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Geometry and physics with geometric algebra

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Dedicated to Miguel Carlos Muñoz Lecanda, in memoriam

The mood when writing this paper has been like that of a continuous dialog with Miguel in which I have contributed my best to address his clever observations and to answer his keen questions. The style of George Mackey's book The Mathematical Foundations of Quantum Mechanics, *which we studied years ago, has been a guiding inspiration for our discourse. The quote at the end of the Prelude, a nod to an illustrious namesake, is in recognition of Miguel's passion, always glowing in our conversations, for a broad spectrum of classical works of all time. I hope the result deserves his approval.*

The aim of this essay is to provide a brief *mathematical* account of the geometric algebra formalism and to revisit a few *outstanding applications* to geometry and physics. The main emphasis is the presentation of geometric algebra as a powerful enrichment of Grassmann' exterior algebra (of metric linear spaces) by the geometric product (see ⋄4), which in our view better connects with Clifford's seminal ideas on this topic and provides a transparent intrinsic formalism in which to cast concepts, applications and effective computations. Hopefully it will be useful to mathematicians wishing to understand geometric algebra in their own terms and to people with other backgrounds to sharpen their awareness of its mathematical underpinnings.

Keywords: Grassmann algebra; geometric algebra; Wessel, Pauli and Dirac algebras; geometric complex numbers and geometric quaternions; spinor regularization for the Kepler problem; symmetries and rotations; Pauli and Dirac matrices; Spinors and Lorentz transformations; Space-time kinematics; The gradient operator; The Riesz-Maxwell equation; monochromatic electromagnetic waves; Dirac-Hestenes equation; matrix representations of geometric algebras.

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Prelude

In our presentation the basic mathematical notions and structures (especially vector spaces and algebras) are assumed to be known, but the field over which vector spaces and algebras are defined is always assumed to be, unless otherwise stated, the field $\mathbb{R}$ of real numbers. Moreover, associative algebras will be assumed to have a multiplication unit, which by default will be denoted by 1. Later on (and in some of the background topics), groups, rings, differential calculus and a bit of topology will also be assumed. Nevertheless, in order to attend readers with different backgrounds, we provide specific details in Appendix A and we refer to them whenever we deem that it may be convenient for those needing a quick refreshing. The most common symbols used in the paper are collected in Appendix B.

The theoretical essentials of geometric algebra (GA) are spelled out in Sec. 1 in a fairly self-contained presentation. Readers wishing to see how GA works in low dimensional cases, including lovely examples of concrete applications, may want to jump to the sections ahead and return to this foundational section whenever they need to fortify their understanding of the general framework to follow the particulars of some concrete issue (back pointers to the GA used in each case are provided throughout). Thus in Sec. 2, devoted to the GA of the Euclidean plane $\mathcal{E}_2$ (Wessel algebra), the reader can find a remarkable explicit formulas for the orthocenter and circumcenter of a given triangle, and a note on the 27 Morley (equilateral) triangles of a given triangle; in Sec. 3 (Pauli algebra), the elegant deduction of Olinde-Rodrigues formulas for the composition of two rotations in the three-dimensional Euclidian space $\mathcal{E}_3$; a crisp presentation of Kepler's two-body dynamics and of Rutherford's scattering formula; and the Kustaanheimo–Stiefel regularization in celestial mechanics with reference to perturbing forces.

In Sec. 4 (Dirac algebra) we review: general Lorentz transformations, including how to get their composition in terms of their defining data; the electromagnetic field and its Maxwell–Riesz (or Maxwell–Heaviside–Riesz) equation; and the Dirac equation about quantum spin 1/2 relativistic particles. All these examples, from the simplest to the more involved, reveal the power and suppleness of the geometric product to express geometry and physics notions and results. The framework revitalizes known concepts, in form and substance, provides means for generating effective and transparent proofs, and favors the adoption of intrinsic approaches to problems and applications. Finally Sec. 5 provides a few pointers for further research.

> [Miguel de] Cervantes' contribution to creation was the quixotic courage—literal, moral, visionary. Cervantes shares with Shakespeare and with Dante a peculiar characteristic of the Kabbalistic *Keter* or crown: the audacity of Adam early in the morning (as Walt Whitman called it), the participation in the divine will or desire that the Kabbalists termed *Ratzon* (רצון). All additional literary emanations radiate from Cervantes, as they do from Shakespeare."
>
> HAROLD BLOOM, from
> *Genius: A Mosaic of One Hundred Exemplary Creative Minds* (2002)

1. An Outline of Geometric Algebra

Every student who enters upon a scientific pursuit, especially if at a somewhat advanced period of life, will find not only that he has much to learn, but much also to unlearn.

John Herschel, from
A Preliminary Discourse on the Study of Natural Philosophy (1830)

In our view, geometric algebra stands on the *Grassmann algebra* $\mathcal{G} = \mathcal{G}E$ of a vector space E of finite dimension n (we bow to its creator by using the symbol $\mathcal{G}$ instead of the conventional $\wedge$, with the bonus that it is also a fitting choice for geometric algebra). The elements of $\mathcal{G}$ are called *multivectors* and its product, which is associative, is the *wedge* product, denoted by $\wedge$.

