INTRODUCTION TO SPINORS AND WAVE EQUATIONS

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AIMS

- Overall aim of my 4 lectures is to introduce you to physical applications of Geometric Algebra (GA)
- Will do this mainly via the new mathematical tools that GA brings
- We will do this over several applications, at the end concentrating on gravitation and cosmology, but today want to show with fairly elementary means how we can begin to understand the nature of spinors and their role in wave equations.
- If work in 2d, then using GA can consider complex numbers as the spinors appropriate to two dimensions.
- This gives GA versions of analycity and the Cauchy-Riemann equations

- In 3d will look at Pauli spinors, and then anticipating the GA of 4d space (the Spacetime Algebra), discuss Weyl and Dirac spinors and their GA versions — allows us to make links both with the Penrose-Rindler formalism, and the wave equations of elementary particles

WHAT ARE SPINORS?

- You are probably familiar with them in the guise of Pauli and Dirac spinors
- Conventionally Pauli spinors are two component single column 'vectors' with each component ψ_1 and ψ_2 a complex number

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

- These are acted on by ‘operators’ such as the Pauli matrices

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- For Dirac spinors, we have 4-component complex column vectors, acted on by combinations of the Dirac gamma matrices (will return to these later)
- The Pauli spinors are crucial in any part of non-relativistic quantum theory involving particle spin, and the Dirac spinors underly all of quantum field theory and quantum electrodynamics
- So what do we think spinors are in a GA approach?
- To start with will work generally
- Let us split a general Clifford space (i.e. the space of general multivectors) into even and odd parts, defined as follows
- Suppose all the basis vectors, will call them $\{\gamma_\mu\}$ for the time being, are

reversed in direction, i.e. $\gamma_\mu \mapsto -\gamma_\mu, \mu = 1, \ldots, n$.

- This operation is called 'inversion'
- Then an 'even' multivector is one which remains invariant under inversion, whereas an 'odd' multivector changes sign
- The total space $\mathcal{C}(R^{p,q})$ splits into a $\mathcal{C}^+(R^{p,q})$ and $\mathcal{C}^-(R^{p,q})$ composed of even and odd multivectors respectively
- This splitting is independent of the basis chosen. IF E and E' are arbitrary elements of $\mathcal{C}^+$ and O and O' of $\mathcal{C}^-$. then it follows

$$EE' \in \mathcal{C}^+, \quad EO, OE' \in \mathcal{C}^-, \quad OO' \in \mathcal{C}^+$$

- Thus an *even* space $\mathcal{C}^+$ is picked out as special, since it forms a sub-algebra under the geometric product
- It is this we will identify with spinors! I.e. spinors are general combinations of the even elements of $\mathcal{C}(R^{p,q})$

- The odd space $\mathcal{C}^-$ does not form a subalgebra, but it can be put into 1-1 correspondence with $\mathcal{C}^+$ via the mapping $\mathcal{C}^- \to \mathcal{C}^+$

$$Ob \mapsto E,$$

where b is any non-null vector of $R^{p,q}$ (same as the grade 1 elements of $\mathcal{C}(R^{p,q})$)

- So dimension of the spinor space is $2^{(n-1)}$ in n-dimensions
- So 2 in 2d, 4 in 3d, 8 in 4d, etc.

2 DIMENSIONS

So let us look in 2d

- Basis vectors γ_1, γ_2, satisfying $\gamma_1^2 = 1$, $\gamma_2^2 = 1$, $\gamma_1\gamma_2 = -\gamma_2\gamma_1$
- The unit pseudoscalar for this space is

$$I = \gamma_1 \wedge \gamma_2 = \gamma_1\gamma_2$$

which is a bivector

- A crucial (though simple) manipulation is

$$I^2 = (\gamma_1\gamma_2)(\gamma_1\gamma_2) = \gamma_1(\gamma_2\gamma_1)\gamma_2 = -\gamma_1(\gamma_1\gamma_2)\gamma_2 = -\gamma_1^2\gamma_2^2 = -1$$

- This reveals that the pseudoscalar of R^2 has exactly the algebraic property required of the unit imaginary i
- Thus suggests that complex numbers, of the form $z = x + iy$, may be something quite different from how we ordinarily view them
- In fact they are the spinors of 2d!
- To show that the identification of complex numbers with $\mathcal{C}^+$ leads to the right results in R^2, we note that a general multivector M can be expanded there as

$$M = x + \boldsymbol{a} + Iy,$$

where x and y are scalars, and $\boldsymbol{a} = a^1\gamma_1 + a^2\gamma_2$

- Since $I = \gamma_1 \wedge \gamma_2$ does not change sign under inversion, whereas $\boldsymbol{a}$ does, the space of even multivectors is precisely $x + Iy$ and that of odd multivectors is just all the ordinary vectors. Note that both have dimension 2 as required
- The 1-1 correspondence mentioned above takes the form

$$\begin{aligned} \boldsymbol{ab} &= \boldsymbol{a.b} + \boldsymbol{a} \wedge \boldsymbol{b} \\ &= \boldsymbol{a.b} + (a^1 b^2 - a^2 b^1) I, \end{aligned}$$

which if the constant vector $\boldsymbol{b}$ is taken as $\boldsymbol{b} = \gamma_1$ say, has the useful form

$$\boldsymbol{a}\gamma_1 = a^1 - a^2 I = a^*,$$

where * is usual complex conjugation and where we adopt the convention that if a symbol for a vector appears in non-bold type, then it refers to the complex number having components equal to that of the vector
- That is, if $\boldsymbol{a} = a^1\gamma_1 + a^2\gamma_2$, then $a = a^1 + ia^2$
- (Note then that $\boldsymbol{a}\gamma_1 = a^*$ involves the identification of i with the I of R^2.

