Toward Autonomous Learners of Mathematics

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In the context provided by the title of this book, the aim of this chapter is to reflect on the importance of fostering autonomous learning of mathematics by means of available technologies, to reflect on the main issues that are relevant for that purpose now and in the coming years, and to discuss the bearing on such questions of some of the developments produced by the WebALT project.[1]

1 Background

They know enough who know how to learn.

Henry Brooks Adams (1838–1918)

The mission of teaching, of mathematics in our case, is to catalyze adequate learning by students.

By "students," we mean all people that are required to take courses with mathematical content, from high school to colleges and universities. In the case of colleges and universities, the great majority of these students are studying for science and engineering degrees. For example, of the 30,000 students at the Universitat Politècnica de Catalunya (Technical University of Catalonia), only about two percent are mathematics or statistics majors, but all of them are required to take one or more semesters of mathematical subjects.

[1] European e-Content project "Web Advanced Learning Technologies," Contract Number EDC-22253.

By “adequate learning,” we mean that the students must acquire, along with knowledge of the relevant subject matter, a number of competencies related to the corresponding degree. In higher education, for instance, the European Commission specifies that the Diploma Supplement should contain “a precise description of the academic career and the competencies acquired during the study period,” and “an objective description of the [student’s] achievements and competencies.” Among the competencies, *critical thinking* is usually regarded as the most fundamental. Its development ties in so well with mathematics and its applications that mastering these goes a long way toward an effective realization of the paramount “learning to learn” capacity.

The teaching of mathematical subjects, therefore, should aim at eliciting from the students the habitual practice of good critical thinking in the context of the current subject matter. The question here, then, is in what ways can the “digital era” assist in these endeavors. Of course, the implicit message of the book title is that it should make a real difference, at least in “communicating mathematics.”

1.1 Are the Old Ways Still the Best?

Some think they are. According to [163], the recipe to fix public education is perhaps simpler than what it could possibly be: “A teacher, a chalkboard and a roomful of willing students.” In more detail:

> The plain truth is we need to return to the method that’s most effective: a teacher in front of a chalkboard and a roomful of willing students. The old way is the best way. We have it from no less a figure than Euclid himself. When Ptolemy I, the king of Egypt, said he wanted to learn geometry, Euclid explained that he would have to study long hours and memorize the contents of a fat math book. The pharaoh complained that that would be unseemly and demanded a shortcut. Euclid replied, “There is no royal road to geometry.”

Admittedly, there are strong reasons to defend the role of good Socratic teachers (assuming those in front of the chalkboard are such). They may indeed be very effective in bringing their students to embrace critical thinking. It has to be remarked, however, that in the present-day circumstances, their impact on large crowds of students must be very limited unless they have means for amplifying dramatically their capacity for interaction with them. It seems safe to say that even the most convincing Socratic teacher cannot do a proper job today (as for example in engineering and science schools) if she or he cannot rely on an environment capable of interacting *autonomously* with most of the students for a good part of the time, in ways that affect them as the direct teacher's contact would.

Since such environments cannot be implemented without advanced technology, the conclusion is that for the job of properly teaching the large number of students taking mathematics courses, there is an ever-increasing need for solutions that boost and magnify the teachers' capacity to inspire and coach their students. Fortunately, we believe that such solutions are slowly becoming available, and we can imagine teachers finally endowed with the capacity to provide a sort of "royal road" to the learning of their subjects.

1.2 Origin of the Digital Era: The Three Main Insights of Shannon

In analyzing what the digital era can bring to the teaching and learning of mathematics, it may be useful to outline the key theoretical advances that made it possible. These advances are basically due to Claude Shannon, and in retrospect they appear with a miraculous aura, for they were generally deemed unfeasible before he formulated his mathematical theory of information (see, for example, [241], which contains a reprint of the original paper [240]). Incidentally, it is interesting to ponder, in connection with "Communicating Mathematics in the Digital Era," that the title of Shannon's far-reaching landmark work is "The Mathematical Theory of Communication" (it was only later that other authors replaced "communication" with "information").

