Multivector differentiation and Linear Algebra

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Overview

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- Functional Differentiation: very briefly...
- Summary

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Recall our definition of the directional derivative in the a direction

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Let us use $*$ to denote taking the scalar part, ie $P * Q \equiv \langle PQ \rangle$. Then, provided A has same grades as X, it makes sense to define:

$$A * \partial_X F(X) = \lim_{\tau \to 0} \frac{F(X + \tau A) - F(X)}{\tau}$$

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With the definition on the previous slide, $e_J * \partial_X$ is therefore the partial derivative in the e_J direction. Giving

$$\partial_X \equiv \sum_J e^J e_J * \partial_X$$

[since $e_J * \partial_X \equiv e_J * \{e^I(e_I * \partial_X\}$].

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Key to using these definitions of multivector differentiation are several important results:

The Multivector Derivative cont...

If $P_X(B)$ is the projection of B onto the grades of X (ie $P_X(B) \equiv e^J \langle e_J B \rangle$), then our first important result is

$$\partial_X \langle XB \rangle = P_X(B)$$

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We can see this by going back to our definitions:

$$e_J * \partial_X \langle XB \rangle = \lim_{\tau \to 0} \frac{\langle (X + \tau e_J)B \rangle - \langle XB \rangle}{\tau} = \lim_{\tau \to 0} \frac{\langle XB \rangle + \tau \langle e_J B \rangle - \langle XB \rangle}{\tau}$$

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Therefore giving us

$$\partial_X \langle XB \rangle = e^J (e_J * \partial_X) \langle XB \rangle = e^J \langle e_J B \rangle \equiv P_X(B)$$

Other Key Results

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$$\partial_X \langle XB \rangle = P_X(B)$$

$$\partial_X \langle \tilde{X}B \rangle = P_X(\tilde{B})$$

$$\partial_{\tilde{X}} \langle \tilde{X}B \rangle = P_{\tilde{X}}(B) = P_X(B)$$

$$\partial_\psi \langle M\psi^{-1} \rangle = -\psi^{-1} P_\psi(M) \psi^{-1}$$

X, B, M, ψ all general multivectors.

Exercises 1

1. By noting that $\langle XB \rangle = \langle (XB)\tilde{} \rangle$, show the second key result
$$\partial_X \langle \tilde{X} B \rangle = P_X(\tilde{B})$$

2. Key result 1 tells us that $\partial_{\tilde{X}} \langle \tilde{X} B \rangle = P_{\tilde{X}}(B)$. Verify that $P_{\tilde{X}}(B) = P_X(B)$, to give the 3rd key result.

3. to show the 4th key result
$$\partial_\psi \langle M \psi^{-1} \rangle = -\psi^{-1} P_\psi(M) \psi^{-1}$$
use the fact that $\partial_\psi \langle M \psi \psi^{-1} \rangle = \partial_\psi \langle M \rangle = 0$. Hint: recall that XAX has the same grades as A.

A Simple Example

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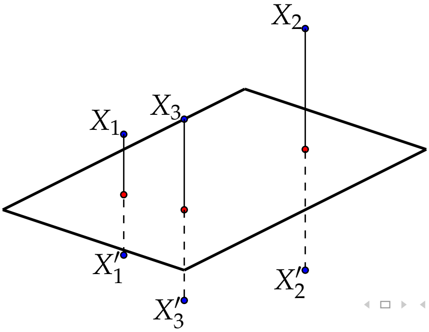
One possible way forward is to find the plane that minimises the sum of the squared perpendicular distances of the points from the plane.



Plane fitting example, cont....

Recall that $\Phi X \Phi$ is the reflection of X in Φ, so that $-X\cdot(\Phi X \Phi)$ is the distance between the point and the plane. Thus we could take as our cost function:

$$S = -\sum_i X_i \cdot (\Phi X_i \Phi)$$

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Now use the result $\partial_X \langle XB \rangle = P_X(B)$ to differentiate this expression wrt Φ

$$\partial_\Phi S = -\sum_i \partial_\Phi \langle X_i \Phi X_i \Phi \rangle = -\sum_i \dot{\partial}_\Phi \langle X_i \dot{\Phi} X_i \Phi \rangle + \dot{\partial}_\Phi \langle X_i \Phi X_i \dot{\Phi} \rangle$$

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$$= -2\sum_i P_\Phi(X_i \Phi X_i) = -2\sum_i X_i \Phi X_i$$

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$\Longrightarrow$ solve (via linear algebra techniques) $\sum_i X_i \Phi X_i = 0$.