Will use them interchangeably for the time being.)

- The magnitude of a vector $\boldsymbol{a}$ will be denoted by $|\boldsymbol{a}|$ to avoid confusion with the spinor a

- Using the basic relations

$$\begin{aligned} \gamma_1 i &= \gamma_1\gamma_1\gamma_2 = +\gamma_2 \\ \gamma_2 i &= \gamma_2\gamma_1\gamma_2 = -\gamma_1 \end{aligned} \tag{1}$$

we can note what happens when we multiply a vector on the right by a complex number $z = x + iy$:

$$\begin{aligned} \boldsymbol{a}z &= (a^1\gamma_1 + a^2\gamma_2)(x + iy) \\ &= (a^1x - a^2y)\gamma_1 + (a^2x + a^1y)\gamma_2 \end{aligned} \tag{2}$$

We thus obtain another vector, as to be expected from the general relation $OE \in \mathcal{C}^-$

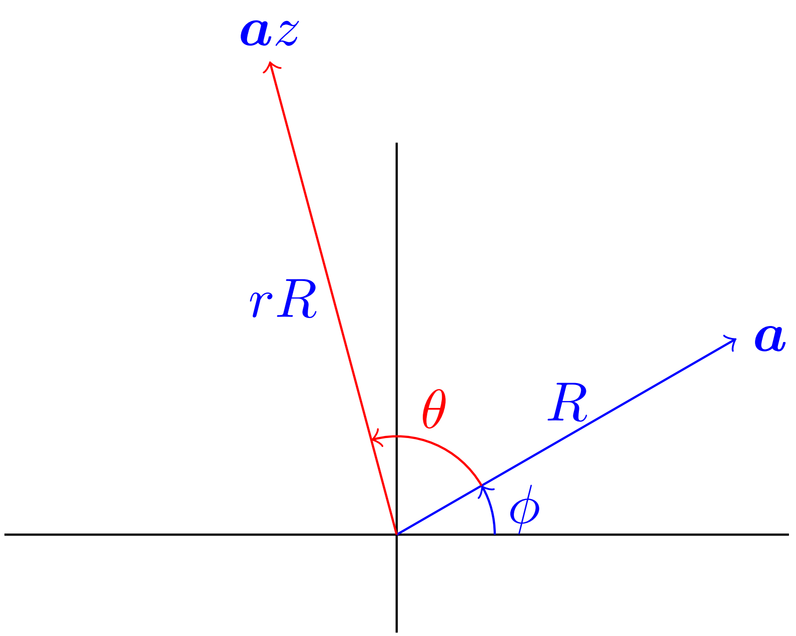
Put

$a^1 = R\cos\phi$, $a^2 = R\sin\phi$

and

$x = r\cos\theta$, $y = r\sin\theta$

in equation (2)



- Get $\boldsymbol{a}z = rR\left[\cos(\theta+\phi)\gamma_1 + \sin(\theta+\phi)\gamma_2\right]$
- This then provides a convenient characterization of the 'spinors' (complex numbers) in R^2 (or indeed more generally) — they rotate and dilate a vector into another vector
- We see that $\boldsymbol{a}$ has been rotated by z anticlockwise through angle θ and magnified by a factor r.
- This ties in very well with the conventional Argand diagram picture of z and

we can now see fully the basis of the distinction between R^2 and the complex plane — one is the space of conventional vectors, and the other is the space of rotations and dilatations comprising the even sub-algebra of $\mathcal{C}(R^2)$

- Some further relations you might like to prove are:

$$\boldsymbol{a} z = z^* \boldsymbol{a} \text{ for any } \boldsymbol{a} \text{ and } z,$$

$$\gamma_1 \boldsymbol{a} = a,$$

$$\boldsymbol{a} = \gamma_1 a,$$

$$\boldsymbol{ab} = a^* b,$$

$$\boldsymbol{ba} = a b^*,$$

$$\text{and so if } \boldsymbol{ab} = z, \quad \text{then} \quad \boldsymbol{ba} = z^*.$$

- All these can be proved quickly by inspection or by expanding in a basis. The last three relations illustrate how the non-commutative geometric product is represented in the commutative complex variable product.

- Note that $\gamma_1 z \gamma_1 = z^*$ for any z, showing how bilinear products with γ vectors, familiar from manipulations with the Dirac γ-matrices and Dirac spinors (or Pauli matrices and Pauli spinors), in fact have much more homely precedents already in complex variable theory!

Generalizing to Higher Dimensions

- Now, this all looks good, but the particular form we are using for the action of a spinor on a vector is partially an accident of 2d
- Let's write our spinor $z = x + Iy$ with $x = r\cos\theta$, $y = r\sin\theta$, in the form

$$z = r\left(\cos\theta + I\sin\theta\right) = re^{I\theta}$$

where the exponential of a multivector M is defined quite generally via a power series in the normal way:

$$\exp(M) = 1 + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots$$

- We can now apply this to rotate the vector $\boldsymbol{a}$

$$\boldsymbol{a}' = \boldsymbol{a}z = \boldsymbol{a}re^{I\theta} = \left(r^{1/2}e^{-I\theta/2}\right)\boldsymbol{a}\left(r^{1/2}e^{I\theta/2}\right)$$

  where we have used the fact that I anticommutes with vectors to have a more symmetrical split form for the transformation

- Let us write ψ for the spinor $r^{1/2}e^{-I\theta/2}$

- Then our transformation law is

$$\boldsymbol{a}' = \psi\boldsymbol{a}\tilde{\psi} \tag{3}$$

  where the reverse of a quantity in the geometric algebra is used, signified with a $\tilde{}$ acting on the object.

