Congreso bienal RSME

Special Session on Geometric Topology

Two-way bridges between Geometric Topology and Mathematical Physics

S. Xambó

UPC

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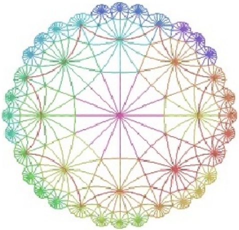
https://mat-web.upc.edu/people/sebastia.xambo/Talks

Abstract. The interaction between geometry and mathematical physics has been a key factor of scientific progress ever since Newton, with a long row of classical contributions and a further stream related to relativity, quantum mechanics and quantum field theory. If the bridge from geometry to physics has seen a clear and an ever increasing flow of ideas and methods, in the last decades it has appeared that theoretical physics is providing a wealth of concepts and structures that have a deep significance for geometry. In this talk some remarkable instances of this two-way bridge will be summarized, particularly in relation to low-dimensional geometric topology, and then a closer attention will be paid to the geometric and physical aspects of Dirac's equation as revealed by the insights of D. Hestenes and a number of researchers that follow his lead.

MTLI

Facultat de Matemàtiques i Estadística

Conferències FME



Volum I

Curs Poincaré

2003-2004

Facultat de Matemàtiques i Estadística
UNIVERSITAT POLITÈCNICA DE CATALUNYA

LA CONJETURA DE POINCARÉ. CIEN AÑOS DE INVESTIGACIÓN.

MARÍA TERESA LOZANO IMÍZCOZ

Henri Poincaré es considerado como el creador de la Topología. Su célebre conjetura sobre la esfera tridimensional ha dado lugar al gran desarrollo de esta rama de las matemáticas en el siglo XX.

Poincaré poseía una mente privilegiada en cuanto a visión geométrica abstracta, y esta capacidad es la base para entender la topología. Recordemos que en topología se estudian las propiedades de los espacios que se conservan bajo deformaciones continuas, es decir, sin permitir cortes o pegados. Es una especie de *geometría blanda*, no rígida. Podemos decir que en topología dos objetos (*espacios topológicos*) son iguales (*homeomorfos*) si uno se obtiene del otro por una deformación continua (existe entre ellos una correspondencia biunívoca y bicontinua). Para un topólogo es lo mismo un poliedro sólido regular, un tornillo o una bola tridimensional. No es difícil imaginar la deformación sin rotura de uno de los dos primeros objetos anteriores en una bola si se suponen hechos de un material moldeable por presión.

Poincaré definió el concepto de homología como abstracción de la siguiente observación: una curva cerrada en una variedad que bordea una superficie nunca será equivalente, por una deformación continua, a una curva que no bordea ninguna superficie, ya que la deformación

1991 *Mathematics Subject Classification.* Primary 57.

91


ZITRUS

Imaginary *Pamplona*

Imaginary *Zaragoza*

JOSÉ MARÍA MONTESINOS AMILIBIA

Nacido en San Sebastián (Guipúzcoa, España, 1944), y actualmente catedrático de Geometría y Topología de la Universidad Complutense de Madrid, José María Montesinos es un experto reconocido internacionalmente en la topología de las variedades de dimensión 3 y 4 utilizando como herramienta fundamental la teoría de nudos y enlaces [1].

Las líneas más significativas en las que se pueden inscribir sus trabajos son las siguientes: cubiertas ramificadas sobre nudos [2], diagramas de Heegaard y cirugía en enlaces y en nudos [3], variedades hiperbólicas y variedades de caracteres [4], grupos aritméticos [5], grupos de automorfismos de formas cuadráticas enteras y sus espacios de órbitas [6] y variedades abiertas y nudos salvajes [7].

Entre las múltiples inquietudes científicas de José María Montesinos, ocupan un lugar preeminente la mineralogía y la cristalografía, ciencias que ha cultivado extensamente, incluyendo asiduos trabajos de campo, y a las que aplica sus conocimientos topológicos y geométricos.

[1] Montesinos-1987
Montesinos-Matsumoto-2011

[2] Montesinos-1974
Montesinos-1983
Hilden-Lozano-Montesinos-Whitten-1987

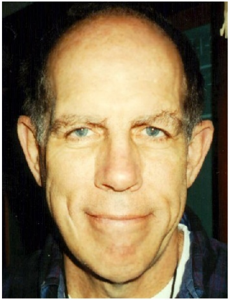
[3] Montesinos-1973
Montesinos-1975
Birman-GonzálezAcuña-Montesinos-1976 (pdf
GonzálezAcuña-Montesinos-1978 (Jstor)

[4] Hilden-Lozano-Montesinos-1996
Hilden-Lozano-Montesinos-2000

[5] Hilden-Lozano-Montesinos-1992
Hilden-Lozano-Montesinos-1995

[6] Montesinos-2014

[7] Montesinos-2003



1996

[77] H. M. Hilden, M. T. Lozano, and J. M. Montesinos. On volumes and Chern-Simons invariants of geometric 3-manifolds. *Journal Mathematical Sciences University Tokyo*, 3(3):723–744, 1996. MR1432115 98h:57056 (D. C. Calegari)

[78] H. M. Hilden, M. T. Lozano, and J. M. Montesinos. Volumes and Chern-Simons invariants of cyclic coverings over rational knots. *Topology and Teichmüller spaces* (Katinkulta, 1995), 31–35, 1996. MR1659808 2000a:57022 (Y. Rong)

1999

[81] H. M. Hilden, M. T. Lozano, and J. M. Montesinos. The Chern–Simons invariants of hyperbolic manifolds via covering spaces. BLMS, 31(3):354–366, 1999. MR1673415 2000c:57022 (D. Auckly)

María Teresa Lozano Imízcoz

Catedrática de Geometría y Topología en el **Departamento de Matemáticas** de la **Universidad de Zaragoza**, María Teresa Lozano Imízcoz (Pamplona, 1946) es especialista en topología de dimensión baja.

Sus aportaciones se pueden inscribir en diversas líneas de investigación: estructuras geométricas singulares (orbifolds) universales [1], invariantes geométricos en variedades tridimensionales [2], la descomposición de nudos en ovillos incompresibles [3], invariantes polinómicos de nudos [4], espacios recubridores virtualmente regulares [5] y superficies incompresibles [6].

