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GA speaks Quantum Esperanto

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- Quantum systems: a brief reminder
- A geometric revisit to the q-bit
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- GA analysis of q-bit registers
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References: [1] and [2], and others at the end.

https://web.mat.upc.edu/sebastia.xambo/99/xambo-QE.pdf

Quantum systems

State vectors
Pure states
The ket map

The fundamental ingredient of a *quantum system* is a complex vector space $\mathcal{H}$, whose non-zero elements are called *state vectors*, or simply *vectors*, satisfying the following conditions:

- $\mathcal{H}$ is endowed with a *Hermitian* scalar product $\langle x|x'\rangle \in \mathbf{C}$ that is:
  - Linear in x'
  - Conjugate-symmetric: $\langle x'|x\rangle = \overline{\langle x|x'\rangle}$. This implies:
    $\langle x|x'\rangle$ is conjugate-linear in x.
    $\langle x|x\rangle \in \mathbf{R}$.
  - Positive-definite: $\langle x|x\rangle > 0$ for $x \neq 0$.
- The norm of $x \in \mathcal{H}$, $|x|$, is defined by $|x| = +\sqrt{\langle x|x\rangle}$.
- The *angle* between non-zero $x, x' \in \mathcal{H}$, $\beta = \beta(x, x') \in [0, \pi/2]$, is defined by the relation $\cos(\beta) = |\langle x|x'\rangle|/|x||x'|$.

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- The relation $x \sim x'$ means that $x' = \alpha x$ for some non-zero $\alpha \in \mathbf{C}$. In words, x and x' are *proportional*. Note: $|x'| = |\alpha||x|$.
- The relation $x \equiv x'$ means that $x' = \alpha x$ for some *unit* $\alpha \in \mathbf{C}$. Equivalently, $|x| = |x'|$
  In words, x and x' are *congruent*.

Unit vectors. Are those $x \in \mathcal{H}$ such that $|x| = 1$.

The *normalization* of a vector $x \in \mathcal{H}$ is the unit vector $\hat{x} = x/|x|$.

Example. $\mathbf{C}^2$ with the scalar product $\langle\xi|\xi'\rangle = \bar{\xi}_0\xi'_0 + \bar{\xi}_1\xi'_1$. The non-zero elements of this space are called *Pauli spinors*.

Thus $|\xi| = +\sqrt{|\xi_0|^2 + |\xi_1|^2}$.

Underlying Euclidean space. Let $\mathcal{H}_{\mathbf{R}}$ be $\mathcal{H}$ considered as a real vector space. This space is an Euclidean vector space with the scalar product $x \cdot x' = \mathrm{re}\langle x|x'\rangle$. Since $\langle x|x\rangle$ is real, the Euclidean norm, $\sqrt{x \cdot x}$, coincides with the Hemitian norm. Note also that the Euclidean ortogonality agrees with Hermitian orthogonality.

The (pure) *states* are the elements of $\Sigma = \mathbf{P}\mathcal{H}$ (*projective space of* $\mathcal{H}$).

This means that each non-zero $x \in \mathcal{H}$ determines a state, which we will denote (momentarily) by $|x\rangle$, and that two non-zero $x, x' \in \mathcal{H}$ determine the same state ($|x\rangle = |x'\rangle$) if and only if $x' \sim x$.

Remark. Each state can be represented by a unit vector, as $x \sim \hat{x}$. But unit vectors representing the same state differ by a *phase* factor (a unit complex number), and hence *sates and unit vectors are quite different concepts*.

