Classical Physics
with
SpaceTime Algebra

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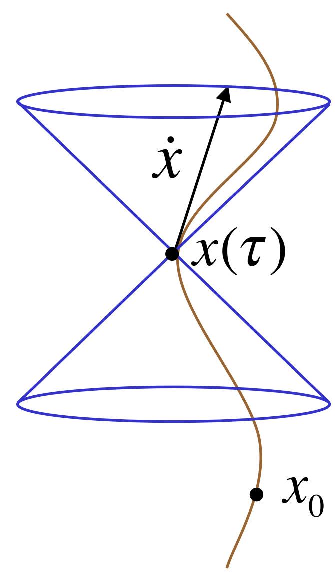


Santalo 2016

Objectives of this talk

To introduce **SpaceTime Algebra (STA)** as *a unified, coordinate-free mathematical language for classical physics:*

- ***the vector derivative*** as the fundamental tool of spacetime calculus and demonstrate its effectiveness ***in electrodynamics***
- vector and ***spinor particle mechanics*** with internal spin and clock.
- ***the spacetime split*** to project 4D invariant physics into 3D geometry of an inertial system.

References

- "Spacetime physics with geometric algebra," *Am. J. Phys.* **71:** 691-704 (2003). <http://geocalc.clas.asu.edu/ >
- *Space Time Algebra* (Springer: 2015) 2nd Ed.

GA unifies and coordinates other algebraic systems

Multivector: $M = \alpha + \mathbf{a} + i\mathbf{b} + i\beta$

Quaternion (spinor): $\psi = \alpha + i\mathbf{b} = \langle M \rangle_+$ (even subalgebra)

Cross product: $\mathbf{a} \wedge \mathbf{b} = i(\mathbf{a} \times \mathbf{b})$

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + i(\mathbf{a} \times \mathbf{b})$$

Gibb's vector algebra

Matrix algebra is subsidiary to and facilitated by GA

Matrix: $a_{ik} = \mathbf{e}_i \cdot \mathbf{a}_k$ (involves only the inner product)

Row vectors

Column vectors

Determinant: $\det(a_{ik}) = (\mathbf{e}_n \wedge \ldots \wedge \mathbf{e}_1) \cdot (\mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_n)$

Complex numbers: $z = \mathbf{ab} = re^{\mathbf{i}\theta}$

Differentiation by vectors

Vector product: $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + i(\mathbf{a} \times \mathbf{b})$

$\Rightarrow$ Vector derivative: $\nabla \mathbf{E} = \nabla \cdot \mathbf{E} + \nabla \wedge \mathbf{E} = \nabla \cdot \mathbf{E} + i(\nabla \times \mathbf{E})$

Names: del (grad) = div + curl

One differential operator!: $\nabla = \partial_{\mathbf{x}} = \boldsymbol{\sigma}_k \partial_k$ $\qquad \partial_k = \dfrac{\partial}{\partial x^k} = \boldsymbol{\sigma}_k \cdot \nabla$

Vector field: $\mathbf{E} = \mathbf{E}(\mathbf{x})$

Electrostatics: $\nabla \mathbf{E} = \rho$ $\quad \Rightarrow \quad$ $\mathbf{E} = \nabla^{-1} \rho$

$\Rightarrow$ $\nabla \cdot \mathbf{E} = \rho \qquad \nabla \wedge \mathbf{E} = 0$

Magnetostatics: $\nabla \mathbf{B} = \frac{1}{c} i \mathbf{J}$ $\quad \Rightarrow \quad$ $\mathbf{B} = \frac{i}{c} \nabla^{-1} \mathbf{J}$

$\Rightarrow$ $\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \frac{1}{c} \mathbf{J}$

Vector derivatives in $\mathcal{R}^n$

Rectangular coordinates: $x^k = x^k(\mathbf{x}) = \boldsymbol{\sigma}_k \cdot \mathbf{x}$ $\qquad \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k = \delta_{jk}$

Position vector: $\mathbf{x} = x^k \boldsymbol{\sigma}_k = x^1\boldsymbol{\sigma}_1 + x^2\boldsymbol{\sigma}_2 + \ldots + x^n\boldsymbol{\sigma}_n$

Vector derivative: $\nabla = \partial_{\mathbf{x}} = \boldsymbol{\sigma}_k \partial_k$ $\qquad \partial_k = \dfrac{\partial}{\partial x^k} = \boldsymbol{\sigma}_k \cdot \nabla$

Basic derivatives for routine calculations:

$$\partial_k \mathbf{x} = \boldsymbol{\sigma}_k \qquad \nabla \mathbf{x} = \boldsymbol{\sigma}_k \partial_k \mathbf{x} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2\boldsymbol{\sigma}_2 + \ldots + \boldsymbol{\sigma}_2\boldsymbol{\sigma}_2 = n$$

$$\mathbf{r} = \mathbf{r}(\mathbf{x}) = \mathbf{x} - \mathbf{x}', \qquad r = |\mathbf{r}| = |\mathbf{x} - \mathbf{x}'|$$

$\Rightarrow$

$$\nabla \mathbf{r} = n \qquad \Rightarrow \qquad \nabla \cdot \mathbf{r} = n \qquad \nabla \wedge \mathbf{r} = 0$$

$$\nabla r = \hat{\mathbf{r}} \qquad \nabla(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a} \cdot \nabla \mathbf{r} = \mathbf{a} \quad \text{for constant } \mathbf{a}$$

$$\nabla \hat{\mathbf{r}} = \tfrac{2}{r} = \nabla r^2 \ ?? \qquad \nabla \times (\mathbf{a} \times \mathbf{r}) = \nabla \cdot (\mathbf{a} \wedge \mathbf{r}) = (n-1)\mathbf{a}$$

$$\nabla r^k = k r^{k-2}\mathbf{r} \qquad \nabla \tfrac{\mathbf{r}}{r^k} = \tfrac{n-k-1}{r^k}$$

One differential operator $\partial = \partial_M$ for all functions $f(M)$ **!!**

