ELECTROMAGNETISM AS A GAUGE THEORY

Start with

$$\nabla\psi I\sigma_3 = m\psi\gamma_0 \tag{33}$$

A global symmetry of this is

$$\psi \mapsto \psi' = \psi e^{I\sigma_3\theta} \tag{34}$$

where $\theta = \text{constant}$. Clearly ψ' is a solution of (33) if ψ is.

But what if $\theta = \theta(x)$? Then, writing $R = e^{I\sigma_3\theta}$, have

$$\nabla\psi' = (\nabla\psi)R + (\nabla\theta)\psi R I\sigma_3.$$

and so

$$\nabla\psi' I\sigma_3 \neq m\psi'\gamma_0.$$

This means the symmetry (34) does not work locally.

Why should we want it to?

— from the structure of the physical statements of the Dirac theory.

These are of two main types:

(i) The values of observables. Formed via inner products

$$\langle\psi|\phi\rangle \leftrightarrow \langle\tilde{\psi}\phi\rangle - \langle\tilde{\psi}\phi I\sigma_3\rangle I\sigma_3$$

(ii) statements of equality like $\psi = \psi_1 + \psi_2$.

The physical content of both these equations is unchanged if all the spinors are rotated by the same locally varying phase factor.

Our theory should be invariant under such changes.

Covariant Derivatives

To achieve this, have to change the ∇ operator to get rid of unwanted term in gradient of R.

Putting $\nabla = \partial_a a\cdot\nabla = \gamma^\mu \partial_\mu$, equation for $\nabla\psi'$ is

$$\nabla\psi' = \partial_a \left(a\cdot\nabla\psi R + \psi a\cdot\nabla R\right).$$

It is the last term which does the damage.

Therefore define a new operator D via

$$D\psi = \partial_a \left(a\cdot\nabla\psi + \tfrac{1}{2}\psi\Omega(a)\right)$$

and a new Dirac equation

$$D\psi I\sigma_3 = m\psi\gamma_0$$

and see what properties $\Omega(a)$ must have to remove unwanted term.

Under $\psi \mapsto \psi' = \psi R$ will have $D \mapsto D'$ where D' should have the same form as D. For general ϕ, set

$$D'\phi = \partial_a \left(a\cdot\nabla\phi + \tfrac{1}{2}\phi\Omega'(a)\right)$$

Our basic requirement is that $\psi' = \psi R$ should solve the Dirac equation with D'

instead of D, if ψ solves the equation with D.

This will work if

$$D'\psi' = D'(\psi R) = (D\psi)R, \tag{35}$$

since then

$$\begin{aligned} D'\psi' I\sigma_3 - m\psi'\gamma_0 &= (D\psi)RI\sigma_3 - m\psi R\gamma_0 \\ &= (-m\psi I\gamma_3)RI\sigma_3 - m\psi R\gamma_0 = 0 \end{aligned}$$

Can see generally that (35) is the right thing
— we want a D that suppresses the differentiation of R.

So let's try our forms for D and D'. We get

$$\begin{aligned} D'(\psi R) &= \partial_a \left(a\cdot\nabla\psi R + \psi a\cdot\nabla R + \tfrac{1}{2}\psi R\Omega'(a)\right) \\ &= D\psi R = \partial_a \left(a\cdot\nabla\psi + \frac{1}{2}\psi\Omega(a)\right) R. \end{aligned}$$

Identifying terms, we must have

$$a\cdot\nabla R + \frac{1}{2}R\Omega'(a) = \frac{1}{2}\Omega(a)R,$$

i.e.

$$\Omega'(a) = \tilde{R}\Omega(a)R - 2\tilde{R}a\cdot\nabla R$$

This gives the transformation property of $\Omega(a)$

— what type of object is it?

$$\tilde{R}R = 1 \implies a\cdot\nabla\tilde{R}R + \tilde{R}a\cdot\nabla R = 0$$

i.e. $\tilde{R}a\cdot\nabla R = -\left(\tilde{R}a\cdot\nabla R\right)^{\sim}$ which is therefore a bivector.

Thus $\Omega(a)$ must be a bivector field. It is called a connection and viewed in group terms belongs to the Lie algebra of the symmetry group.

In general $\Omega(a)$ will not be expressible as the derivative of a rotor field. This is the essence of the gauging step. Take something arising from a derivative, and generalize it to a term that cannot be formed this way.