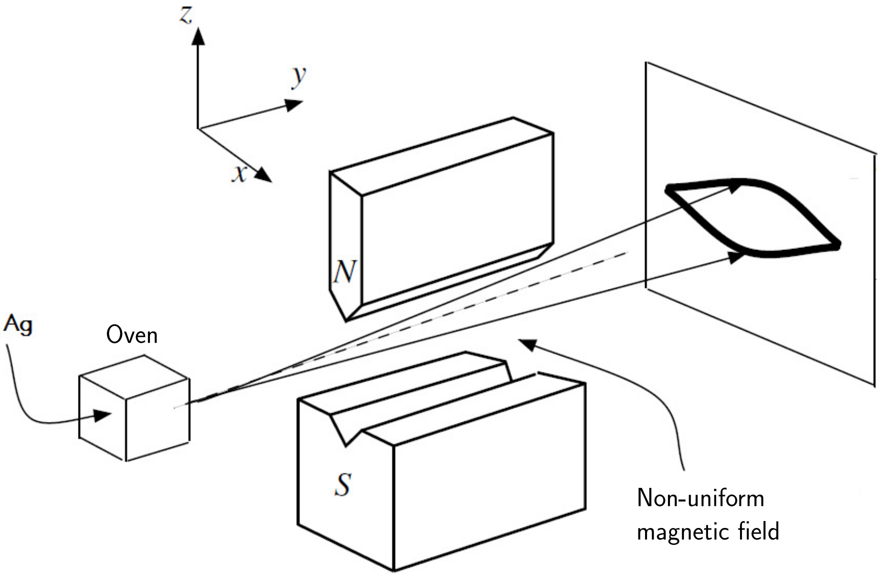
Example. $\mathbf{PC}^2 \simeq \mathbf{C} \sqcup \{\infty\} = \widehat{\mathbf{C}}$: For $\xi = (\xi_0, \xi_1) \in \mathbf{C}^2 - \{(0,0)\}$,

$$|\xi\rangle = \begin{cases} |(1, \xi_1/\xi_0)\rangle & \text{if} \quad \xi_0 \neq 0 \\ |(0,1)\rangle & \text{otherwise} \end{cases}$$

In the other direction, $\widehat{\mathbf{C}} \to \mathbf{PC}^2$: $\zeta \mapsto |(1,\zeta)\rangle$ for $\zeta \in \mathbf{C}$, and $\infty \mapsto |(0,1)\rangle$.

A geometric revisit to the q-bit

Stern-Gerlach experiments
Composition of S-G experiments
Pure states of a q-bit and its state vectors


z
y
x
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Ag
Oven
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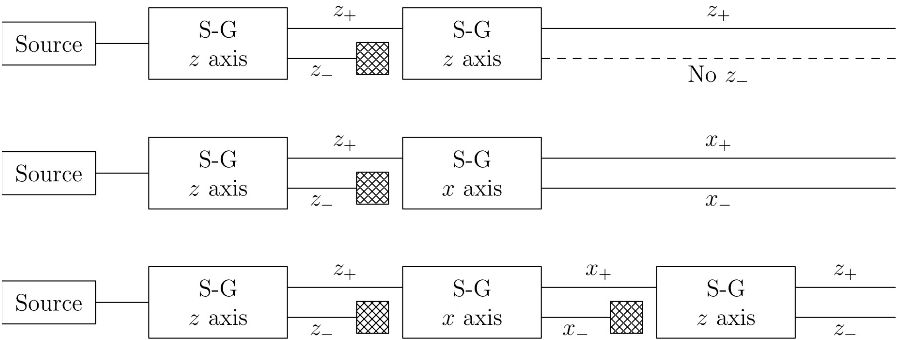
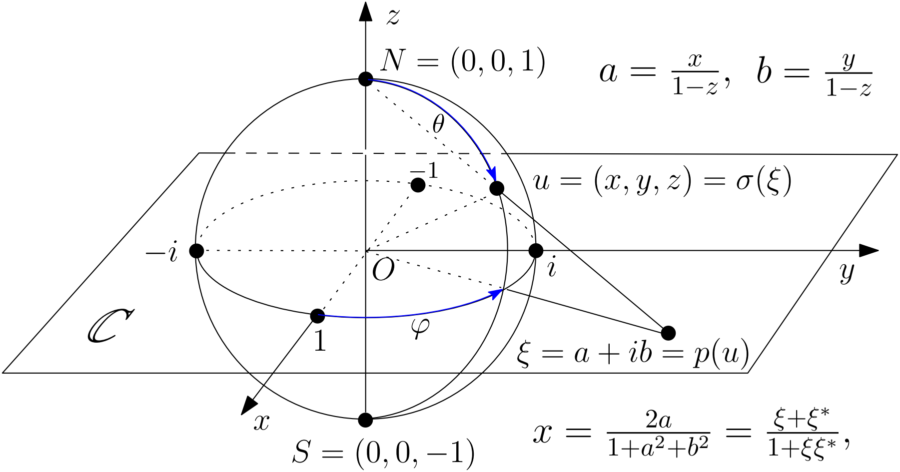
Non-uniform
magnetic field



Composition of S-G experiments
(from WP, Stern–Gerlach_experiment)