Vector derivative: $\nabla = \partial_x$

scalar (field) $\varphi = \varphi(\mathbf{x})$: **Gradient**: $\nabla\varphi = \partial_x \varphi$

vector (field) $\mathbf{A} = \mathbf{A}(\mathbf{x})$: **Grad**: $\nabla\mathbf{A} = \nabla\cdot\mathbf{A} + \nabla\wedge\mathbf{A}$

Divergence: $\nabla\cdot\mathbf{A}$

Curl: $\nabla\wedge\mathbf{A} = i(\nabla\times\mathbf{A})$

multivector $\mathbf{F} = \mathbf{F}(\mathbf{x})$: **Grad**: $\nabla\mathbf{F} = \nabla\cdot\mathbf{F} + \nabla\wedge\mathbf{F}$

Divergence: $\nabla\cdot\mathbf{F}$

Curl: $\nabla\wedge\mathbf{F}$

Scalar derivative: $\partial_t = \frac{\partial}{\partial t}$

Multivector derivative: ∂_M

Chain Rule

Electromagnetic Fields

One Electromagnetic Field!: $F = \mathbf{E} + i\mathbf{B}$

One Maxwell's Equation: $\left(\tfrac{1}{c}\partial_t + \nabla\right)F = \rho - \tfrac{1}{c}\mathbf{J}$

$$\tfrac{1}{c}\partial_t\mathbf{E} + i\tfrac{1}{c}\partial_t\mathbf{B} + \nabla\mathbf{E} + i\nabla\mathbf{B} = \rho - \tfrac{1}{c}\mathbf{J}$$

Use $\nabla\mathbf{E} = \nabla\cdot\mathbf{E} + i(\nabla\times\mathbf{E})$ and separate k-vector parts:

1	*Scalar*	$\nabla\cdot\mathbf{E} = \rho$
3	*Vector*	$\tfrac{1}{c}\partial_t\mathbf{E} + i^2\nabla\times\mathbf{B} = -\tfrac{1}{c}\mathbf{J}$
3	*Bivector*	$i\tfrac{1}{c}\partial_t\mathbf{B} + i\nabla\times\mathbf{E} = 0$
1	*Pseudoscalar*	$i\nabla\cdot\mathbf{B} = 0$

Energy-momentum density:

$$\tfrac{1}{2}FF^\dagger = \tfrac{1}{2}(\mathbf{E} + i\mathbf{B})(\mathbf{E} - i\mathbf{B}) = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \mathbf{E}\times\mathbf{B}$$

Invariants: $F^2 = (\mathbf{E} + i\mathbf{B})^2 = \mathbf{E}^2 - \mathbf{B}^2 + 2i(\mathbf{E}\cdot\mathbf{B})$

Redundancy in conventional mathematics

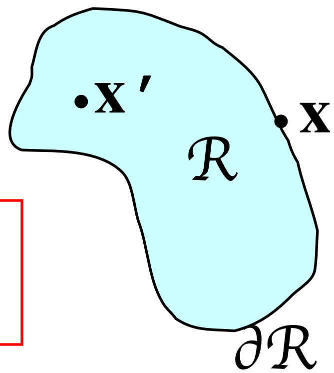
Fund. Thm of calculus: $\int_a^b f'(x)dx = f(b) - f(a)$

Green's Thm: $\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x} \right) dxdy = \oint (Pdx + Qdy)$

Stokes' Thm: $\int \mathbf{n} \cdot (\nabla \times \mathbf{B}) dA = \oint \mathbf{B} \cdot d\mathbf{x}$

Gauss' Thm: $\int \nabla \cdot \mathbf{E} dV = \oint \mathbf{E} \cdot \mathbf{n} dA$

Generalized Stokes' Thm: $\int d \wedge \omega = \oint \omega$



Unification in a single **Fundamental Theorem:** $\int d^k\mathbf{x} \cdot \nabla F = \oint d^{k-1}\mathbf{x} F$

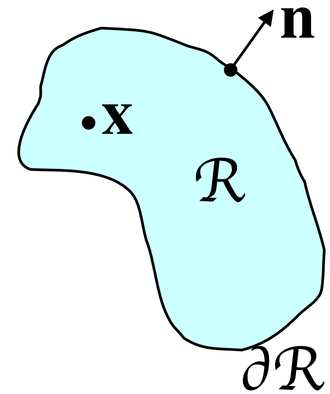
k-vector directed measure: $d^k\mathbf{x} = d_1\mathbf{x} \wedge d_2\mathbf{x} \wedge \ldots \wedge d_k\mathbf{x}$

Generalized Cauchy Thm: $\nabla F = 0 \quad \Leftrightarrow \quad \oint d^n\mathbf{x} F = 0$

Cauchy-Riemann Eqn.

Antiderivatives: $\nabla F = s \quad \Rightarrow \quad F = \nabla^{-1} s$

$$F(\mathbf{x}) = \int_{\mathcal{R}} G(\mathbf{x},\mathbf{x}')d^n\mathbf{x}'s(\mathbf{x}') + \int_{\partial\mathcal{R}} G(\mathbf{x},\mathbf{x}')d^{n-1}\mathbf{x}'F(\mathbf{x}')$$



$$\nabla G(\mathbf{x},\mathbf{x}') = \delta^n(\mathbf{x}-\mathbf{x}')I_n^{-1}$$

$$d^n\mathbf{x} = I_n d^n x$$

$$d^{n-1}\mathbf{x} = I_n \mathbf{n} d^{n-1} x$$

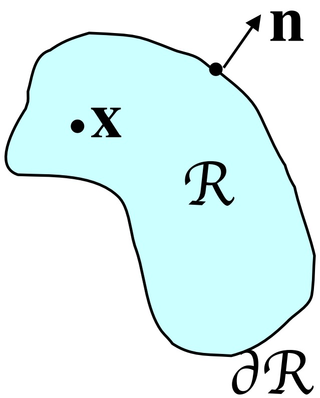
$F = \mathbf{E} + i\mathbf{B}$ solves electrostatic and magnetostatic problems