MINIMALLY-COUPLED DIRAC EQUATION

Now restrict rotation to $\gamma_2\gamma_1$ plane using $R = e^{I\sigma_3\theta}$. Then

$$-2\tilde{R}a\cdot\nabla R = -2e^{-I\sigma_3\theta}a\cdot(\nabla\theta)e^{I\sigma_3\theta}I\sigma_3$$
$$= -2a\cdot(\nabla\theta)I\sigma_3.$$

Generalizing this, we can deduce $\Omega(a) = \lambda a\cdot A I\sigma_3$ where A is a general 4-vector and λ a coupling constant.

Note if A were equal to $\nabla\theta$, then $\nabla\wedge A = 0$. Will generalise this when we look at the field strength tensor. Now have

$$D\psi = \partial_a(a\cdot\nabla\psi + \tfrac{1}{2}\lambda a\cdot A\,\psi I\sigma_3) = \nabla\psi + \tfrac{1}{2}\lambda A\psi I\sigma_3.$$

and so if $\lambda = 2e$ then we get the 'minimally coupled' Dirac equation

$$\nabla\psi I\sigma_3 - eA\psi = m\psi\gamma_0.$$

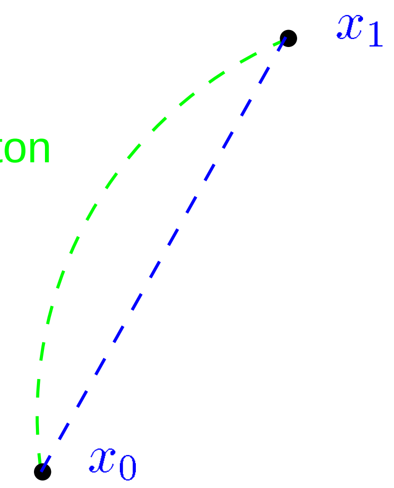
This is simplest (minimal) possible modification to original equation. No extra terms in $F\psi$ or $F^2\psi$ (all acceptable). Nature appears to be ‘minimal’ in its principles.

Gauge Principles for Gravitation

Aim: To model gravitational interactions in terms of (gauge) fields defined in the STA.

A radical departure from GR! The STA is the geometric algebra of flat spacetime. Extra fields cannot change this.

But what about standard arguments that spacetime is curved? These all involve light paths, or measuring rods. All modeled with interacting fields. So photon paths need not be ‘straight’.



The STA vector $x_1 - x_0$ has no measurable significance now. This will follow if we ensure that all physical predictions are independent of the absolute position and orientation of fields in the STA. Only relations between fields are important.

Becomes clearer if we consider fields. Take spinors $\psi_1(x)$ and $\psi_2(x)$. A sample physical statement is

$$\psi_1(x) = \psi_2(x).$$

At a point where one field has a particular value, the second field has the same value. This is independent of where we place the fields in the STA. And independent of where we choose to locate other values of the fields. Could equally well introduce two new fields

$$\psi_1'(x) = \psi_1(x'), \quad \psi_2'(x) = \psi_2(x'),$$

with x' an arbitrary function of x. Equation $\psi_1'(x) = \psi_2'(x)$ has precisely the same physical content as original.

Same is true if act on fields with a spacetime rotor

$$\psi_1' = R\psi_1, \quad \psi_2' = R\psi_2$$

Again, $\psi_1' = \psi_2'$ has same physical content as original equation. Same true of observables, eg $J = \psi\gamma_0\tilde{\psi}$. Now $\psi \mapsto \psi'$ produces the new vector $J' = RJ\tilde{R}$. Hence absolute direction irrelevant.

DISPLACEMENTS

We write

$$x' = f(x)$$

for an arbitrary (differentiable) map between spacetime position vectors. A rule for relating position vectors in same space — not a map between manifolds.

