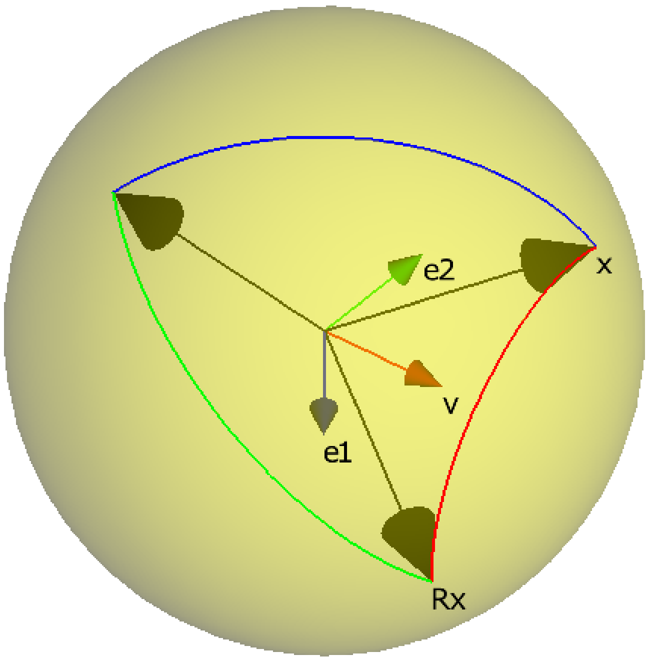
so characterized by the rotor:

$$e^{-\mathbf{J}\phi/2}\;e^{-\mathbf{I}\psi/2}$$

This is *not* in general equal to $e^{(\mathbf{I}\phi/2+\mathbf{J}\psi/2)}$ – although it *is* when $\mathbf{I}$ and $\mathbf{J}$ commute.

(Footnote: in linear algebra, a rotation matrix contains not only the rotation, but also the consequences of wanting to make it act in a location-representation: so it also depends on the coordinate system. A rotor does not! In rotors, no need for an eigenvector-analysis to see what it actually does.)

6 Example of Rotation Composition



tutorial/DEMOrotex()

Rotation over $\pi/2$ around $\mathbf{e}_1$ followed by rotation over $\pi/2$ around $\mathbf{e}_2$. What is the total rotation?

$$
\begin{aligned}
& e^{-\mathbf{e}_3\mathbf{e}_1\pi/4}\, e^{-\mathbf{e}_2\mathbf{e}_3\pi/4} = \\
&\quad = \tfrac{1}{\sqrt{2}}(1-\mathbf{e}_3\mathbf{e}_1)\,\tfrac{1}{\sqrt{2}}(1-\mathbf{e}_2\mathbf{e}_3) \\
&\quad = \tfrac{1}{2}\left(1-(\mathbf{e}_3\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_3-\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)\right) \\
&\quad = \tfrac{1}{2}\left(1-(\mathbf{e}_2\mathbf{e}_3+\mathbf{e}_3\mathbf{e}_1-\mathbf{e}_1\mathbf{e}_2)\right)
\end{aligned}
$$

with the axis $\mathbf{v} = \frac{\mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_3}{\sqrt{3}}$. This represents a rotation over $\mathbf{v}$ over $2\pi/3$.

A lot more work with rotation matrices!

But it looks non-intuitive, can we visualize this?