If $v, v' \in E$, $v \wedge v'$ represents an oriented area, and so $v' \wedge v = -v \wedge v'$ because the orientation of $v' \wedge v$ is the reverse of the orientation $v \wedge v'$. In particular, $v \wedge v = 0$.

When E is endowed with a *metric* q (i.e. a non-degenerate quadratic form), there is a natural extension of q to $\mathcal{G}E$, and the result is a structure that we denote by $\mathcal{G}_q E$. For example, if $v, v' \in E$, $q(v \wedge v') = v^2 v'^2 - q(v, v')^2$. In the case of the Euclidean space $\mathcal{E}_n$, this expression is equal to $A(v, v')^2$, where $A(v, v')$ is the area of the parallelogram $[v, v']$.

In addition to the wedge product $x \wedge x'$, $\mathcal{G}_q E$ has two additional natural bilinear products: the *dot* or *inner product*, $x \cdot x'$ and the *geometric product*, xx' (no specific symbol for it—its factors are simply juxtaposed). The inner product is a natural generalization of the (left and right) contractions of multivectors with a vector; it is neither associative nor commutative. The *geometric product* is associative and is defined by means of the outer and inner products. For example, for any $v \in E$ and any multivector x, the following equalities hold:

$$vx = v \cdot x + v \wedge x \quad \text{and} \quad xv = x \cdot v + x \wedge v.$$

In principle these relations can be used recursively to compute any geometric product. For example, if $x = v' \in E$, we get the relation $vv' = q(v, v') + v \wedge v'$. This relation tells us that $vv' = v'v$ if and only if $v \wedge v' = 0$ (that is, if and only if v and v' are linearly dependent, a fact that for $v, v' \neq 0$ is expressed by $v \sim v'$) and that $vv' = -v'v$ if and only if $q(v, v') = 0$ (that is, if and only if v and v' are orthogonal, $v \perp v'$). In particular we have $v^2 = q(v)$ (which we call *contraction rule*). As we will repeatedly see, these two observations are crucial to carry on reasonings and computations; readers are therefore advised to appreciate them as such from the very beginning.

$\diamond$**1** (Grassmann algebra)**.** Our ground structure is a vector space E of finite dimension n endowed with its *Grassmann algebra* $\mathcal{G} = (\mathcal{G}E, \wedge)$:

$$\mathcal{G}E = \bigoplus_{k=0}^{n} \mathcal{G}^k E = \mathcal{G}^0 E \oplus \mathcal{G}^1 E \oplus \mathcal{G}^2 E \oplus \cdots \oplus \mathcal{G}^n E,$$

where $\mathcal{G}^k E$ $(0 \leq k \leq n)$ is the k-the exterior power of E and $\wedge$ is the *wedge product.* It is associative and is also called *outer* (or *exterior*) *product.* With $\wedge$, $\mathcal{G}$ is a *graded* *algebra*, which means that

$$x \wedge x' \in \mathcal{G}^{k+k'}E$$

when $x \in \mathcal{G}^k E$ and $x' \in \mathcal{G}^{k'}E$, with the convention that $\mathcal{G}^r E = \{0\}$ for $r > n$. The wedge product is *skew-commutative* (or *super-commutative*): for $x \in \mathcal{G}^k E$ and $x' \in \mathcal{G}^{k'}E$,

$$x \wedge x' = (-1)^{kk'} x' \wedge x.$$

In particular, $v \wedge v' = -v' \wedge v$ for all $v, v' \in E$, hence also $v \wedge v = 0$ for all $v \in E$.

A fundamental property of the wedge product is that for $v_1, \ldots, v_k \in E$ the relation $v_1 \wedge \cdots \wedge v_k \neq 0$ is equivalent to say that $v_1, \ldots, v_k$ are linearly independent. For example, if $e_1, \ldots, e_n$ is a basis of E, the $\binom{n}{k}$ exterior products

$$e_{j_1} \wedge \cdots \wedge e_{j_k} \quad (1 \leq j_1 < \cdots < j_k \leq n)$$

are nonzero, and in fact they form a basis of $\mathcal{G}^k E$. In particular,

$$\dim \mathcal{G}^k E = \binom{n}{k} \quad \text{and} \quad \dim \mathcal{G}E = 2^n.$$

The elements of $\mathcal{G}$ are called *multivectors*, and the elements of $\mathcal{G}^k$, *k-vectors*. Since $\mathcal{G}^0 = \mathbb{R}$ and $\mathcal{G}^1 = E$, the 0-vectors are *scalars* and the 1-vectors are *vectors* (elements of E). Moreover, 2-vectors and 3-vectors are usually termed *bivectors* and *trivectors*, respectively. The n-vectors are called *pseudoscalars*; this is because $\dim \mathcal{G}^n = 1$, but note that while $1 \in \mathbb{R}$ is a distinguished nonzero scalar, there is no distinguished nonzero pseudoscalar. Said differently, there are as many linear isomorphisms $\mathbb{R} \simeq \mathcal{G}^n$ as nonzero elements in $\mathcal{G}^n$, but there is no way to distinguish one from the other in geometric terms.