- Quite generally, this is defined by reversing the order of Clifford products within the sum over blades which make up the object, e.g. $(\gamma_1\gamma_2)\tilde{} = \gamma_2\gamma_1$, etc.

- Writing it the transformation this way, rather than as $\boldsymbol{a} r e^{I\theta}$, makes no difference in 2d, but the double-sided version is key to generalizing to higher dimensions

- E.g., suppose we wish to include a third dimension, with vector γ_3

- We would want this to be left alone by rotations in the (x, y) plane, and we already know these can be written (taking $r = 1$, since just considering the rotational rather than dilatational part) as any of

$$\boldsymbol{a}' = e^{-I\theta}\boldsymbol{a} = \boldsymbol{a}a^{I\theta} = e^{-I\theta/2}\boldsymbol{a}e^{I\theta/2}$$

  for a vector $\boldsymbol{a}$ with just γ_1 and γ_2 components

- It's only the last form, however, that satisfies the desired extension that $\boldsymbol{a}' = \boldsymbol{a}$ when $\boldsymbol{a} = \gamma_3$! (This follows from γ_3 commuting with $\gamma_1\gamma_2$.)

- So it's (3) that provides the blueprint for how we transform using spinors in higher dimensions

Spinors and Calculus

- In the elementary Gibbs vector calculus of R^3, the basic differential operator is written
$$\boldsymbol{\nabla} = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z},$$
where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the x, y, z directions

- With the usual summation convention and noting $\gamma^\mu = \gamma_\mu$ for R^3, we can write this more compactly as $\boldsymbol{\nabla} = \gamma^\mu \frac{\partial}{\partial x^\mu}$

- This vector differential operator, , is precisely the right object to allow generalization of vector and complex variable calculus to general spaces $R^{p,q}$

- As long as correct ordering is maintained, it can be substituted in place of a non-differential vector in any multivector identity

- In particular, if $\boldsymbol{r}$ is the usual position vector ($\boldsymbol{r} = x^\mu \gamma_\mu$) and $\boldsymbol{A}(\boldsymbol{r})$ a vector field, then we have

$$\boldsymbol{\nabla} \boldsymbol{A} = \boldsymbol{\nabla} \boldsymbol{.} \boldsymbol{A} + \boldsymbol{\nabla} \wedge \boldsymbol{A}$$

in any $R^{p,q}$, generalizing the notions of divergence and curl to arbitrary dimension

- Most importantly, this unites them into a single quantity $\boldsymbol{\nabla} \boldsymbol{A}$ for which an inverse operator $\boldsymbol{\nabla}^{-1}$ exists, and so from which $\boldsymbol{A}$ can be recovered
- This is not true for either $\boldsymbol{\nabla} \boldsymbol{.} \boldsymbol{A}$ or $\boldsymbol{\nabla} \wedge \boldsymbol{A}$ separately

SPECIALISATION TO 2D

- The immediate use we have for $\boldsymbol{\nabla}$ is in application to spinors, as in the Dirac equation, and in our case we wish to do this first in the Euclidean plane
- Specialising the notation in order to emphasize an (x, y) dependence, we rename γ_1 as $\boldsymbol{\sigma_x}$ and γ_2 as $\boldsymbol{\sigma_y}$
- Thus $\boldsymbol{\nabla} = \boldsymbol{\sigma_x} \partial / \partial x + \boldsymbol{\sigma_y} \partial / \partial y$

- A general spinor function of position in the plane will be written $\psi = \psi(\boldsymbol{r})$ with real and imaginary (i.e. scalar and bivector) parts given by $\psi = u + iv$, where each of u and v is a real valued function of x and y, as in the usual theory

- If one looks at the collection of useful results in complex variable theory, it soon becomes apparent that the aspect of differentiability they all rely on for their proof is that the real and imaginary parts of the function satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{4}$$

- So what is this in our terms? In fact it is very simple. It is just the requirement that $\boldsymbol{\nabla}\psi = 0$!! We will call a spinor satisfying this, analytic. Working this out, get

$$
\begin{aligned}
\boldsymbol{\nabla}\psi &= \left(\boldsymbol{\sigma_x}\frac{\partial}{\partial x} + \boldsymbol{\sigma_y}\frac{\partial}{\partial y}\right)(u+iv) \\
&= \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\boldsymbol{\sigma_x} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\boldsymbol{\sigma_y},
\end{aligned}
\tag{5}
$$

using equation (1). Setting the right hand side to zero then yields the Cauchy-Riemann equations (4).

- It is well known that if u and v satsify the Cauchy-Riemann equations then each of u and v is a real harmonic function, i.e. satisfies Laplace's equation in 2-d.

- We can also see this directly from (5), where if we apply $\boldsymbol{\nabla}$ to both sides we obtain (using the symmetry of the second derivative)

$$
\boldsymbol{\nabla}^2\psi = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + i\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{6}
$$

- ψ analytic, i.e. $\boldsymbol{\nabla}\psi = 0$, then implies Laplace's equation for each of u and v

separately

- Alternatively we may note directly that

$$\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \nabla^2,$$

the scalar Laplacian operator. Note that the transition from (5) to (6) corresponds exactly to the transition from the Dirac to Klein-Gordon equations in spacetime!