It is remarkable that Shannon's theoretical breakthroughs, brought forth by his deep mathematical insight, were conceived and established long before we had word processors, digital music, and cellular phones; digital cameras and digital TV; or PDAs combining a variety of powerful information and communication features. His theories were clearly stated and proved before anyone could dream about the multimedia world and its great potential in teaching and learning (cf. [44]), and are briefly summarized in the next few paragraphs.

Principle of digitization. Text, sound, and images (including video) can be represented by a string of bits in such a way that each can be reconstructed from its representation and is indistinguishable from the original by the human senses.

The working of this principle is straightforward for text, since text is a string of characters and characters can be encoded (say in the Unicode standard) for practically all writing systems (see [url:174]). Notice, however, that the final visual rendering of the text characters amounts to manipulations of digital images.

The principle is a bit more involved for sound. In this case, it was shown by Shannon that it is enough to sample the air pressure at twice the maximum frequency that is audible by the human ear (about 20 KHz), and to quantize each sample by (say) a 16-bit binary number (this amounts to 65,536 possible intensity levels, and in the near future, the standard number of levels will be more than 100 times higher).

For images, the sampling units are the *pixels*, and the quantizing is done for each of the three fundamental colors. Video is represented as a sequence of digital images. Again, if the pixels are sufficiently small, as in high definition TV or high resolution cameras, then the human eye cannot discriminate the original image from the image reconstructed from the binary representation.

Optimal compressibility. The raw stream of bits delivered by the digitization process usually contains much redundancy, as for example, in areas of uniform color in an image. Shannon found a precise way to measure this redundancy, or (equivalently) the information content of the stream, and proved that it can be compressed into another binary sequence that has the same information as the original one and that cannot be further compressed without loss.

Possibility of fast transmission without errors. When binary-coded information is sent from one place to another, it is transported by some system that is generically called the *channel*. This channel has a *transmission rate*, say, in bits per second. But the "noise" in the channel corrupts some proportion of the bits sent, in a random way, and thus there is an information loss. To protect against this loss, there is the notion of error-correcting codes. The idea is to add redundancy to the information before transmission in such a way that it can be used at the receiving end of the channel to discover the corrupted bits. The problem is that in this way, the information transmission rate is lower, and for a long time it was believed that smaller and smaller proportions of bit errors could only be achieved by lower and lower information rates. Shannon's celebrated channel-coding theorem, a really fundamental breakthrough, states that it is possible to use coding schemes that guarantee as small a proportion of erroneous bits as wanted and yet have an information rate as close to the channel rate as desired.

1.3 The Other Two Engines of the Digital Era

The realization of Shannon's theoretical insights has been made possible by parallel developments that are at the heart of the digital era and that have to do with computing and communication networks.

Increasing computing power. We have witnessed the spread of ever more powerful computing devices at lower and lower prices. As recently noted by a leading expert in the field of computer architectures, there is more computing power in one modern cell phone than in any computer at the time Armstrong went to the Moon. It is foreseen that the evolution in the near future will follow similar trends, to a great extent driven by advances in parallel computing.

Wideband networks. Increasing bandwidths (which amount to higher transmission rates) and decreasing access costs, have been, and will continue to be, the dominating trends in communications networks. The blending and synergy of these technologies with computing, which among many other things make possible remote and distributed computing, are to a great extent the traits that distinguish the digital era from any era in the past.

In this world, the three main insights of Shannon work at their fullest. Terabytes of digital information are daily compressed and decompressed, sent through high-speed networks or stored and retrieved to and from hard disks and other media, and protected in effective ways against corruption by channel noise. An interesting final remark in that sense is that the theoretical limit that Shannon established for the capacity to correct errors was approached rather slowly in the last half of the twentieth century, through the work of a multitude of researchers, and that today's best error-correcting codes work practically on that limit. Altogether, these achievements herald today's convergence of theoretical work with computer science and technology and clearly have a major impact on today's society.