The k-vectors of the form $x = v_1 \wedge \cdots \wedge v_k$ $(v_1, \ldots, v_k \in E)$ are said to be *split* (or *decomposable*), and nonzero split k-vectors are called *k-blades*. A k-blade x determines the linear subspace $\langle x \rangle = \{v \in E : v \wedge x = 0\}$ and it turns out that $\langle x \rangle$ is equal to the linear span $\langle v_1, \ldots, v_k \rangle$ of the vectors $v_1, \ldots, v_k$. Indeed, the inclusion $\langle v_1, \ldots, v_k \rangle \subseteq \langle x \rangle$ is clear, as $v_j \wedge x = 0$ for any $j = 1, \ldots, k$. Conversely, if $v \notin \langle v_1, \ldots, v_k \rangle$, then $v, v_1, \ldots, v_k$ are linearly independent, so $v \wedge x \neq 0$ and $v \notin \langle x \rangle$.

Moreover, the map $x \mapsto \langle x \rangle$ has the property that $\langle x \rangle = \langle x' \rangle$ (where x' is another k-blade) if and only if $x' = \lambda x$ for some scalar λ (if $\langle x \rangle = \langle x' \rangle = V$, x and x' are pseudo-scalars of V and hence they are proportional; conversely, it is clear that for any nonzero scalar λ we have $\langle \lambda x \rangle = \langle x \rangle$). Therefore, we have an injective map $\mathrm{Gr}_k(E) \hookrightarrow \mathbf{P}(\mathcal{G}^k E)$, $\langle x \rangle \mapsto [x]$, where $\mathbf{P}(\mathcal{G}^k E)$ is the projective space of $\mathcal{G}^k E$ and $\mathrm{Gr}_k(E)$ is the set of k-dimensional subspaces of E. With the structure inherited from $\mathbf{P}(\mathcal{G}^k E)$, the $\mathrm{Gr}_k(E)$ are called *Grassmann manifolds* of E, or *grassmannians* $(k = 0, \ldots, n)$. Note that $\mathrm{Gr}_0(E) = \{\{0\}\}$, $\mathrm{Gr}_n(E) = \{E\}$ and $\mathrm{Gr}_1(E) = \mathbf{P}(E)$. $\triangleright \boxed{\mathbf{1}}$.

Involutions of $\mathcal{G}$. Let $x = \sum_{j=1}^n x_j$ $(x_j \in \mathcal{G}^j)$ and set $\hat{x} = \sum_{j=1}^n (-1)^j x_j$, $\tilde{x} = \sum_{j=1}^n (-1)^{j /\!\!/ 2} x_j$, where $j /\!\!/ 2 = \lfloor \frac{j}{2} \rfloor$. Then $x \mapsto \hat{x}$ in an automorphism of $\mathcal{G}$, and it is an involution, i.e. $\hat{\hat{x}} = x$. This involution is called the *grade involution* of $\mathcal{G}$ (or also the *principal involution*). The map $x \mapsto \tilde{x}$ is also an involution, $\tilde{\tilde{x}} = x$ for all $x \in \mathcal{G}$, but it is an anti-automorphism, that is, $\widetilde{x \wedge x'} = \widetilde{x}' \wedge \widetilde{x}$. This follows easily from the following fact: if $v_1, \ldots, v_k \in E$, then $(v_1 \wedge \cdots \wedge v_k)^{\sim} = (-1)^{k/\!/2} v_1 \wedge \cdots \wedge v_k = v_k \wedge \cdots \wedge v_1$, where we use the fact that $k/\!/2$ and $\binom{k}{2}$ have the same parity. Because of the latter property, we say that $x \mapsto \tilde{x}$ is the *reverse involution*. The composition of these involutions, namely $x \mapsto \hat{\tilde{x}} = \tilde{\hat{x}}$, is called the *Clifford involution*, and is denoted by $\bar{x}$. It is an anti-automorphism of $\mathcal{G}$ and it is easy to check that for $x \in \mathcal{G}^k$ we have $\bar{x} = (-1)^{(k+1)/\!/2} x$.

The even Grassmann algebra. Since the grade involution is an algebra automorphism, $\mathcal{G}^+ = \{x \in \mathcal{G} : \hat{x} = x\}$ is a subalgebra of $\mathcal{G}$. It consists of multivectors with no odd components and it is called the *even Grassmann algebra*. We also set $\mathcal{G}^- = \{x \in \mathcal{G} : \hat{x} = -x\}$, which consists of multivectors with no even components. It is clear that $\mathcal{G} = \mathcal{G}^+ \oplus \mathcal{G}^-$ but $\mathcal{G}^-$ is not more than a vector subspace of $\mathcal{G}$.

Note that for $x \in \mathcal{G}^+$ we have $\bar{x} = \tilde{x}$, while $\bar{x} = -\tilde{x}$ if $x \in \mathcal{G}^-$.

$\diamond$**2** (Metric Grassmann algebra $\mathcal{G}_q$)**.** When E is endowed with a metric q, so that (E, q) is a *quadratic space* ($\triangleright$ $\boxed{\mathbf{2}}$), q can be extended in a natural way to $\mathcal{G}$. The resulting structure is denoted by $\mathcal{G}_q = \mathcal{G}_q E$ and we call it the *metric Grassmann algebra*.