7 Visualizing the Composition of 3D Rotations

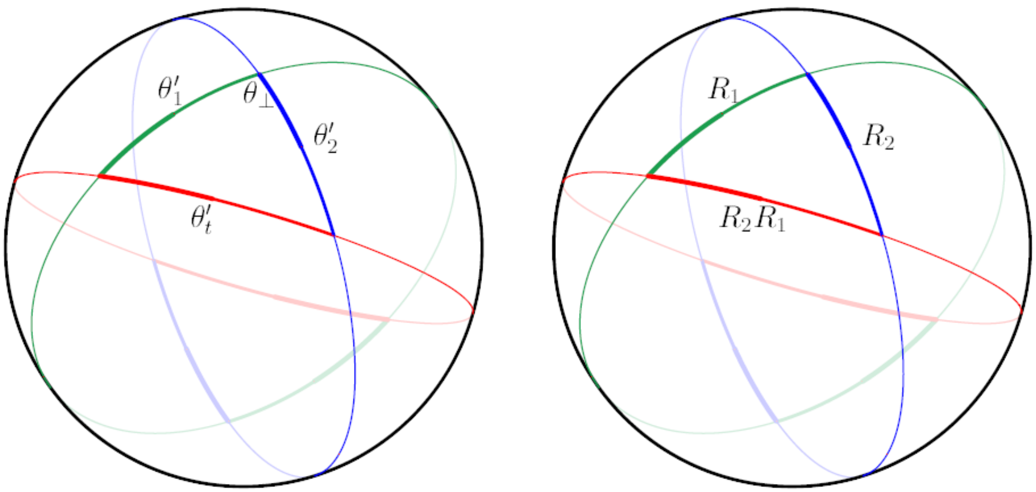
The product $R_t = R_2 R_1$ of rotor R_1 followed by R_2, expressed in their (halved) rotor angles:

$$\begin{aligned} c'_t - \mathbf{I}_t s'_t &= (c'_2 - \mathbf{I}_2 s'_2)\,(c'_1 - \mathbf{I}_1 s'_1) \\ &= c'_2 c'_1 + s'_2 s'_1 c_\perp - c'_1 s'_2 \mathbf{I}_2 - c'_2 s'_1 \mathbf{I}_1 - s'_2 s'_1 s_\perp \mathbf{I}_\perp \end{aligned}$$

where $\mathbf{I}_\perp \theta_\perp$ is the bivector angle between rotation planes $\mathbf{I}_2$ and $\mathbf{I}_1$. Note:

$$\cos\theta'_t = \cos\theta'_1 \cos\theta'_2 + \sin\theta'_1 \sin\theta'_2 \cos\theta_\perp.$$

This is the 'cosine rule for sides' from spherical trigonometry. Aha!



FIG(7,6)

So, *multiplicative composition of rotors is additive composition of (half angle) rotor arcs*.

Some care: arcs are free to slide, to compose them bring them to the common point.

8 Quaternions Subsumed (the details)

Rotors are like unit quaternions, but embedded in algebra of real vectors.

$$\text{quaternion } q_0 + \mathbf{q} \quad \leftrightarrow \quad \text{rotor } q_0 - \mathbf{I}_3\mathbf{q}.$$

Here $\mathbf{I}_3$ is the volume element of 3-dimensional space, and $\mathbf{I}_3^2 = -1$. It makes duals: a rotation axis $\mathbf{q}$ becomes a rotation 2-blade $\mathbf{I}_3\mathbf{q}$.

The 'complex vector part' of a quaternion is actually a 'real bivector part'. All quaternion math then follows from geometric algebra.

$$\begin{aligned}
qp &= (q_0 - \mathbf{I}_3\mathbf{q})\,(p_0 - \mathbf{I}_3\mathbf{p}) \\
&= q_0p_0 + \langle \mathbf{I}_3\mathbf{q}\mathbf{I}_3\mathbf{p}\rangle_0 - \mathbf{I}_3(\mathbf{q}p_0 + \mathbf{p}q_0 - \mathbf{I}_3^{-1}\langle \mathbf{I}_3\mathbf{q}\mathbf{I}_3\mathbf{p}\rangle_2) \\
&= q_0p_0 - \langle \mathbf{q}\mathbf{p}\rangle_0 - \mathbf{I}_3(\mathbf{q}p_0 + \mathbf{p}q_0 + \mathbf{I}_3^{-1}\langle \mathbf{q}\mathbf{p}\rangle_2) \\
&= p_0q_0 - \mathbf{p}\cdot\mathbf{q} + \mathbf{I}_3(p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{q}\times\mathbf{p}),
\end{aligned}$$

well-known in quaternion literature, but somewhat *ad hoc* there.

Quaternions are very real – not involving 'imaginary vectors' but 'real bivectors'.