En 2016 le ha sido otorgada la Medalla de la RSME por haber abarcado "durante más de 40 años de manera excelente todas las facetas de la profesión matemática: investigación, docencia, gestión, divulgación y servicio a la comunidad. Destacan sus trabajos con Hilden y Montesinos sobre teoría de nudos y variedades tridimensionales, su vocación docente y de servicio a través de la gestión universitaria y su labor en la divulgación de las matemáticas".

---

[1] **Hilden-Lozano-Montesinos-Whitten-1987**

[2] **Hilden-Lozano-Montesinos-1996**

[3] **Lozano-1983**
Lozano-1987

[4] **Lozano-Morton-1990**
Hilden-Lozano-Montesinos-1995

[5] **Lozano-Safont-1989**

[6] **Lozano-Przytycki-1985**

Otras informaciones

9/1/2017 (SX $\rightarrow$ MTLI)

Muchísimas gracias por los materiales (MTLI-1996 [1], MTLI-2014 [2]).

No me has presentado todavía a la física matemática que va contigo. ¿O sí?

11/1/2017 (MTLI $\rightarrow$ SX)

Lo cierto es que no se casi nada de física matemática. Solo me he interesado a nivel de aficionada en conferencias y he hablado a veces con físicos sobre la variedad de Poincaré, teoría de cuerdas, diagramas e integrales de Feynman, ... Lo que sí tengo es buenos amigos que son físicos teóricos, por eso he impartido todas las conferencias que me han solicitado sobre lo que sé un poco más, las 3-variedades. Un ejemplo es la conferencia que te envié de Jaca (MTLI-1996 [1]).

Physics

Durch die Untersuchungen über die Grundlagen der Geometrie wird uns die Aufgabe nahe gelegt, nach diesem Vorbilde diejenigen physikalischen Disciplinen axiomatisch zu behandeln, in denen schon heute die Mathematik eine hervorragende Rolle spielt ...

The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part ....

$s = \frac{1}{2}gs^2$ (Galileo, terrestrial gravity dynamics, Galileo relativity)

$\boldsymbol{f} = m\boldsymbol{a}$ (Newton's second law)

$\boldsymbol{f} = -G\dfrac{Mm}{r^3}\boldsymbol{r} = -G\dfrac{Mm}{r^2}\boldsymbol{u}$ (celestial dynamics, EDOs)

$\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}} - \dfrac{\partial L}{\partial q} = 0$ (Euler-Lagrange equations, differential geometry)

$(\dot{q}, \dot{p}) = \left(\dfrac{\partial H}{\partial p}, -\dfrac{\partial H}{\partial q}\right)$ (Hamilton, symplectic geometry).

EDPs: $\Delta\varphi = f$, $\dfrac{\partial^2 u}{\partial t^2} = c^2\Delta u$, $\dfrac{\partial u}{\partial t} = \alpha\Delta u$,

$\dfrac{\partial}{\partial t}(\rho\boldsymbol{u}) + \nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u} + p\boldsymbol{I}) = \nabla\cdot\boldsymbol{\tau} + \rho\boldsymbol{g}$

(functional analysis, operator theory, Hilbert space)

$\partial_t \rho = \nabla \cdot \boldsymbol{j}$ (conservation of electric charge)

$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ (Lorentz force law)

$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0$ (Coulomb-Gauss)

$\nabla \cdot \boldsymbol{B} = 0$ (Gauss)

$\nabla \wedge \boldsymbol{E} + \partial_t \boldsymbol{B} = 0$ (Faraday)

$\nabla \wedge \boldsymbol{B} - \mu_0 \varepsilon_0 \partial_t \boldsymbol{E} = \mu_0 \boldsymbol{j}$ (Ampère-Maxwell).

$\mu_0 \varepsilon_0 = 1/c^2$

Wave theory of light.

Einstein 1905: Sets up a relativistic mechanics (that for small speeds agrees with classical mechanics) which is compatible with Maxwell's equations. Guiding principle: Lorentz invariance.

New features: time dilation, length contraction, $E = mc^2$, $\boldsymbol{B}$ is a relativistic effect of $\boldsymbol{E}$. If the electromagnetic field is encoded as $F = (E_x dx + E_y dy + E_z dz) \wedge dt + B_x dy \wedge dz + B_y dz \wedge dy + B_z dx \wedge dy$. then ME's are equivalent to $dF = d^*F = 0$, where $*$ is the Hodge duality isomorphism.

Potential: $F = dA$.

Planck, 1900: $E = h\nu$ ($h = 6.62607 \times 10^{-34}$ J s)

Einstein 1905: Photoelectric effect (the photon was born, but christened much later).

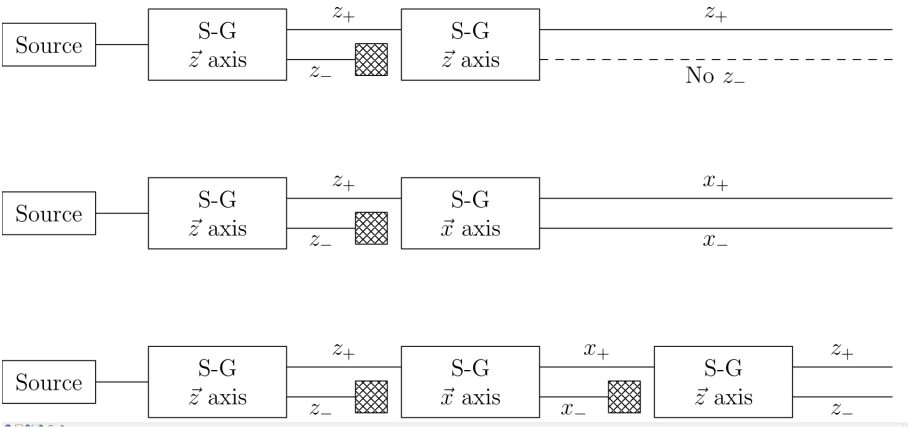
De Broglie: A particle of mass m and speed v has an associated wave whose wave-length is $\lambda = h/p$ ($p = mv$ the momentum). The closest to the equation $mc^2 = h\nu$ is replacing $\nu = v/\lambda$ (which is clear) and c by v (which is ad hoc), giving

$$\lambda = h/p.$$

But the relation was observed experimentally in 1927 by C. J. Davisson and L. H. Germer, and in a different way by G. P. Thomson.