Use this to move field $\psi(x)$ to new field

$$\psi'(x) \equiv \psi(x').$$

Call this a displacement or translation

Now consider behaviour of derivative of ψ, $\nabla\psi = \partial_a \, a\cdot\nabla\psi$. See that

$$
\begin{aligned}
a\cdot\nabla\psi'(x) &= a\cdot\nabla\psi[f(x)] \\
&= \lim_{\epsilon\to 0} \tfrac{1}{\epsilon}\left(\psi[f(x+\epsilon a)] - \psi[f(x)]\right) \\
&= \lim_{\epsilon\to 0} \tfrac{1}{\epsilon}\left(\psi[f(x)+\epsilon \mathsf{f}(a)] - \psi[f(x)]\right).
\end{aligned}
$$

where

$$\mathsf{f}(a) = \mathsf{f}(a;x) = a\cdot\nabla f(x)$$

and have Taylor expanded $f(x+\epsilon a)$ to first order. $\mathsf{f}(a)$ is linear on a. Suppress position dependence where possible. Now have

$$a\cdot\nabla\psi'(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\psi[x'+\epsilon \mathsf{f}(a)] - \psi(x')\right).$$

But this is the vector derivative with respect to x' in $\mathsf{f}(a)$ direction

$$a\cdot\nabla\psi'(x) = \mathsf{f}(a)\cdot\nabla_{x'}\psi(x'),$$

where $\nabla_{x'}$ is derivative with respect to the new vector position variable x'. Since

$$\mathsf{f}(a)\cdot\nabla_{x'} = a\cdot\bar{\mathsf{f}}(\nabla_{x'})$$

get operator relation

$$\nabla_x = \bar{\mathsf{f}}(\nabla_{x'})$$

$\mathsf{f}(a)$ is coordinate-free form of Jacobian.

Now suppose we have a physical relation such as

$$\nabla\phi = A.$$

Scalar ϕ and vector A. (Eg A is pure electromagnetic gauge). Now replace $\phi(x)$ by $\phi'(x) = \phi(x')$ and $A(x)$ by $A'(x) = A(x')$. Left-hand side becomes

$$\nabla\phi'(x) = \bar{\mathsf{f}}(\nabla_{x'})\phi(x') = \bar{\mathsf{f}}[A(x')] = \bar{\mathsf{f}}(A')$$

so no longer equal to A'.

Gauge field must assemble with ∇ to form object which, under displacements, re-evaluates to derivative with respect to the new position vector. Replace ∇ with $\bar{\mathsf{h}}(\nabla)$, with

$$\bar{\mathsf{h}}(a) = \bar{\mathsf{h}}(a;x)$$

an arbitrary function of position, and a linear function of a. Property we require is that

$$\bar{\mathsf{h}}'(\nabla\phi') = \bar{\mathsf{h}}'[\bar{\mathsf{f}}(A')] = \bar{\mathsf{h}}(A';x')$$

Suppressing position dependence, basic requirement is

$$\bar{\mathsf{h}}'(a) = \bar{\mathsf{h}}\bar{\mathsf{f}}^{-1}(a)$$

for general vector a. Now systematically replace ∇ by $\bar{\mathsf{h}}(\nabla)$. Get all equations invariant under displacements. Eg. Dirac equation is now

$$\bar{\mathsf{h}}(\nabla)\psi I\sigma_3 = m\psi\gamma_0.$$

$\bar{\mathsf{h}}$-field not a connection in conventional Yang-Mills sense. But $\bar{\mathsf{h}}$-field does ensure that a symmetry is local, so still called a gauge field.

ROTATIONS

Second symmetry we require is invariance under

$$\psi \mapsto \psi' = R\psi, \tag{36}$$

where R is an arbitrary, position-dependent rotor in spacetime. (Say that R

generates rotations. Understood that this includes boosts.) Back in familiar territory now! Write

$$\bar{\mathsf{h}}(\nabla)\psi = \bar{\mathsf{h}}(\partial_a)\, a\cdot\nabla\psi.$$

To make (36) a symmetry, modify $a\cdot\nabla$ by adding a bivector connection $\Omega(a)$,

$$D_a\psi = a\cdot\nabla + \tfrac{1}{2}\Omega(a)\psi$$

where $\Omega(a)$ has the transformation law

$$\Omega(a) \mapsto \Omega'(a) = R\Omega(a)\tilde{R} - 2a\cdot\nabla R\tilde{R}.$$

Since R is an arbitrary rotor, now no constraint on the terms in $\Omega(a)$. Has $\Omega(a)$ has $6 \times 4 = 24$ degrees of freedom.