9 The Advantage of Rotors: Structure Preservation

Remember from my Monday talk: using rotors (versors) provides geometric covariance.

$$\begin{aligned} X = x_1\, x_2 \cdots x_k \;\rightarrow\; & (V\, x_1\, V^{-1})\,(V\, x_2\, V^{-1}) \cdots (V\, x_k\, V^{-1}) \\ = \; & V\,(x_1\, x_2 \cdots x_k)\, V^{-1} \\ = \; & V\, X\, V^{-1}. \end{aligned}$$

So we do not need to know how X was made from $x_1, \cdots x_k$ to transform it!

Since outer prdouct and inner product are linear combinations of geometric products, they are also covariant; and so is taking grades.

If you can write your construction in terms of these basic products, they will transform covariantly under your versors.

Used in this multiplicative way (and not using the bare addition of elements of arbitrary grades), all your objects can be interpreted geometrically. And vice versa, all truly geometric objects in the algebra can be made this way.

We have unit quaternions that can rotate more than just 3D vectors and other unit quaternions at the origin! We have 'generalized quaternions' that can even do conformal transformations.

10 Are All Rotors the Exponentials of Bivectors?

Rotor composition:

$$e^{-\mathbf{J}\phi/2}\, e^{-\mathbf{I}\psi/2}.$$

In Euclidean 3D, this is a new element of the form $e^{-\mathbf{K}\alpha/2}$ with $\mathbf{K}$ a 2-blade(and we showed how to compute it).
In Euclidean 4D, this is more subtle: $\mathbf{K}$ is in general not a 2-blade but a bivector; yet a general rotor is still exponential of bivector.
In general n-D, not even this holds!

Only in Euclidean and Minkowski spaces (i.e., $\mathbb{R}^{n,0}$, $\mathbb{R}^{0,n}$, $\mathbb{R}^{n,1}$, $\mathbb{R}^{1,n}$)
can every orthogonal transformation continuously connected to the identity
be written as a rotor of the form ‘exponential of a bivector’.

Moreover, only in those spaces can any bivector be written as the sum of commuting 2-blades.

So only in those spaces can we make all orthogonal transformation continuously connected to the identity from a suite of ‘simple rotors’ (exponentials of 2-blades).

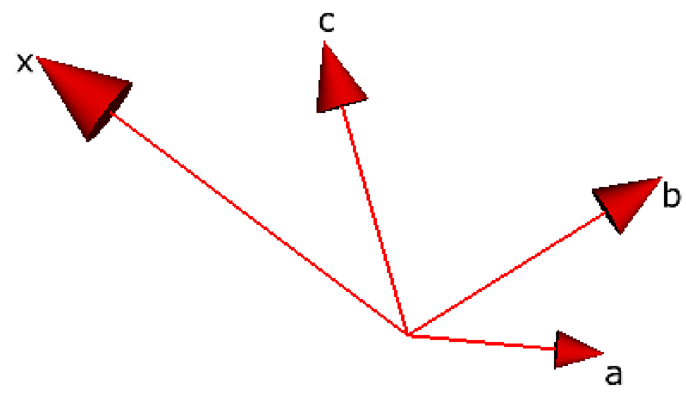
Only on those spaces do all* rotors have logarithms.

* For a subtlety in Minkowski spaces of dimension less than 4, see Section 7.4.3 of GA4CS.

11 A Rotor is the Square Root of the Ratio of Unit Vectors

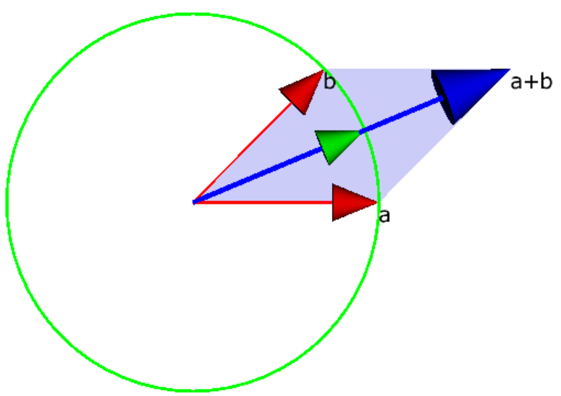
Remember how we started?