1.4 Mathematics and e-Learning

Here we quote just a few points from the introductory part of [31]:

> The expectations created by e-Learning are certainly high, at all levels, and we may wonder how much of it is going

to be true, and up to what point can it help in the case of mathematics.

The reasons behind the high expectations on e-Learning stem from well-known characteristics of the e-Learning systems:

- In principle, access is possible from anywhere and at any time, thus making possible flexible (even just-for-me) and just-in-time courses of learning.
- The teacher can also be anywhere and do most of his teaching job at any time (preparing materials or following up and coaching his students).
- It allows for synchronous activities of a teacher and a group (at an agreed time), but again without restriction on the location of the people involved, and, what is more, with the possibility of addressing a much larger audience than a conventional class.
- Assessment can be automated to a large extent and final grading can be integrated seamlessly into the institution's information system.
- The learning materials and experiences can be richer in many ways, and they can be easily maintained and updated (as compared to preparing, say, a new edition of a paper book).
- There are also indications that e-Learning may induce deeper understanding and stronger retention.

For a general view of the main issues involved in e-Learning, see [153].

2 The Content Revolution

Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances student's learning.

USA National Council of Teachers of Mathematics

There is as much a need for the teacher to communicate mathematics to the students, as for students to communicate mathematics to the teacher. Since teaching methods will be increasingly measured by objectively proven learning outcomes in the students, and this through a variety of ways and circumstances, communication will increase in both directions.

It is thus desirable that the new learning environments have tools for the production and management of mathematical content that are available to everyone, and reasonably straightforward to use. Such tools should assist students in the task of formulating their required productions in the same way that they should help teachers with the job of encoding the learning materials.

How can this possibly be realized?

2.1 Encoding Mathematics

When doing mathematics on a computer, one has to take the view that mathematics is just like any ordinary language and, as such, it is meant to support communication. As Confucius said, "If language is not correct, then what is said is not what is meant. If what is said is not what is meant, then what ought to be done remains undone."

Any piece of computational software handles a well-defined proprietary internal representation of mathematics most suited for the kind of manipulations expected on the objects, be they complex numbers, equations, polynomial ideals, or formal proofs.

Computer-internal representations of mathematical entities often choose to employ the essential, necessary arguments required to identify an abstract object uniquely, for instance, a complex number $a + ib$ as the pair $\langle a, b \rangle$. The internal representation is chosen so that, being unambiguous, manipulations ("what ought to be done") can be carried out exactly and efficiently. However, computational software converts the internal representation to a more natural form whenever user intervention is required, usually by a "prettyprint" functionality.

While it is possible for a mathematician to interact with a computer program using a specific way to express mathematical objects, it is not so simple for different computer programs to exchange their internal representations. Clearly, the typeset expressions rendered at the user interface cannot be a candidate for unambiguous language, since different mathematical objects may be printed in the same way, e.g., a closed interval and a two-dimensional vector.

OpenMath [url:151] has been explicitly designed to support the communication of mathematics among different computer programs and to be pretty printed on a computer monitor via MathML presentation [url:180] or on paper via LaTeX.

Mathematical markup languages like OpenMath and MathML offer the possibility to represent mathematical content in a level of abstraction that is not dependent on localized information about notation and culture. This representation typically focuses on the semantics of the mathematical object and postpones localization aspects of mathematics to the rendering process of the markup. While the typesetting of mathematical markup has been the object of numerous efforts, from MathML presentation to SVG converters, the rendering of mathematics in a “verbalized” jargon has not yet received similar attention. The WebALT EU e-Content project has been devoted to the application of language technologies that automatically generate text from mathematical markup.

Mathematical jargon is an important aspect of the education of students. Not only does a teacher train pupils in problem solving skills, but she also makes sure that they acquire a proper way of expressing mathematical concepts. To our knowledge, digital e-Learning resources have used a representation in which text is intermixed with mathematical expressions, even in situations where the actual abstract representation, for instance of the statement of a theorem, can be reduced to a single mathematical object. One reason for this representation choice is that the rendering process would otherwise produce a symbolic, typeset mathematical formula that might prove too difficult to understand for the students or simply just too hard to read. However, by representing this kind of mathematical text in a language-independent format such as the one provided by markup languages, it is possible to apply language technologies and generate a “pretty printed” version that mixes text and symbolic notation to adapt to the native language and sophistication level of the reader. Languages covered so far include English, Spanish, Finnish, Swedish, French, and Italian.