Equation now reads

$$D\psi I\sigma_3 = \bar{\mathsf{h}}(\partial_a)D_a\psi I\sigma_3 = m\psi\gamma_0. \tag{37}$$

Replace ψ by ψ' and $\Omega(a)$ by $\Omega'(a)$, find that the left-hand side becomes

$$\bar{\mathsf{h}}(\partial_a)D_a'(R\psi)I\sigma_3 = \bar{\mathsf{h}}(\partial_a)RD_a\psi I\sigma_3$$

But right-hand side is simply $mR\psi\gamma_0$. Need to transform the $\bar{\mathsf{h}}$-field as well,

$$\bar{\mathsf{h}}(a) \mapsto \bar{\mathsf{h}}'(a) = R\bar{\mathsf{h}}(a)\tilde{R}.$$

This is sensible. Recall $\bar{\mathsf{h}}(\nabla\phi) = A$. Invariant under displacements. Also invariant if both vectors are rotated. But rotation of $\bar{\mathsf{h}}(\nabla\phi)$ must be driven by transforming $\bar{\mathsf{h}}$.

Dirac equation now invariant under both rotations and displacements. Achieved by introducing two new gauge fields, $\bar{\mathsf{h}}(a)$ and $\Omega(a)$. A total of $16 + 24 = 40$ degrees of freedom!

The key to deriving the field equations in a gauge theory is the covariant field strength tensor.

THE FIELD STRENGTH

Form commutator of covariant derivatives. First take electromagnetism, $\psi \mapsto \psi R$. With a and b constant vectors, get

$$[D_a, D_b]\psi = \tfrac{1}{2}\psi[a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) - \Omega(a)\times\Omega(b)]$$

All derivatives of ψ have canceled. (Try this as an exercise.)
Restricting to $\Omega(a) = -2a\cdot A\, I\sigma_3$, get term in

$$b\cdot\nabla(a\cdot A\, I\sigma_3) - a\cdot\nabla(b\cdot A\, I\sigma_3) - 2a\cdot A\, b\cdot A\, I\sigma_3\times I\sigma_3$$
$$= (a\wedge b)\cdot(\nabla\wedge A) I\sigma_3 = (a\wedge b)\cdot F\, I\sigma_3$$

Maps bivector $a\wedge b$ linearly onto a pure phase term. In electromagnetism lose mapping and extract $F = \nabla\wedge A$. This is physical field. Vanishes if A is pure gauge.

Rotation Gauge

For rotations, rotors multiply ψ from left, so

$$[D_a, D_b]\psi = \tfrac{1}{2}\mathsf{R}(a\wedge b)\psi$$

where

$$\mathsf{R}(a\wedge b) = a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) + \Omega(a)\times\Omega(b)$$

Right-hand side is antisymmetric on a, b, so a linear function of the bivector $a\wedge b$. Extend to general bivectors

$$\mathsf{R}(a\wedge b + c\wedge d) = \mathsf{R}(a\wedge b) + \mathsf{R}(c\wedge d).$$

Can write the field strength as,

$$\mathsf{R}(B) = \mathsf{R}(B;x)$$

- A position dependent, linear function of the bivector B. Returns a general bivector, so $6\times 6 = 36$ Degrees of freedom.
- Term in $\Omega(a)\times\Omega(b)$ is non-linear. Cannot superpose two solutions to get a third. Much more difficult than electromagnetism.

Transformation properties easy to establish

$$\begin{aligned}[][D'_a, D'_b,]\psi' &= \tfrac{1}{2}\mathsf{R}'(a\wedge b)R\psi \\ &= R[D_a, D_b]\psi = \tfrac{1}{2}R\mathsf{R}(a\wedge b)\psi\end{aligned}$$

so read off that

$$\mathsf{R}'(a\wedge b) = R\mathsf{R}(a\wedge b)\tilde{R}.$$

Field strength now transforms under gauge transformations.

DISPLACEMENT GAUGE FIELD STRENGTH

This is slightly more complicated, so we won't go through it here, but just state the form of the result

The commutator of the $a\cdot\overline{h}(\nabla)$ and $b\cdot\overline{h}(\nabla)$ directional derivatives leads to an object called the Torsion tensor

This is the field-strength corresponding to demanding invariance under translations

We can write it as $\mathcal{S}(a)$ where

$$\mathcal{D}\wedge\overline{h}(a) \equiv \mathcal{S}\left(\overline{h}(a)\right)$$

and $\mathcal{D} \equiv \overline{h}\,(\partial_a)(a\cdot\nabla + \Omega(a)\times)$ is the appropriate derivative for objects that transform two-sidedly under R (i.e. all classical objects)

$\mathcal{S}(a)$ maps vectors to bivectors. If we want to match to General Relativity, then the torsion needs to vanish, i.e. GR would correspond to $\mathcal{D}\wedge\overline{h}(a) = 0$

COVARIANT FIELD STRENGTHS

The $\mathcal{S}(a)$ just defined is already covariant under both the rotation and displacement gauges (we jumped straight to the answer in this form).