$$\mathbf{r} = \mathbf{r}(\mathbf{x}) = \mathbf{x} - \mathbf{x}', \qquad r = |\mathbf{r}| = |\mathbf{x}-\mathbf{x}'|$$

$$\nabla \frac{\mathbf{r}}{r^n} = \delta^n(r)\Omega_n$$

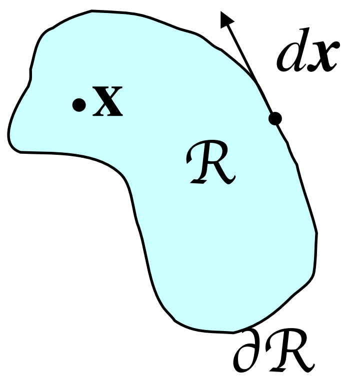
solid angle	pseudoscalar
$\Omega_2 = 2\pi$	$I_2 = \mathbf{i} = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1$
$\Omega_3 = 4\pi$	$I_3 = i = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$

$$\mathbf{E}(\mathbf{x}) = \frac{1}{\Omega_n}\left\{\int_{\mathcal{R}} \frac{\mathbf{r}}{r^n} d^n x' \rho(\mathbf{x}') + \int_{\partial\mathcal{R}} \frac{\mathbf{r}}{r^n} d^{n-1}x' \, \mathbf{n}\mathbf{E}(\mathbf{x}')\right\}$$



dim $\mathcal{R} = n = 3$:

$$\mathbf{E}(\mathbf{x}) = \int_{\mathcal{R}} \frac{d^3x'}{4\pi} \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \rho(\mathbf{x}') + \int_{\partial\mathcal{R}} \frac{d^2x'}{4\pi} \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \mathbf{n}\,\mathbf{E}(\mathbf{x}')$$



dim $\mathcal{R} = n = 2$:

$$d\mathbf{x}' = \mathbf{i}\mathbf{n}dx'$$

$$\mathbf{E}(\mathbf{x}) = \frac{1}{2\pi} \int_{\mathcal{R}} d^2x' \frac{1}{\mathbf{x}-\mathbf{x}'} \rho(\mathbf{x}') + \frac{1}{2\pi\mathbf{i}} \int_{\partial\mathcal{R}} \frac{1}{\mathbf{x}-\mathbf{x}'} d\mathbf{x}'\,\mathbf{E}(\mathbf{x}')$$

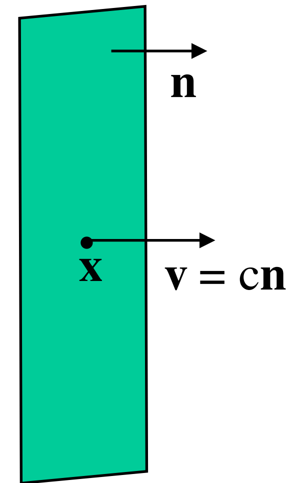
Complex variable: $z = \mathbf{a}\mathbf{x}$ $\quad d\mathbf{x} = \mathbf{a}dz$

Generalised Cauchy Integral Formula

$$F(z) = \frac{1}{2\pi} \int_{\mathcal{R}} d^2x' \frac{1}{z-z'} \rho(z') + \frac{1}{2\pi\mathbf{i}} \int_{\partial\mathcal{R}} \frac{dz'}{z-z'} F(z')$$

EM Plane Waves: Solutions of the form $F(\mathbf{x},t) = F(s)$

$s = t - \frac{\mathbf{x}\cdot\mathbf{n}}{c}$ constant on moving plane $\mathbf{x} = \mathbf{x}(t)$



$$\frac{ds}{dt} = 1 - \frac{\dot{\mathbf{x}}\cdot\mathbf{n}}{c} = 0 \quad \rightarrow \quad \dot{\mathbf{x}}\cdot\mathbf{n} = \mathbf{vn} = c \quad \rightarrow \quad \boxed{\mathbf{v} = c\mathbf{n}}$$

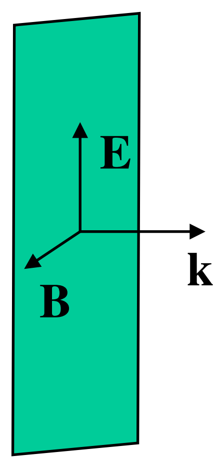
$$\left(\tfrac{1}{c}\partial_t + \nabla\right)F = \boxed{0 = (1+\mathbf{n})\frac{dF}{ds}}$$

Shock wave!

Monochromatic: $F(\mathbf{x},t) = f\,e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}$

$$\Rightarrow \left(\frac{\omega}{c} - \mathbf{k}\right)Fi = 0 \quad \Rightarrow \quad \left(\frac{\omega}{c} + \mathbf{k}\right)\left(\frac{\omega}{c} - \mathbf{k}\right)F = \left(\frac{\omega^2}{c^2} - \mathbf{k}^2\right)F = 0$$

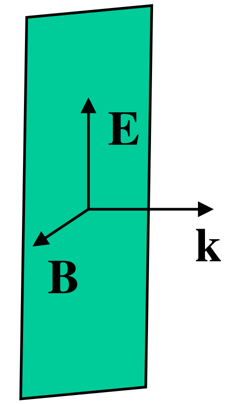
$$\Rightarrow \boxed{\mathbf{n}F = F} \qquad F \neq 0 \quad \Rightarrow \quad |\mathbf{k}| = \frac{\omega}{c}, \quad \omega > 0$$



$$\mathbf{n}(\mathbf{E} + i\mathbf{B}) = \mathbf{E} + i\mathbf{B} \quad \Rightarrow \quad \boxed{\mathbf{nE} = i\mathbf{B}}$$

$$F = \mathbf{E} + i\mathbf{B} = (1+\mathbf{n})\mathbf{E} = \mathbf{E}(1-\mathbf{n}) = \mathbf{E}_0(1-\mathbf{n})e^{i\omega s}$$