2.2 Mathematical e-Content

Digital mathematics content, and the advanced tools for its creation, are among the main pillars in the communication of mathematics. We have to stress, however, that this is not as a straightforward as it may seem at first glance.

A commonly accepted estimate is that the cost of preparing high-quality interactive online learning materials is 200–300 hours of labor for one hour of student learning. A dedicated group of experts is needed to accomplish such a task. The preparation of a full one-semester course (56 hours) will typically require more than 10,000 hours of work by experts. This is months of labor for 100 people. With overheads, the total cost of the production of a 56-hour one-semester online course in mathematics amounts to about one million euros. This is prohibitively expensive. It is no wonder that, the teaching of sciences still mostly happens in the traditional way, which, in the long run, is even more expensive.

3 A Sample of Experiences

The wildness we all need to live, grow, and define ourselves is alive and well, and its glorious laws are all around.

Robert B. Laughlin, in *A Different Universe*

In this section, we will give a brief presentation of some of the experiences undergone within the WebALT project, or in closely related endeavors, that are related to communicating mathematics in the digital era. For more information on some of the topics, we refer the reader to [55] and [56].

3.1 WIRIS at Edu365

For our purposes here, it will be just enough to quote two paragraphs from [96]:

> Generically, WIRIS is an Internet platform which, on one hand, performs general mathematical computations solicited by its users and, on the other hand, supports the creation of Web-accessible interactive documents and materials.
> WIRIS is one of the main services offered by the Internet portal **edu365** [url:9] of the Education Department of the Catalan Government.[2] Access to the site is unrestricted and only a standard Web browser is needed.

Users of WIRIS facilities are teachers and students both in secondary schools and in the universities.

3.2 Teaching and Learning Error-Correcting Codes

One of the earliest uses of WIRIS was to set up a web lab as a resource for the computational and practical aspects of the theory of error-correcting codes (see [284]). It contains over 140 entries, structured according to the table of contents of the paper book. These entries can be accessed by the links provided in the PDF book or directly from the webpage. In any case, the system can be used both by a teacher in a lecture hall equipped with an Internet connection and a

[2] At present, there are many similar WIRIS servers in several countries; see [url:188].

videoprojector, and by the students in the PC rooms. This model of digital content may promote autonomy in learning for some subjects, but in the coding theory class the benefit is limited to those who are really interested in the computational side of the mathematical subjects. The explanation of why the interest is not general is that, so far, the lab activities have not been taken directly into account in the final assessment.

3.3 Refreshing Mathematics at UPC

One of the needs at universities today, and especially in engineering schools, is to provide suitable means to help reinforce the mathematical background of first year students in topics like plane geometry, trigonometry, and basic facts about linear equations.

To solve these kinds of problems, we can only rely on suitable digital content and web labs. One possible design is the one followed by the EVAM project at UPC [url:58]. The virtual tools used in this project started with WIRIS technology integrated in Moodle and continued with the production of MapleTA exercises in connection with the WebALT project.

One advantage for users and for the academic community is that only a standard web browser with Java is required by the end user. In other words, users need no additional software. The interface and the computational engine can also be adjusted according to the specifications of the school involved. In any case, the architecture of WIRIS enhances the adjustment to the computer and the communications facilities available.

The methodology we have just sketched fits perfectly in the framework of the European Credit Transfer System. For professional engineers-to-be, this methodology helps students in gaining competences for working both individually and in teams, for managing time effectively, and for using computer resources appropriately.