- The SG experiments suggest to model the states of q-bit by the points of $\Sigma = S^2$



$$a=\frac{x}{1-z},\quad b=\frac{y}{1-z}$$

$$x=\frac{2a}{1+a^2+b^2}=\frac{\xi+\xi^*}{1+\xi\xi^*},$$

$$y=\frac{2b}{1+a^2+b^2}=-i\frac{\xi-\xi^*}{1+\xi\xi^*},$$

$$z=\frac{1-a^2-b^2}{1+a^2+b^2}=\frac{\xi\xi^*-1}{\xi\xi^*+1}$$

$u=u_{\varphi,\theta}=$

$(\cos\varphi\sin\theta,\sin\varphi\sin\theta$

A GA portrait of a q-bit

The geometric quaternions $\mathbf{H} = \mathcal{G}_3^+$

Complex structures of $\mathbf{H}$

Hermitian product of $\mathbf{H}$

The ket map $\kappa : \mathbf{H} \to S^2 \subset E_3$

- $S^2 \simeq \widehat{\mathbf{C}}$ (P.10).
- $\widehat{\mathbf{C}} \simeq \mathbf{PC}^2$ (P.6)
- So $S^2 \simeq \mathbf{PC}^2$, and we could set $\mathcal{H} = \mathbf{C}^2$ (after Pauli).

Desiderata. To replace $\mathbf{C}^2$ by $\mathbf{H} = \mathcal{G}_3^+$
This requires *choosing a complex structure* for $\mathbf{H}$, *defining a Hermitian structure* relative to that structure, and *reproducing the measurement statistics*.

Let u_x, u_y, u_z is an orthonormal basis of E_3
(think of u_z as aligned with the SG magnetic field).
Set $\mathrm{i} = u_x u_y$, $\mathrm{j} = u_x u_z$, $\mathrm{k} = u_y u_z$.

Complex structure. Let $\mathbf{C} = \langle 1, \mathrm{i} \rangle \subset \mathbf{H}$. Then $\mathbf{H}$ is a $\mathbf{C}$-vector space and $\{1, \mathrm{j}\}$ is a $\mathbf{C}$-basis:
$\mathfrak{q} = a1 + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} = (a + b\mathrm{i})1 + (c + d\mathrm{i})\mathrm{j}$.

- $\mathbb{H} \leftrightarrow \mathbf{H} = \mathcal{G}_3^+$ (as real spaces).

- If $\mathfrak{q} = \xi_0 + \xi_1 \mathrm{j}$, $\xi_0, \xi_1 \in \mathbf{C}$, then
  $\xi_0 = \frac{1}{2}(\mathfrak{q} - \mathrm{i}\mathfrak{q}\,\mathrm{i}) = \mathrm{cx}(\mathfrak{q})$ and $\xi_1 = -\frac{1}{2}(\mathfrak{q}\,\mathrm{j} + \mathrm{i}\mathfrak{q}\,\mathrm{k})$.

Hermitian product. (1) The scalar product $\mathbf{H} \times \mathbf{H} \to \mathbf{C}$ defined by the formula $\langle \mathfrak{q} | \mathfrak{q}' \rangle = \mathrm{cx}(\mathfrak{q}'\bar{\mathfrak{q}})$ is Hermitian.
(2) For $\xi_0, \xi_1, \xi_0', \xi_1' \in \mathbf{C}$ we have $\langle \xi_0 + \xi_1 \mathrm{j} | \xi_0' + \xi_1' \mathrm{j} \rangle = \bar{\xi}_0 \xi_0' + \bar{\xi}_1 \xi_1'$.
(3) $\langle \mathfrak{q} | \mathfrak{q} \rangle = \mathrm{cx}(\mathfrak{q}\bar{\mathfrak{q}}) = \mathfrak{q}\bar{\mathfrak{q}} = |\mathfrak{q}|^2$.
(4) If $v \in E_3$, $|v^*|^2 = |v|^2$ (v^* the dual of v).

Note. $\bar{\mathfrak{q}}$ is the Clifford involution of $\mathfrak{q}$ (it coincides with the reverse involution). The customary symbol $\tilde{x}$ for the reverse involution will be used for other purposes below (P.15).