Given **a** and **b**, find a vector **x**
that is to **c**
what **b** is to **a**?



For unit vectors:

$$\begin{aligned}\mathbf{x} &= (\mathbf{b}/\mathbf{a})\,\mathbf{c} \\ &= e^{-\mathbf{I}\phi}\,\mathbf{c} \\ &= e^{-\mathbf{I}\phi/2}\,\mathbf{c}\,e^{\mathbf{I}\phi/2} \qquad \text{much preferred form!} \\ &= \left(\sqrt{\mathbf{b}/\mathbf{a}}\right)\mathbf{c}\left(\sqrt{\mathbf{b}/\mathbf{a}}\right)^{-1}.\end{aligned}$$



So we need the square root $\sqrt{\mathbf{b}/\mathbf{a}}$.

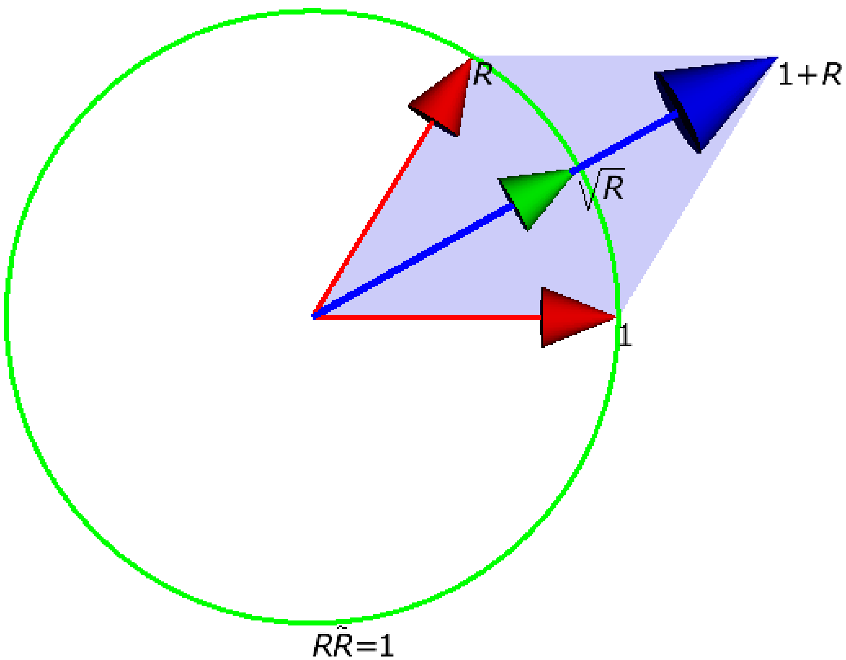
We could employ logarithms, but there is a good trick here:

$$\sqrt{\mathbf{b}/\mathbf{a}} = \frac{\mathbf{a}+\mathbf{b}}{\sqrt{2(1+\mathbf{a}\cdot\mathbf{b})}}/\mathbf{a} = \frac{1+\mathbf{b}/\mathbf{a}}{\sqrt{2(1+\mathbf{a}\cdot\mathbf{b})}}.$$

simpleroot1()

12 Rotor Square Root Intuition in Euclidean 2D and 3D

A 2D Euclidean rotor R is like an marker on a unit circle representing the rotor manifold $R\widetilde{R} = 1$. Taking its square root involves constructing the half angle, which may be done by a parallellogram spanning $1 + R$.



This extends to 3D Euclidean GA, so:

$$\begin{aligned}\sqrt{R} &= \text{normalize}[1+R] \\ &= \frac{1+R}{\sqrt{(1+R)(1+\widetilde{R})}} \\ &= \frac{1+R}{\sqrt{2(1+\langle R\rangle_0)}}.\end{aligned} \tag{1}$$

Small print: $R \neq -1$.

simpleroot()

Note that we first produce something off the rotor manifold (but in the linear space of rotor elements), then rescale by a scalar to get a (normalized) rotor.