In detail, there is a unique metric on $\mathcal{G}_q E$, still denoted by q, such that the spaces $\mathcal{G}^k E$ are pairwise q-orthogonal and (*Gram determinants*)

$$q(v_1 \wedge \cdots \wedge v_k, v_1' \wedge \cdots \wedge v_k') = \Delta_q(v_1, \ldots, v_k, v_1', \ldots, v_k'),$$

where $\Delta_q(v_1, \ldots, v_k, v_1', \ldots, v_k') = \det(q(v_i, v_j'))$.

In particular, $q(v_1 \wedge \cdots \wedge v_k) = \Delta_q(v_1, \ldots, v_k)$, where $\Delta_q(v_1, \ldots, v_k) = \Delta_q(v_1, \ldots, v_k, v_1, \ldots, v_k)$.

We observe that $q(v_1 \wedge \cdots \wedge v_k, v_1' \wedge \cdots \wedge v_k')$ vanishes if one of the vectors v_i is orthogonal to all vectors v_j', or the other way around. Moreover, if $v_1, \ldots, v_k$ are pairwise orthogonal, then the Gram determinant is diagonal and therefore

$$q(v_1 \wedge \cdots \wedge v_k) = q(v_1) \cdots q(v_k). \tag{1}$$

Furthermore, for a k-blade $x = v_1 \wedge \ldots \wedge v_k$, $q(x) = 0$ if and only if the associated space $\langle x \rangle$ is singular. Indeed, if $\langle x \rangle$ is singular, there is a nonzero vector $v_1 \in \langle x \rangle \cap \langle x \rangle^\perp$, which can be completed to a basis $v_1, \ldots, v_k$ of $\langle x \rangle$, and then it is clear that $q(x) = 0$. Conversely, if $q(x) = 0$, then for any orthogonal basis $v_1, \ldots, v_k$ of $\langle x \rangle$ we have $q(x) = q(v_1) \cdots q(v_k)$, and hence $q(v_j) = 0$ for some $j \in [n]$, which says that $v_j \in \langle x \rangle^\perp$.

Remark. Any blade can be written as the wedge of pairwise orthogonal vectors. Indeed, if x is a k-blade, $\langle x \rangle$ admits an orthogonal basis (even if $\langle x \rangle$ is singular; see the Remark at the end of $\triangleright$ $\boxed{\mathbf{2}}$), say $u_1, \ldots, u_k$. Then x and $u_1 \wedge \cdots \wedge u_k$ are pseudoscalars of $\langle x\rangle$ and hence there is a nonzero scalar λ such that $x = \lambda u_1\wedge\cdots\wedge u_k$ and this proves the claim (redefine u_1 to be λu_1).

$\diamond$**3** (Inner product of $\mathcal{G}_q$). An important feature of $\mathcal{G}_q$ is that it is endowed with a bilinear product $x \cdot x' \in \mathcal{G}_q$, called *dot* or *inner product*, that extends the left and right *inner contraction* linear operators $v\lrcorner, \llcorner v : \mathcal{G}_q \to \mathcal{G}_q$ (any $v \in E$), which are uniquely determined by the rules

$$v\lrcorner(v_1 \wedge \cdots \wedge v_k) = \sum_{j=1}^{k}(-1)^{j-1}q(v, v_j)v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_k, \tag{2}$$

$$(v_1 \wedge \cdots \wedge v_k)\llcorner v = \sum_{j=1}^{k}(-1)^{k-j}q(v_j, v)v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_k. \tag{3}$$

To note that for $x \in \mathcal{G}_q^k$, $x\llcorner v = (-1)^{k-1}v\lrcorner x$. In terms of the inner product (defined below), it happens that $v\lrcorner x = v \cdot x$ and $x\llcorner v = x \cdot v$ for all $x \in \mathcal{G}_q$.

For all $x, x' \in \mathcal{G}$, $x \cdot x' \in \mathcal{G}$ is a *bilinear* product, but it is *neither unital nor associative*, and it is **not** *commutative*, a set of features that hopefully will not bewilder readers accustomed to the scalar-valued dot product of vectors. In fact, the rules obeyed by the inner product are quite easy to apply and to implement in a computer system. For $v, v', v_1, \ldots, v_k \in E$, the rules governing $x \cdot x'$ are the following:

(1) $\lambda \cdot x = x \cdot \lambda = \lambda x_0$ for any $x \in \mathcal{G}$ and any scalar λ. In other words, $\lambda \cdot x = \lambda x$ for scalars x and $\lambda \cdot x = 0$ for scalar free multivectors x.
(2) $v \cdot (v_1 \wedge \cdots \wedge v_k) = \sum_{j=1}^{k}(-1)^{j-1}q(v, v_j)v_1 \wedge \cdots \wedge v_{j-1} \wedge v_{j+1} \wedge \cdots \wedge v_k$, $k \geq 1$. So $v\cdot = v\lrcorner$ and $v \cdot v' = q(v, v')$ for any $v, v' \in E$.
(3) If $x \in \mathcal{G}^k$, $x' \in \mathcal{G}^j$ and $k \geq j \geq 1$, then $x \cdot x' = (-1)^{(k-j)j}x' \cdot x$ (commutation rule). For example $(v_1 \wedge \cdots \wedge v_k) \cdot v = (-1)^{k-1}v \cdot (v_1 \wedge \cdots \wedge v_k)$. So $\cdot v = \llcorner v$.
(4) If $1 < k \leq j$, $x' \in \mathcal{G}^j$, $(v_1 \wedge v_2 \wedge \cdots \wedge v_k) \cdot x' = (v_1 \wedge \cdots \wedge v_{k-1}) \cdot (v_k \cdot x')$ (recursive rule).

