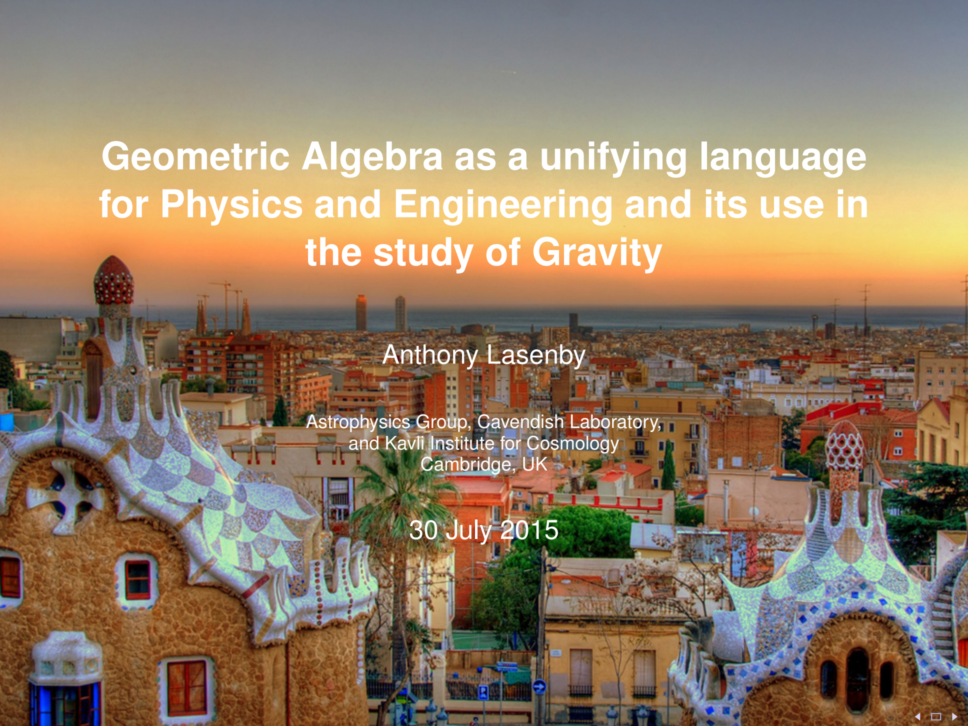

Geometric Algebra as a unifying language for Physics and Engineering and its use in the study of Gravity

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Overview

- Geometric Algebra is an extremely useful approach to the mathematics of physics and engineering, that allows one to use a common language in a huge variety of contexts
- E.g. complex variables, vectors, quaternions, matrix theory, differential forms, tensor calculus, spinors, twistors, all subsumed under a common approach
- Therefore results in great efficiency — can quickly get into new areas
- Also tends to suggest new geometrical (therefore physically clear, and coordinate-independent) ways of looking at things
- Despite title, not going to attempt a survey (too many good talks by experts in each field to attempt that!)
- Instead want to look briefly at why GA is so useful, then at a couple of areas — Electromagnetism and Acoustic Physics — in a bit more detail, and then pass on to consider the use of the tools of GA in Gravity

Some main features of the usefulness of GA I

- One of the major aspects for me, is that one can do virtually everything with just geometric objects in spacetime
- Here is what we need:

$$1 \cdots\cdots \{\gamma_\mu\} \cdots\cdots \{\boldsymbol{\sigma}_i, I\boldsymbol{\sigma}_i\} \cdots\cdots \{I\gamma_\mu\} \cdots\cdots I \qquad 4-d$$

$$1 \qquad \{\boldsymbol{\sigma}_i\} \qquad \{I\boldsymbol{\sigma}_i\} \qquad I \qquad 3-d$$

- As an example of how different this approach can be from the usual ones, consider the Pauli algebra
- David spent some time discussing yesterday how the σ_i vectors provide a representation-free version of the Pauli matrices $\hat{\sigma}_i$
- But how about what they operate on?
- Conventionally the $\hat{\sigma}_i$ act on 2-component Pauli spinors

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

ψ_1, ψ_2 complex

Some main features of the usefulness of GA II

- In GA approach, something rather remarkable happens, we can replace both objects (operators and spinors), by elements of the same algebra

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k I\sigma_k$$

For spin-up $|+\rangle$, and spin-down $|-\rangle$ get

$$|+\rangle \leftrightarrow 1 \qquad |-\rangle \leftrightarrow -I\sigma_2$$

Action of the quantum operators $\{\hat{\sigma}_k\}$ on states $|\psi\rangle$ has an analogous operation on the multivector ψ:

$$\hat{\sigma}_k|\psi\rangle \leftrightarrow \sigma_k\psi\sigma_3 \quad (k = 1, 2, 3)$$

This view offers a number of insights.