- Although we now have a definition of ‘analytic’, it is not clear what the derivative *is*, in the sense of conventional complex variable theory, in which we define a quantity

$$\frac{d\psi}{dz} = \lim_{\Delta z \to 0} \frac{\psi(z+\Delta z) - \psi(z)}{\Delta z} \tag{7}$$

and from which the Cauchy-Riemann equations are derived by requiring that the limit be independent of the direction in which Δz approaches zero

- One can translate this into GA, but it turns out this standard approach is less useful, and carries less information than the 'vector operator acting on a spinor' approach we have highlighted, so won't go through the details

- Note e.g. the relation $\frac{d}{dz}(z) = 1$ is just $\boldsymbol{\nabla} \boldsymbol{r} = 2$, and everything can be translated in a similar way

- Why the d/dz machinery is less useful, is partially because while conventional complex variable theory does have a lot to say about functions which are analytic everywhere except at isolated poles (meromorphic functions), it doesn't have much to say about functions that are 'non-analytic' almost everywhere

- For example, a standard non-analytic function in conventional theory is z^* (which we can write as $\tilde{z}$), about which the theory has little to say other than that it fails the Cauchy-Riemann equations and directional derivative test

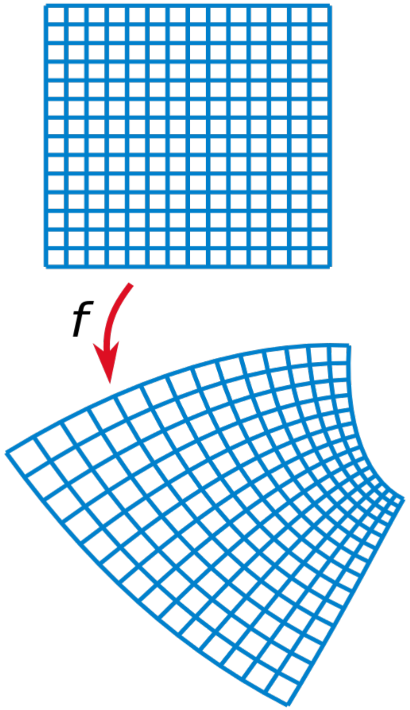
- In Geometric Calculus we can compute

$$\boldsymbol{\nabla} z^* = 2\boldsymbol{\sigma_x}. \tag{8}$$

- This measure of the failure of the function to be analytic can then be used in theorems such as Stokes' Theorem and the generalized Cauchy Theorem (see below) which apply perfectly well to non-analytic functions, and possibly gives some insight into the origin of the mass terms in the Dirac and Klein-Gordon equations.

- As a final illustration of the way that familiar notions can be viewed in an entirely new light, and thereby become clearer and often simpler, we comment briefly on conformal transformations

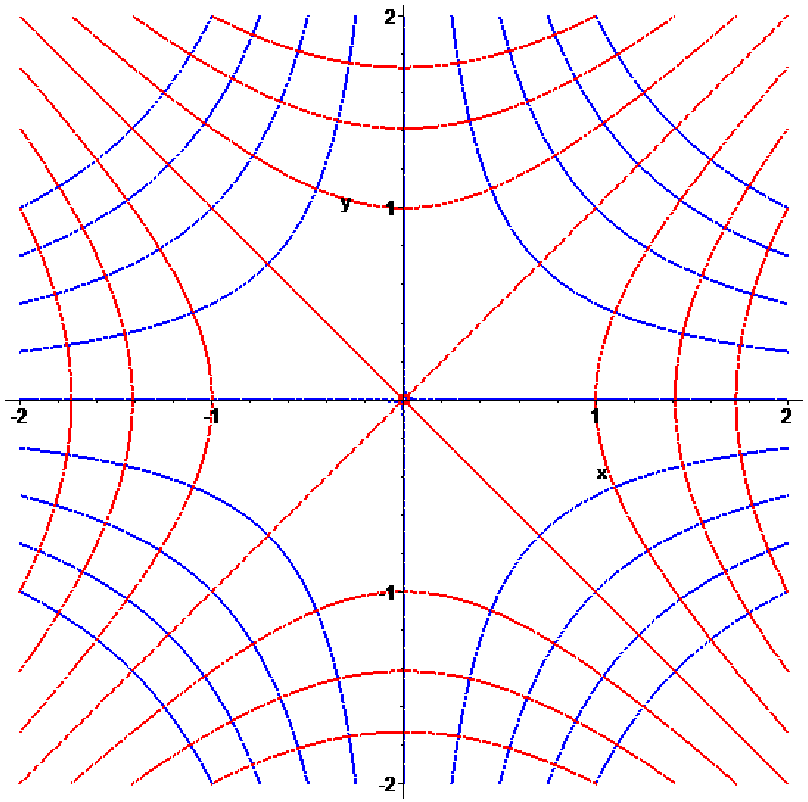
- In R^2 we are familiar with the fact that for an analytic function $\psi = u + iv$, the contour surfaces of u and v form orthogonal families of curves at equal spacings



- The usual proof of this consists of manipulations with the Cauchy-Riemann equations which show $\boldsymbol{\nabla} u \boldsymbol{.} \boldsymbol{\nabla} v = 0$ and $|\boldsymbol{\nabla} u| = |\boldsymbol{\nabla} v|$
- Note that information about the handedness of the orientation of the u surfaces with respect to the v surfaces is lost in this route. In Geometric Calculus we have

$$\boldsymbol{\nabla}\psi = 0 \Rightarrow \boldsymbol{\nabla} u + \boldsymbol{\nabla}(iv) = 0 \Rightarrow \boldsymbol{\nabla} u = -(\boldsymbol{\nabla} v)i$$