For $k = j$ in (3), the commutation rule says that $x \cdot x' = x' \cdot x$, and it is not hard to see that in this case $x \cdot x' = (-1)^{k/\!\!/2}q(x, x')$ $\triangleright$ $\boxed{\mathbf{3}}$. Note that this implies, for $x, x' \in \mathcal{G}_q^k$, that

$$q(x, x') = \tilde{x} \cdot x'. \tag{4}$$

The behavior of grade, reverse and Clifford involutions with respect to the inner product is given by the following relations, where x and x' are arbitrary multivectors ($\triangleright$ $\boxed{\mathbf{4}}$):

$$\widehat{x \cdot x'} = \widehat{x} \cdot \widehat{x}', \quad \widetilde{x \cdot x'} = \widetilde{x}' \cdot \widetilde{x}, \quad \overline{x \cdot x'} = \bar{x}' \cdot \bar{x}. \tag{5}$$

$\diamond$**4** (Geometric product of $\mathcal{G}_q$). As discovered by Clifford and Lifschitz, $\mathcal{G}_q$ can be further enriched with a *bilinear*, *associative* product, with *unit* 1. Here, we can phrase this fundamental result as follows.

Theorem. *In* $\mathcal{G}_q E$ *there is a* unique *bilinear product* $xx' \in \mathcal{G}_q$ *for all* $x, x' \in \mathcal{G}_q$ *satisfying the following two rules*:

(1) *Contraction*: $v^2 = q(v)$ *for all* $v \in E$.
(2) *Transference*: $v(v_1 \wedge \cdots \wedge v_k) = v \wedge v_1 \wedge \cdots \wedge v_k$ *if* $q(v, v_j) = 0$ *for* $j = 1, \ldots, k$.
Furthermore, *this product is* associative *and* unital (*its unit is* $1 \in \mathbb{R}$).

We will proceed by deriving a number of useful consequences of the rules (1) and (2) that will suffice to establish uniqueness, and then we will cap the argument by defining a bilinear product satisfying rules (1) and (2) and showing that it is associative and unital.

Remark. In our presentation, a pivotal role is played by *Artin's formula* (Eq. (8) below), one of the consequences of the rules (1) and (2), for it will be the stepping stone toward establishing the associativity of the geometric product. The formula is named after Emil Artin because in [1, §V.4] it is postulated as a means for defining the product of the *Clifford algebra* of a non-singular orthogonal geometry with a view to use it to elicit the structure of their orthogonal groups. Although far from our concerns in this paper, it may be worth mentioning that Artin's quite general mathematical goals can be reached perfectly by starting with the Grassmann algebra of a non-singular orthogonal geometry and carrying on the scheme we have chosen. This makes sense because of the utmost generality of the Grassmann algebra construction, and in so doing geometric algebra has the potential of being relevant in situations in which the base field is a finite field (in coding and information theory), or the p-adic numbers (for p-adic physics).

(1′) The rule $v^2 = q(v)$ implies that v is invertible (for the geometric product) if and only if $q(v) \neq 0$, and in this case $v^{-1} = v/q(v)$. To a good extend, the magic of the geometric product is rooted in this property. For example, for the n-dimensional Euclidean space $\mathcal{E}_n$ all nonzero vectors are invertible. To note also that the contraction rule is equivalent to *Clifford's relations*:

$$vv' + v'v = 2q(v, v') = 2v \cdot v' \quad \text{for all } v, v' \in E. \tag{6}$$

Indeed, on one hand $q(v + v') = (v + v')^2 = v^2 + v'^2 + vv' + v'v$ and on the other $q(v + v') = q(v + v', v + v') = q(v) + q(v') + 2q(v, v')$, hence the claim.

(2′) If $v_1, \ldots, v_k \in E$ are pairwise orthogonal, then $v_1 \cdots v_k = v_1 \wedge \cdots \wedge v_k$. This follows from the transference rule and a simple induction argument. Actually, this property implies (2), and hence (2) and (2′) are equivalent. Indeed, in (2) we may write $v_1 \wedge \cdots \wedge v_k = u_1 \wedge \cdots \wedge u_k$, where $u_1, \ldots, u_k$ is an orthogonal basis of $\langle v_1, \ldots, v_k \rangle$ (see the Remark at the end of ⋄2) and then we have

$$\begin{aligned} v(v_1 \wedge \cdots \wedge v_k) &= v(u_1 \wedge \cdots \wedge u_k) \\ &= v\, u_1 \cdots u_k = v \wedge u_1 \wedge \cdots \wedge u_k = v \wedge v_1 \wedge \cdots \wedge v_k, \end{aligned}$$

where we have used (2′) in the second and third equalities.