- The spin-vector $\boldsymbol{s}$ defined by

$$\langle\psi|\hat{\sigma}_k|\psi\rangle = \sigma_k \cdot \boldsymbol{s}.$$

can now be written as

$$\boldsymbol{s} = \rho R \sigma_3 \tilde{R}.$$

The double-sided construction of the expectation value contains an instruction to rotate the fixed σ_3 axis into the spin direction and dilate it.

- Also, suppose that the vector $\boldsymbol{s}$ is to be rotated to a new vector $R_0 \boldsymbol{s} \tilde{R}_0$. The rotor group combination law tells us that R transforms to $R_0 R$.
This induces the spinor transformation law

$$\psi \mapsto R_0 \psi.$$

This explains the 'spin-1/2' nature of spinor wave functions.
- Similar things happen in the relatvistic case

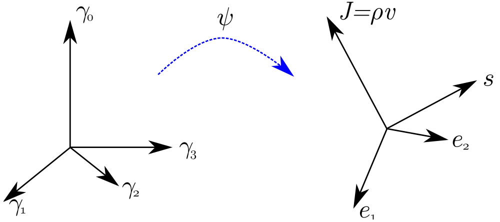
Some main features of the usefulness of GA IV

- Instead of the wavefunction being a weighted spatial rotor, it's now a full Lorentz spinor:

$$\psi = \rho^{1/2} e^{I\beta/2} R$$

 with the addition of a slightly mysterious β term related to antiparticle states.

- Five observables in all, including the current, $J = \psi\gamma_0\psi = \rho R\gamma_0\tilde{R}$, and the spin vector $s = \psi\gamma_3\psi = \rho R\gamma_3\tilde{R}$



So we can see very interesting link with GA treatment of rigid body mechanics!

Some main features of the usefulness of GA V

- Another huge unification is from the nature of the derivative operator

$$\nabla \equiv \gamma^\mu \partial_\mu$$

- Someone used to using this in the Maxwell equations

$$\nabla F = J$$

perhaps in an engineering application, where $F = \boldsymbol{E} + I\boldsymbol{B}$ is the Faraday bivector, can immediately proceed to understanding the wave equation for the neutrino

$$\nabla \psi = 0$$

- In fact their only problem is they won't know whether neutrinos are Majorana, in which case

$$\psi = \phi \frac{1}{2}(1 + \sigma_2)$$

- Here ϕ is a Pauli spinor, and the idempotent $\frac{1}{2}(1+\sigma_2)$ removes 4 d.o.f.
- Or a full Dirac spinor ψ, since currently no one knows this!
- Could then proceed to the Dirac equation

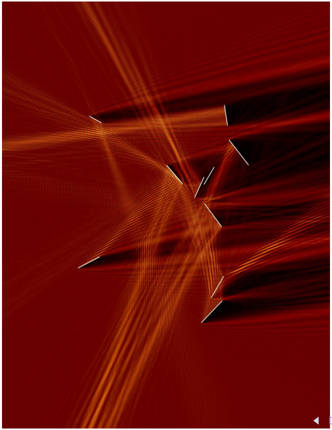
$$\nabla\psi I\sigma_3 = m\psi\gamma_0$$

  where the $I\sigma_3$ at the right of ψ reveals a geometrical origin for the unit imaginary of QM
- Might then wonder about generalising this choice, and allowing spatial rotations at the right of ψ to transform between $I\sigma_1$, $I\sigma_2$ and $I\sigma_3$
- This would then be the $SU(2)$ part of electroweak theory!
- So we would have succeeded in getting quite a long way into High Energy physics with exactly the same tools as needed for e.g. rigid body mechanics and electromagnetism!

Electromagnetism

- Have already said that defining the Lorentz-covariant field strength $F = \boldsymbol{E} + I\boldsymbol{B}$ and current $J = (\rho + \boldsymbol{J})\gamma_0$, we obtain the single, covariant equation

$$\nabla F = J$$



- The advantage here is not merely notational - just as the geometric product is invertible, unlike the separate dot and wedge product, the geometric product with the vector derivative is invertible (via Green's functions) where the separate divergence and curl operators are not
- This led to the development of a new method for calculating EM response of conductors to incoming plane waves
- Was possible to change the illumination in real time and see the effects

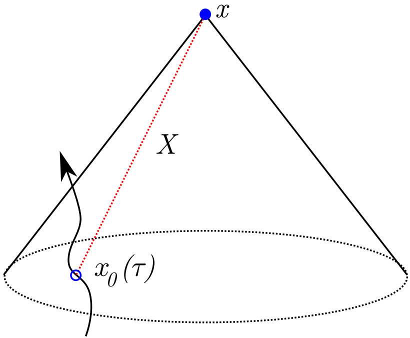
Electromagnetism

- For more detailed example want to consider radiation from a moving charge
- David pioneered the techniques on this some years ago, but want to show how one can use the approach in a different field, so need to remind about the EM case first
- Since $\nabla \wedge F = 0$, we can introduce a vector potential A such that $F = \nabla \wedge A$
- If we impose $\nabla \cdot A = 0$, so that $F = \nabla A$, then A obeys the wave equation

$$\nabla F = \nabla^2 A = J$$

Point Charge Fields

- Since radiation doesn't travel backwards in time, we have the electromagnetic influence propagating along the future light-cone of the charge.