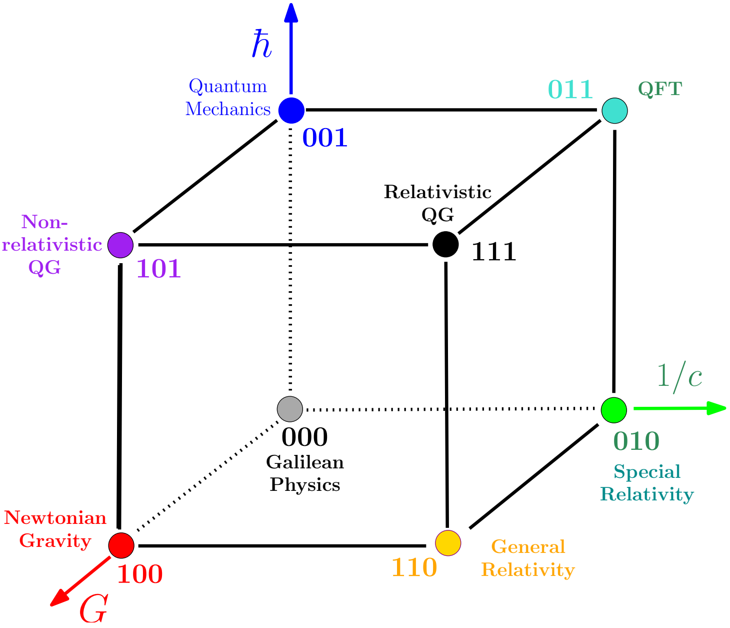
Schrödinger equation:

$$\widehat{H}|\psi(t)\rangle = i\hbar\partial_t|\psi(t)\rangle$$



Discovery of the electron spin: Stern-Gerlach experiment.

In his theory, Pauli rediscovered the Clifford algebra of Euclidean space. Mathematically: the deep relationship of S^2 with the groups SO_3 and SU_2.


$\hbar$
Quantum Mechanics
001
011
QFT
Relativistic QG
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Non-relativistic QG
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$1/c$
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Galilean Physics
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Special Relativity
Newtonian Gravity
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110
General Relativity
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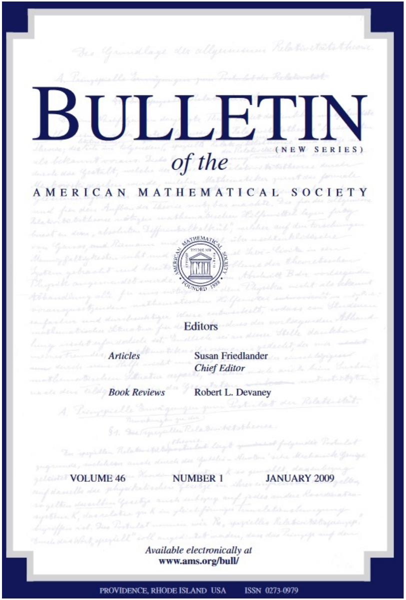
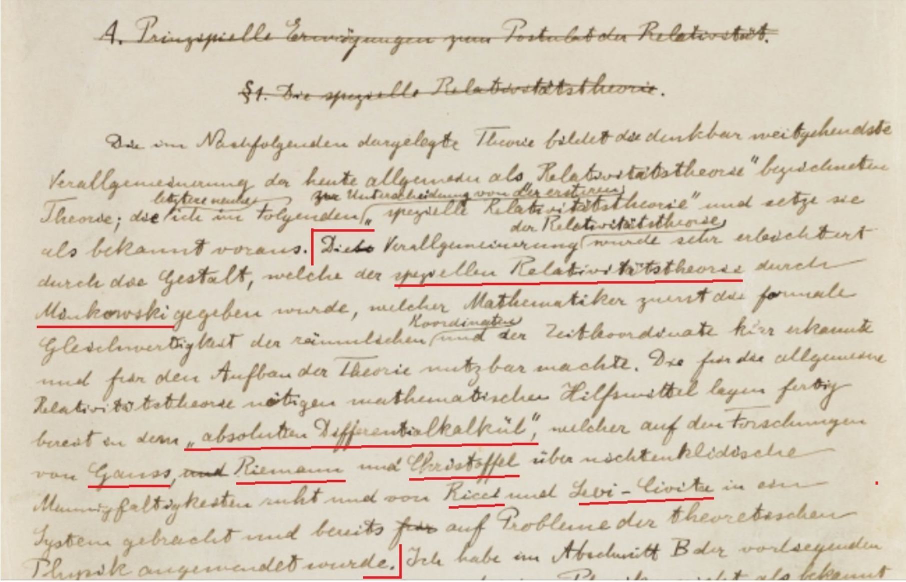


The generalization of the theory of relativity has been facilitated considerably by Minkowski, a mathematician who was the first one to recognize the formal equivalence of space coordinates and the time coordinate, and utilized this in the construction of the theory. The mathematical tools that are necessary for general relativity were readily available in the "**absolute differential calculus**," which is based upon the research on non-Euclidean manifolds by Gauss, Riemann, and Christoffel, and which has been systematized by Ricci and Levi-Civita and has already been applied to problems of theoretical physics.

See paper by Alicia Dickenstein in the same issue (pp 120-129):