Plane Waves: $F = F(s) = F(t - \mathbf{x}\cdot\hat{\mathbf{k}}c^{-1})$



$$F = \mathbf{E} + i\mathbf{B} = (1+\mathbf{n})\mathbf{E} = \mathbf{E}(1-\mathbf{n})$$

$$F^2 = \mathbf{E}^2 - \mathbf{B}^2 + 2i(\mathbf{E}\cdot\mathbf{B}) = 0$$

Monochromatic

Wave: $F(\mathbf{x},t) = f\,e^{\pm i(\omega t - \mathbf{k}\cdot\mathbf{x})} = (\mathbf{E}_0 + i\mathbf{B}_0)e^{\pm i\mathbf{n}(\omega t - \mathbf{k}\cdot\mathbf{x})}$

Wave packet: $F = fz(s), \quad f = (1+\mathbf{n})\mathbf{e}, \quad |f|^2 = f^\dagger f = 2$

$$z(s) = \int_{-\infty}^{\infty} d\omega\, \alpha(\omega) e^{i\omega s} = \int_0^{\infty} d\omega \left[\alpha_+(\omega)e^{i\omega s} + \alpha_-(\omega)e^{-i\omega s}\right]$$

$\alpha_\pm(\omega) = \alpha(\pm|\omega|)$ Sign of ω = helicity

Energy density: $\frac{1}{2}\left\langle FF^\dagger \right\rangle = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) = |z|^2$

$\Rightarrow$ $z(s)$ describes the energy, frequency and *polarization* structure of the plane wave

Electromagnetic Fields in Continuous Media

$$\left(\tfrac{1}{c}\partial_t + \nabla\right)(\mathbf{E} + i\mathbf{B}) = \rho - \tfrac{1}{c}\mathbf{J} = \rho_\mathrm{f} + \rho_\mathrm{b} - \tfrac{1}{c}(\mathbf{J}_\mathrm{f} + \mathbf{J}_\mathrm{b})$$

Bound charge density: $\rho_\mathrm{b} = -\nabla \cdot \mathbf{P}$ Free charge density ρ_f

Bound current: $\mathbf{J}_\mathrm{b} = \partial_t \mathbf{P} + c\nabla \times \mathbf{M}$ Free charge density $\mathbf{J}_\mathrm{f}$

Constitutive equations for linear, homogeneous, isotropic media:

$\mathbf{P} = (\varepsilon - 1)\mathbf{E}$ ε = permitivity (dielectric constant)

$\mathbf{M} = (1 - \mu^{-1})\mathbf{B}$ μ = (magnetic) permeability

$\mathbf{J}_\mathrm{f} = \sigma \mathbf{E}$ σ = conductivity (Ohm's law)

Multiply even part of Max. eqn. by $\sqrt{\dfrac{\mu}{\varepsilon}}$ to get

$$\left(\frac{\sqrt{\mu\varepsilon}}{c}\partial_t + \nabla\right)\left(\mathbf{E} + \frac{i}{\sqrt{\mu\varepsilon}}\mathbf{B}\right) = -\frac{\sigma}{c}\sqrt{\frac{\mu}{\varepsilon}}\,\mathbf{E}$$

Gaussian Units

Or: $\left(\tfrac{1}{v}\partial_t + \nabla\right)G = 0$

$G = \mathbf{E} + \tfrac{i}{n}\mathbf{B}$ $n = \sqrt{\mu\varepsilon} = \tfrac{c}{v}$

What is free space?

Maxwell's equation for a homogeneous, isotropic medium

ε = permitivity (dielectric constant)

μ = (magnetic) permeability

$$G = \mathbf{E} + \frac{i}{\sqrt{\mu\varepsilon}}\mathbf{B}$$

$$(\sqrt{\mu\varepsilon}\,\partial_t - \nabla)G = 0 \qquad \text{Maxwell's Equation}$$

$$(\sqrt{\mu\varepsilon}\,\partial_t + \nabla)\times(\sqrt{\mu\varepsilon}\,\partial_t - \nabla)G = 0$$

$$= (\mu\varepsilon\partial_t^2 - \nabla^2)G = 0$$

$$(c^{-2}\partial_t^2 - \nabla^2)G = 0 \qquad \text{Wave Equation}$$

$c = 1/\sqrt{\mu\varepsilon}$ = velocity of light in the medium = free space

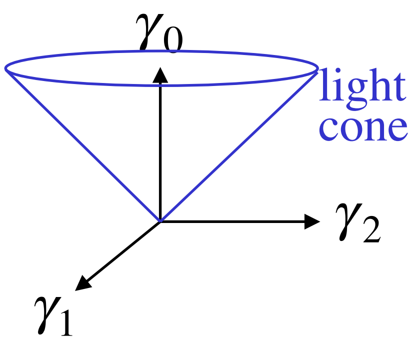
D'Alembertian: $\square^2 = c^{-2}\partial_t^2 - \nabla^2$ Wave operator

Invariant under Lorentz transformations

$\Longrightarrow$ *Theory of relativity* $\qquad$ *But* $\sqrt{\dfrac{\mu}{\varepsilon}} = \rho(x) = ??$

SpaceTime Algebra (STA): $\mathcal{R}_{1,3} = \mathcal{G}(\mathcal{R}^{1,3})$

Generated by a frame of vectors $\{\gamma_\mu\}$



STA $\xrightarrow[\text{rep}]{\text{matrix}}$ (Real) Dirac Algebra

Product: $\gamma_\mu \gamma_\nu = \gamma_\mu \cdot \gamma_\nu + \gamma_\mu \wedge \gamma_\nu$

Metric: $g_{\mu\nu} \equiv \gamma_\mu \cdot \gamma_\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu)$ $\qquad \gamma_0{}^2 = 1$

