Reciprocal Frames, the Vector Derivative and Curvilinear Coordinates.

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Overview

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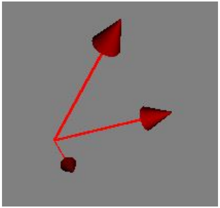

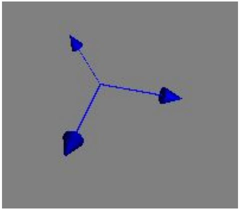

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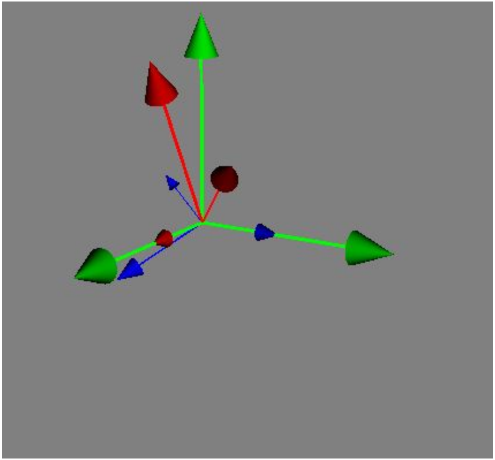

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We call such a frame a reciprocal frame. Note that since any vector a can be written as $a = a^k e_k \equiv \sum a^k e_k$ (ie we are adopting the convention that repeated indices are summed over), we have

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So how do we find a reciprocal frame?

We need, for example, e^1 to be orthogonal to the set of vectors $\{e_2, e_3, ..., e_n\}$. ie e^1 must be perpendicular to the hyperplane $e_2 \wedge e_3 \wedge .... \wedge e_n$.

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α is a scalar found by dotting with e_1:

$$e_1 \cdot e^1 = 1 = e_1 \cdot (\alpha e_2 \wedge e_3 \wedge ... \wedge e_n I) = \alpha (e_1 \wedge e_2 \wedge ... \wedge e_n) I$$

(this uses a useful GA relation $a \cdot (BI) = (a \wedge B) I$).

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These reciprocal frames are remarkably useful!

Exercises 1

1. Show that $a\cdot(BI) = (a\wedge B)I$. [Hint: make use of the fact that $a\cdot(B_r I_n) = \langle aB_r I_n\rangle_{n-r-1}$].

2. For $\{f_1, f_2, f_3\} = \{e_1,\ e_1 + 2e_3,\ e_1 + e_2 + e_3\}$ show, using the given formulae, that the reciprocal frame is given by

$$\{f^1, f^2, f^3\} = \{e_1 - \frac{1}{2}(e_2 + e_3),\ \frac{1}{2}(e_3 - e_2),\ e_2\}$$

[these are the reciprocal frames shown in the earlier pictures]

Exercises 2

1. Interchanging the role of frame and reciprocal frame, verify that we can write the frame vectors as

$$e_k = (-1)^{k+1} e^1 \wedge e^2 \wedge \ldots \wedge \check{e}^k \wedge \ldots \wedge e^n \{E^n\}^{-1}$$

where $E^n = e^1 \wedge e^2 \wedge \ldots \wedge e^n \neq 0$.

2. Now show that we can move vectors through each other (changing sign) to give

$$e_k = (-1)^{k-1} e^n \wedge e^{n-1} \wedge \ldots \wedge \check{e}^k \wedge \ldots \wedge e^1 \{IV\}$$

where $\{E^n\}^{-1} = IV$, and V is therefore a *volume* factor.

Example: Recovering a Rotor in 3-d

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A very easy way of recovering rotations.

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ie the derivative with respect to the first coordinate, keeping the second and third coordinates constant.

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We will see later that the definition of ∇ is independent of the choice of frame.

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See later discussions of electromagnetism.

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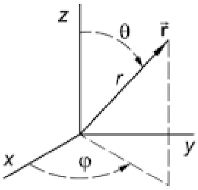

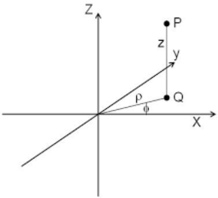

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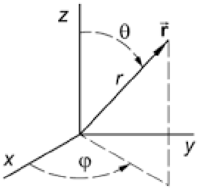

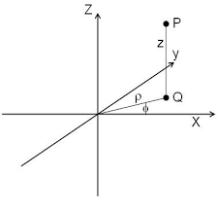

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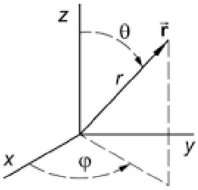

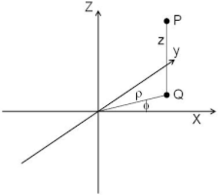

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$$e^j = \nabla x^j \qquad [\text{note}: \quad \nabla \wedge e^j = \nabla \wedge \nabla x^j = 0]$$

Curvilinear Coordinates : Summary

Given coordinates $\{x^i,\ i = 1, ..., n\}$, which any position vector, $\mathbf{r}$ [note, use boldface to distinguish from distance from origin], can be expressed in terms of, we can define a set of frame vectors as

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We see therefore that the vector derivative is crucial in relating coordinates to frames – and we will see how this simplifies manipulations in curvilinear coordinates.

Div, Grad, Curl in Curvilinear Coordinates

Gradient of a Scalar Function, ψ

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Div, Grad, Curl cont....

Now, we can write the expressions for div and curl in a way which makes them easier to relate to the standard expressions for derivatives in curvilinear coordinates.