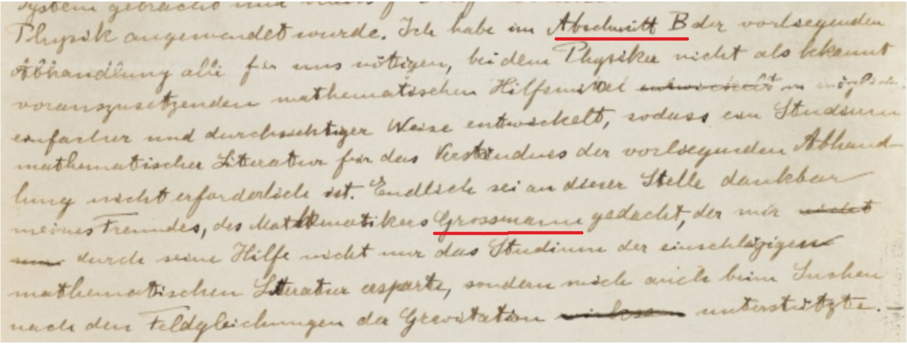
A hidden praise of Mathematics



A. ~~Prinzipielle Erwägungen zum Postulat der Relativität.~~

§1. ~~Die spezielle Relativitätstheorie.~~

Die im Nachfolgenden dargelegte Theorie bildet die denkbar weitgehendste Verallgemeinerung der heute allgemein als „Relativitätstheorie" bezeichneten Theorie; die letztere nenne ich im Folgenden zur Unterscheidung von der ersteren „spezielle Relativitätstheorie" und setze sie als bekannt voraus. Die Verallgemeinerung der Relativitätstheorie wurde sehr erleichtert durch die Gestalt, welche der speziellen Relativitätstheorie durch Minkowski gegeben wurde, welcher Mathematiker zuerst die formale Gleichwertigkeit der räumlichen Koordinaten und der Zeitkoordinate klar erkannte und für den Aufbau der Theorie nutzbar machte. Die für die allgemeine Relativitätstheorie nötigen mathematischen Hilfsmittel lagen fertig bereit in dem „absoluten Differentialkalkül", welcher auf den Forschungen von Gauss, Riemann und Christoffel über nichteuklidische Mannigfaltigkeiten ruht und von Ricci und Levi-Civita in ein System gebracht und bereits auf Probleme der theoretischen Physik angewendet wurde. Ich habe im Abschnitt B der vorliegenden



System gebracht und …
Physik angewendet wurde. Ich habe im Abschnitt B der vorliegenden
Abhandlung alle für uns nötigen, bei dem Physiker nicht als bekannt
vorauszusetzenden mathematischen Hilfsmittel ~~entwickelt~~ in möglichst
einfacher und durchsichtiger Weise entwickelt, sodass ein Studium
mathematischer Literatur für das Verständnis der vorliegenden Abhand-
lung nicht erforderlich ist. Endlich sei an dieser Stelle dankbar
meines Freundes, des Mathematikers Grossmann gedacht, der mir ~~nicht~~
~~…~~ durch seine Hilfe nicht nur das Studium der einschlägigen
mathematischen Literatur ersparte, sondern mich auch beim Suchen
nach den Feldgleichungen der Gravitation ~~…~~ unterstützte.

In section B of the present paper I developed all the necessary mathematical tools –which cannot be assumed to be known to every physicist– and I tried to do it in as simple and transparent a manner as possible, so that a special study of the mathematical literature is not required for the understanding of the present paper. Finally, I want to acknowledge gratefully my friend, the mathematician Grossmann, whose help not only saved me the effort of studying the pertinent mathematical literature, but who also helped me in my search for the field equations of gravitation.

The matrices

$$\sigma_1 = \sigma_x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \ \sigma_2 = \sigma_y = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \ \sigma_3 = \sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \quad (1)$$

were introduced by W. Pauli in 1927 in his study of spin $1/2$ particles.

They satisfy *Clifford's relations* $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{j,k}$.

Remark. The eigenvalues of the σ_j are ± 1 and the corresponding eigenvectors in $\mathbf{C}^2$ (Pauli's *spinor space*) are $[1, \pm 1]$, $[1, \pm i]$, and $\{[1, 0], [0, 1]\}$, respectively. These eigenvectors become the unit points on the x, y, z axes under the *spinor map* $S^3 \to S^2$ (*Hopf fibration* in mathematics): if $\psi = [\xi_0, \xi_1] \in S^3 \subseteq \mathbf{C}^2$ (so that $\xi_0\bar{\xi}_0 + \xi_1\bar{\xi}_1 = 1$), then

$$x = \xi_0\bar{\xi}_1 + \bar{\xi}_0\xi_1, \ y = i(\xi_0\bar{\xi}_1 - \bar{\xi}_0\xi_1), \ z = \xi_1\bar{\xi}_1 - \xi_0\bar{\xi}_0. \quad (2)$$

If ψ is not normalized, which means that ψ is any non-zero vector of $\mathbf{C}^2$, $(x, y, z) \in S^2_{|\psi|}$ (cf. [3], p. 248-249).

The rest mass μ of a particle of mass m and momentum p in the lab frame is given by

$$\mu^2 = m^2 - p^2.$$

Schrödinger's quantization trick ($m \mapsto i\hbar\partial_t$, $\boldsymbol{p} \mapsto -i\hbar\boldsymbol{\nabla}$)[1] leads to the equation

$$(i\hbar)^2\Box\psi = \mu^2\psi, \tag{3}$$

where $\psi : E_{1,3} \to \mathbf{C}$ is the *wave function* and

$$\Box = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$$

is the *dalambertian*. If we set $M = \mu/\hbar$, then (3) is equivalent to

$$(\Box + M^2)\psi = 0, \tag{4}$$

which is the *Klein-Gordon* wave equation.

---

1 $\boldsymbol{\nabla} = \boldsymbol{i}\partial_x + \boldsymbol{j}\,\partial_y + \boldsymbol{k}\,\partial_z$, $\boldsymbol{\nabla}^2 = -(\partial_x^2 + \partial_y^2 + \partial_z^2) = -\Delta$.

In 1928 Dirac introduced his famous γ-matrices:

$$\gamma_0 = \begin{pmatrix} \sigma_0 & \\ & -\sigma_0 \end{pmatrix},\ \gamma_k = \begin{pmatrix} & -\sigma_k \\ \sigma_k & \end{pmatrix}, \quad (5)$$

with $\sigma_0 = I_2$. These matrices satisfy Clifford's relations for the metric $\eta = (+,-,-,-)$:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}. \quad (6)$$

These relations are tailored to guarantee that $\partial^2 = \Box$, where ∂ (the *Dirac operator*)[1] is given by

$$\partial = \gamma^0 \partial_0 + \gamma^1 \partial_1 + \gamma^2 \partial_2 + \gamma^3 \partial_3, \quad (7)$$

where $\gamma^0 = \gamma_0$ and $\gamma^k = -\gamma_k$ ($k = 1, 2, 3$).