13 Square Roots of General Rotors in 3D CGA

Take Home Message:

The same formula

$$\sqrt{R} = \text{normalize}[1 + R]$$

works for all rotors in 3D conformal geometric algebra – when we adapt the normalization to a proper 'projection onto the rotor manifold'.

Sketch: Since $\sqrt{R}$ is a rotor, we can rewrite:

$$1 + R = \sqrt{R}\,(\sqrt{R}^{\sim} + \sqrt{R}). \quad (2)$$

The first factor is a rotor. The final factor is self-reverse. But it is not necessarily a scalar, in 3D CGA it can have a grade-4 part.

We merely need a method to split general elements of the linear rotor space of 3D CGA into a rotor factor, and a self-reverse factor (this is a polar decomposition).

14 Root of 3D CGA Rotor

The common case is rather involved:

$$\sqrt{R} = (1+R)\,\frac{1+R_0-R_4}{2\big((1+R_0)^2-R_4^2\big)}\,\frac{1+R_0+R_4+\sqrt{(1+R_0)^2-R_4^2}}{\sqrt{1+R_0+\sqrt{(1+R_0)^2-R_4^2}}}$$

and there are some more cases to extend the square root to all cases of R.

Details in Dorst & Valkenburg [2011].

15 Special Case: The Square Root of Various Motors in 3D CGA

Motors M represent rigid body motions.

$$\text{square root of motor } M: \qquad \sqrt{M} = \frac{1+M}{\sqrt{2(1+\langle M\rangle)}}\left(1 - \frac{\langle M\rangle_4}{2(1+\langle M\rangle)}\right). \tag{3}$$

The formula explicitly shows how a self-reverse element consisting of only a scalar and 4-vector part 'nudges $(1+M)$ back onto the motor manifold'.

For a pure rotation $\mathbf{R}$ at the origin in 3D, the motor has no grade-4 part, so:

$$\text{square root of pure rotation } \mathbf{R}: \qquad \sqrt{\mathbf{R}} = \frac{1+\mathbf{R}}{\sqrt{2(1+\langle \mathbf{R}\rangle)}}. \tag{4}$$

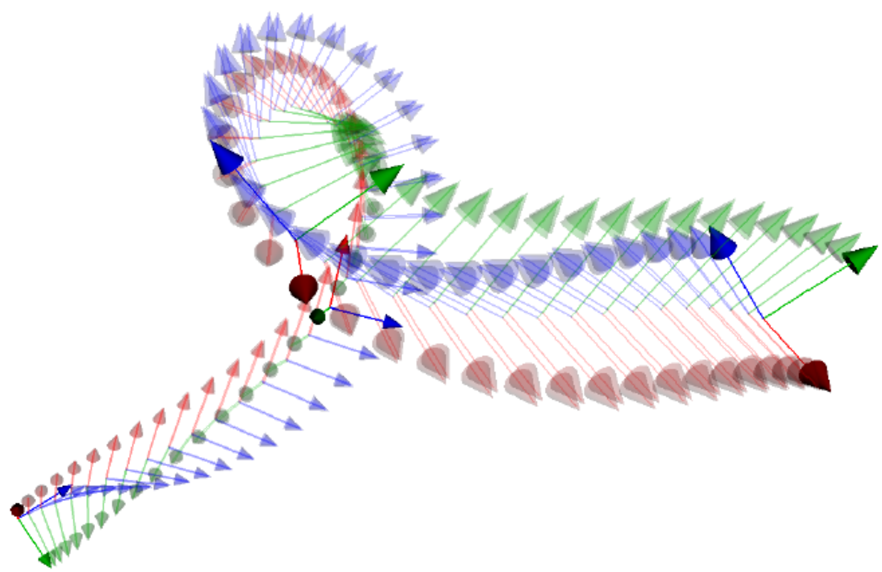
This is a formula to interpolate unit quaternions (which are after all merely 3D rotors).

For a pure translation T, there is no grade-4 part, and $\langle T\rangle = 1$, so we obtain:

$$\text{square root of pure translation } T: \qquad \sqrt{T} = \tfrac{1}{2}(1+T). \tag{5}$$

Of course $\sqrt{T_{\mathbf{t}}}$ is simply $T_{\mathbf{t}/2}$, but it is nice to see it included in the pattern.