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Now, take the pseudovector ($n-1$-blade) $P = (-1)^{k-1} e^n \wedge e^{n-1} \wedge \ldots \wedge \check{e}^k \wedge \ldots \wedge e^1$, and recall that $e_i = PIV$ [See Exercises 2]. So that (where $\langle X \rangle$ denotes the scalar part of X)

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$$\nabla \cdot e_i = \langle \nabla (PIV) \rangle = \langle (\nabla P) IV \rangle + \langle PI (\nabla V) \rangle$$

Div, Grad, Curl cont....

After some manipulation (which will be outlined in the following exercises) we are able to write

$$\nabla \cdot J = e_i \cdot (\nabla J^i) + J^i (e_i \cdot \nabla (\ln V))$$

$$= e_i \cdot (\nabla J^i) + J \cdot (\nabla (\ln V)) = \frac{1}{V} \frac{\partial}{\partial x^i} (V J^i)$$

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Curl of a Vector Function, J

$$\boxed{\nabla \wedge J = (\nabla J_i)\wedge e^i} \qquad [\text{bivector}]$$

Exercises 3

1. Since IV is a pseudoscalar, show that

$$\langle(\nabla P)IV\rangle = \langle(\nabla\wedge P)IV)\rangle$$

2. Using the fact that $e_i = PIV$, show that

$$PI(\nabla V) = e_i\nabla(\ln V)$$

3. Verify that $\nabla\wedge a\wedge b = (\nabla\wedge a)\wedge b - a\wedge(\nabla\wedge b)$,
   and then, using our previous result of $\nabla\wedge e^i = 0$, show that

$$\nabla\wedge P = 0$$

   and therefore that

$$\nabla\cdot e_i = e_i\cdot\nabla(\ln V)$$

Exercises 4

1. By expanding $\nabla \wedge J$ as

$$\nabla \wedge J = \nabla \wedge (J_i e^i) = \dot{\nabla} \wedge (\dot{J}_i e^i) + \dot{\nabla} \wedge (J_i \dot{e}^i)$$

explain how we obtain the result $\nabla \wedge J = (\nabla J_i) \wedge e^i$

An Example: Spherical Polars in 3d

Recall our coordinates are (r, θ, ϕ), and we also have an orthogonal set of unit vectors $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$ as shown in the diagram. Thus, we can define a frame via $e_i = \frac{\partial \mathbf{r}}{\partial x^i}$ to be

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Which agrees with the formula given in tables etc.

An Example: Spherical Polars in 3d cont...

Divergence

$$\nabla \cdot J = \frac{1}{V}\frac{\partial (V J^i)}{\partial x^i} = \frac{1}{r^2 \sin\theta}\frac{\partial (r^2 \sin\theta J^r)}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial (r^2 \sin\theta J^\theta)}{\partial \theta} + \frac{1}{r^2 \sin\theta}\frac{\partial (r^2 \sin\theta J^\phi)}{\partial \phi}$$

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Since $V = -r^2\sin\theta$ (see exercises).

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Now, note that

$$J = J^r e_r + J^\theta e_\theta + J^\phi e_\phi = J^r\hat{e}_r + J^\theta(r\hat{e}_\theta) + J^\phi(r\sin\theta)\hat{e}_\phi = \hat{J}_r\hat{e}_r + \hat{J}_\theta\hat{e}_\theta + \hat{J}_\phi\hat{e}_\phi$$

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An Example: Spherical Polars in 3d cont...

Curl

$$\nabla \wedge J = (\nabla J_i) \wedge e^i =$$

$$\left[\frac{\partial J_r}{\partial \theta} - \frac{\partial J_\theta}{\partial r}\right] (e^r \wedge e^\theta) + \left[\frac{\partial J_\phi}{\partial \theta} - \frac{\partial J_\theta}{\partial \phi}\right] (e^\theta \wedge e^\phi) + \left[\frac{\partial J_r}{\partial \phi} - \frac{\partial J_\phi}{\partial r}\right] (e^\phi \wedge e^r)$$

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Now look at the second component, noting that
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which can be written to agree with conventional tabulated form (though we have a bivector and not a vector):

An Example: Spherical Polars in 3d cont...

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$$\frac{1}{r \sin\theta}\left[\frac{\partial (\sin\theta \hat{J}_\phi)}{\partial \theta} - \frac{\partial (\hat{J}_\theta)}{\partial \phi}\right]\hat{e}_r I$$

Exercises 5

1. For 3d spherical polars, show that $V = -r^2 \sin\theta$, where $VI = (E^n)^{-1}$ and $E^n = e^r \wedge e^\theta \wedge e^\phi$.
2. Show that the $e^\phi \wedge e^r$ component of $\nabla \wedge J$ can be written as:
$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial(\hat{J}_r)}{\partial\phi} - \frac{\partial(r\hat{J}_\phi)}{\partial r}\right]\hat{e}_\theta I$$
3. Show that the $e^r \wedge e^\theta$ component on $\nabla \wedge J$ can be written as:
$$\frac{1}{r}\left[\frac{1}{r}\frac{\partial(r\hat{J}_\theta)}{\partial r} - \frac{\partial(\hat{J}_r)}{\partial\theta}\right]\hat{e}_\phi I$$

Check these against standard tabulated formulae.

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All expressions of div, grad, curl etc in terms of the h_is, can then be directly related to the expressions we derive in GA.

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- Shown how to relate coordinates, frame vectors and reciprocal frame vectors in curvilinear coordinate systems.