The ket map. Given $\mathfrak{q} = \xi_0 + \xi_1 j \in \mathbf{H}$, let $r = |\mathfrak{q}|$ and

$$x = (\xi_1\bar{\xi}_0 + \xi_0\bar{\xi}_1)/r^2, \quad y = i(\xi_0\bar{\xi}_1 - \xi_1\bar{\xi}_0)/r^2, \quad z = (\xi_1\bar{\xi}_1 - \xi_0\bar{\xi}_0)/r^2.$$

Then $x^2 + y^2 + z^2 = 1$ and the map he $\kappa : \mathbf{H} - \{0\} \to S^2 \subset E_3$ defined by $\kappa(\mathfrak{q}) = xu_x + yu_y + zu_z$ is onto and satisfies that $\kappa(\mathfrak{q}) = \kappa(\mathfrak{q}')$ if and only if $\mathfrak{q} \sim \mathfrak{q}'$. These statements follow from the formulas on page P.10.

Examples

(1) $\kappa(e^{i\varphi}\mathfrak{q}) = \kappa(\mathfrak{q})$, for any $\varphi \in \mathbf{R}$. Thus $\kappa(\mathfrak{q}') = \kappa(\mathfrak{q})$ if $\mathfrak{q}' \equiv \mathfrak{q}$.

(2) $\kappa(1) = \kappa(\mathrm{i}) = -u_z$ (the south pole of S^2) and $\kappa(\mathrm{j}) = \kappa(\mathrm{k}) = u_z$ (the north pole of S^2). Note that $\mathrm{i} \equiv 1$ and $\mathrm{k} = \mathrm{ij} \equiv \mathrm{j}$, so it is enough to check that $\kappa(1) = -u_z$ and $\kappa(\mathrm{j}) = u_z$.

(3) If $\kappa(\mathfrak{q}) = u_z$ then $\mathfrak{q} \sim \mathrm{j}$.

Notation. The vectors $\mathrm{j} = \mathrm{j}^1$ and $1 = \mathrm{j}^0$ represent the parallel anti-parallel states of the q-bit. As we will see, this notations are handy to represent the state vectors of q-bit registers. For the q-bit, the equation $\mathfrak{q} = \xi_0 \mathrm{j}^0 + \xi_1 \mathrm{j}^1$ expresses the fact that any state is a *superposition* of the states $u_z = \kappa(\mathrm{j}^1)$ and $-u_z = \kappa(\mathrm{j}^0)$.

In the literature those two state vectors are often denoted by $|\uparrow\rangle$, $|e_1\rangle$, or $|1\rangle$, in the case of j^1, and by $|\downarrow\rangle$, $|e_0\rangle$, or $|0\rangle$, in the case of $1 = \mathrm{j}^0$.

The map $\tilde{}: S^2 \to \mathbf{H}$. Given a state $u = xu_x + yu_y + zu_z \in S^2$, define $\tilde{u} \in \mathbf{H}$ to be j if $u = u_z$ and otherwise

$$\tilde{u} = \frac{1}{\sqrt{2(1-z)}}(1 - z + x\,\mathrm{j} + y\,\mathrm{k}).$$

Then $|\tilde{u}| = 1$ and $\kappa(\tilde{u}) = u$.

- If we express u in spherical coordinates, we get

$$\begin{aligned}\tilde{u} &= \sin\tfrac{\theta}{2} + e^{\mathrm{i}\varphi}\cos\tfrac{\theta}{2}\,\mathrm{j}\\ &\equiv e^{-\mathrm{i}\varphi/2}\sin\tfrac{\theta}{2} + e^{\mathrm{i}\varphi/2}\cos\tfrac{\theta}{2}\,\mathrm{j}.\end{aligned}$$

The behaviour of this parametrization in spinor space is illustrated on P.27 (see also P.26 about the behavior of spherical coordinates).

- $\widetilde{-u} = \frac{1}{\sqrt{2(1+z)}}(1 + z - x\,\mathrm{j} - y\,\mathrm{k})$, which is orthogonal to $\tilde{u}$. So the Hermitian angle between $\tilde{u}$ and $\widetilde{-u}$ is $\pi/2$, while the Euclidean angle between u and $-u$ is π.

Statistics. The probability $p_u(v)$ of observing the spin aligned with $u \in S^2$, assuming that the state before measurement is $v \in S^2$, is given by the formula

$$p_u(v) = |\langle \tilde{u} | \tilde{v} \rangle|^2.$$

Using the trig expressions for $\tilde{u}$ and $\tilde{v}$, it follows that

$$p_u(v) = \cos^2(\tfrac{\alpha}{2}),$$

where $\alpha = \alpha(u, v)$ is the Euclidean angle between u and v.