- An observer at x receives an influence from the intersection of their past light-cone with the charge's worldline, x_0, so the separation vector down the light-cone $X = x - x_0$ is null.
- In the rest frame of the charge, the potential is pure $1/r$ electrostatic, so covariance tells us

$$A = \frac{q}{4\pi}\frac{v}{r} = \frac{q}{4\pi}\frac{v}{X \cdot v}$$

(the Liénard-Wiechert potential)

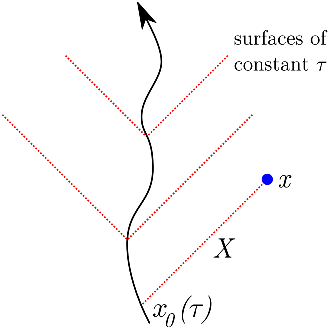
Point Charge Fields

- Now we want to find $F = \nabla A$
- One needs a few differential identities
- Following is perhaps most interesting
- Since $X^2 = 0$,

$$
\begin{aligned}
0 = \mathring{\nabla}(\mathring{X}\cdot X) &= \mathring{\nabla}(\mathring{x}\cdot X) - \mathring{\nabla}(\mathring{x_0}(\tau)\cdot X) \\
&= X - \gamma^\mu(X\cdot\partial_\mu x_0(\tau)) \\
&= X - \gamma^\mu(X\cdot(\partial_\mu\tau)\partial_\tau x_0) \\
&= X - (\nabla\tau)(X\cdot v)
\end{aligned}
$$

$$\Rightarrow \quad \nabla\tau = \frac{X}{X\cdot v} \quad (*)$$

where we treat τ as a scalar field, with its value at $x_0(\tau)$ being extended over the charge's forward light-cone



- Proceeding using this result, and defining

$$\Omega_v = \dot{v}\wedge v$$

which is the acceleration bivector, then result for F itself can be found relatively quickly

Point Charge Fields

$$F = \frac{q}{4\pi} \frac{X \wedge v + \frac{1}{2} X \Omega_v X}{(X \cdot v)^3}$$

- Equation displays clean split into Coulomb field in rest frame of charge, and radiation term

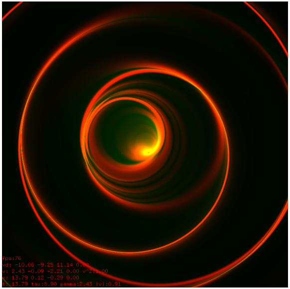
  $$F_{rad} = \frac{q}{4\pi} \frac{\frac{1}{2} X \Omega_v X}{(X \cdot v)^3}$$

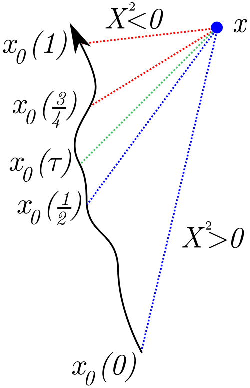
  proportional to rest-frame acceleration projected down the null vector X.
- $X \cdot v$ is distance in rest-frame of charge, so F_{rad} goes as $1/\mathrm{distance}$, and energy-momentum tensor $T(a) = -\frac{1}{2} F a F$ drops off as $1/\mathrm{distance}^2$. Thus the surface integral of T doesn't vanish at infinity - energy-momentum is carried away from the charge by radiation.

Point Charge Fields

For a numerical solution:

- Store particle's history (position, velocity, acceleration)
- To calculate the fields at x, find the null vector X by bisection search (or similar)
- Retrieve the particle velocity, acceleration at the corresponding τ - above formulae give us A and F



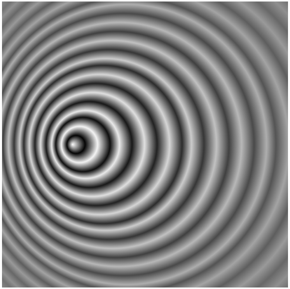


Acoustic physics

- Now look at a perhaps surprising application of these techniques
- Look at the wave equation for linearised perturbations in a stationary fluid

$$\frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} = 0$$

where c_0 is the speed of sound in the fluid.