STA basis: $1, \quad \gamma_\mu, \quad \gamma_\mu \wedge \gamma_\nu, \quad i\gamma_\mu, \quad i = \gamma_0\gamma_1\gamma_2\gamma_3$ $\qquad \gamma_k{}^2 = -1$

scalar, vector, bivector, pseudovector, pseudoscalar

Vector: $a = a^\mu \gamma_\mu$ $\qquad$ Bivector: $F = \frac{1}{2} F^{\nu\mu} \gamma_\mu \wedge \gamma_\nu$

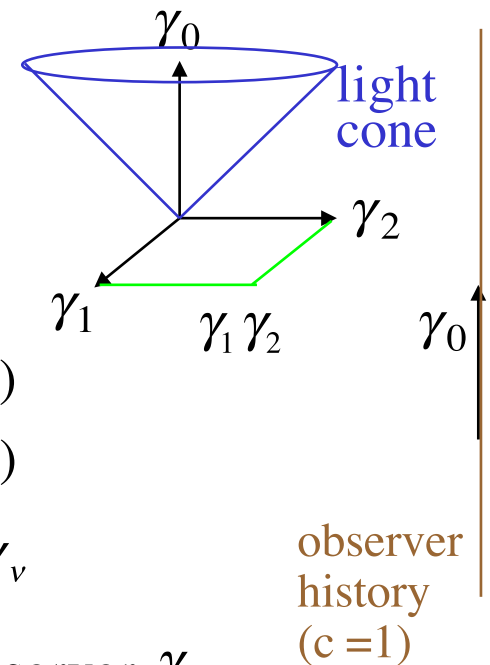
Unit pseudoscalar: i $\qquad$ $i^2 = -1, \quad ia = -ai$

Multivector: $M = \alpha + a + F + ib + i\beta$ $\qquad$ $\gamma_0 i = -i\gamma_0 = \gamma_1\gamma_2\gamma_3$

$$1 + 4 + 6 + 4 + 1 = 16 = 2^4$$

Dual: $iM = i\alpha + ia + iF - b - \beta$

SpaceTime Algebra (STA):



STA $\xrightarrow[\text{rep}]{\text{matrix}}$ **(Real) Dirac Algebra**

Generated by a frame of vectors: $\{\gamma_\mu\}$

Geometric product: $\gamma_0{}^2 = 1, \qquad \gamma_k{}^2 = -1 \quad (k = 1,2,3)$

bivector: $\gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu \equiv \gamma_\mu \wedge \gamma_\nu \quad (\mu \neq \nu)$

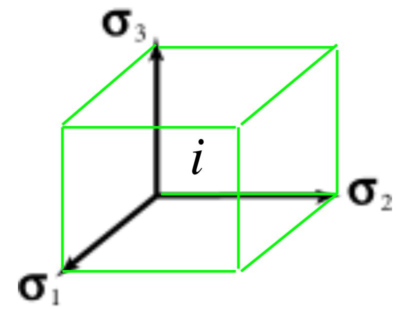
$p = p^\mu \gamma_\mu = mv \qquad p^2 = m^2 \qquad\qquad F = \tfrac{1}{2} F^{\nu\mu} \gamma_\mu \wedge \gamma_\nu$

SpaceTime Split:

Inertial system determined by observer γ_0

Spatial vector frame: $\{\boldsymbol{\sigma}_k = \gamma_k \gamma_0\}$

Unit pseudoscalar: $i = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$



$p\gamma_0 = (E\gamma_0 + p^k \gamma_k)\gamma_0 = E + \mathbf{p}$

$F = F^{0k} \boldsymbol{\sigma}_k + \tfrac{1}{2} F^{jk} \boldsymbol{\sigma}_k \boldsymbol{\sigma}_j = \mathbf{E} + i\mathbf{B} \qquad i^2 = -1$

[D. Hestenes (2003), Am. J. Phys. **71**: 691-714]

Geometric Calculus & Electrodynamics

Spacetime point: $x = x^\mu \gamma_\mu$

Coordinates: $x^\mu = x \cdot \gamma^\mu$

Derivative: $\nabla = \partial_x = \gamma^\mu \partial_\mu$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \gamma_\mu \cdot \partial$$

EM field: $F = F(x) = \frac{1}{2} F^{\nu\mu} \gamma_\mu \wedge \gamma_\nu$

ST split: $F = \mathbf{E} + i\mathbf{B}$

Current: $J = J(x) = J^\mu \gamma_\mu$

$$\gamma_0 J = \rho - \mathbf{J}$$

Maxwell's equation: $\boxed{\nabla F = J}$

$$\gamma_0 \nabla F = (\partial_t + \partial_\mathbf{x})(\mathbf{E} + i\mathbf{B}) = \rho - \mathbf{J}$$

$$\nabla = \partial_\mathbf{x}$$

$$\nabla \mathbf{E} = \nabla \cdot \mathbf{E} + \nabla \wedge \mathbf{E} = \nabla \cdot \mathbf{E} + i\nabla \times \mathbf{E} \quad \Rightarrow \quad \begin{aligned} &\nabla \cdot \mathbf{E} = \rho \\ &\partial_t \mathbf{E} + i(i\nabla \times \mathbf{B}) = -\mathbf{J} \\ &\nabla \wedge \mathbf{E} + i\partial_t \mathbf{B} = 0 \\ &i(\nabla \cdot \mathbf{B}) = 0 \end{aligned}$$

Lorentz Force: $\boxed{m\dot{v} = qF \cdot v}$

$$v\gamma_0 = v \cdot \gamma_0 (1 + v \wedge \gamma_0 / v \cdot \gamma_0) = \gamma(1 + \mathbf{v})$$

ST split: $m\dot{\gamma} = q\mathbf{E} \cdot \mathbf{v}$ $\qquad m\dot{\mathbf{v}} = q\gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ $\qquad (c = 1)$

Summary for rotations in 2D, 3D and beyond

Thm. I: Every rotation can be expressed in the canonical form:

$$\mathbf{x} \rightarrow \quad \boxed{\mathbf{x}' = U\mathbf{x}U^\dagger} \qquad \text{where } UU^\dagger = 1 \text{ and } U \text{ is even}$$