GA analysis of q-registers

The algebra of a register of q-bits

Basis of $\mathbf{H}^{(n)}$ derived from $\{1, \mathbf{j}\}$

Split elements and the Segre conditions

The state space $\Sigma_n = \mathbf{PH}^{(n)}$

Split and entangled states

The $\mathcal{H}$-space of n qbits, considered as a single quantum system, is [3]

$$\mathbf{H}^{(n)} = \mathbf{H}^{\otimes n},$$

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where the n factors $\mathbf{H}$ refer to the ordered array formed by the qbits. This description has an *important feature that is not present in the conventional treatment* of q-registers: $\mathbf{H}^{(n)}$ is a *unital associative* $\mathbf{C}$*-algebra*. Its structure is determined by $\mathbf{C}$-multilinearity and the rule

$$(\mathfrak{q}_1 \otimes \cdots \otimes \mathfrak{q}_n)(\mathfrak{q}'_1 \otimes \cdots \otimes \mathfrak{q}'_n) = (\mathfrak{q}_1\mathfrak{q}'_1) \otimes \cdots \otimes (\mathfrak{q}_n\mathfrak{q}'_n).$$

The Hermitian scalar product of $\mathbf{H}^{(n)}$ is determined by the rule

$$\langle \mathfrak{q}_1 \otimes \cdots \otimes \mathfrak{q}_n | \mathfrak{q}'_1 \otimes \cdots \otimes \mathfrak{q}'_n \rangle = \langle \mathfrak{q}_1 | \mathfrak{q}'_1 \rangle \cdots \langle \mathfrak{q}_n | \mathfrak{q}'_n \rangle$$

and $\bar{\mathbf{C}}/\mathbf{C}$-multilinearity. We note that $\mathfrak{q}_1 \otimes \cdots \otimes \mathfrak{q}_n$ and $\mathfrak{q}'_1 \otimes \cdots \otimes \mathfrak{q}'_n$ are orthogonal if and only if $\mathfrak{q}_k$ and $\mathfrak{q}'_k$, for some $k \in 1..n$, are orthogonal.

Note also that $|\mathfrak{q}_1 \otimes \cdots \otimes \mathfrak{q}_n|^2 = |\mathfrak{q}_1|^2 \cdots |\mathfrak{q}_n|^2$.

Let $B = \{0, 1\}$. For each sequence $\nu = (\nu_1, \ldots, \nu_n) \in B^n$, set

$$\mathrm{j}^{\nu} = \mathrm{j}^{\nu_1} \otimes \cdots \otimes \mathrm{j}^{\nu_n} \in \mathbf{H}^{(n)}.$$

Then $\{\mathrm{j}^{\nu} \mid \nu \in B^n\}$ is an *orthonormal basis* of $\mathbf{H}^{(n)}$ and hence a general element of $\mathbf{H}^{(n)}$ has the form

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$$\xi = \sum_{\nu \in B^n} \xi_{\nu} \mathrm{j}^{\nu}, \ \xi_{\nu} \in \mathbf{C}.$$

We have $\mathrm{j}^{\nu} \mathrm{j}^{\nu'} = \epsilon(\nu, \nu') \mathrm{j}^{\nu+\nu'}$, where $\epsilon(\nu, \nu')$ is the parity of the number of $k \in 1..n$ such that $\nu_k = \nu'_k = 1$, that is, the parity of the number of 1's in $\nu\nu'$ (component-wise binary product). This rule provides an *algorithm for the computation of products* $\xi\xi'$ in $\mathbf{H}^{(n)}$.

Remark. The set $J_n = \{\pm \mathrm{j}^{\nu} \mid \nu \in B^n\}$ has cardinal 2^{n+1} and forms a commutative group with the product. It can be seen that it is the product of the cyclic group of order 4 generated by $\mathrm{j}^{10\cdots0}$ and the $n-1$ groups or order 2 generated by $\mathrm{j}^{110\cdots0}, \ldots, \mathrm{j}^{0\cdots011}$. Note that $J_n/\{\pm 1\} \leftrightarrow B^n$.