- Thus $\boldsymbol{\nabla} u$ is of equal magnitude to $\boldsymbol{\nabla} v$ and rotated by 90° clockwise with respect to it, giving first of all a very short proof of the basic result, but also giving the handedness missing in the usual route



- Here's an example. We'll let the analytic function be $\psi(z) = z^2$. This gives $u = x^2 - y^2$, $v = 2xy$
- Exercise: show this is analytic, i.e. $\nabla\psi = 0$
- Diagram shows contour lines of u (red) and v (blue) at equal increments
- Exercise: Label the contour lines with the values of u and v and demonstrate the handedness result just described
- Several other results in complex variable and conformal transformation theory are speedier in GA than via conventional theory, but want to move forward now to unusual aspects of the GA approach, and to higher dimensions

The transition to higher dimensions

- We have shown how GA spinors in 2d are in fact the complex numbers, and how the equation for an analytic function is just

$$\boldsymbol{\nabla}\psi = 0$$

  for a spinor ψ

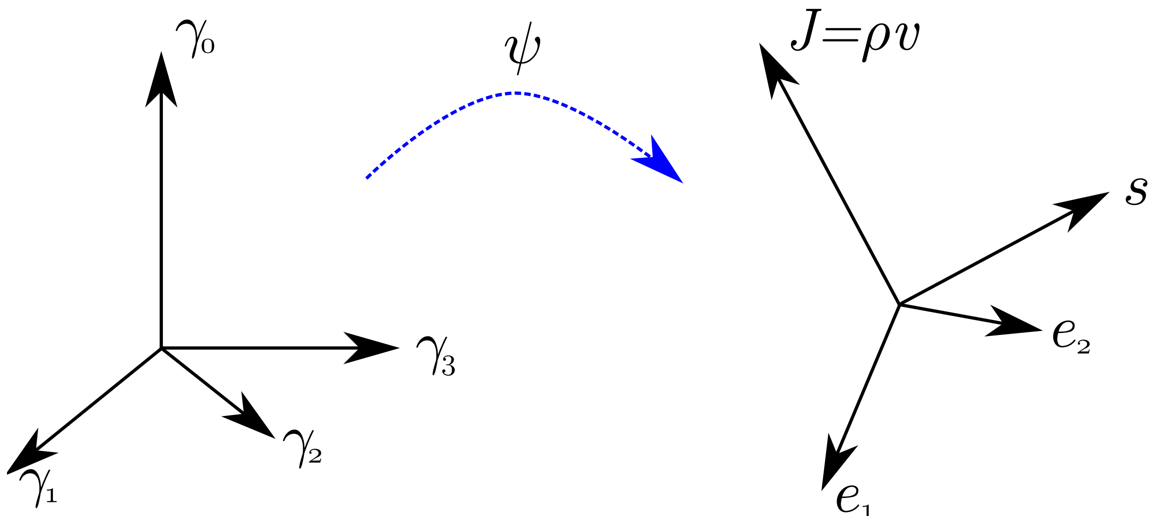
- We have defined spinors more generally as the even subalgebra of the Geometric Algebra in n-dimensions, and the derivative operator is defined generally as $\boldsymbol{\nabla} = \gamma^\mu \partial/\partial x^\mu$

- So (new bit) in n dimensionsm the generalisation of 2d analytic functions is those spinors ψ satisfying exactly the same equation $\boldsymbol{\nabla}\psi = 0$!

- These are called monogenic functions, and we can think of $\boldsymbol{\nabla}\psi = 0$ as the generalisation of the Cauchy-Riemann equations to n-d

- In fact, this seems to be the basic building block for the wave equations of elementary particles!
- E.g., the wave equation for a (massless) neutrino, is exactly $\boldsymbol{\nabla}\psi = 0$ in 4d
- The wave equation for the electron, is the Dirac equation, which now has $\boldsymbol{\nabla}\psi$ given by a term proportional to the mass m, and ψ itself (will discuss this further later).
- So what we have been doing to this point, although it looks as though it is just a new version of complex variable theory, is in fact deeply connected with elementary particles!
- The result we found in 2d, that the spinor is an instruction to rotate and dilate vectors also turns out to generalise immediately
- E.g., the spinors of 4d, the Dirac wavefunctions ψ, are indeed instructions to rotate and dilate the 4 basis vectors of spacetime, via