16 Interpolation and Extrapolation of (Conformal) Motions



Conformal motions (such as rigid body movements) are characterized by rotors in the conformal model $\mathcal{G}_{4,1}$.

Interpolation of such motions in a coordinate-independent manner involves breaking up the relative rotor R_2/R_1 between two 'stances' R_1 and R_2 into equal parts. For instance:

$$r = \sqrt[n]{R_2/R_1}, \quad R_i = r^i R_1, \quad X_i = R_i X_0 \widetilde{R}_i.$$

Thus we need n-th roots of rotors in 3D CGA.

We give a sketch, details in Dorst & Valkenburg [2011].

17 Logarithms of Rotors in 3D CGA

Rotors of $\mathcal{G}_{4,1}$ are exponentials of bivectors. To get their logarithms, follow Hestenes CAGC procedure (extended to CGA) to retrieve a bivector and split it into manageable 2-blades:

1. Given rotor R, take exterior derivative $\frac{1}{2}\partial_x \wedge (Rx\widetilde{R})$ to find associated bivector F.[1]
2. Decompose that bivector F into two commuting 2-blades $F = F_+ + F_-$ (next slide).
3. Linearize 2-blade parametrization: $B_+ = \operatorname{asinh}(F_+)$ and $B_- = \operatorname{asinh}(F_-)$.[2]
4. Now R can be decomposed into commuting factors: $R = e^{B_+ + B_-} = e^{B_+}e^{B_-} = e^{B_-}e^{B_+}$.
5. The logarithm of R is the sum of the logarithms of e^{B_+} and e^{B_-}.
6. Since B_+ and B_- are 2-blades of 3D CGA, those logarithms are standard (and may be found in GA4CS).

[1] Using coordinates $\{e_i\}$ for $\mathcal{G}^+_{4,1}$, compute this as $F = \frac{1}{2}\sum_{i,j}\langle e_i R e_j \widetilde{R}\rangle\, e^i \wedge e^j$, see CAGC.

[2] Extend asinh to general 2-blades through: $\operatorname{asinh}(B) = \begin{cases} \frac{\operatorname{asin}(\sqrt{-B^2})}{\sqrt{-B^2}}\,B & \text{if } B^2 < 0 \\ B & \text{if } B^2 = 0 \\ \frac{\operatorname{asinh}(\sqrt{B^2})}{\sqrt{B^2}}\,B & \text{if } B^2 > 0 \end{cases}$.

18 Bivector Splitting in 3D CGA

A bivector F can be decomposed into two commuting 2-blades $F = F_+ + F_-$ through:

$$F_\pm = \tfrac{1}{2}F\,(1 \pm \frac{\|F\|^2}{F^2}) \tag{6}$$

where $\|\cdot\|$ is a rather unusual ' norm' of a bivector:

$$\|F\| = \sqrt[4]{\langle \widetilde{F}F\rangle^2 - \langle \widetilde{F}F\rangle_4^2}. \tag{7}$$

The square $F_\pm^2 = \frac{1}{2}(\langle F^2\rangle \pm \|F\|^2)$ is indeed scalar.

- This decomposition is almost always unique.
- The 2-blade F_- is always an imaginary point pair (or zero).
- The 2-blade F_+ can be imaginary, null or real.

I found it in [CA2GC].

19 Summary of CGA Logarithm

Given rotor, to write: $R = e^{-(B_+ + B_-)/2} = e^{-B_+/2}\, e^{-B_-/2} = e^{-B_-/2}\, e^{-B_+/2}$.