For the rotation gauge field strength, $\mathsf{R}(a\wedge b)$, we've just seen that this transforms covariantly (i.e. like $R \ldots \tilde{R}$) under rotation gauge changes, but we need to modify it to make it p.g. covariant.

Won't derive it, but turns out the correct covariant field strength is

$$\mathcal{R}(B) = \mathsf{R}[\mathsf{h}(B)]$$

Factor of $\mathsf{h}(B)$ alters rotation gauge properties.

$$\overline{h}(a) \mapsto \overline{h}'(a) = R\overline{h}(a)\tilde{R}.$$

so adjoint goes as

$$\mathsf{h}(a) \mapsto \mathsf{h}'(a) = \partial_b \langle a R \overline{h}(b) \tilde{R} \rangle = \mathsf{h}(\tilde{R} a R).$$

Summarise transformation properties of $\mathcal{R}(B)$ by:

$$\text{Displacements:} \qquad \mathcal{R}'(B, x) = \mathcal{R}(B, x')$$

$$\text{Rotations:} \qquad \mathcal{R}'(B) = R\mathcal{R}(\tilde{R} B R)\tilde{R}.$$

Just what we want for a covariant tensor. Call $\mathcal{R}(B)$ the Riemann tensor.

Similarly, for the torsion tensor, find

$$\text{Displacements:} \qquad \mathcal{S}'(a, x) = \mathcal{S}(a, x')$$

$$\text{Rotations:} \qquad \mathcal{S}'(a) = R\mathcal{S}(\tilde{R} a R)\tilde{R}.$$

so have succeeded in finding successful forms of the field strengths in our theory

The next step is to set up a Lagrangian, so that we can derive the equations of motion, but let's have a brief interlude where we look at the forms of Riemann in the GTG approach for standard GR solutions

Examples

I. The Schwarzschild Solution

Spherically symmetric source, mass M at rest in γ_0 frame, has

$$\mathcal{R}(B) = -\frac{M}{2r^3}(B + 3\boldsymbol{\sigma}_r B \boldsymbol{\sigma}_r)$$

where $r = |x\wedge\gamma_0|$, $\boldsymbol{\sigma}_r = x\wedge\gamma_0/r$'. $M/2r^3$ controls the tidal force.

II. The Kerr Solution

Outside a rotating black hole get

$$\mathcal{R}(B) = -\frac{M}{2(r + IL\cos\theta)^3}(B + 3\boldsymbol{\sigma}_r B \boldsymbol{\sigma}_r).$$

Get Schwarzschild by $r \mapsto r + IL\cos\theta$. Explains complex structure in Kerr solution!

III. Cosmic Strings

Infinite, pressure-free string along γ_3, density ρ has

$$\mathcal{R}(B) = 8\pi\rho\langle BI\sigma_3\rangle I\sigma_3$$

Get tidal forces in $I\sigma_3$ plane only. Magnitude determined by density.

IV. Cosmology

Isotropic, homogeneous cosmology has

$$\mathcal{R}(B) = 4\pi(\rho + P)B\cdot e_t\, e_t - \frac{1}{3}(8\pi\rho + \Lambda)B.$$

P and ρ are pressure and density, Λ is the cosmological constant, and e_t is 'rest-frame' of the universe (defined by the cosmic microwave background radiation). No other direction present.

The ability to represent the Riemann so compactly (and informatively!) in these ways is unique to GA and GTG. E.g. the Kerr form normally takes pages to describe in conventional approaches, and these miss the essential role of the spacetime pseudoscalar.