$$\gamma_\mu \mapsto \psi\gamma_\mu\tilde{\psi}$$

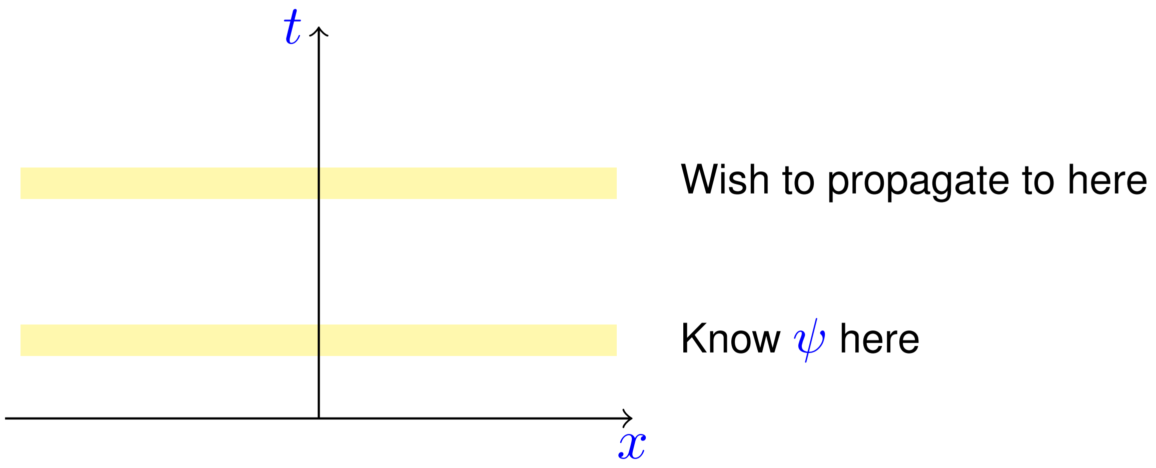


- This pins down all but one of the 8 degrees of freedom in ψ

Propagation and the Cauchy Integral Formula in GA

- As a bit of an aside, though certainly interesting, can pause to look at a further link between complex variable theory, reinterpreted in terms of GA spinors, and elementary particle theory
- We know a central task in quantum theory, e.g. in Quantum Electrodynamics

(QED), is to be able to propagate a solution forwards or backwards in time from information on some given timeslice



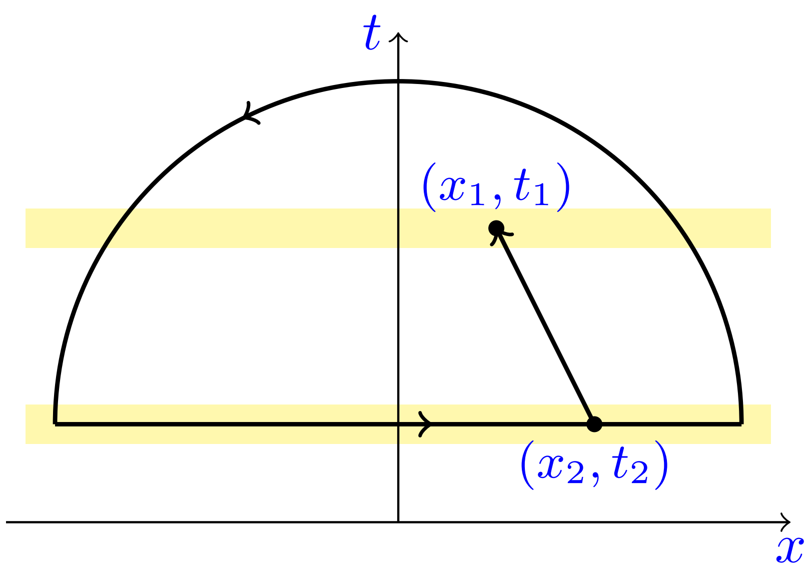
- This is what the machinery of retarded, advanced and Feynman propagators achieves
- But also, in 2d, this is what the Cauchy Integral Formula achieves!
- This tells us that for the boundary ∂M of a region M in which a function

$f(z)$ is analytic, and for z_1 an interior point, then

$$f(z_1) = \frac{1}{2\pi i} \oint_{\partial M} \frac{f(z_2)}{z_2 - z_1} dz_2 \qquad (9)$$

where the integral is taken with z_2 tracing out the boundary in an anticlockwise sense

- So, thinking about a case where we have a Euclidean time (i.e. a t axis but where the corresponding basis vector has the same square as the spatial axis), we could propagate a 2d spinor function $\psi(x, t)$ as follows



- As long as (a), there were no singularities of the function within the contour, and (b) the function decayed sufficiently rapidly to the 'future' that the integral over the semicircle could be ignored, then we would merely have to integrate at fixed t_2 from $x_2 = -\infty \ldots \infty$ to get

$$\psi(x_1, t_1) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{(x_2 - x_1) + i(t_2 - t_1)} \psi(x_2, t_2)\, dx_2$$

- So in fact

$$G(x_1, t_1; x_2, t_2) = \frac{1}{2\pi i} \frac{1}{z_2 - z_1} = \frac{1}{2\pi i} \frac{1}{(x_2 - x_1) + i(t_2 - t_1)}$$

  is what we would call the retarded propagator for this problem

- In GA, we can go a step better than the Cauchy Theorem in the form (9), and extend it to deal with non-analytic functions as well!

- This comes out of the treatment of Stokes Theorem in GA, as discussed e.g. in Chapter 7 of Hestenes & Sobczyk

- Don't have time to go through the details not, but here's the GA version of Cauchy's Integral Formula, applicable to any spinor function ψ

$$\psi(z_1) = \frac{1}{2\pi} \boldsymbol{\sigma_x} \left\{ \int_{\partial M} I^{-1} \, \boldsymbol{dS}(z_2) \, \frac{1}{z_2 - z_1} \, \psi - \int_M |dX(z_2)| \, \frac{1}{z_2^* - z_1^*} \boldsymbol{\nabla}_2 \psi \right\} \tag{10}$$

- Here $\boldsymbol{dS}(z_2)$ is a vector along the path (the bounding 'surface') and $|dX(z_2)|$ is the scalar 'volume element' of the interior
- The second term is non-zero if $\boldsymbol{\nabla}_2\psi(z_2)$ is non-zero, i.e. ψ is non-analytic
- This probably looks very abstract, but let's show how it works in some particular case
- E.g., let's consider the functions $\psi(z)$ satisfying not $\boldsymbol{\nabla}\psi = 0$ but