- To make things look as simple as possible, and emphasise the tie-in with special relativity, we will henceforth use units of length such that $c_0 = 1$. (So for propagation in air, the unit of length is about 330 metres.)
- Wave equation is then identical to

$$\nabla^2\phi = 0$$

where ∇^2 is the usual relativistic Laplacian, and we are using a Special Relativistic (SR) metric of the form $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$

- We have written the wave equation without a source, but we want solutions corresponding to a δ-function source which follows a given path.
- It is known already that the solution for ϕ for this case corresponds to the electrostatic part of the electromagnetic Liénard-Wiechert potential (see e.g. S. Rienstra and A. Hirschberg, An Introduction to Acoustics (2013) Online version at http://www.win.tue.nl/~sjoerdr/papers/boek.pdf)
- For a fluid source with variable strength $Q(t)$, then

$$\phi(t,x,y,z) = \frac{1}{4\pi}\frac{Q_s}{R_s\,(1 - M_s\cos\theta_s)}$$

- s means that the corresponding quantity is evaluated at the *retarded* position
- As in EM this is where backwards null cone from the observer's position ((t,x,y,z)) intersects the world line of the source, and *null cone* is defined in terms of the above SR metric.

Acoustic physics II

- M is the Mach number of the moving source, i.e. the ratio of its speed to the speed of sound in the fluid (assumed subsonic)
- θ is the angle between the source velocity vector and the observer's position, seen from the source, and R is the distance between the source position and the observer's position
- Can we tie this in what what we've just looked at for EM?
- An immediate aspect we need to deal with, is that of course, in this Newtonian case, there is no concept of particle proper time
- Instead, the time of the particle is the same as the time recorded by any observer, in whatever state of motion
- This is essentially the defining characteristic of Newtonian time, which is universal, and flows equably and imperturbably, unaffected by anything else.
- We can still define a retarded time τ, however, by exactly the same construction as above. This is the Newtonian time at the point where the backward nullcone from the observer's position intersects the worldline of the particle

- The key is to note that we can define a covariant Newtonian 4-velocity as follows.
- Consider the particle moving in Newtonian time. We use the 'projective spilt' in which relative vectors in a 3d frame orthogonal to γ_0 are bivectors in the overall spacetime. Suppose the 3d track of the particle as a function of Newtonian time τ is $\boldsymbol{x}_0(\tau)$. The particle position in 4d Newtonian spacetime is then given by

$$x_{0N} = (\tau + \boldsymbol{x}_0(\tau))\, \gamma_0$$

so that we can define its Newtonian 4-velocity as

$$v_N = \frac{dx_{0N}}{d\tau} = (1 + \boldsymbol{M})\, \gamma_0$$

where $\boldsymbol{M}$ is the (relative) ordinary velocity divided by the sound speed.

- The two key observations, on which the entire equivalence rests, are that (a), for a given particle path, although it will in general not have the same length, v_N is *in the same direction* as the relativistic 4-velocity v, and
- (b), the velocity v appears 'projectively' in the formula for the EM 4-potential. 'Projectively' here means that it appears only linearly, and any scale associated with it will cancel out between numerator and denominator.
- Putting these observations together, means that an equally good expression for A in EM is

$$A = \frac{q}{4\pi} \frac{v_N}{X \cdot v_N}$$

in which just the Newtonian 4-velocity appears.

- Time part of this provides the solution for the potential due to a moving source in a fluid

The Doppler Effect

- Starting with the fluid case, suppose we have a wave with modulation function $f(t, x, y, z)$, and a moving observer, with Newtonian 4-velocity

$$V_N = (1 + \boldsymbol{N})\,\gamma_0$$

say. ($\boldsymbol{N}$ here is being used to indicate the observer's ordinary velocity, divided by the sound speed.) The (4d) gradient in the V_N direction is

$$V_N \cdot \nabla = \frac{\partial}{\partial t} + \boldsymbol{N} \cdot \boldsymbol{\nabla}$$

which we recognise as the 'convective derivative' for the given observer. We then claim that $i V_N \cdot \nabla \ln f$ provides a covariant definition of the 'effective frequency' observed. (Apologies for use of i!)

- As an example, if the modulation is purely harmonic:

$$f(t,x,y,z) = \exp\left(i\left(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t\right)\right)$$

  with $|\boldsymbol{k}| = \omega$, then we get

$$iV_N\cdot\nabla \ln f = \omega\left(1 - \hat{\boldsymbol{k}}\cdot\boldsymbol{N}\right)$$

  as expected.

- We now apply this to our function $f(\tau)$ of retarded time. This gives

$$iV_N\cdot\nabla \ln f = \frac{i\left(V_N\cdot\nabla\tau\right) df/d\tau}{f(\tau)}$$

  where we have as usual used the chain rule in evaluating the ∇ applied to τ.

- But $i(df/d\tau)/f(\tau)$ is the effective frequency as observed at the emitter. Also we can use equation (*) with v replaced by v_N for evaluating $\nabla\tau$, since nothing in its derivation depended on $v^2 = 1$.