1. $S = 2(\langle R\rangle_4 - \langle R\rangle_0)\, \langle R\rangle_2 = \sinh(B_+) + \sinh(B_-)$, then split.
2. The 'bivector split' of any bivector S of $\mathbb{R}^{4,1}$ can be computed as: $S_\pm = \frac{1}{2} S\big(1 \pm \|S\|^2/S^2\big)$, with $\|S\| = \sqrt[4]{(2\langle S^2\rangle_0 - S^2)\, S^2}$, and for $\|S\| \neq 0$.
   When $\|S\| = 0$, no split, or no unique split, see CA2GC.
3. $S_\pm = \sinh(B_\pm)$ thus found; $C_\pm = \cosh(B_\pm) = -\langle R^2\rangle_2 / S_\pm$.
   Then $B_\pm = \operatorname{atanh2}(S_\pm, C_\pm)$
4. Done: $\operatorname{Log}(R) = -\frac{1}{2} B_+ - \frac{1}{2} B_-$.

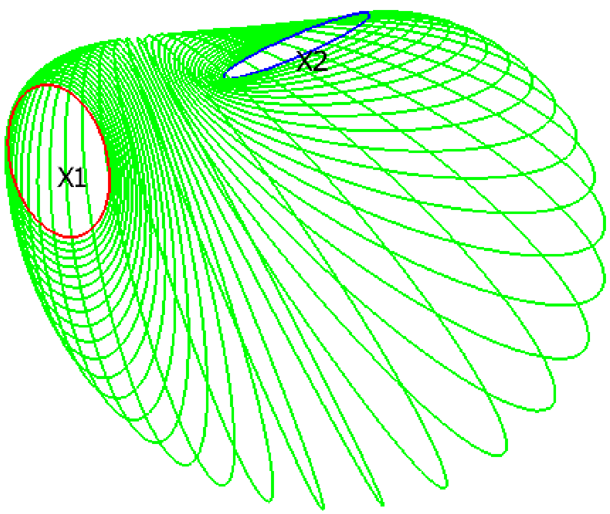
Dorst & Valkenburg, *Square Root and Logarithm of Rotors in 3D Conformal Geometric Algebra Using Polar Decomposition*, in: Guide to GA in Practice 2011.

20 Back to Interpolation

- The logarithm of any rotor in 3D CGA can now be computed.
- Therefore any relative rotor R can be linearly interpolated in its parameters:

$$R(t) = \exp(\log(R)\, t), \qquad t \in [0, 1].$$

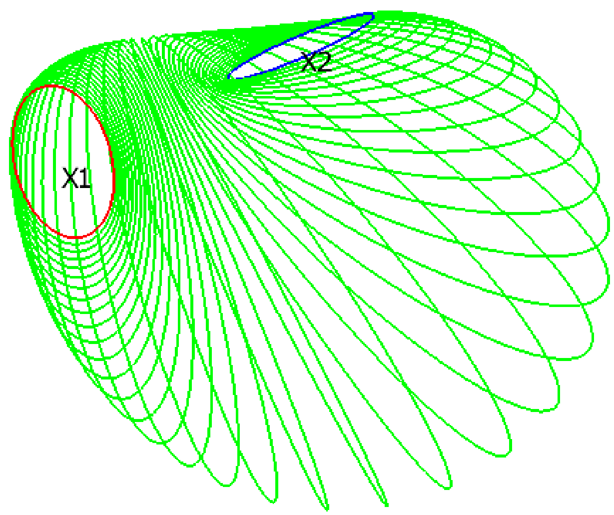
- The square root permits linear interpolation by iterative halving.
- The 2-blades in the logarithm can be used in more advanced interpolation schemes (such as B-splines).



roots-and-rbm/Vratio()

21 Wrap-up

- You can make versors as (square root of) ratios of geometric objects.
- But the form 'exponential of a bivector' is a more pleasant parametrization.
- In tomorrow's talk, I will show what those bivectors mean, and how you can construct your rotors from them directly (in 3D CGA).
- The exponents do *not* add under versor multiplication!
- A logarithm of a rotor can be made (for any CGA rotor), and used for interpolation.