An element $\xi \in \mathbf{H}^{(n)}$ is said to be *split*, or *decomposable*, if it is of the form $\xi = \mathfrak{q}_1 \otimes \cdots \otimes \mathfrak{q}_n$, $\mathfrak{q}_1, \ldots, \mathfrak{q}_n \in \mathbf{H}$. The ν component of this element is $\xi_\nu = \xi_{\nu_1}(\mathfrak{q}_1) \cdots \xi_{\nu_n}(\mathfrak{q}_n)$, where we write, for $\mathfrak{q} \in \mathbf{H}$, $\mathfrak{q} = \xi_0(\mathfrak{q}) + \xi_1(\mathfrak{q})\mathsf{j}$. Now *these ξ_ν are not independent*. Indeed, we can write relations among them as follows. Partition the ν's into those that begin with 0 and those that beging with 1. Then form the $2 \times 2^{n-1}$ matrix whose rows correspong to the ξ_ν's of these two groups. Since the two rows are proportional, all the 2×2 minors of the matrix vanish. These are the *Segre relations* and it happens that they are also sufficient (and in general redundant) to insure that a vector $\xi \in \mathbf{H}^{(n)}$ is split. N

For $n = 2$, we get a single relation: $\det \begin{bmatrix} \xi_{00} & \xi_{01} \\ \xi_{10} & \xi_{11} \end{bmatrix} = 0$. For $n = 3$ we have the matrix $\begin{bmatrix} \xi_{000} & \xi_{001} & \xi_{010} & \xi_{011} \\ \xi_{100} & \xi_{101} & \xi_{110} & \xi_{111} \end{bmatrix}$ and 6 relations.

Entanglement. The vectors that are not split are said to be *entangled*. For $n = 2$, the vector $\xi^{\mathrm{EPR}} = \mathsf{j}^{00} + \mathsf{j}^{11}$ is entangled. A random element of $\mathbf{H}^{(n)}$ is entangled.

By definition, $\Sigma_n = \mathbf{H}^{(n)} - \{0\}/\sim$, a space of complex dimension $2^n - 1$. Let $\kappa : \mathbf{H}^{(n)} - \{0\} \to \Sigma_n$ be the *ket* map, which by definition is onto and with the property that $\kappa(\xi) = \kappa(\xi')$ if and only if $\xi \sim \xi'$.

The condition for $\xi \in \mathbf{H}^{(n)}$ to be a unit vector is that

$$\sum_{\nu \in B^n} |\xi_\nu|^2 = 1.$$

This equation represents the *unit sphere* of the Euclidean space $\mathbf{H}^{(n)}_{\mathbf{R}}$. Since this Euclidean space has of dimension $2 \times 2^n = 2^{n+1}$, that sphere is denoted by $S^{2^{n+1}-1}$, and thus

$$\Sigma_n = S^{2^{n+1}-1}/\equiv.$$

The map $\kappa : S^{2^{n+1}-1} \to \Sigma_n$ is onto and with the property that $\kappa(\xi) = \kappa(\xi')$ if and only if $\xi \equiv \xi'$.

For $n = 1, 2, 3, 4$ the (real) dimension of these spheres is $3, 7, 15, 31$ and hence the real dimension of Σ_n is $2, 6, 14, 30$.

A state $\kappa(\xi)$ is said to be *split* if ξ is a split vector. This is well defined, because if ξ is split and $\xi \sim \xi'$, then ξ' is split.

Let $\Sigma'_n \subset \Sigma_n$ be the set of split states. We have an onto map $(S^2)^n \to \Sigma'_n$ defined by

$$(v_1, \ldots, v_n) \mapsto \kappa(\tilde{v}_1 \otimes \cdots \otimes \tilde{v}_n).$$

This shows that entangled states are specified with $2n$ real parameters, or n complex parameters, whereas general states are specified with $2^n - 1$ complex parameters. This again confirms the assertion that *a random state is entangled*.

Conclusions and outlook

- Introduction to Hermitian spaces, with highlightings of some important concepts, like the Hermitian angle.

- Hermitian structure of $\mathbf{H} = \mathcal{G}_3^+$ (the *geometric quaternions*) and its use for a geometric algebra account of a q-bit.

- The *algebras* $\mathbf{H}^{(n)}$, that supply a transparent formalism for modeling q-bit registers of arbitrary length, both conceptually and computationally. Especially apt to encode quantum gates.

- Segre relations and entanglement. One step further would be to tackle in this formalism references such as [4] and some related works of Jordi Tura (`https://jtura.cat/`), particularly his memoir [5].

- The algebras $\mathbf{H}^{(n)}$ may be a suitable resource for research in discreet mathematics, as for example quantum error-correcting codes.