4 The WebALT Content Architecture

Our course, which we called the Laboratory in Mathematical Experimentation *and which students called "the Lab", succeeded beyond any of our expectations.*

G. Cobb *et al.*, in *Mathematical experimentation.*

The high cost of the production of online learning materials can be partly offset by preparing reusable materials. In the WebALT project, we have chosen to produce a large number of “educational lego pieces,” i.e., modules that explain only one concept or method. Such lego pieces can then be combined in a variety of ways to assemble courses that serve different needs.

A module consists of the following components:

1. A short lecture (“Ten Minute Talk”) that is a slideshow that can be used both in traditional contact instruction and in e-learning. A Ten Minute Talk may be a PowerPoint presentation or a slide show prepared by LaTeX (using, e.g., the Beamer class). For example, the various ways to define functions could be the topic of a module.
2. A set of solved problems pertinent to the Ten Minute Talk.
3. A set of unsolved problems.
4. Various laboratories that allow students to experiment with the mathematical concept at hand. Such laboratories may use a variety of techniques. In the WebALT Project, we have developed both WIRIS and Maplet laboratories and drills (java applets powered by the computer algebra system Maple).
5. A set of automatically graded problems administered to students by a system like Web Work or MapleTA that can automatically grade the students’ responses.

Clearly, not all modules would contain all of the above. In fact, this architecture allows an incremental production of content and a continuous improvement of the quality. When teaching a course, for example, typically the first priority will be to have slides covering 1 and 2. Having a good set of unsolved problems is also part of the regular business of the content developer.

The really difficult parts, but also the most crucial ones for advanced learning environments, are those specified in 4, for producing good labs and question banks is time consuming and (for many) not easy at all, if only because doing so requires many different sorts of skills that generally cannot be found in a single person. In any case, laboratories have a great potential in education in general, because the applications that they enable render mathematics live. Maplets offer an alternative technology. Clearly, such laboratories can also be created using java. Systems like WIRIS make the creation of laboratories much faster and more fun.

4.1 Single Variable Calculus

This is an elaborate instance of this modular-structure organization [url:186]. The modules are accessed either through a category index or through an interactive labeled planar graphical tree whose leaves correspond to the modules and whose nodes correspond to category subdivisions of the contents. The navigation of the tree is easily accomplished by dragging nodes around.

5 Concluding Remarks

Who has seen the wind?
Neither you nor I:
But when the trees bow down their heads
The wind is passing by.

Christina Rossetti, in *Poems*.

In conclusion, we stress the following points:

1. *Have clear aims.* Instruction has to be driven by aims and should be successful for most students. This is not currently the case. Overcoming this shortcoming must be regarded as one of the great challenges to today's educational systems, and we do not see a way out without widespread access to good technology-based learning environments.
2. *Diversity.* Learning environments should be able to cope with the great diversity of students in background, cognitive styles, attitudes, mental maps, and so on. This goal is not easy to reach, but again we think that this is the direction to take.
3. *Learning autonomy.* Educating students to become more and more autonomous, the learning environments should promote self-study and self-assessment, with as little learning overhead as possible.
4. *Assessments.* They will continue to be very important, since it is safe to say that students are basically driven by what will be required of them in the final assessments. This can be a good opportunity to enhance learning through a wide spectrum of self-assessment tests provided by the learning environment.

In the context of these desiderata, let us finish by pointing out some of the things that work and some of those that do not, as this gives an indication of what we should try to bring about and on what we can rely to that end.

Here are a few of the things that work reasonably well, and that can be expected to keep improving in the near future:

- computing and communications technologies,
- content formats,
- authoring tools,
- presentation tools.

And here are a few of the deficiencies that should be overcome:

- Suitable technology infrastructure in lecture halls, study rooms, and libraries is more the exception than the rule.
- The use of technology, when it is available, still represents a large knowledge overhead for teachers and students. Because of this, many simply refuse to spend the required effort. This resistance is often magnified by the lack of suitable academic rewards.
- Most teaching is still the same for all, delivered at the same time, and driven by subject matter—and not by the student's learning. This is a consequence of the preceding point.
- Use of technology, including present-day forms of content, tends to be dispersive for students. Overcoming this, by means of smart feedback systems, should be one of the most urgent concerns.