The Doppler Effect III

- So

$$\nabla\tau = \frac{X}{X \cdot v_N}$$

- Putting these two facts together, we can deduce

$$\frac{\text{effective frequency measured by observer}}{\text{effective frequency at transmitter}} = \frac{V_N \cdot X}{v_n \cdot X}$$

- So have a nice compact, covariant, expression for the Doppler effect, given solely in terms of 4d geometric quantities.
- Could of course have worked with purely harmonically varying quantities at a single frequency, but we wanted to illustrate that the essence of it rests with the action of $V_N \cdot \nabla$, and could in principle be applied to any time-varying quantity, to give an effective 'stretching' effect
- Also, can now go back to EM case, and can recover an interesting result there

The Doppler Effect IV

- Now need to work with the relativistic 4-velocities v and V instead of the Newtonian 4-velocities v_N and V_N, and the retarded proper time instead of retarded Newtonian time
- Main difference arises in the 'convective derivative', which acquires a factor $\cosh\alpha$, where $\tanh\alpha = |\boldsymbol{N}|$, compared to the Newtonian case. However, this is exactly what's required to convert the 'laboratory frame' time t to the proper time of the observer. Thus we now get the result for EM

$$\frac{\text{effective frequency measured by observer}}{\text{effective frequency at transmitter}} = \frac{V\cdot X}{v\cdot X} \qquad (**)$$

- This is an interesting expression for the redshift in special relativity, which I haven't seen before!

The Doppler Effect

- The usual expression, and one which works in general relativity (GR) as well, is derived by working with a photon with 4-momentum p. There one finds

$$\frac{\text{photon energy measured by observer}}{\text{photon energy at emission}} = \frac{V \cdot p}{v \cdot p} \qquad (***)$$

  (In the GR version, the p in the numerator may be different from the p in the denominator, despite referring to the same photon, due to gravitational redshift.)
- We see that in the current approach, the role of the null-momentum p is taken over by the retarded null vector X, which is an interesting equivalence
- Equation (**) is more general than (***), since it refers to the stretching or compression in time of any information flow from source to receiver.

Gauge Theory Gravity

- For rest of talk, want to concentrate on gravity, and in particular the Gauge Theory approach to gravity
- Don't have time to explain this properly, but continuing the theme of seeking to show how Spacetime Algebra is able to reach deep into modern physics using just the same tools (and entities!) as useful in classical physics and engineering applications, will illustrate it in action in a very simple setting where one can see all the details
- Hopefully will convince you that you could do this and play with it yourself!
- The setting with be Gravity in 2 dimensions!
- Want to show how one can recover the results of differential geometry via a gauge theory approach
- Then if time after, will consider an extension (in 4d) to a larger class of gauge symmetries I'm excited about

What is Gauge Theory Gravity?

- This is a version of gravity that aims to be as much like our best descriptions of the other 3 forces of nature:
  - The strong force (nuclei forces)
  - The weak force (e.g. radioactivity etc.)
  - electromagnetism
- These are all described in terms of Yang-Mills type gauge theories (unified in quantum chromodynamics) in a flat spacetime background
- In the same way, Gauge Theory Gravity (GTG) is expressed in a flat spacetime
- Has two gauge fields $\overline{h}(a)$ and $\Omega(a)$
- 1. $\overline{h}(a)$: this allows an arbitrary remapping of position to take place $x \mapsto f(x)$ (position gauge change) — vector function of vectors (16 d.o.f)
- 2. $\Omega(a)$: this allows Lorentz rotations to be gauged locally (rotation gauge change) — bivector function of vectors (24 d.o.f)

Rapid explanation of GTG notation I

- Standard GR cannot even see changes of the latter type, since metric corresponds to $g_{\mu\nu} = \underline{h}^{-1}(e_\mu)\cdot\underline{h}^{-1}(e_\nu)$ and is invariant under such changes
- Covariant derivative in a direction is

$$\mathcal{D}_a \equiv a\cdot\nabla + \Omega(a)\times$$

- The $\times$ operator is defined by $A\times B = \frac{1}{2}(AB - BA)$
- If A is a bivector, then $A\times$ preserves grade of object being acted upon
- Thus $\mathcal{D}_a$ is actually a scalar operator!
- Get full vector covariant derivative via $\mathcal{D} \equiv \overline{h}(\partial_a)\mathcal{D}_a$
- ∂_a is the multivector derivative w.r.t. a (e.g. $\partial_a a = 4$. Note $\nabla \equiv \partial_x$!)

Rapid explanation of GTG notation II

- Field strength tensor got by commuting covariant derivatives:

$$[\mathcal{D}_a, \mathcal{D}_b]M = R(a \wedge b) \times M \quad M \text{ some multivector field}$$

- This leads to the Riemann tensor

$$R(a \wedge b) = \partial_a \Omega(b) - \partial_b \Omega(a) + \Omega(a) \times \Omega(b)$$

- Note this is a mapping of bivectors to bivectors
- Ricci scalar (rotation gauge and position gauge invariant) is

$$\mathcal{R} = \left[\overline{h}(\partial_b) \wedge \overline{h}(\partial_a)\right] \cdot R(a \wedge b)$$