Note: $$(\mathbf{x}')^2 = U\mathbf{x}U^\dagger U\mathbf{x}U^\dagger = U\mathbf{x}^2U^\dagger = UU^\dagger\mathbf{x}^2 = \mathbf{x}^2$$

Thm. II: Every rotation in 3D can be expressed as product of two reflections:

$$\left.\begin{aligned} U &= \mathbf{ba} \\ U^\dagger &= \mathbf{ab} \end{aligned}\right\} \qquad UU^\dagger = \mathbf{baab} = \mathbf{a}^2 = 1$$

Generalizations:

III. Thm I applies to **Lorentz transformations of spacetime**

IV. Cartan-Dieudonné Thm (Lipschitz, 1880): Every orthogonal transformation can be represented in the form: $\boxed{U = \mathbf{a}_n \ldots \mathbf{a}_2\mathbf{a}_1}$

Advantages over matrix form for rotations:

— coordinate-free

— composition of rotations: $\boxed{U_2U_1 = U_3}$

— parametrizations (see NFCM)

Lorentz rotations without matrices or coordinates

Rotation of a frame: $\gamma_\mu \rightarrow \boxed{e_\mu = R\gamma_\mu \tilde{R}} = a_\mu^\eta \gamma_\eta$

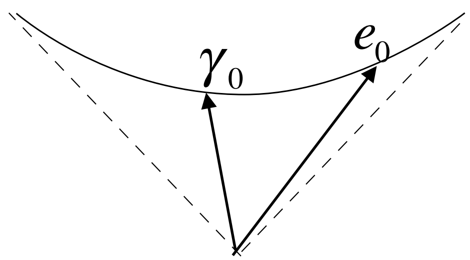
Matrix representation: $a_\mu^\eta = \gamma^\eta \cdot e_\mu = \langle \gamma^\eta R \gamma_\mu \tilde{R} \rangle$

Spin representation: $R = \pm \dfrac{A}{(\tilde{A}A)^{1/2}}$ $\quad A \equiv e_\mu \gamma^\mu = a_\mu^\eta \gamma_\eta \gamma^\mu$

Rotor R defined by: $R\tilde{R} = 1$ $\quad Ri = iR$ $\quad$ or: $R = e^{\frac{1}{2}B}$ $\quad \tilde{R} = e^{-\frac{1}{2}B}$

Orthogonality: $e_\mu \cdot e_\nu = \langle R\gamma_\mu \tilde{R} R \gamma_\nu \tilde{R} \rangle = \langle R \gamma_\mu \gamma_\nu \tilde{R} \rangle = \gamma_\mu \cdot \gamma_\nu$

SpaceTime Split: $\boxed{R = LU}$



Boost: $e_0 = R\gamma_0 \tilde{R} = L\gamma_0 \tilde{L} = L^2 \gamma_0$

$$\boxed{L = (e_0 \gamma_0)^{1/2}}$$

Spatial rotation: $U\gamma_0 \tilde{U} = \gamma_0$

$$\Rightarrow \quad \mathbf{e}_k \equiv U \boldsymbol{\sigma}_k \tilde{U} = U\gamma_k \gamma_0 \tilde{U} = U \gamma_k \tilde{U} \gamma_0 = \tilde{L} e_k e_0 L$$

SpaceTime-splits and particle kinematics

Inertial observer defined by a unit timelike vector: γ_0

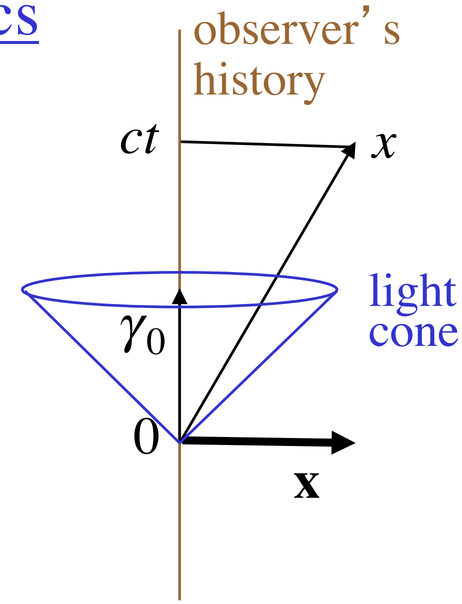
ST-split of a spacetime point x: $\gamma_0^{\,2} = 1$

$$x\gamma_0 = x\cdot\gamma_0 + x\wedge\gamma_0$$

$$x\gamma_0 = ct + \mathbf{x} \qquad \Leftrightarrow \qquad ct = x\cdot\gamma_0$$

$$\gamma_0 x = ct - \mathbf{x} \qquad \mathbf{x} = x\wedge\gamma_0$$

$$x^2 = (x\gamma_0)(\gamma_0 x) = (ct+\mathbf{x})(ct-\mathbf{x}) = c^2t^2 - \mathbf{x}^2$$



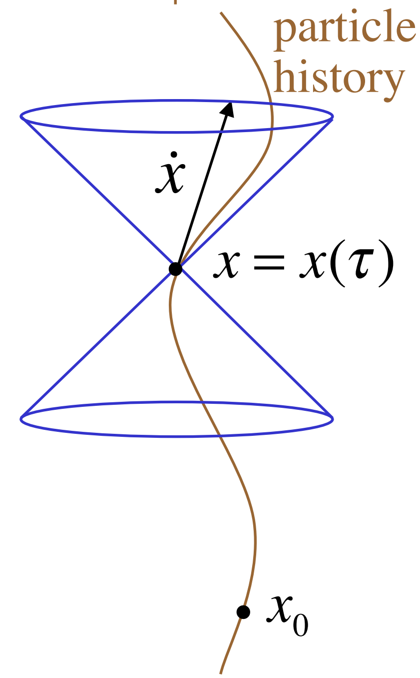
Particle history: $x = x(\tau)$ $\quad |dx| = d\tau$ $\quad$ c = 1

proper velocity: $v = \dot{x} = \dfrac{dx}{d\tau}$

ST-split: $v\gamma_0 = v_0(1+\mathbf{v}) \implies v^2 = 1 = v_0^{\,2}(1-\mathbf{v}^2)$

time dilation factor: $v_0 = v\cdot\gamma_0 = \dfrac{dt}{d\tau} = (1-\mathbf{v}^2)^{-\frac{1}{2}}$

relative velocity: $\mathbf{v} = \dfrac{d\mathbf{x}}{dt} = \dfrac{d\tau}{dt}\dfrac{d\mathbf{x}}{d\tau} = \dfrac{v\wedge\gamma_0}{v_0}$