---

1 In the physics literature it is often denoted $\not\partial$ (*Feynman's notation*).

The Klein-Gordon equation $(\Box + M^2)\psi = 0$ can now be writen as $(\partial^2 + M^2)\psi = 0$, or $(\partial - iM)(\partial + iM)\psi = 0$. The *Dirac equation* is $(\partial + iM)\psi = 0$, which can be written as

$$i\hbar\partial\psi = \mu\psi. \tag{8}$$

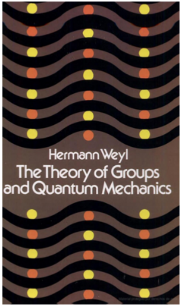
In the presence of an electromagnetic field given by a potential A, $\boldsymbol{p} \leftrightarrow -i\hbar(\boldsymbol{\nabla} - eA)$ and the Dirac equation is

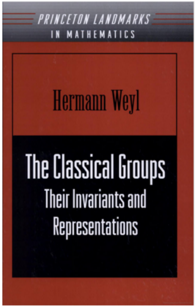
$$i\hbar(\partial - eA)\psi = \mu\psi. \tag{9}$$

"Even if Dirac had known of Clifford algebras before, this would not have dimmed the brilliance of the realization that *such entities are of importance for the quantum mechanics of a spinning electron* —this constituting *a major and unexpected advance in physical understanding*" (Penrose-2005 [4], p. 619).

"A somewhat surprising feature ... is that QFT seems to tie up with deep properties of low-dimensional geometry, i.e. in dimensions 2, 3 and 4" Atiyah-1990 [5].

Two perspectives: Weyl & Dirac



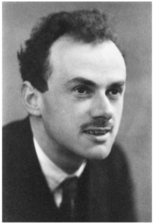
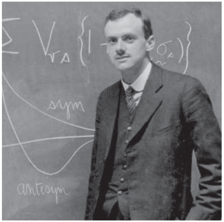


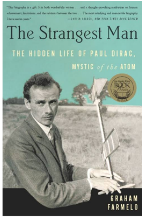
Herrmann Weyl (1885-1955)

1918. *Raum, Zeit, Materie*.

1928. *Gruppentheorie und Quantenmechanik* (English: 1931)

1938: *The Classical Groups* (key aspects of the theory were developed as early as 1925).



1926: PhD (Cambridge, supervised by Ralph Fowler, "first thesis on quantum mechanics to be submitted anywhere". Postdoc at Copenhagen (Bohr) and Göttingen (Hilbert, Weyl).

1928: Dirac's equation: $i\gamma \cdot \partial\psi = m\psi$ (*Quantum elecdrodynamics)*.

1930: *Principles of Quantum Mechanics*.

1932: Lucasian Professor of Mathematics.

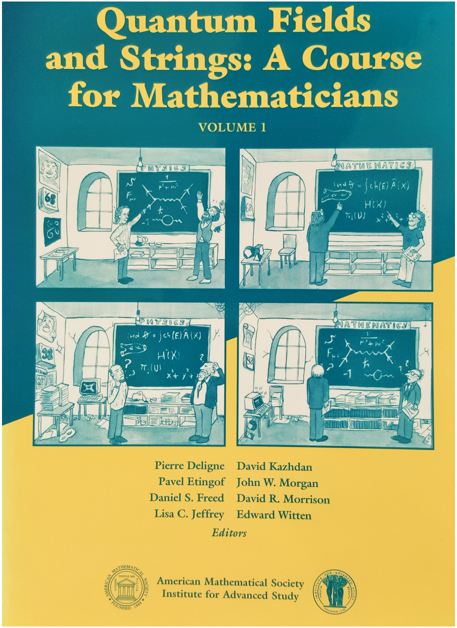
1933: Nobel Prize (shared with Erwin Schrödinger).

Paul A. M. Dirac (1902-1984) Dirac-1931 [6]:

[...] the modern physical developments have **required a mathematics that continually shifts its foundations and gets more abstract**. Non-euclidean geometry and non-commutative algebra, [...] have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation. [...] The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities.

1918 M. Planck (1858-1947). **1921** Albert Einstein (1879-1955). **1922** Niels Bohr (1885-1962). **1932** Werner Heissenberg (1901-1976). **1933** Erwin Schrödinger (1887-1961) & Dirac. **1945** Wolfgang Pauli (1900-1958). **1949** Hideyuki Yukawa (1907-1981). 1963 Eugen Wigner (1902-1995). **1965** Richard Feynman (1918-1988), Julian Schwinger (1918-1994), Sin-Itiro Tomonaga (1906-1979). **1969** Murray Gell-Mann (1929). **1979** S. Glashow (1932), A. Salam (1926-1996), S. Weinberg (1933). **1999** G. T'Hooft (1946), M. Veltman (1931).

1950 L. Schwartz (1915-2002) **1954** J.-P. Serre (1926) **1966** M. Atiyah (1929) **1974** D. Mumford (1937) **1982** A. Connes (1947) **1986** S.-T. Yau (1949) **1986** S. Donaldson (1957) **1990** V. Jones (1952) **1990** E. Witten (1951) **1994** J. Bourgain (1954) **1998** M. Kontsevich (1964) **2014** A. Avila (1979)


Quantum Fields and Strings: A Course for Mathematicians
VOLUME 1
PHYSICS
MATHEMATICS
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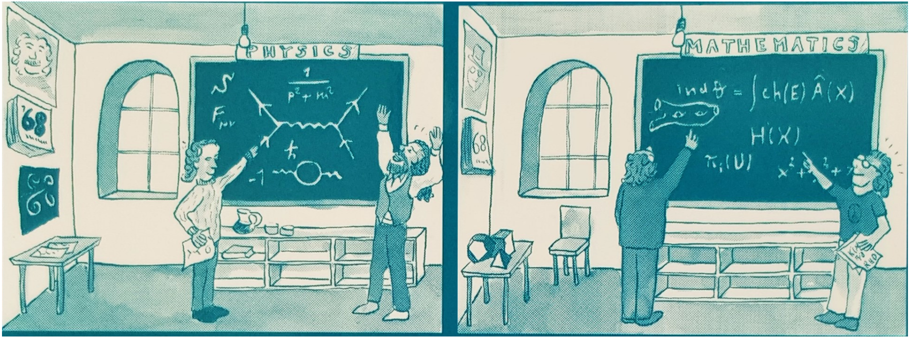
Pierre Deligne
Pavel Etingof
Daniel S. Freed
Lisa C. Jeffrey
David Kazhdan
John W. Morgan
David R. Morrison
Edward Witten
Editors
American Mathematical Society
Institute for Advanced Study