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Suppose we wished to create a Kalman filter-like system which tracked bivectors (not simply their components in some basis) – this might involve evaluating expressions such as

$$\partial_{B_n} \sum_{i=1}^{L} \langle v_n^i R_n u_{n-1}^i \tilde{R}_n \rangle$$

where $R_n = \mathrm{e}^{-B_n}$, u, v s are vectors.

Differentiation cont....

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$$\partial_{B_n} \langle v_n R_n u_{n-1} \tilde{R}_n \rangle = -\Gamma(B_n) + \frac{1}{|B_n|^2} \langle B_n \Gamma(B_n) \tilde{R}_n B_n R_n \rangle_2$$

$$+ \frac{\sin(|b_n|)}{|B_n|} \left\langle \frac{B_n \Gamma(B_n) \tilde{R}_n B_n}{|B_n|^2} + \Gamma(B_n) \tilde{R}_n \right\rangle_2$$

where $\Gamma(B_n) = \frac{1}{2}[u_{n-1} \wedge \tilde{R}_n v_n R_n] R_n$.

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We can now extend f to act on any order blade by (outermorphism)

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Note that the resulting blade has the same grade as the original blade. Thus, an important property is that these extended linear functions are grade preserving, ie

$$\mathrm{f}(A_r) = \langle \mathrm{f}(A_r) \rangle_r$$

Linear Algebra cont....

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The product of linear functions is associative.

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from which we get the first result.

Exercises 2

1. For a matrix F

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$$

Verify that $F_{ij} = e_i \cdot f(e_j)$, where $e_1 = [1,0]^T$ and $e_2 = [0,1]^T$, for $i,j = 1,2$.

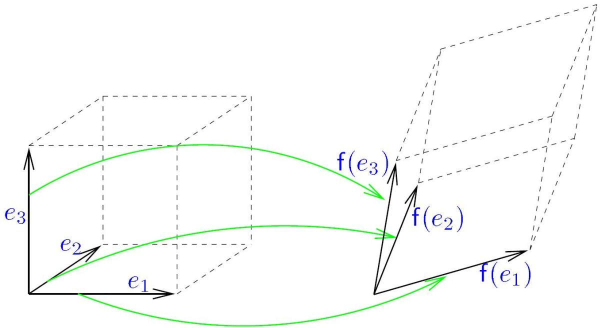
2. Rotations are linear functions, so we can write $R(a) = Ra\tilde{R}$, where R is the rotor. If A_r is an r-blade, show that

$$RA_r\tilde{R} = (Ra_1\tilde{R}) \wedge (Ra_2\tilde{R}) \wedge \ldots \wedge (Ra_r\tilde{R})$$

Thus we can rotate any element of our algebra with the same rotor expression.

The Determinant

Consider the action of a linear function f on an orthogonal basis in 3d:



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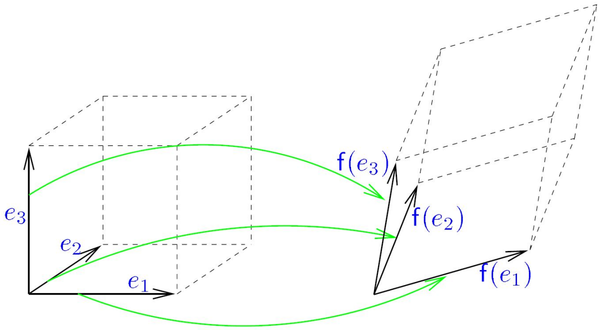

The unit cube $I = e_1 \wedge e_2 \wedge e_3$ is transformed to a parallelepiped, V

$$V = \mathsf{f}(e_1) \wedge \mathsf{f}(e_2) \wedge \mathsf{f}(e_3) = \mathsf{f}(I)$$

The Determinant cont....

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Let us define the determinant of the linear function f as the volume scale factor V. So that

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This enables us to find the form of the determinant explicitly (in terms of partial derivatives between coordinate frames) very easily in any dimension.