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22 Appendix: A Sense of Rotation

It is often said that the rotors form a 'double cover' of the rotation group. It seems that a rotation over ϕ can be represented in two ways:

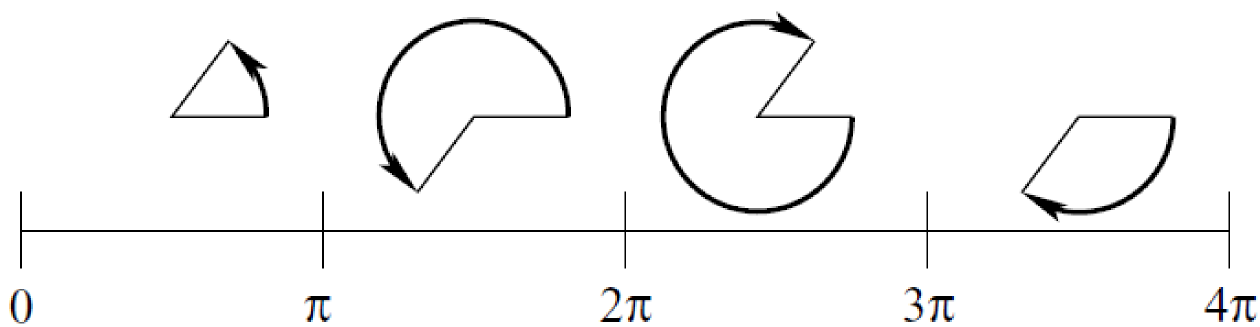
$$\exp(-\mathbf{I}\phi/2) \quad \text{and} \quad -\exp(-\mathbf{I}\phi/2) = \exp(-\mathbf{I}(2\pi+\phi)/2),$$

because the sign disappears in the sandwiching product.

On objects like vectors, rotations are indeed periodic with 2π, so rotation over ϕ is indistinguishable from rotation over $2\pi + \phi$.

But rotations on general elements are actually periodic with 4π, not with 2π. Any Balinese dancer knows this.

So we need the whole range of angles $[0, 4\pi)$; there is no double cover of physical rotations by rotors.



23 Appendix: Linear Interpolation of Orientations

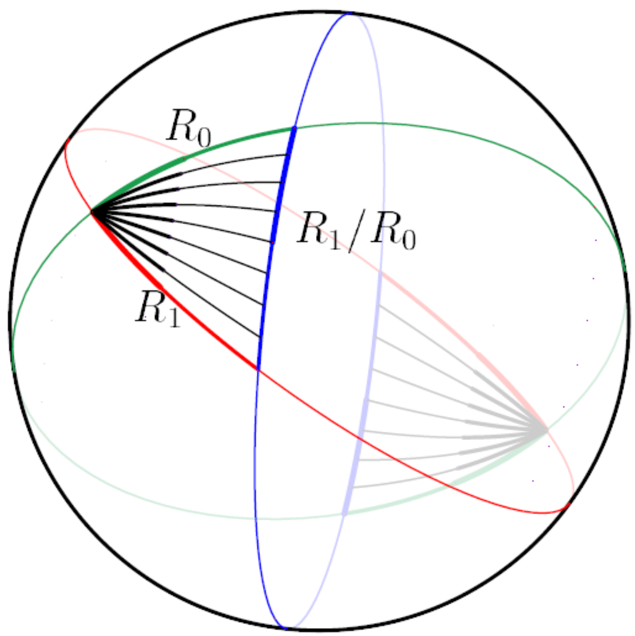
For translations $\mathbf{x}_0$ and $\mathbf{x}_1$, linear interpolation is

$$\boxed{\mathbf{x}_\lambda = \mathrm{lerp}(\mathbf{x}_0, \mathbf{x}_1; \lambda) \equiv (1-\lambda)\, \mathbf{x}_0 + \lambda\, \mathbf{x}_1, \qquad \lambda \in [0, 1]}$$

For rotations characterized by rotors R_0 and R_1 (relative angle θ'):

$$\boxed{R_\lambda = \mathrm{slerp}(R_0, R_1; \lambda) \equiv \frac{\sin(1-\lambda)\theta'}{\sin\theta'}\, R_0 + \frac{\sin\lambda\theta'}{\sin\theta'}\, R_1, \quad \lambda \in [0, 1]}$$

(There is nothing new under the sine...)



You can also use this for prediction, by taking $\lambda > 1$.