$$\boldsymbol{\nabla}\psi = \boldsymbol{k}\psi$$

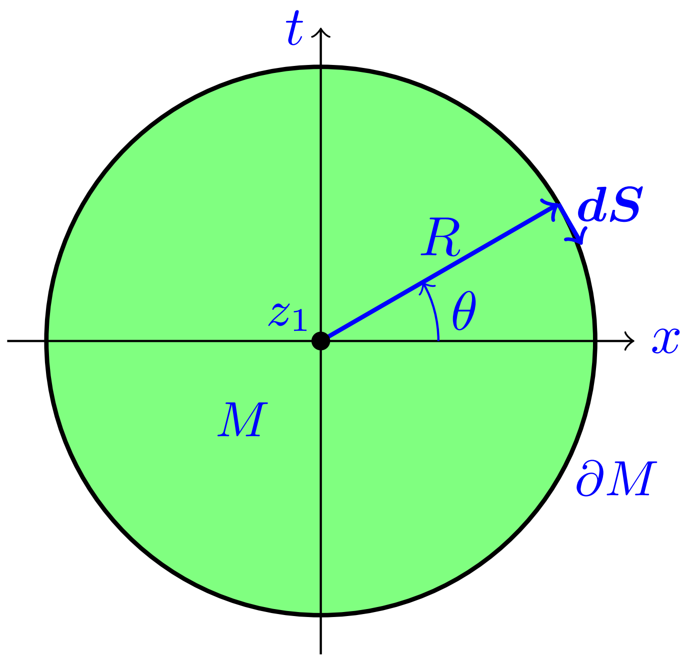
for some fixed vector $\boldsymbol{k}$. (We can think of such a ψ as a type of 'momentum eigenstate'.)

- A solution is $\psi = e^{Ik_2 x} e^{-Ik_1 t}$ (try this out — remember our coordinates are (x, t), which are identical operationally to (x, y))

- By construction this ψ is non-analytic and so provides a test of the full formula

- Write $x = r\cos\theta$, $t = r\sin\theta$, $k_1 = |\boldsymbol{k}|\cos\phi$, $k_2 = |\boldsymbol{k}|\sin\phi$, then $\psi = e^{Ir|\boldsymbol{k}|\sin(\phi-\theta)}$

- If we restrict attention to the case where M is a disc radius R and $z_1 = 0$, then the setup becomes the following



- The first integral of (10) becomes

$$\int_{\partial M} I^{-1}\, \boldsymbol{dS}(z_2)\, \frac{1}{z_2 - z_1}\, \psi = \oint_{\partial M} I^{-1} \left(-R\boldsymbol{\sigma_y} e^{I\theta} d\theta\right) \frac{1}{e^{I\theta}} \psi$$
$$= \int_0^{2\pi} d\theta\, \boldsymbol{\sigma_x}\, e^{IR|\boldsymbol{k}|\sin(\phi-\theta)}$$
$$= 2\pi \boldsymbol{\sigma_x}\, J_0(R|\boldsymbol{k}|)$$

while in the second we can replace $e^{i\theta}\boldsymbol{k}$ by $|\boldsymbol{k}|\boldsymbol{\sigma_x} e^{i(\phi-\theta)}$ to get

$$2\pi|\boldsymbol{k}|\boldsymbol{\sigma_x} \int_0^R J_1(r|\boldsymbol{k}|)\, dr = 2\pi\boldsymbol{\sigma_x} \left[J_0(R|\boldsymbol{k}|) - 1\right].$$

- The overall result from the r.h.s. of the generalized Cauchy Integral Formula is thus 1, which is the correct value of ψ at the origin!

- We can get further insight by going to the propagator approach for getting from one t slice to another

- Again, won't go into the details, but here is the answer

$$G_{\mathrm{ret}}(\boldsymbol{r}_1;\boldsymbol{r}_2) = \frac{-1}{2\pi I}\,\frac{1}{z_2 - z_1}\,e^{(\boldsymbol{r}_2-\boldsymbol{r}_1)\wedge\boldsymbol{k}}$$

which is quite neat, and extends the previous result in an intuitive way

SPACETIME AND 3D SPINORS

- At this point we could continue by looking at the properties of 3d spinors (in Quantum Mechanics these would be the Pauli spinors)
- However, it is easier and more instructive to go first to spacetime
- Can then see how the algebra of 3d space can be seen as a subset of the 4d space, with a natural interpretation of their relation, and how 3d and 4d spinors fit together

The spacetime algebra or STA is defined by

1	$\{\gamma_\mu\}$	$\{\gamma_\mu \wedge \gamma_\nu\}$	$\{I\gamma_\mu\}$	I
1 scalar	4 vectors	6 bivectors	4 trivectors	1 pseudoscalar

where the set of 4 vectors $\{\gamma_\mu\}$ ($\mu = 0, \ldots, 3$) is used for preferred orthonormal frame, $\gamma_0^2 = +1$, $\gamma_i^2 = -1$ for $i = 1, 2, 3$ and $I \equiv \gamma_0\gamma_1\gamma_2\gamma_3$

- The $\{\gamma_\mu\}$ satisfy

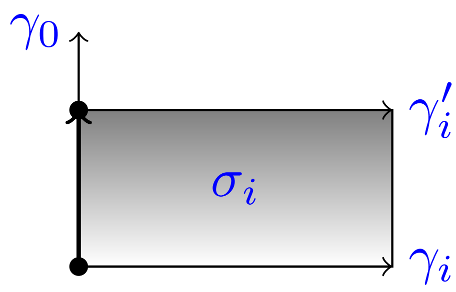
$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$$

- If we reinterpret this equation with the γs being Dirac matrices, then this is the Dirac matrix algebra

- Explains notation, but of course here the $\{\gamma_\mu\}$ are vectors, not a set of matrices in 'isospace'.

- A key insight of GA approach is relation of this Dirac algebra to the Pauli algebra:

- Each observer sees set of relative vectors. Model these as spacetime bivectors. Take timelike vector γ_0, relative vectors $\sigma_i = \gamma_i\gamma_0$.