A Key Result

As before, let $\mathrm{h} = \mathrm{fg}$, then

$$\mathrm{h}(I) = \det(\mathrm{h})\, I = \mathrm{f}(\mathrm{g}(I)) = \mathrm{f}(\det(\mathrm{g})\, I)$$

$$= \det(\mathrm{g})\, \mathrm{f}(I) = \det(\mathrm{g})\, \det(\mathrm{f})(I)$$

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As before, let $h = fg$, then

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A very easy proof!

The Adjoint/Transpose of a Linear Function

For a matrix F and its transpose, F^T we have (for any vectors a, b)

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The Adjoint cont....

It is not hard to show that the adjoint extends to blades in the expected way

$$\bar{\mathsf{f}}(a_1 \wedge a_2 \wedge ... \wedge a_n) = \bar{\mathsf{f}}(a_1) \wedge \bar{\mathsf{f}}(a_2) \wedge .... \wedge \bar{\mathsf{f}}(a_n)$$

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See following exercises to show that

$$a \cdot f(b \wedge c) = f[\bar{f}(a) \cdot (b \wedge c)]$$

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$$a \cdot \mathsf{f}(b \wedge c) = \mathsf{f}[\bar{\mathsf{f}}(a) \cdot (b \wedge c)]$$

This can now be generalised to

$$A_r \cdot \bar{\mathsf{f}}(B_s) = \bar{\mathsf{f}}[\mathsf{f}(A_r) \cdot B_s] \quad r \leq s$$

$$\mathsf{f}(A_r) \cdot B_s = \mathsf{f}[A_r \cdot \bar{\mathsf{f}}(B_s)] \quad r \geq s$$

Exercises 3

1. For any vectors p, q, r, show that

$$p \cdot (q \wedge r) = (p \cdot q) r - (p \cdot r) q$$

2. By using the fact that $a \cdot \mathsf{f}(b \wedge c) = a \cdot [\mathsf{f}(b) \wedge \mathsf{f}(c)]$, use the above result to show that

$$a \cdot \mathsf{f}(b \wedge c) = (\bar{\mathsf{f}}(a) \cdot b)\mathsf{f}(c) - (\bar{\mathsf{f}}(a) \cdot c)\mathsf{f}(b)$$

and simplify to get the final result

$$a \cdot \mathsf{f}(b \wedge c) = \mathsf{f}[\bar{\mathsf{f}}(a) \cdot (b \wedge c)]$$

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Now put $B_s = I$ in this formula:

$$A_r \cdot \bar{\mathrm{f}}(I) = A_r \cdot \det(\mathrm{f})(I) = \det(\mathrm{f})(A_r I)$$

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We can now write this as

$$A_r = \bar{\mathrm{f}}[\mathrm{f}(A_r) I] I^{-1} [\det(\mathrm{f})]^{-1}$$

The Inverse cont...

Repeat this here:

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The next stage is to put $A_r = \mathrm{f}^{-1}(B_r)$ in this equation:

$$\mathrm{f}^{-1}(B_r) = \bar{\mathrm{f}}[B_r I]I^{-1}[\det(\mathrm{f})]^{-1}$$

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This leads us to the important and simple formulae for the inverse of a function and its adjoint

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Let us see if this works for rotations

$$\mathrm{R}(a) = Ra\tilde{R} \quad \text{and} \quad \bar{\mathrm{R}}(a) = \tilde{R}aR$$

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$$= [\det(\mathrm{f})]^{-1}\tilde{R}(AI)RI^{-1} = \tilde{R}AR$$

since $\det(R) = 1$. Thus the inverse is the adjoint ... as we know from $R\tilde{R} = 1$.

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That we won't look at.....

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- Decompositions such as Singular Value Decomposition
- Tensors - we can think of tensors as linear functions mapping r-blades to s-blades. Thus we retain some physical intuition that is generally lost in index notation.

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In engineering, this, in particular, enables us to differentiate wrt to structured matrices in a way which is very hard to do otherwise.

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- Linear Algebra : we will see applications of the GA approach to linear algebra – using, in particular, the beautiful expressions for the inverse of a function.
- Functional Differentiation : used widely in physics, scope for much more use in engineering.