- Gravitational action is then $\mathcal{L}_{\mathrm{grav}} = \det h^{-1} \mathcal{R}$
- The dynamical variables are $\overline{h}(a)$ and $\Omega(a)$ and field equations correspond to taking $\partial_{h(a)}$ and $\partial_{\Omega(a)}$
- Further details and full description in Lasenby, Doran & Gull, Phil.Trans.Roy.Soc.A. (1998), **356**, 487

- Want to give a simple illustration of the Gauge Theory approach to differential geometry
- Let's specialise to 2 Euclidean dimensions — not really a gravity theory here (for interesting reasons we discuss), but instructive
- So have an h-function which we write as

$$\bar{\mathrm{h}}(e^1) = f_1(x,y)e^1 + f_2(x,y)e^2$$
$$\bar{\mathrm{h}}(e^2) = g_2(x,y)e^1 + g_1(x,y)e^2$$

together with an Ω function

$$\Omega(e_1) = A_1(x,y)I$$
$$\Omega(e_2) = A_2(x,y)I$$

Here $I = e_1 e_2$ is the pseudoscalar of the 2d space, and the functions f_i, g_i and A_i are all scalar functions of position in the 2d space

An illustration — 2d differential geometry II

- Note, defining a vector field $A = A_i e^i$ we have

$$\Omega(a) = (a{\cdot}A)I$$

- We can now find the Riemann:

$$R(a\wedge b) = \partial_a \Omega(b) - \partial_b \Omega(a) + \Omega(a) \times \Omega(b)$$

and via a double contraction then create the Ricci scalar

$$\mathcal{R} = \left(\bar{\mathsf{h}}(\partial_b)\wedge\bar{\mathsf{h}}(\partial_a)\right) \cdot R(a\wedge b)$$

- This works out to something nice-looking in terms of the vector field A:

$$\mathcal{R} = 2\,(\det h)\;(\nabla\wedge A)\;I$$

and we are guaranteed that this is position-gauge and rotation-gauge covariant

An illustration — 2d differential geometry III

- However, disastrous as regards being a suitable Lagrangian for gravity!
- First know that we should multiply by $\det h^{-1}$ to form the p.g. covariant Lagrangian, so full action is

$$\mathcal{A} = \int d^2x \; \det h^{-1} \det h \; (\nabla \wedge A) \, I = \int d^2x \; (\nabla \wedge A) \, I$$

- Thus this doesn't even depend on h! Moreover, things are even worse. Can write

$$\mathcal{A} = \int d^2x \; (\nabla \wedge A) \, I = \int d^2x \, \nabla \cdot A' = \int dl \, n \cdot A'$$

  where $A' = AI$ is the vector dual to A, the final integral is around the 'boundary' in 2d space and n is a vector normal to the boundary
- We see from this that we won't get any equations of motion — the action consists of just a 'topological' boundary term

An illustration — 2d differential geometry IV

- So Einstein-Hilbert gravity, based on just the first power of the Ricci scalar doesn't work in 2d
- So what should one do instead to get 2d gravity? (This is the subject of current research in Quantum Gravity — 2d can provide a test bed for more complicated theories.)
- What if instead of $\mathcal{R}$ we used $\mathcal{R}^2$?
- Suddenly everything looks more sensible, get

$$\mathcal{A} = \int d^2x \,\det h^{-1} \mathcal{R}^2 = -\int d^2x \,\det h \, F \cdot F$$

where we have written $F = \nabla \wedge A$. In fact can go further. Let's define $\mathcal{F} = \overline{\mathrm{h}}(\nabla \wedge A)$, which is the 'covariant' version of F. Then

$$\mathcal{A} = -\int d^2x \,\det h^{-1} \, \mathcal{F} \cdot \mathcal{F}$$

An illustration — 2d differential geometry V

- So this is exactly the $\mathcal{F}\cdot\mathcal{F}$ Lagrangian of electromagnetism within Gauge Theory gravity! So can tell that doing gravity in 2d is going to be a lot like doing electromagnetism
- (Note that original $\mathcal{R}$ Lagrangian now looks very strange — it's equal to

$$\int d^2x \det h^{-1} \mathcal{F} I$$

  which would be an odd way of doing electromagnetism.)
- Won't pursue the full setup here, but suffice to say that to determine e.o.m. for both h and Ω we need to bring in the torsion defined by

$$\mathcal{S}(\bar{\mathsf{h}}(a)) \equiv \mathcal{D}\wedge\bar{\mathsf{h}}(a)$$

  and then this provides another term (specifically $\mathcal{S}(\partial_a)\cdot\mathcal{S}(a)$) we can put in the Lagrangian, and which 'stiffens up' the equations for h
- This is effectively at the boundary of what people are working on for 2d gravity!