Relativistic Physics

invariant vectors vs. covariant paravectors

$$\mathrm{x} \longrightarrow \mathrm{x}\gamma_0 = \mathrm{x} \cdot \gamma_0 + \mathrm{x} \wedge \gamma_0 = ct + \mathbf{x} \quad \Leftrightarrow \quad X = ct + \mathbf{x}$$

Recommended exercise: Undo the original paravector
treatment of relativistic mechanics,
written in 1980, published in 1999
in NFCM (chapter 9 in 2nd Ed.)

The chief advocate: W. E. Baylis.
Geometry of Paravector Space with Applications
to Relativistic Physics (Kluwer: 2004)

Electrodynamics (W. E. Baylis, Birkhäuser, 1999)

Exercise: Translate from covariant to invariant

Spinor particle dynamics

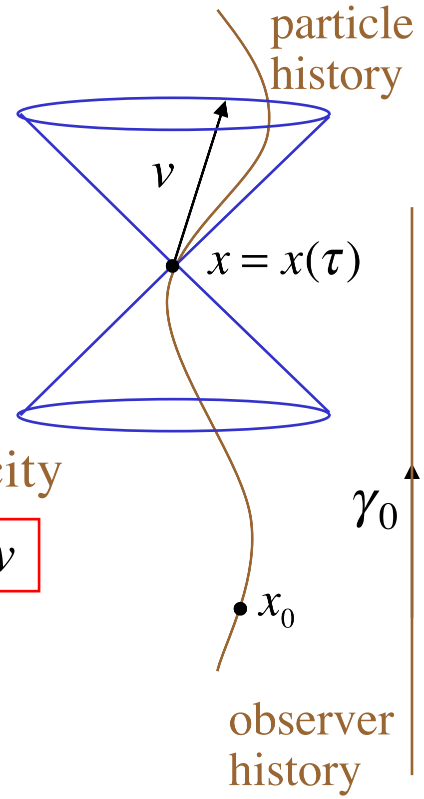
$v^2 = 1 \quad \Rightarrow$ particle velocity $v = \dot{x} = v(\tau)$ can only rotate as the particle traverses its history $x = x(\tau)$

$\Rightarrow$ $\boxed{v = R\gamma_0\tilde{R}}$ Rotor: $R = R(\tau)$

Rotor eqn. of motion: $\dfrac{dR}{d\tau} = \boxed{\dot{R} = \dfrac{1}{2}\Omega R} \quad \Leftrightarrow \quad \Omega = 2\dot{R}\tilde{R} = -2R\dot{\tilde{R}}$

$\Omega =$ rotational velocity

$$\frac{dv}{d\tau} = \dot{v} = \dot{R}\gamma_0\tilde{R} + R\gamma_0\dot{\tilde{R}} = \frac{1}{2}(\Omega v + v\Omega) \quad \Rightarrow \quad \boxed{\dot{v} = \Omega \cdot v}$$



What does the rotor equation buy us?

- $\Omega = \dfrac{q}{m}F \quad \Rightarrow \quad \boxed{m\dot{v} = qF \cdot v}$ Lorentz force!
- $\dot{\Omega} = 0 \quad \Rightarrow$ general solution: $R = e^{\frac{1}{2}\Omega\tau}R_0$
- Comoving frame: $e_0 = R\gamma_0\tilde{R} = v = \dot{x} \quad \boxed{e_\mu = R\gamma_\mu\tilde{R}} \Rightarrow \boxed{\dot{e}_\mu = \Omega \cdot e_\mu}$

Spin: $s = \dfrac{\hbar}{2}e_3 = \dfrac{\hbar}{2}R\gamma_3\tilde{R} \quad \Rightarrow \quad \dot{s} = \dfrac{q}{m}F \cdot s \quad \boxed{g = 2}$

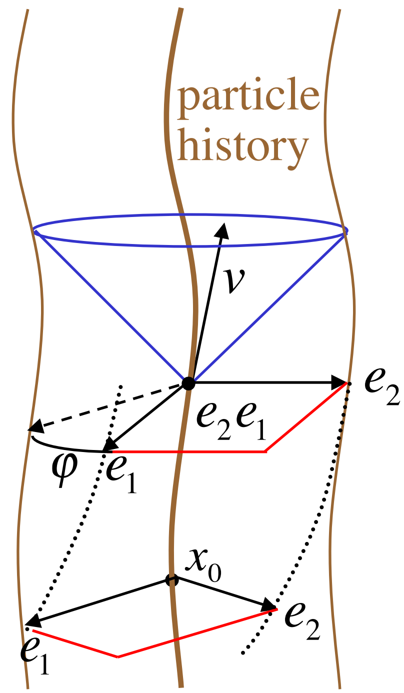
Classical limit of Dirac equation!

Real spinors are natural & useful in both classical & quantum theory!

Implications of Real Dirac Theory: the geometry of electron motion with

de Broglie's electron clock in quantum mechanics!