1968

Physics office and Mathematics office



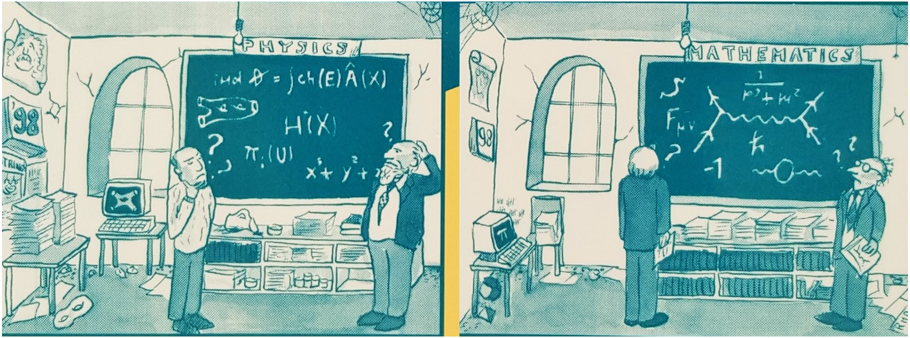
Feynman-Schwinger-Tomonaga (Nobel Prize 1965)
Atiyah-Singer index theorem (1962), Fields Medal 1966 (Atiyah),
Abel Prize 2004.

Tomonaga-Schwinger-Feynman

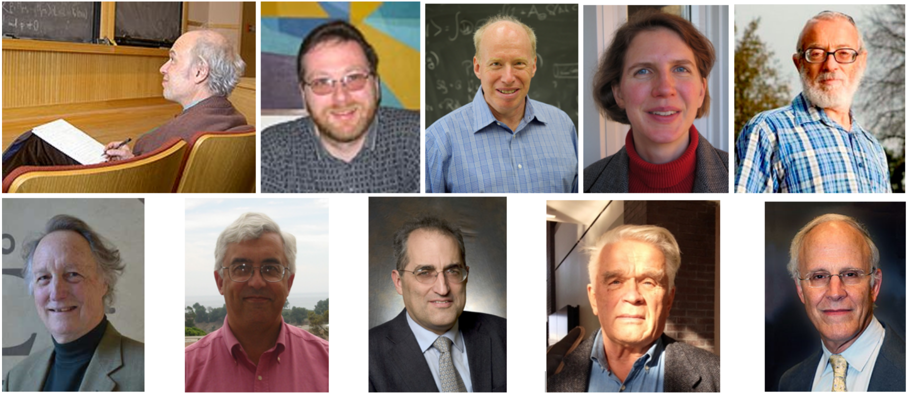
Atiyah-Singer

1998

Physics office and Mathematics office



That metamorphosis happened in Enumerative Geometry: Mirror symmetry (ca. 1990), Manin-Kontsevich (1994), ...
(cf. Xambo-2014-sas [7]). Also in low dimensional topology (Solution of the Poincaré conjecture, ca. 2003)



1 Pierre Delighe (IAS). **2** Pavel Etingof (MIT). **3** Daniel S. Freed (UT Austin). **4** Lisa C. Jeffrey (Toronto). **5** David Kazhdan (HUJ, Harvard). **6** John W. Morgan (Simons Center for Geometry and Physics). **7** David Morrison (UCSB). **8** Ed Witten (IAS). **9** Ludwig Fadeev (S. Petersburg). **10** David Gross (UCSB; Nobel 2004, shared with H. D. Plotizer and F. Wilczek –asymptotic freedom of s. f.).

Volume 1

Part 1: Classical fields and supersymmetry

Part 2: Formal aspects of QFT

Volume 2

Part 3: Conformal Field Theory and Strings

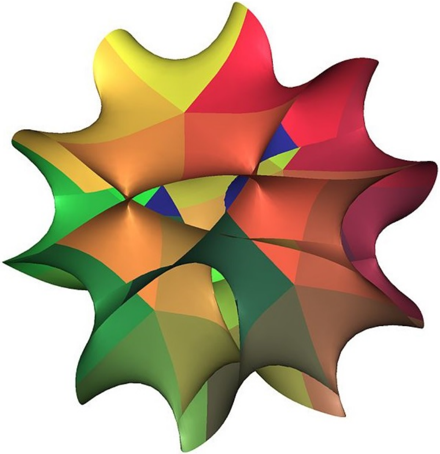
Part 4: Dynamical Aspects of QFT

Physics $\rightarrow$ Mathematics

- Gauge theories (Yang-Mills) and Principal bundles. Donaldson and 4-manifolds.
- Morse theory and supersymmetry (Witten 1989)
- New invariants of knots and links. New invariants of manifolds.
- **Mirror symmetry and enumerative geometry**
- New mathematics arising from Feynmann diagrams
- Did a mathematical physicist go with Grothendieck?

"It is in trying to go beyond the limitations of quantum field theory that physicists have really begun to meet mathematical frontiers" (Witten-1986 [8]).

Mirror symmetry



(Slice of a) quintic hypersurface in $\mathbf{P}^4$. Has complex dimension 3 and is an instance of a Calabi-Yau 3-fold. Example:

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0.$$

Hori-Katz-Klemm-et-5-2003 [9]: "Mirror symmetry is an example of a general phenomenon known as duality, which occurs when two seemingly different physical systems are isomorphic in a non-trivial way".

CY Mirror symmetry: "when two Calabi–Yau manifolds look very different geometrically but are nevertheless equivalent when employed as extra dimensions of string theory" (Wikipedia)

COGP-1991 [10] *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*

Among the physics conjectures here we are going to consider only those that predicted the number of rational curves on certain Calabi–Yau varieties, for example on a general quintic hypersurface in $\mathbf{P}^4_{\mathbf{C}}$. String theorists calculate a 'Yukawa coupling' series $f(q)$ in two different ways, using a principle called 'mirror symmetry', and get the following two expressions:

$$f(q) = 5 + 2875q + 4876875q^2 + \dots$$

and

$$\begin{aligned} f(q) &= 5 + \sum_{k\geqslant 1} n_k k^3 \frac{q^k}{1-q^k} \\ &= 5 + n_1 q + (2^3 n_2 + n_1)q^2 + \dots \end{aligned}$$

where n_k is the number of rational curves of degree k in the quintic threefold. The second expression comes, roughly speaking, from a quantum correction called 'sum over instantons' (which here we may take to mean rational curves).