- Here to proceed further, we are just going back to the differential geometry aspects, and will assume for the rest of this development that torsion=0, i.e.

$$\mathcal{D}\wedge\bar{\mathsf{h}}(a) = 0$$

- Remembering that

$$\mathcal{D}\wedge\bar{\mathsf{h}}(a) = \bar{\mathsf{h}}(\nabla)\wedge\bar{\mathsf{h}}(a) + \bar{\mathsf{h}}(\partial_b)\wedge\left(\Omega(b)\cdot\bar{\mathsf{h}}(a)\right)$$

  this gives a relation between $\bar{\mathsf{h}}$ and Ω that we can solve for Ω (i.e. A in this 2d case) in terms of $\bar{\mathsf{h}}$

- Details not very instructive, so will just jump straight to the answer we get for $\mathcal{R} = 2\det h\,(\nabla\wedge A)\,I$. Also, will further specialise to where h, and therefore implied metric $g_{\mu\nu}$ is diagonal, i.e. $g_{11} = 1/f_1^2$, $g_{22} = 1/g_1^2$

- We get

$$\mathcal{R} = \frac{1}{g_{11}g_{22}}\left\{-\frac{\partial^2 g_{11}}{\partial x_2^2} - \frac{\partial^2 g_{22}}{\partial x_1^2} + \frac{1}{2g_{11}}\left[\frac{\partial g_{11}}{\partial x_1}\frac{\partial g_{22}}{\partial x_1} + \left(\frac{\partial g_{11}}{\partial x_2}\right)^2\right] + \frac{1}{2g_{22}}\left[\frac{\partial g_{11}}{\partial x_2}\frac{\partial g_{22}}{\partial x_2} + \left(\frac{\partial g_{22}}{\partial x_1}\right)^2\right]\right\}$$

- Those used to differential geometry, will recognise this as the quantity which appears in (the diagonal version of) **Gauss' Theorema Egregium:** No matter what coordinate transformations we carry out (thereby changing the $g_{\mu\nu}$ of course), then in two dimensions the quantity that we have just found is invariant, and its value is twice the Gaussian curvature, K!
- This is a very useful result in General Relativity!

- Since we can work out lots of problems in terms of two-dimensional hypersurfaces, e.g. the (r, ϕ) plane for spherically symmetric systems, or the (t, r) plane for cosmology, 2d is often all we need. Also since the metric tensor is symmetric, we can always diagonalise it, so we have derived the essential formula
- Note in terms of the covariant $\mathcal{F}$, we have quite generally

$$K = \mathcal{F} I$$

which gives an interesting view of the Gaussian curvature

- A special case worth looking at, where value of A becomes transparent, is for a conformal metric, i.e. where

$$\bar{\mathsf{h}}(a) = f(x, y)\, a$$

for some scalar function f. Quickly find that

$$A = (\nabla \ln f)\, I \quad \text{which note means} \quad \nabla \cdot A = 0$$

so A is already in Lorenz gauge.

- For a constant curvature space (two-dimensional version of de Sitter space, or of the spatial sections of any Friedmann-Robertson-Walker metric) we find a possible solution is

$$f = \frac{1}{2}\left(1 + Kr^2\right)$$

where $r^2 = x^2 + y^2$, so have here recovered the spatial part of the line element for constant curvature universes.

- Note A has form analogous for what we would expect for a 'constant magnetic field', but modified by the conformal factor:

$$A = \frac{2K}{1 + Kr^2}\left(-y, x\right)$$

- Overall hope this has given you a feel for Gauge Theory Gravity, and how despite working in a flat space and without tensor calculus, it can recover standard differential geometry results

Progressing to scale invariance

- Now want to add an additional symmetry to those of position gauage and rotation gauge covariance
- This is scale invariance.
- We want to be able to rescale the h-function by an arbitrary function of position
$$\overline{h}(a) \mapsto e^{\alpha(x)}\overline{h}(a)$$
- Then the 'metric' obeys
$$g_{\mu\nu} = \underline{h}^{-1}(e_\mu)\cdot\underline{h}^{-1}(e_\nu) \mapsto g'_{\mu\nu} = \Omega(x) g_{\mu\nu} \quad \text{with} \quad \Omega(x) = e^{-2\alpha(x)}$$
- Want physical quantities to respond covariantly under this change
- Note that the change where we remap x to an arbitrary function of x ($x \mapsto f(x)$), is already included in the position-gauge freedom
- So we are not talking about $x \mapsto e^\alpha x$
- Instead we are talking about a change in the standard of length at each point (original Weyl idea)

Scale invariance

- There are a variety of ways of going about this
- Have been working (in the background!) on a novel approach to this for the last 8 years
- Gave a preliminary account in the Brazil ICCA meeting in 2008, but a lot has changed since then
- (Didn't manage to write up the talk, but see http://www.ime.unicamp.br/icca8/videos.html for a video of the talk if interested.)
- With a colleague (Mike Hobson) have nearly finished writing up the theoretical foundations of the work — unfortunately not in GA notation to start with!
- In fact hardest bit has been converting to conventional notation!
- Won't give details here, but want to give a flavour of it by considering a subset of full theory, which ties into discussion we've just had of 2d