In particular, if $\boldsymbol{e} = e_1, \ldots, e_n \in E$ is any *orthogonal* basis, then

$$e_{i_1} \cdots e_{i_k} = e_{i_1} \wedge \cdots \wedge e_{i_k} \tag{7}$$

for every multiindex $I = i_1, \ldots, i_k$ of distinct indices, but not necessarily in increasing order. To ease notation we will write e_I to denote this blade and $q_I = q(e_{i_1}) \cdots q(e_{i_k})$. Note that the basis $\{e_I\}$ of $\mathcal{G}_q$ is q-orthonormal.

Artin's formula: If $\boldsymbol{e}$ is orthogonal and I, J are increasing multiindices, then

$$e_I e_J = (-1)^{t(I,J)} q_{I \cap J}\, e_{I \Delta J}, \tag{8}$$

where $I \Delta J$ is the *symmetric difference* of I and J sorted in increasing order and $t(I, J)$ is the number of order reversing pairs in the sequence I, J.

To prove this formula, reorder the indices of $e_I e_J = e_{I,J}$ in non-decreasing order, which amounts to $t(I, J)$ changes of sign. Then for each $j \in I \cap J$ there will be the scalar factor $e_j^2 = q(e_j)$, which together produce the factor $q_{I \cap J}$. The product of the remaining factors is just $e_{I \Delta J}$.

Commutation rule: With the same notations as in Artin's formula:

$$e_J e_I = (-1)^c (-1)^{|I| \times |J|} e_I e_J, \tag{9}$$

where $c = |I \cap J|$. So e_I and e_J commute if $|I| \times |J|$ and c have the same parity and otherwise anticommute.

For the proof, observe that there are $|I| \times |J|$ pairs (i, j) with $i \in I$ and $j \in J$. The number of pairs such that $i > j$ is $t(I, J)$; similarly, the number of pairs such that $j > i$ is $t(J, I)$; and there are c pairs such that $i = j$. Thus $|I| \times |J| = t(I, J) + t(J, I) + c$, which implies that $t(J, I) \equiv |I| \times |J| + c + t(I, J) \bmod 2$. The assertion is now an immediate consequence of Artin's formula and the fact that $J \cap I = I \cap J$ and $J \vartriangle I = I \vartriangle J$.

Proof of the theorem. Artin's formula shows the existence of a unique bilinear product xx' satisfying rules (1) and (2) in the statement of the theorem, for the bilinear condition reduces xx' to the products $e_I e_J$, which are determined by the formula. Note that $1 = e_\emptyset$ is a unit element of this product: clearly $e_\emptyset e_I = e_I e_\emptyset = e_I$ and this and the bilinearity condition establish the claim.

To prove that the product is associative, it suffices to check that for any increasing multiindices I, J, K the relation $(e_I e_J) e_K = e_I (e_J e_K)$ holds. This will be achieved by regarding the multiindices I in a different way. The idea is that there is a bijective map between the set of increasing multiindices I of length k and the binary strings of length n and weight k (i.e. with exactly k 1's). In fact, this is the usual description of subsets of $[n]$ as binary strings of length n in which the cardinal of the subset corresponds to the weight of the binary string. Now if we look at I and J in Artin's formula as binary strings, the formula looks nicer, because $I \cap J$ is represented by the binary product IJ and the symmetric difference is represented by the binary sum $I+J$, so that

$$e_Ie_J = (-1)^{t(I,J)}q_{IJ}\,e_{I+J}. \tag{10}$$

Using this we have

$$\begin{aligned}(e_Ie_J)e_K &= (-1)^{t(I,J)}q_{IJ}\,e_{I+J}\;e_K\\ &= (-1)^{t(I,J)}(-1)^{t(I+J,K)}q_{IJ}\,q_{IK+JK}\,e_{I+J+K}.\end{aligned}$$

In this expression, $(I+J)+K$ was been replaced by $I+J+K$, as the binary sum is associative. Now we claim: (a) $t(I+J,K)$ has the same parity as $t(I,K)+t(J,K)$ and (b) $q_{IK+JK} = q_{IK}q_{JK}/q_{IJK}^2$, which allow us to conclude, assuming (as we may) that the basis $\boldsymbol{e}$ is orthonormal, that

$$(e_Ie_J)e_K = (-1)^{t(I,J)+t(I,K)+t(J,K)}q_{IJ}\;q_{IK}\;q_{JK}\;e_{I+J+K}$$

and we get the same result if we proceed similarly with $e_I(e_Je_K)$. This proves that the product xx' is associative.

Proof of claim (a). In fact we have the relation

$$t(I,K)+t(J,K) = t(I+J,K)+2t(IJ,K).$$

Indeed, $t(I,K) = t(I+IJ,K)+t(IJ,K)$, as $I+IJ$ and IJ are disjoint. Similarly $t(J,K) = t(J+IJ,K)+t(IJ,K)$. So

$$t(I,K)+t(J,K) = t(I+IJ,K)+t(J+IJ,K)+2t(IJ,K)$$

and the claim follows because $I+IJ$ and $J+IJ$ are disjoint and so

$$t(I+IJ,K)+t(J+IJ,K) = t(I+IJ+J+IJ,K) = t(I+J,K).$$

Proof of claim (b). Given multiindices I and J, the product q_Iq_J contains twice the terms $q(e_j)$ for $j\in IJ$, and the remaining terms are precisely the factors of q_{I+J}. Thus $q_Iq_J = q_{IJ}^2\,q_{I+J}$. If the basis $\boldsymbol{e}$ is orthonomal, then $q_Iq_J = q_{I+J}$.

