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Physical Applications: Are there Conservation Laws of Energy-Momentum and Angular-Momentum in General Relativity Theory?

Summary

In this Lecture I explain why in the General Relativity theory (**GRT**), where a gravitational field generated by an energy-momentum tensor $\boldsymbol{T} \in \sec T_1^1 M$ is represented by a Lorentzian spacetime structure $(M, \boldsymbol{g}, \boldsymbol{D}, \tau_{\boldsymbol{g}}, \uparrow)$, there are no conservation laws of energy-momentum and angular momentum for a given physical system modeled by $\boldsymbol{T}$ and also that there are no conservation laws (mentioned above) for the physical system plus the gravitational field. Despite that, it is often said that $(M, \boldsymbol{g})$ possesses one timelike and three spacelike Killing vector fields then there exists conserved quantities which may be identified with the total energy and total momentum of a given physical system. I show that this inference is equivocated, i.e., the existence of the mentioned Killing vector fields does not permit us the construction of a *genuine* conserved energy-momentum *covector*[1] for a given physical system. Note that construction of a conserved covector is always possible for special relativistic consistent field theories in Minkowski spacetime.

Moreover, I show that even if the Lorentzian spacetime structure does not possesses Killing vector fields there exists in **GRT** an infinity of conserved currents. This result is disclosed by introducing the so called Komar currents in a Lorentzian spacetime modelling a gravitational field generated by a given energy momentum tensor $\boldsymbol{T}$ and showing how any diffeomorphism associated to a one parameter group generated by a vector field $\mathbf{A}$ leads to a conserved current. I show moreover using the Clifford bundle formalism that $F = dA \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ where $A = \boldsymbol{g}(\mathbf{A},\ ) \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ satisfy, (with $\boldsymbol{\partial}$ denoting the Dirac operator acting on sections of the Clifford bundle $\mathcal{C}\ell(M, \mathtt{g})$ of differential forms) a Maxwell like equation $\boldsymbol{\partial} F = \boldsymbol{J}_A$ (equivalent to $dF = 0$ and $\underset{\boldsymbol{g}}{\delta} F = -\boldsymbol{J}_A$). The explicit form of $\boldsymbol{J}_A$ as a function of the

energy-momentum tensor is derived and a explicit scalar invariant is exhibited. I establish that[2] $\boldsymbol{\partial} F = \boldsymbol{J}_A$ encode the contents of Einstein equation. Moreover, I show that even if it is possible to get four conserved currents given one time like and three spacelike vector fields and thus get four scalar invariants, these objects cannot be associated to the components of a momentum *covector* for the system of fields producing the energy-momentum tensor $\boldsymbol{T}$. I also give the form of $\mathcal{J}_A$ when $\mathbf{A}$ is a Killing vector field and emphasize that even if the Lorentzian spacetime under consideration has one time like and three spacelike Killing vector fields it is not possible to find a conserved momentum covector for the system of fields.

The above results show that the culprit for the non existence of conservations laws in **GRT** is the association of the gravitational field with a Lorentzian spacetime structure. Thus, taking into account that: **(i)** the necessity of having as arena for theories with genuine conservation laws a parallelizable manifold and **(ii)** the well known Geroch theorem saying that existence of spinor fields in a 4-dimensional Lorentzian structure $(M, \boldsymbol{g})$ implies the trivialization of the orthonormal frame bundle, I present a theory of the gravitational field in Minkowski spacetime where that field is represented by four 1-form fields $\mathfrak{g}^{\mathbf{a}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. A Lagrangian for the potentials is postulated and the equations of motions are derived for $\mathfrak{F}^{\mathbf{a}} = d\mathfrak{g}^{\mathbf{a}}$. In particular, I evaluated for the case of a system consisting of matter and gravitational field $\delta\mathfrak{F}^{\mathbf{a}} = \boldsymbol{t}^{\mathbf{a}} + T^{\mathbf{a}}_m$, thereby identifying a legitimate energy-momentum tensor for the gravitational field where the energy-momentum 1-form fields $\boldsymbol{t}^{\mathbf{a}}$ are given by really nice formulas. I show that the equations of motion can easily be shown to be equivalent to Einstein equation and thus may describe an effective Lorentzian spacetime structure with fixed topology. Also, another interpretation of the field equations saying that they may describe an effective teleparallel spacetime structure (or even a more general geometric structure) is given.

I conclude that the usual interpretation of the gravitational field as spacetime geometry is not a necessity (since indeed very different geometrical structures can be associated with that field) and that it is perhaps time to try to understand the real nature of the gravitational field as a physical field in the sense of Faraday as suggested by several authors.

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[1] The energy-momentum *covector* is an element of a vector space and is not a covector field.

[2] The symbol $\boldsymbol{\partial}$ denotes the the Dirac operator acting on sections of the Clifford bundle $\mathcal{C}\ell(M, \mathtt{g})$.