Riemann squared theory

- We can immediately get a version of scale invariance in 4d, by using as Lagrangian, not the Ricci scalar $\mathcal{R}$, but the 'square' of the Riemann, which in GA form we can write

$$\mathcal{A} = \int d^4x \det h^{-1} \beta \mathcal{R}(\partial_b \wedge \partial_a) \cdot \mathcal{R}(a \wedge b)$$

- Point about this, is that if we think of h transforming as $\exp(\alpha)$, then the Riemann transforms as $\exp(2\alpha)\mathcal{R}(B)$
- Thus overall integrand is of right 'conformal weight' to be scale invariant
- The (standard) Ricci scalar version

$$\mathcal{A} = \int d^4x \det h^{-1} \frac{\mathcal{R}}{2\kappa}, \quad \text{where} \quad \kappa = 8\pi G$$

fails this test, and so can't lead to a scale-invariant theory
- Also Riemann^2 version has the right weight in terms of dimensions for β coupling factor to be dimensionless (again Ricci scalar version fails this test)

Riemann squared theory (contd.)

- Very importantly as well, the Riemann^2 Lagrangian is exactly what we'd expect if we were to model gravity as a gauge theory just like the electroweak and strong forces!
- The Riemann is the gravitational version of the 'field strength tensor' of the other theories, which is always found by communting covariant derivatives (as here)
- In electroweak and QCD, we then form an invariant Lagrangian, by contracting the field-strength tensor with itself — again as here
- So directly parallels e.g. the Maxwell structure $\mathcal{F}\cdot\mathcal{F}$, which we saw above emerging as a viable candidate for 2d gravity
- A very interesting feature of this approach, is that torsion, i.e.

$$\mathcal{D}\wedge\bar{\mathsf{h}}(a) \neq 0$$

becomes inevitable in general, and that quantum spin becomes a source not just for torsion (as happens in standard Einstein-Cartan type theories), but for the Riemann itself

Riemann squared theory (contd.)

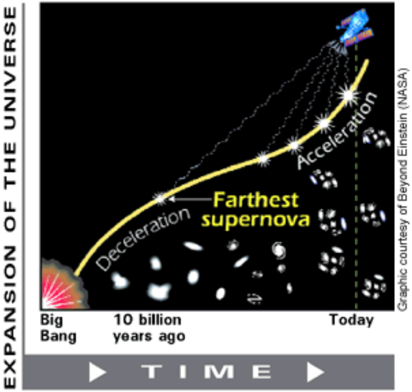
- In this connection, a beautiful feature is that the gravitational field equations then become ‘Maxwell-like’ in form, e.g. Ω equation is (schematically)

$$\dot{\mathcal{D}}\dot{\mathcal{R}}(B) = \frac{1}{\beta}\overline{\mathcal{S}}(B)$$

  where $\overline{\mathcal{S}}(B)$ is the adjoint of the ‘spin source’ tensor
- So quantum spin can feed through directly to give gravitational effects
- Perhaps most interesting to me as a cosmologist, is that this approach gives unique insights into the ‘cosmological constant’ problem

Riemann squared theory (contd.)

- We now know that on the largest scales in the universe we see not extra attraction, but 'repulsion'
- The universe is accelerating, as measured by the brightness of distant supernovae
- Is this the cosmological constant Λ?
- Big problems with the physics of this as vacuum energy, and we wish to put this as a source term in the Einstein equations — particle physics predictions are too big by about 10^{120} compared to the Λ we observe!



Riemann squared theory (contd.)

- Find something remarkable with Riemann squared:
- Have proved (a) that all vacuum solutions of GR with Λ are solutions of Riemann^2 without a Λ!
- (b) All cosmological solutions (technically those with vanishing Weyl tensor) of GR with Λ and a certain type of 'matter', are solutions of Riemann^2 without a Λ!
- The effective Λ which is simulated in each case, is given by

$$\Lambda_{\rm eff} = -\frac{3}{8\beta G}$$

where β is the coupling constant mentioned above

- So Λ can arise from our modified gravity theory, and does not have to do with vacuum fluctuations as a source
- All very good. However, big catch is the type of matter this works with

Riemann squared theory (contd.)

- This also has to be scale-invariant — e.g. in cosmology can use radiation, and have a radiation-filled universe (which was like ours for the first tens of thousands of years), but cannot use ordinary baryonic matter, such as dominates the universe today
- It was to get around this problem that have been exploring a more general scale-invariant theory, that can incorporate ordinary matter
- The foundational aspects are now clear (see Lasenby & Hobson, Gauge theories of gravity and scale invariance. I. Theoretical foundations to be submitted next month)
- But coherently knitting together the applications so that one can be clear whether it can evade all the current constraints on departures from GR, whilst doing useful things for Dark energy and Λ is still not quite clear
- Hopefully on right lines, however!