Thank you!

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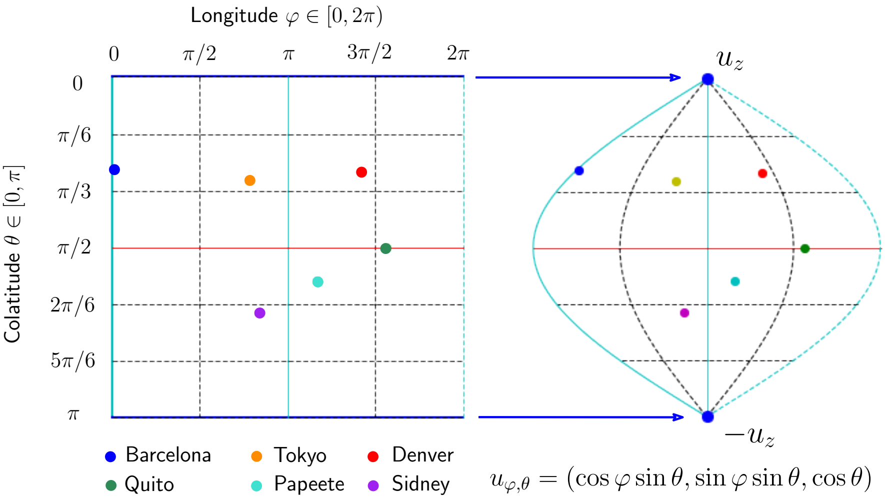
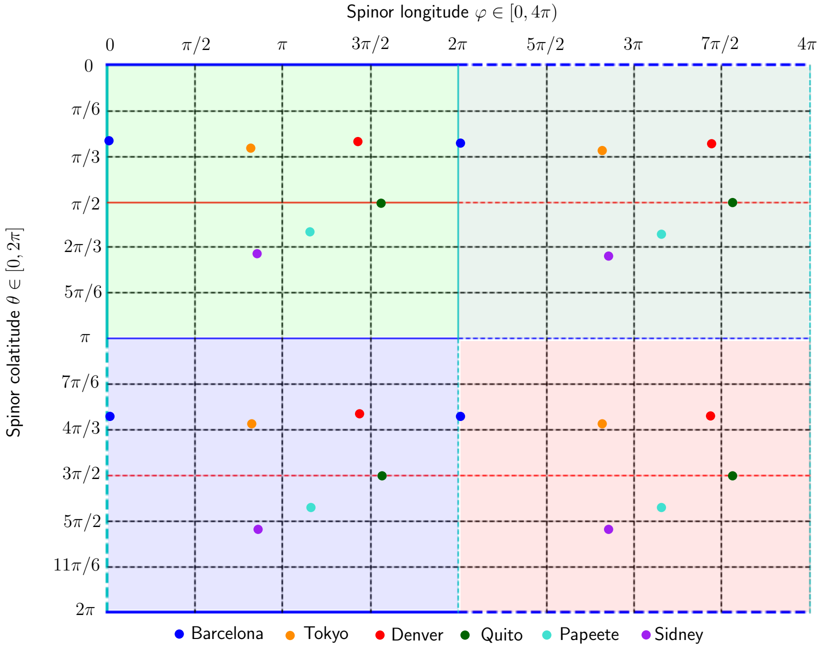


Figure 12.1: On the left, we have the rectangle of points (φ, θ) with $\varphi \in [0, 2\pi)$ and $\theta \in [0, \pi]$. The parameter φ is the *longitude* measured eastward from a given meridian (vertical segment on the left, in cyan), say the Greenwich one. The parameter θ is the *colatitude* and is measured along a meridian from the north pole u_z to the south pole $-u_z$.


Spinor longitude $\varphi \in [0, 4\pi)$
0 $\pi/2$ π $3\pi/2$ 2π $5\pi/2$ 3π $7\pi/2$ 4π
Spinor colatitude $\theta \in [0, 2\pi]$
0 $\pi/6$ $\pi/3$ $\pi/2$ $2\pi/3$ $5\pi/6$ π $7\pi/6$ $4\pi/3$ $3\pi/2$ $5\pi/2$ $11\pi/6$ 2π
Barcelona
Tokyo
Denver
Quito
Papeete
Sidney