Let us end this subsection with a few useful facts.

Recursive formulas: For any $v\in E$ and $x\in\mathcal{G}_q$,

$$vx = v\cdot x + v\wedge x \quad\text{and}\quad xv = x\cdot v + x\wedge v. \tag{11}$$

In particular

$$vv' = q(v,v') + v\wedge v' \in \mathcal{G}_q^0\oplus\mathcal{G}_q^2$$

for all $v,v'\in E$ (cf. Eq. (6)). Note that if x is a scalar λ, then $\lambda v = v\lambda = \lambda\wedge v = v\wedge\lambda$, so that the validity of Eq. (11) requires that $v\cdot\lambda = \lambda\cdot v = 0$ for all v (cf. $\diamond 3\,(1)$).

Since q is symmetric and $\wedge$ is skew-symmetric, it follows that

$$vv'+v'v = 2q(v,v') \quad\text{and}\quad vv'-v'v = 2v\wedge v'. \tag{12}$$

Thus $vv' = v'v = q(v,v')$ precisely when v and v' are proportional (i.e. $v\wedge v' = 0$) and $vv' = -v'v = v\wedge v'$ precisely when v and v' are orthogonal.

For the proof of Eq. (11) we can use bilinearity to reduce it to the case e_je_I (and e_Ie_j) relative to an orthonormal basis $e_1,\ldots,e_n$. If $j\notin e_I$, $e_je_I=e_j\wedge e_I$, $e_j\cdot e_I=0$; and if $j\in I$, $e_je_I=(-1)^{t(j,I)}q(e_j)e_{I-\{j\}}=e_j\cdot e_I$, $e_j\wedge e_I=0$ (and similarly for e_Ie_j).

Dot product in terms of the geometric product. Let I and J be non-empty multi-indicies. If $I\subseteq J$ or $I\supseteq J$, then

$$e_I\cdot e_J=e_Ie_J. \tag{13}$$

In particular, if $\mathfrak{j}=e_{[n]}$ is a pseudoscalar and $e_I\neq 1$, then $e_I\cdot\mathfrak{j}=e_I\mathfrak{j}$ and $\mathfrak{j}\cdot e_I=\mathfrak{j}e_I$. In general, if $x\in\mathcal{G}_q$ is scalar free (i.e. $x_0=0$), then

$$\mathfrak{j}x=\mathfrak{j}\cdot x \quad\text{and}\quad x\mathfrak{j}=x\cdot\mathfrak{j}. \tag{14}$$

First consider the case $I\subseteq J$ and let us proceed by induction on $k=|I|$. If $k=1$, say $I=\{i\}$, then $i\in J$ and $e_i\cdot e_J=e_ie_J$ (by Eq. (11), as $e_i\wedge e_J=0$). If $k>1$, then $e_I\cdot e_J=e_{I-i_k}\cdot(e_{i_k}e_J)$ (by $\diamond$3(4) and the case $k=1$). Since $e_{i_k}e_J\sim e_{J-i_k}$, we can apply induction and conclude that the last expression is equal to $e_{i_1}\cdots e_{i_{k-1}}e_{i_k}e_J=e_Ie_J$.

It remains the case $e_I\cdot e_J$ when $I\supseteq J$. If we set $l=|J|$, then $e_I\cdot e_J=(-1)^{(k-l)l}e_J\cdot e_I=(-1)^{(k-l)l}e_Je_I$ (by $\diamond$3(2) and the first case) and $e_Ie_J=(-1)^{l+kl}e_Je_I$, by the commutation rule of the geometric product (Eq. (9)). But $(k-l)l=kl-l^2$ and $l+kl$ have the same parity and therefore $e_I\cdot e_J=e_Ie_J$.

Involutions of a geometric product. The behavior of grade, reverse and Clifford involutions with respect to the geometric product is given by the following relations, where x and x' are arbitrary multivectors ($\triangleright\boxed{\mathbf{5}}$):

$$\widehat{xx'}=\hat{x}\,\hat{x}',\quad \widetilde{xx'}=\tilde{x}'\,\tilde{x},\quad \overline{xx'}=\bar{x}'\bar{x}. \tag{15}$$

The metric of $\mathcal{G}_q$ in terms of the geometric product. If $x,x'\in\mathcal{G}_q$, then

$$q(x,x')=(\tilde{x}x')_0=(x\tilde{x}')_0 \quad\text{and}\quad q(x)=(\tilde{x}x)_0=(x\tilde{x})_0. \tag{16}$$

Since both expressions are bilinear, we may assume that $x=e_I\in\mathcal{G}_q^i$ and $x'=e_J\in\mathcal{G}_q^j$. If $I=J$, then $\tilde{e}_Ie_I=q_I=q(e_I)$, and if $I\neq J$, $(\tilde{e}_Ie_J)_0=\pm(e_Ie_J)_0=0$ by Artin's formula (the blade e_Ie_J has grade $|I+J|>0$) and $q(e_I,e_J)=0$ as the basis $\{e_I\}$ of $\mathcal{G}_q$ is orthonormal.