LAGRANGIAN AND FIELD EQUATIONS

We seek the simplest scalar object that is fully invariant under our position gauge and rotation gauge transformations. We know that $\mathcal{R}(a\wedge b)$ is fully covariant under both, so we contract $\mathcal{R}(a\wedge b)$ and define the Ricci tensor

$$\mathcal{R}(b) = \partial_a \cdot \mathcal{R}(a\wedge b)$$

(NB. Same symbol. Grade of argument distinguishes type.)
Contract again to get Ricci scalar

$$\mathcal{R} = \partial_a \cdot \mathcal{R}(a)$$

Our first scalar observable. Note can write in a form where the dependence on the gauge functions is clearer:

$$\mathcal{R} = \langle \overline{h}(\partial_c \wedge \partial_b)\mathsf{R}(b \wedge c)\rangle$$

((Exercise: prove this.) This will be our Lagrangian density which we will now use in a

LAGRANGIAN-BASED MULTIVECTOR DERIVATIVE APPROACH

- Will use our multivector derivative approach to deriving the e.o.m.
- Since don't have too much time, will illustrate this just for the equations involving the Einstein tensor and stress-energy tensor
- These are what's called the Einstein equations anyway, (GR effectively just assumes *a priori* that the torsion vanishes) and will give us the chance to illustrate some points about SETs that we alluded to earlier
- However, will state the results for the torsion field, since this will enable us to see how torsion can be induced by quantum spin

We take the overall action integral to be of the form

$$S = \int |d^4x| \; \det \underline{h}^{-1} (\tfrac{1}{2}\mathcal{R} + \Lambda - \kappa \mathcal{L}_m),$$

where $\mathcal{L}_m$ describes the matter content and $\kappa = 8\pi G$.

The $\det \underline{h}^{-1}$ part is included to make the $\int |d^4x|$ invariant under remappings (see GTG paper for details).

We have also included the cosmological constant Λ, which on invariance grounds is another term we are free to add in

The independent dynamical variables are $\overline{h}(a)$ and $\Omega(a)$, and we assume that $\mathcal{L}_m$ contains no second-order derivatives, so that $\overline{h}(a)$ and $\Omega(a)$ appear undifferentiated in the matter Lagrangian.

The $\overline{h}$-field is undifferentiated in the entire action, so the Euler–Lagrange equation for $\overline{h}$ is simply

$$\partial_{\overline{h}(a)} \big(\det \underline{h}^{-1} (\mathcal{R}/2 + \Lambda - \kappa \mathcal{L}_m)\big) = 0.$$

Employing the results on multivector differentiation from above we find that

$$\partial_{\overline{h}(a)} \det \underline{h}^{-1} = -\det \underline{h}^{-1} \underline{h}^{-1}(a)$$

and

$$\partial_{\overline{h}(a)} \mathcal{R} = \partial_{\overline{h}(a)} \langle \overline{h}(\partial_c \wedge \partial_b) \mathsf{R}(b \wedge c) \rangle$$
$$= 2\overline{h}(\partial_b) \cdot \mathsf{R}(b \wedge a).$$

It follows that

$$\partial_{\overline{h}(a)} (\mathcal{R} \det \underline{h}^{-1}) = 2\mathcal{G}\big(\underline{h}^{-1}(a)\big) \det \underline{h}^{-1},$$

where $\mathcal{G}$ is the Einstein tensor,

$$\mathcal{G}(a) = \mathcal{R}(a) - \tfrac{1}{2} a \mathcal{R}. \qquad (38)$$

We now define the *functional* matter energy-momentum tensor $\mathcal{T}(a)$ by

$$\det \underline{h} \partial_{\overline{h}(a)} (\mathcal{L}_m \det \underline{h}^{-1}) = \mathcal{T}\big(\underline{h}^{-1}(a)\big).$$

We therefore arrive at the equation

$$\mathcal{G}(a) - \Lambda a = \kappa \mathcal{T}(a).$$

This is the gauge theory statement of Einstein's equation.

The source term in the Einstein equations is the functional energy-momentum tensor, not the canonical one, and we'll look at examples of it shortly

The other field equation arises from taking the derivative of the action w.r.t. the $\Omega(a)$ bivector field, and the result we end up with relates the torsion $\mathcal{S}\overline{h}(a) = \mathcal{D}\wedge\overline{h}(a)$ to the matter spin tensor

$$\mathcal{S}_{\mathrm{spin}}(a) = \partial_{\Omega(a)} \mathcal{L}_m$$

and is the simple relation

$$\mathcal{D}\wedge\overline{h}(a) = \kappa \mathcal{S}_{\mathrm{spin}} \overline{h}(a)$$

THE MATTER CONTENT

To illustrate the structure of the source terms we return to the covariant Maxwell and Dirac Lagrangian densities. First consider free-field electromagnetism. Under displacements, the vector potential A transforms as

$$A(x) \mapsto A'(x) = \bar{\mathsf{f}}\big(A(x')\big), \tag{39}$$

and the field strength F transforms as

$$F \mapsto F'(x) = \nabla \wedge A'(x) = \bar{\mathsf{f}}\big(F(x')\big). \tag{40}$$