Dirac equation determines a congruence of streamlines,
each a potential particle history $x = x(\tau)$
with particle velocity $\dot{x} = v(\tau) = R\gamma_0\tilde{R}$



Spinning frame picture of electron motion

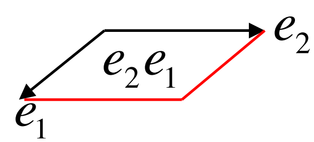
Dirac wave function $\Psi = (\rho e^{i\beta})^{\frac{1}{2}} R$ determines

Rotor: $R = R(\tau) = R[x(\tau)] = R_0 e^{-\frac{1}{2}\varphi\gamma_2\gamma_1}$

comoving frame: $e_\mu = R\gamma_\mu\tilde{R}$ phase $\varphi/2$

velocity: $e_0 = R\gamma_0\tilde{R} = v$

Spin: $S = \frac{\hbar}{2} e_2 e_1$



$$e_2 e_1 = R\gamma_2\gamma_1\tilde{R} = R_0\gamma_2\gamma_1\tilde{R}_0$$

Plane wave solution: $R = R_0 e^{-\frac{1}{2}\varphi\gamma_2\gamma_1} = R_0 e^{-\frac{p\cdot x}{\hbar}\gamma_2\gamma_1}$ $\tau = \tau(x) = v \cdot x$

$$p = m_e c^2 v \quad \Rightarrow \quad \frac{1}{2}\varphi = \frac{p\cdot x}{\hbar} = \frac{m_e c^2}{\hbar} v \cdot x = \omega_B \tau$$

$$\omega_B = \frac{m_e c^2}{\hbar} = \frac{1}{2}\frac{d\varphi}{d\tau}$$

Summary and comparison

	Tensor form	STA		
Field strength:	$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$	$F = \nabla \wedge A$	$F = \mathbf{E} + i\mathbf{B}$	(ST-split)
Maxwell's Equations:	$\partial_\mu F^{\mu\nu} = J^\nu$ $\partial_{[\alpha} F_{\mu\nu]} = 0$	$\left.\begin{array}{l}\nabla \cdot F = J \\ \nabla \wedge F = 0\end{array}\right\}$	$\boxed{\nabla F = J}$	$\Rightarrow \quad F = \nabla^{-1} J$
Lorentz force:	$m \dfrac{dv^\mu}{d\tau} = qF^{\mu\nu} v_\nu$	$\boxed{m\dot{v} = qF \cdot v}$		
	coordinate-dependent	coordinate-free		
Spinors:	None!	$e_\mu = R\gamma_\mu \tilde{R}$	$\boxed{\dot{R} = \frac{1}{2}\Omega R}$	rigid body precession!
Quantum Mechanics:	Dirac matrices superimposed	Real spinors natural & useful in both classical & quantum theory!		
General Relativity:	• General covariance • Equivalence principle	• Displacement gauge invariance • Rotation gauge covariance		

Current Status of GA & STA

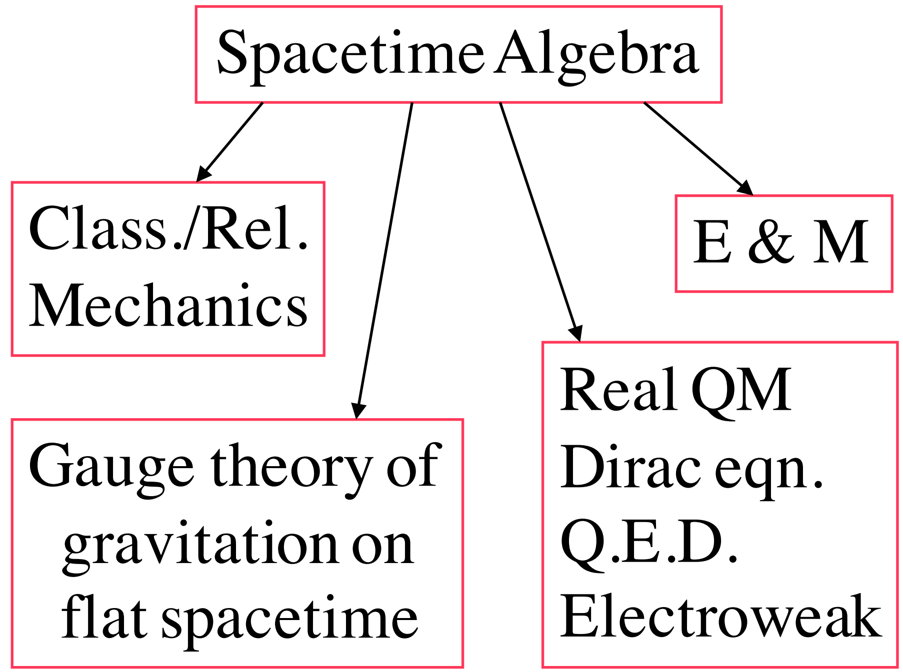
Mathematical scope

— greater than any other system!

Linear algebra
Multilinear algebra
Differential geometry
Hypercomplex function theory
  (unifying and generalizing
  real and complex analysis)
Lie groups as spin groups
Crystallographic group theory
Projective geometry
Computational geometry
  distance geometry
  line geometry & screw theory

Physics scope

— covers all major branches!



Coordinate-free
– Lasenby, Gull & Doran

Geometric basis for complex wave functions in QM!

Outline and References

<http:\\modelingnts.la.asu.edu>   <http://www.mrao.cam.ac.uk>

I. Intro to GA and non-relativistic applications

- Oersted Medal Lecture 2002 (Web and AJP)
- NFCM (Kluwer, 2nd Ed.1999)
- *New Foundations for Mathematical Physics* (Web)
  1. Synopsis of GA   2. Geometric Calculus

II. Relativistic Physics (covariant formulation)

- NFCM (chapter 9 in 2nd Ed.)
- *Electrodynamics* (W. E. Baylis, Birkhäuser, 1999)

III. Spacetime Physics (invariant formulation)

- Spacetime Physics with Geometric Algebra (Web & AJP)
- Doran, Lasenby, Gull, Somaroo & Challinor,
  Spacetime Algebra and Electron Physics (Web)

Lasenby & Doran, *Geometric Algebra for Physicists*
(Cambridge: The University Press, 2002).

THE END