Notes

For *Euclidean spaces*, where the (real symmetric) inner product is denoted $x \cdot x'$, the Cauchy-Schwarz inequality says that $|x \cdot x'| \leqslant |x||x'|$. Since $x \cdot x'$ is *real*, this is equivalent to the inequalities $-|x||x'| \leqslant x \cdot x' \leqslant |x||x'|$. This implies that $-1 \leqslant (x \cdot x')/|x||x'| \leqslant 1$ if $x, x' \neq 0$, and therefore there exists a unique real number $\alpha = \alpha(x, x') \in [0, \pi]$ such that $\cos(\alpha) = x \cdot x'/|x||x'|$. This α is the (Euclidean) *angle* between x and x'. With a bit more attention, it can be seen that $\alpha = 0$ if and only if $x' = tx$, with $t > 0$, and that $\alpha = \pi$ if and only if $x' = tx$, with $t < 0$. In any case, the angle does not vary if we rescale the vectors: if t, t' are positive real numbers, then $\alpha(tx, t'x') = \alpha(x, x')$. But note that $\alpha(-x, x') = \alpha(x, -x') = \pi - \alpha(x, x')$.

P

"The classical theory predicts that the atomic magnets assume all possible directions with respect to the direction of the magnetic field. On the other hand, the quantum theory predicts that we shall find only two directions parallel and antiparallel to the field (new theory, the old one gave also the direction perpendicular to the field)" (from Stern's Nobel lecture).

P

Each point on this rectangle determines a point $u = u_{\varphi,\theta}$ on the sphere S^2 by the formula $u = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$. Outside of the two blue segments, the map $(\varphi, \theta) \mapsto u_{\varphi,\theta}$ is bijective, while it maps all the points of the blue upper segment, which have the form $(\varphi, 0)$, to u_z, and all the points of the blue lower segment, which have the form (φ, π), to $-u_z$. The image on the right is formed by rescaling the parallel of each collatitude according to its length, which is maximum at the equator (red segments) and decreases to 0 on approaching the poles.

P

The paper [3] (aerts-daubechies-1978) provides a "physical justification for using the tensor product to describe two quantum systems as one joint system".

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Classical computations happen in B^n. Quantum computations happen in $\mathbf{H}^{(n)}$, where the classical arena appears only as indices for the basis $\{\mathrm{j}^{\nu}\}$ of $\mathbf{H}^{(n)}$.

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The number of Segre 2×2 determinants is $2^{2n-3} - 2^{n-2}$. The minimum number of sufficient conditions turns out to be $2^n - n - 1$ and for $n \geqslant 2$, $2^{2n-3} - 2^{n-2} \geqslant 2^n - n - 1$, with equality ($= 1$) only for $n = 2$. For $n = 3$, the values are 6 and 4 (so 2 redundant equations); for $n = 4$, 28 and 11, so 17 redundant equations; and for large n, the number of redundant equations is asymptotically equal to the number of equations. For $n = 10$, for example, the two numbers are 130816 and 1013, which means 129803 redundancies.

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Rectangle of spinor spherical coordinates. The green rectangle covers the rectangle of the sphere spherical coordinates, so that spinors $\widetilde{u}$ in this region give all the points of the sphere, one-to-one except at the poles. Spinors in the grey rectangle (top right) have the form $-\widetilde{u}$. This explains why in spinor space it is required to go round a parallel twice to come back to the starting spinor. Spinors in the blue rectangle have the form $-\widetilde{u}^{\perp}$. This means that if we follow a meridian and continue after going through the south pole, in spinor space it is like moving along the antipodal points, which on the sphere amounts to moving on the oposite meridian from south to north. In the reddish region, spinors have the form $\widetilde{u}^{\perp}$.

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References I

[1] S. Xambó-Descamps, "Geometric Algebra Speaks Quantum Esperanto, I," 2022.

Submitted to the ICACGA-2022 Conference.

[2] S. Xambó-Descamps, "Geometric Algebra Speaks Quantum Esperanto, II," 2022.

In preparation.

[3] D. Aerts and I. Daubechies, "Physical justification for using the tensor product to describe two quantum systems as one joint system," *Helvetica Physica Acta*, vol. 51, no. 5-6, 1978.

References II

[4] M. Gharahi, S. Mancini, and G. Ottaviani, "Fine-structure classification of multiqubit entanglement by algebraic geometry," *Physical Review Research*, vol. 2, no. 4, p. 043003, 2020.

[5] J. Tura, *Characterizing entanglement and quantum correlations constrained by symmetry*.

Springer, 2017.