The values gotten for the first four n_k are the following:

k	n_k
1	2875
2	609 250
3	317 206 375
4	242 467 530 000

In Candelas et al. [1991], the work where such numbers were published for the first time, there is a table for $1 \leqslant k \leqslant 10$, and in principle string theorists can calculate n_k up to any value of k because it is possible to calculate, in principle, as many terms of the first form of the q-expansion of $f(q)$.

This result is very striking, even if we disregard the magic of such calculations and do not bother about what is the precise meaning of rational curves which is used. Indeed, if we write a_k for the k-th coefficient of the first form of the q-expansion, then $a_2 - a_1$ must be an integer divisible by 8, $a_3 - a_1$ must be and integer divisible by 27, and so on, which are unlikely properties at the very least. In fact there is no known a priori reason for the a_k to be integers and proving that they indeed are so would be very interesting.

There is another reason why the results are so striking. They tacitly say that on a general quintic threefold there are only finitely many rational curves for each degree $k \geqslant 1$. This was in fact conjectured by Clemens (see Clemens [1983, 1984]), but its truth is far from known at present: Katz [1986 b] showed that the conjecture is correct for $k \leqslant 7$ and recently the cases $k = 8$ and $k = 9$ have been settled by Johnsen and Kleiman [1993].

(These authors show that there are finitely many irreducible reduced rational curves on a general quintic 3-fold of degree $k \in \{8, 9\}$, that its number is positive, and that they are necessarily smooth. For comparison let us add that Clemens' conjecture states not only that on a general quintic threefold there are finitely many rational curves, but also that each such curve is smooth, that *its normal bundle is* $\mathcal{O}_X(-1) \oplus \mathcal{O}_X(-1)$, where $\mathcal{O}_X(-1)$ is the tautological line bundle on $\mathbf{P}^1_{\mathbf{C}}$, and that *two distinct rational curves do not intersect*.)

Remarks

$n_1 = 2875$ was already known by Schubert over a century ago (it has been worked out using modern methods by many authors)

n_2 had been calculated by S. Katz [1986 b, 1988], and confirmed by other authors (see next reference).

n_3 was calculated, after the string theorists discovery, by Ellingsrud-Strømme [1991].[1]

---

[1] For a quick and clear introduction to the geometry behind these and other calculations, see Piene [1993]. For a historical perspective and later developments, see Xambo-2014-sas [7].

For a compact Kähler manifold X, let $\omega = \omega_X$ be the corresponding 1-form, T_X the holomorphic tangent bundle, $\Omega_X = T_X^*$ (locally free sheaf of holomorphic 1-forms on X) and $\Omega_X^p = \Lambda^p \Omega_X$ the sheaf of holomorphic p-forms. If n is the dimension of X, then $\mathcal{K}_X = \Omega_X^n$ is called the *canonical sheaf*. By $H^{p,q}(X)$ we denote the cohomology space $H^q(X, \Omega_X^p)$. Its dimension, which is finite, will be denoted $h^{p,q}(X)$, or just $h^{p,q}$.

The first Chern class of Ω_X, which coincides with the first Chern class of $\mathcal{K}_{\mathcal{X}}$, is called the *canonical class* of X. From the definition of Ω_X it follows that the canonical class is $-c_1(X)$, where we set $c_i(X) = c_i(T_X)$ to denote the Chern classes of X.

There is a *Hodge decomposition*

$$H^k(X,\mathbf{C}) \simeq \bigoplus_{\substack{p+q=k\\ p,q\geqslant 0}} H^{p,q}(X)$$

This decomposition follows from the Hodge theorem for the de Rham cohomology,

$$H^k_{\mathrm{dR}} \simeq \bigoplus_{\substack{p+q=k\\ p,q\geqslant 0}} H^{p,q}_{\bar\partial}(X)\ ,$$

and the Dolbeault isomorphism

$$H^{p,q}_{\bar\partial}(X) \simeq H^q(X,\Omega^p_X) = H^{p,q}(X)\ .$$

Since $H^{q,p}(X) = \overline{H^{p,q}(X)}$, where the bar denotes complex conjugation, we see that $h^{q,p}(X) = h^{p,q}(X)$.

On the other hand, there is a natural pairing

$$H^{i,j}(X) \times H^{n-i,n-j}(X) \to \mathbf{C} ,$$

given by integrating along X the exterior product, which leads to an isomorphism $H^{i,j}(X) \simeq H^{n-i,n-j}(X)^*$ (*Hodge-Poincaré duality*). Hence

$$h^{i,j}(X) = h^{n-i,n-j}(X) .$$

Notice also that $h^{0,0}(X) = h^{n,n}(X) = 1$. The numbers $h^{i,j}$ are often arranged on the *Hodge diamond*. Since the numbers $h^{i,j}$ vanish if i or j is negative or greater than n, it looks like (say for 3-folds) as follows:

$$
\begin{array}{ccccccc}
 & & & h^{00} & & & \\
 & & h^{10} & \vdots & h^{01} & & \\
 & h^{20} & & h^{11} & & h^{02} & \\
h^{30} & \cdots & h^{21} & \cdot\vdots\cdot & h^{12} & \cdots & h^{03} \\
 & h^{31} & & h^{22} & & h^{13} & \\
 & & h^{32} & \vdots & h^{23} & & \\
 & & & h^{33} & & &
\end{array}
$$

By what we have just said, only the numbers in, say, the left top quarter (h^{10}, h^{20}, h^{30}, h^{11} and h^{21}) are independent, for the diamond is symmetrical with respect to the middle horizontal line (by the duality relation) and also with respect to the middle vertical line (by complex conjugation).

In particular we have that

$$b_k(X) = \sum_{i=0}^{k} h^{i,k-i}(X) \ ,$$

where $b_k(X)$, called the k-th Betti number of X, is the dimension of $H^k(X, \mathbf{C})$. The *Euler characteristic* of X, $\chi = \chi(X)$, which by definition is the alternating sum $\sum_{i=0}^{2n}(-1)^i b_i(X)$, can thus be computed in terms of the Hodge numbers. For example, for a 3-fold one gets

$$\chi(X) = 2 - 4h^{0,1} + 4h^{0,2} - 2h^{0,3} + 2h^{1,1} - 2h^{1,2} \ .$$

By the Gauss–Bonnet theorem, the Euler characteristic of X is also equal to $\int c_n(X)$.