The covariant field strength is therefore defined by

$$\mathcal{F} = \overline{h}(F) = \overline{h}(\nabla \wedge A), \tag{41}$$

and the covariant Lagrangian density for the electromagnetic field is

$$\mathcal{L}_{\mathrm{EM}} = \tfrac{1}{2}\mathcal{F} \cdot \mathcal{F}. \tag{42}$$

The functional energy-momentum tensor is defined by

$$\mathcal{T}_{\mathsf{em}}\big(\mathsf{h}^{-1}(a)\big) = \det \underline{h}\partial_{\overline{h}(a)}(\tfrac{1}{2}\mathcal{F}\cdot\mathcal{F} \det \underline{h}^{-1})$$
$$= \overline{h}(a\cdot F)\cdot\mathcal{F} - \underline{h}^{-1}(a).$$

(exercise) so we obtain

$$\mathcal{T}_{\mathsf{em}}(a) = (a\cdot\mathcal{F})\cdot\mathcal{F} - a = -\tfrac{1}{2}\mathcal{F}a\mathcal{F}. \tag{43}$$

This is precisely the form we would expect for the covariant generalisation of the electromagnetic field strength.

Unlike the canonical definition mentioned above, there is no issue about the tensor being electromagnetic gauge invariant, and the tensor is automatically symmetric.

Furthermore, there is no coupling to $\Omega(a)$, so the electromagnetic spin density is zero (it turns out the canonical angular momentum tensor is non-zero for EM — actually this has some interest — photons are spin-1!)

As an example of a field with non-vanishing spin density we next consider the

Dirac theory. With the electromagnetic coupling included, the fully covariant Dirac Lagrangian reads

$$S = \int d^4x \, \det(\mathsf{h})^{-1} \langle \bar{\mathsf{h}}(\partial_a)(a\cdot\nabla\psi + \tfrac{1}{2}\Omega(a)\psi)I\gamma_3\tilde{\psi} - e\bar{\mathsf{h}}(A)\psi\gamma_0\tilde{\psi} - m\psi\tilde{\psi}\rangle.$$

The functional energy-momentum tensor is simply

$$\mathcal{T}_D(a) = \langle a\cdot\bar{\mathsf{h}}(\partial_b)D_b\psi I\gamma_3\tilde{\psi}\rangle_1 - ea\cdot\mathcal{A}\,\psi\gamma_0\tilde{\psi}. \qquad (44)$$

This is manifestly a covariant tensor, though it is not necessarily symmetric. The spin density is

$$\mathsf{S}_{\text{spin}}(a) = \tfrac{1}{2}\overline{h}(a)\cdot(\psi I\gamma_3\tilde{\psi})$$

or, covariantly,

$$\mathcal{S}_{\text{spin}}(a) = \tfrac{1}{2}a\cdot(\psi I\gamma_3\tilde{\psi}) = \tfrac{1}{2}a\cdot S$$

where S is the spin trivector. We can then use this as the source term in the equation for torsion.

The SET given in (44) settles the question of the correct SET within Dirac theory, and as stated, will not in general be symmetric.

One can show that the antisymmetric part is given by the covariant divergence of the spin trivector

This is general — i.e. for any SET one can demonstrate that its antisymmetric part is a total divergence, thus won't show up in integral quantities constructed from the SET, but could e.g. require a non-symmetric Einstein tensor, which is possible if there is torsion.