Satisfy

$$\begin{aligned}\sigma_i\cdot\sigma_j &= \tfrac{1}{2}(\gamma_i\gamma_0\gamma_j\gamma_0 + \gamma_j\gamma_0\gamma_i\gamma_0) \\ &= \tfrac{1}{2}(-\gamma_i\gamma_j - \gamma_j\gamma_i) = \delta_{ij}\end{aligned}$$

- Generators for a 3-d algebra! Moreover have

$$\tfrac{1}{2}(\sigma_i\sigma_j - \sigma_j\sigma_i) = \epsilon_{ijk} I\sigma_k$$

so together with the preceding relation, the σ_i satisfy the same algebra as the Pauli matrices, but we have got them in a neat way from the γ vectors

- Moreover, the volume element for the GA of the 3-d relative space in rest frame of γ_0 is

$$\sigma_1\sigma_2\sigma_3 = (\gamma_1\gamma_0)(\gamma_2\gamma_0)(\gamma_3\gamma_0) = -\gamma_1\gamma_0\gamma_2\gamma_3 = I$$

which means 3-d subalgebra shares the same pseudoscalar as spacetime — very economical!

- We can see how things fit together as follows

$$\begin{array}{cccccccccc} 1 & \cdots & \{\gamma_\mu\} & \cdots & \{\boldsymbol{\sigma}_i, I\boldsymbol{\sigma}_i\} & \cdots & \{I\gamma_\mu\} & \cdots & I & \quad 4-d \\ 1 & & \{\boldsymbol{\sigma}_i\} & & \{I\boldsymbol{\sigma}_i\} & & I & & & \quad 3-d \end{array}$$

- The 6 spacetime bivectors split into relative vectors and relative bivectors. This split is observer dependent. A very useful technique

PAULI THEORY

With this in hand, now drop down to 3d, to look at Pauli spinors.

This works conventionally by regarding the Pauli matrices as being matrix operators on column vectors, the latter being the Pauli spinors.

Pauli matrices are

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Matrix operators (with hats). The $\{\hat{\sigma}_k\}$ act on 2-component Pauli spinors

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

ψ_1, ψ_2 complex. (Use bras and kets to distinguish from multivectors.)

$|\psi\rangle$ in two-dimensional complex vector space

- In GA approach, something rather remarkable happens, we can replace both objects (operators and spinors), by elements of the same algebra. Thus spacetime objects, and relations between them, can replace all (single particle) quantum statements!

- Crucial aspect we have to understand is how to model the Pauli and Dirac spinors within STA. For Pauli spinors (2 complex entries in the column spinor), we put $\psi_1 = a^0 + ia^3$, $\psi_2 = -a^2 + ia^1$ ($a^0, \ldots, a^3$ real scalars) and then the translation (conventional on left, STA on right) is

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k I\sigma_k \tag{11}$$

- For spin-up $|+\rangle$, and spin-down $|-\rangle$ get

$$|+\rangle \leftrightarrow 1 \qquad |-\rangle \leftrightarrow -I\sigma_2$$

- Action of the quantum operators $\{\hat{\sigma}_k\}$ on states $|\psi\rangle$ has an analogous operation on the multivector ψ:

$$\hat{\sigma}_k|\psi\rangle \leftrightarrow \sigma_k\psi\sigma_3 \quad (k = 1, 2, 3).$$

σ_3 on the right-hand side ensures that $\sigma_k \psi \sigma_3$ stays in the even subalgebra

- Verify that the translation procedure is consistent by computation; e.g.

$$\hat{\sigma}_1 |\psi\rangle = \begin{pmatrix} -a^2 + ia^1 \\ a^0 + ia^3 \end{pmatrix}$$

translates to

$$-a^2 + a^1 I\sigma_3 - a^0 I\sigma_2 + a^3 I\sigma_1 = \sigma_1 \psi \sigma_3.$$

- Also need translation for multiplication by the unit imaginary i. Do this via noting

$$\hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

- See multiplication of both components of $|\psi\rangle$ achieved by multiplying by the product of the three matrix operators.

- Therefore arrive at the translation

$$i|\psi\rangle \leftrightarrow \sigma_1\sigma_2\sigma_3\psi(\sigma_3)^3 = \psi I\sigma_3.$$

- Unit imaginary of quantum theory is replaced by right multiplication by the bivector $I\sigma_3$. (Same happens in Dirac case.)

- Now define the scalar

$$\rho \equiv \psi\tilde{\psi}.$$

- The spinor ψ then decomposes into

$$\psi = \rho^{1/2}R,$$

  where $R = \rho^{-1/2}\psi$.

- The multivector R satisfies $R\tilde{R} = 1$, so is a rotor. In this approach, Pauli spinors are simply unnormalised rotors!

This view offers a number of insights.

- The spin-vector $\boldsymbol{s}$ defined by

$$\langle\psi|\hat{\sigma}_k|\psi\rangle = \sigma_k \cdot \boldsymbol{s}.$$

can now be written as

$$\boldsymbol{s} = \rho R \sigma_3 \tilde{R}.$$

The double-sided construction of the expectation value contains an instruction to rotate the fixed σ_3 axis into the spin direction and dilate it

- Also, suppose that the vector $\boldsymbol{s}$ is to be rotated to a new vector $R_0 \boldsymbol{s} \tilde{R}_0$. The rotor group combination law tells us that R transforms to $R_0 R$.

  This induces the spinor transformation law

$$\psi \mapsto R_0 \psi.$$

This explains the 'spin-1/2' nature of spinor wave functions

TWO REFERENCES

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