A *Calabi–Yau manifold* is a compact Kähler manifold with trivial canonical bundle and such that $h^{i,0}(X) = 0$ for $1 \leqslant i \leqslant n-1$.

The triviality of the canonical bundle is equivalent to the vanishing of the first Chern class of X, because $c_1(X) = -c_1(\omega_X)$. Moreover, since the existence of an isomorphism $\Omega_X^n \simeq \mathcal{O}_X$ implies the existence of an isomorphism $T_X \simeq \Omega_X^{n-1}$, we have, taking sections, $h^0(T_X) = h^{n-1,0} = 0$. So a Calabi–Yau manifold has no global non-zero holomorphic vector fields. Moreover, we also have that

$$H^1(T_X) \simeq H^1(\Omega_X^{n-1}) = H^{n-1,1}(X)$$

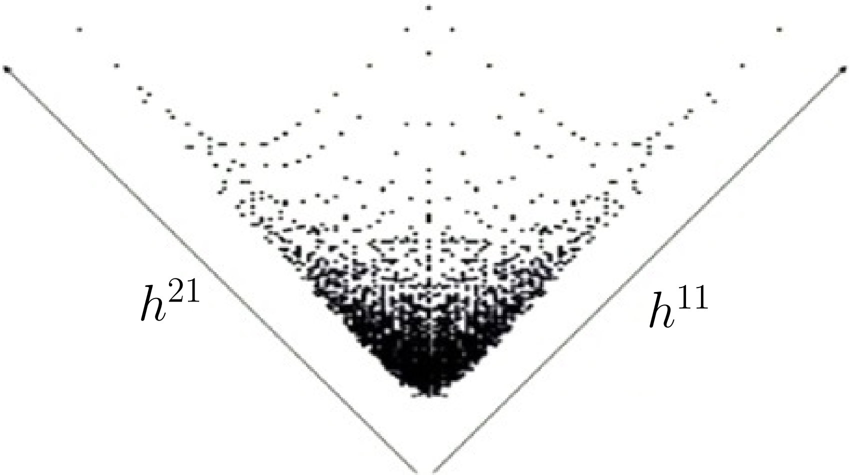
Another remark is that $h^{n,0} = 1$, for

$$h^{n,0} = \dim_{\mathbf{C}} H^0(X, \Omega_X^n) = \dim_{\mathbf{C}} H^0(X, \mathcal{O}_X) = h^{0,0}(X) = 1\ .$$

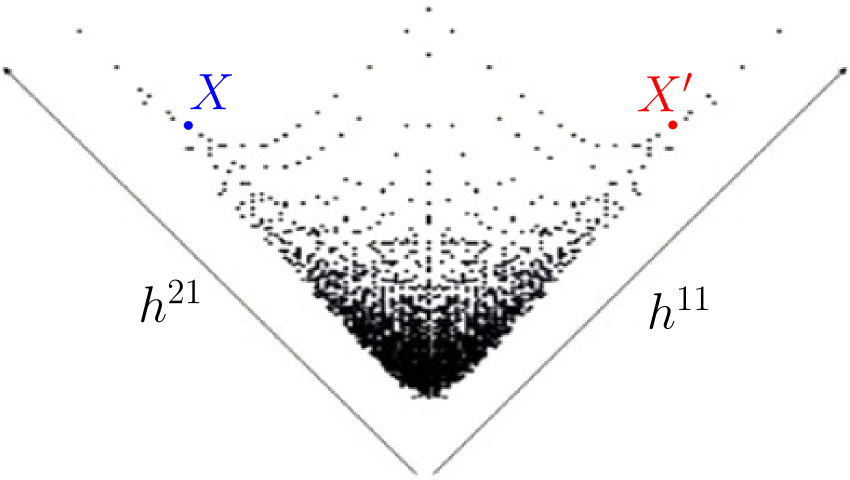
Hence the value at the four vertices of the Hodge diamond of a Calabi–Yau manifold is 1 and the remaining entries around the edge are 0. In particular we see that for a Calabi–Yau 3-fold there are only 2 independent Hodge numbers, which henceforth we will choose to be $h^{1,1}(X)$ and $h^{2,1}(X)$. The Euler characteristic of a Calabi–Yau 3-fold is

$$\chi(X) = 2\left(h^{1,1}(X) - h^{1,2}(X)\right) .$$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & \vdots & 0 & & \\
 & 0 & & h^{11} & & 0 & \\
1 & \cdots & h^{21} & \cdot\vdots\cdot & h^{21} & \cdots & 1 \\
 & 0 & & h^{11} & & 0 & \\
 & & 0 & \vdots & 0 & & \\
 & & & 1 & & &
\end{array}
$$



Mirror symmetry conjecture


X
X'
h^{21}
h^{11}

"The mathematical community has benefited from this interaction in two ways. First, and more conventionally, mathematicians have been spurred into learning some of the relevant physics and collaborating with colleagues in theoretical physics. Second, and more surprisingly, many of the ideas emanating from physics have led to significant new insights in purely mathematical problems, and remarkable discoveries have been made in consequence. The main input from physics has come from quantum field theory. While the analytical foundations of quantum field theory have been intensively studied by mathematicians for many years the new stimulus has involved the more formal (algebraic, geometric, topological) aspects.

From this very brief summary of Witten's achievements it should be clear that he has made a profound impact on contemporary mathematics. In his hands physics is once again providing a rich source of inspiration and insight in mathematics". Atiyah-1990 [5]

"Perhaps I should conclude by briefly explaining my view of the significance of the mathematical and physical work that I have been involved in. It actually is simpler to explain my opinion on the mathematical side. Quantum field theory and String Theory contain many mathematical secrets. I believe that they will play an important role in mathematics for a long time. For various technical reasons, these subjects are difficult to grapple with mathematically. Until the mathematical world is able to overcome some of these technical difficulties and to grapple with quantum fields and strings per se, and not only with their implications for better-established areas of mathematics, physicists working in these areas will continue to be able to surprise the mathematical world with interesting and surprising insights. I have been lucky to be at the right place at the right time to contribute to part of this". Witten-2014 [11] (Kyoto Prize 2014 for "Outstanding Contributions to the Development of Mathematical Sciences through the Exploration of Superstring Theory")

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