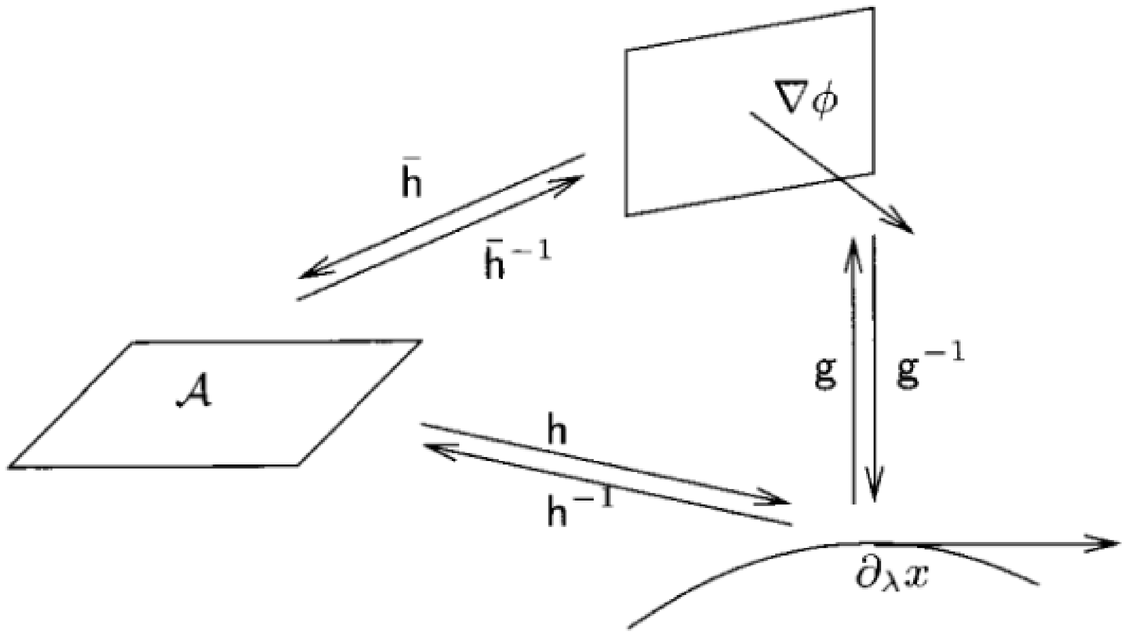
A NEW SPACE

It is worth considering an aspect of the novelty of our gauge theory approach

The space that our covariant vectors live in simply doesn't exist in any conventional treatments of differential geometry

Vectors transform like $\partial_\mu x$ (with $\underline{f}(a)$) or like ∇ (with $\overline{f}^{-1}(a)$). Mathematically these are the vectors and 1-forms, and are thought of as separate spaces

We use the $\overline{h}$-field to make all same type — covariant vectors. Then just have rotor group transformations.



g here is our version of the metric tensor, which conventionally (and here) maps between the tangent (vector) and cotangent (1-form) spaces. We can write

$$g = \bar{h}^{-1}\underline{h}^{-1}$$

and in components recover the standard GR metric as

$$g_{\mu\nu} = \underline{\mathsf{h}}^{-1}(e_\mu) \cdot \underline{\mathsf{h}}^{-1}(e_\nu)$$

but in fact never any need to do this! In practice better to work in terms of the $\bar{\mathsf{h}}$ function

As David said, there is probably a lot to explore in relation to the rest of differential geometry, not least of course the fact that everything we have been doing here in gravity is in a flat space!

Final point, am currently working on introducing a further invariance into the theory

SCALE INVARIANCE

- Here add an additional symmetry to those of position gauge and rotation gauge covariance
- This is scale invariance.

- Want to be able to rescale the h-function and other fields by an arbitrary function of position and want physical quantities to respond covariantly under this change
- Note that the change where we remap x to an arbitrary function of x ($x \mapsto f(x)$), is already included in the position-gauge freedom
- So we are not talking about $x \mapsto e^{\alpha} x$
- Instead we are talking about a change in the standard of length at each point (original Weyl idea)
- There are a variety of ways of going about this
- Have been working (in the background!) on a novel approach to this for the last 9 years
- Gave a preliminary account in the Brazil ICCA meeting in 2008, but a lot has changed since then
- (Didn't manage to write up the talk, but see

http://www.ime.unicamp.br/icca8/videos.html for a video of the talk if interested.)

- With a colleague (Mike Hobson) we have written up the theoretical foundations of the work, but unfortunately not in GA notation to start with!
- In fact hardest bit has been converting to conventional notation!
- So if interested have a look at Lasenby & Hobson, Gauge theories of gravity and scale invariance. I. Theoretical foundations, arXiv:1510.06699 (provisionally accepted in JMP)
- And also see some discussion in the write-up of last year's talk by me at the AGACSE meeting in Barcelona, courtesy of Sebastia's efforts as editor:

  Lasenby, A.N. Adv. Appl. Clifford Algebras (2016),' Geometric Algebra as a Unifying Language for Physics and Engineering and Its Use in the Study of Gravity', doi:10.1007/s00006-016-0700-z

  which is freely available online