If x is a blade, then $\tilde{x}x$ is a scalar (we may assume that x is the product of orthogonal vectors), and therefore

$$q(x)=\tilde{x}x,\quad x^2=(-1)^{k//2}q(x),\quad x^{-1}=\tilde{x}/q(x)\quad\text{if } q(x)\neq 0. \tag{17}$$

$\diamond$**5** (Pseudoscalars of $\mathcal{G}_q=\mathcal{G}_{r,s}$)**.** Let $\boldsymbol{e}=e_1,\ldots,e_n$ be an q-orthonormal basis of $E_q=E_{r,s}$ and define

$$\mathfrak{j}_{\boldsymbol{e}}=e_1\wedge\cdots\wedge e_n\in\mathcal{G}_q^n.$$

We will say that $\mathrm{j} = \mathrm{j}_{\boldsymbol{e}}$ is the *pseudoescalar* associated to $\boldsymbol{e}$. Note that the metric formula for a blade, Eq. (17), gives us that

$$q(\mathrm{j}_{\boldsymbol{e}}) = q(e_1) \cdots q(e_n) = (-1)^s.$$

If j' is another pseudoscalar associated to another orthonormal basis, then $\mathrm{j}' = \pm\mathrm{j}$. Indeed, there is a nonzero scalar δ such that $\mathrm{j}' = \delta\mathrm{j}$ and $(-1)^s = q(\mathrm{j}') = q(\delta\mathrm{j}) = \delta^2 q(\mathrm{j}) = (-1)^s\delta^2$, hence $\delta = \pm 1$. We say that $\pm\mathrm{j}$ are the *unit* pseudoscalars.

Choosing one of the unit pseudoscalars is equivalent to pick an orientation of E, in the following sense: For any basis $\boldsymbol{v} = v_1, \ldots, v_n$ of E, $v_1 \wedge \cdots \wedge v_n = \lambda\mathrm{j}$, for some nonzero scalar λ. If $\lambda > 0$ ($\lambda < 0$), the orientation of $\boldsymbol{v}$ is equal to (opposite of) the orientation of $\boldsymbol{e}$.

Let $\mathrm{j} \in \mathcal{G}^n$ be a unit pseudoscalar. Then

(1) $\mathrm{j}^2 = (-1)^{n/\!/2}(-1)^s$ and $\mathrm{j}^{-1} = (-1)^{n/\!/2}(-1)^s\mathrm{j}$ (use Eq. (17)).
(2) *Hodge duality.* For any $x \in \mathcal{G}^k$, $\mathrm{j}x, x\mathrm{j} \in \mathcal{G}^{n-k}$ (follows from Artin's formula) and the maps $x \mapsto \mathrm{j}x$ and $x \mapsto x\mathrm{j}$ are linear isomorphisms $\mathcal{G}^k \to \mathcal{G}^{n-k}$. The inverse maps are given by $x \mapsto \mathrm{j}^{-1}x$ and $x \mapsto x\mathrm{j}^{-1}$, respectively. Note that Eq. (13) implies that $x\mathrm{j} = x \cdot \mathrm{j}$ and $\mathrm{j}x = \mathrm{j} \cdot x$, as $\mathrm{j} = e_{[n]}$.
(3) If n is odd, j commutes with all the elements of $\mathcal{G}$ *(this is expressed by saying that* j *is a* central *element of* $\mathcal{G}$*).* If n is even, j commmutes (anticommutes) with even (odd) multivectors. These claims follow from linearity and the commutation rule applied to $\mathrm{j}e_k$ $(k = 1, \ldots, n)$.
(4) If $q(\mathrm{j}) = 1$ ($q(\mathrm{j}) = -1$), the Hodge duality maps are *isometries* (*antiisometries*). Indeed,

$$q(x\mathrm{j}) = (x\mathrm{j}\,\widetilde{x\mathrm{j}}\,)_0 = (x\mathrm{j}\,\widetilde{\mathrm{j}}\widetilde{x})_0 = (xq(\mathrm{j})\widetilde{x})_0 = q(\mathrm{j})q(x).$$

That $q(\mathrm{j}x) = q(\mathrm{j})q(x)$ is shown in a similar way:

$$q(\mathrm{j}x) = (\widetilde{\mathrm{j}x}\,\mathrm{j}x)_0 = (\widetilde{x}\widetilde{\mathrm{j}}\,\mathrm{j}x)_0 = (\widetilde{x}q(\mathrm{j})x)_0 = q(\mathrm{j})q(x).$$

Remark. Let j be a pseudoscalar of $\mathcal{G} = \mathcal{G}_{r,s}$. If $x \in \mathcal{G}^k$, then

$$\widetilde{x} \wedge (x\mathrm{j}) = q(x)\mathrm{j}. \tag{18}$$

This is clear if $x = e_I$, because $\tilde{e}_I \wedge (e_I\mathrm{j}) = \widetilde{e}_I(e_I\mathrm{j}) = q(e_I)\mathrm{j}$ (the first equality is a direct consequence of the transference rule, as $e_I\mathrm{j} \sim e_{[n]-I}$). If $x = \sum_I \lambda_I e_I$, then $x\mathrm{j} = \sum_I \lambda_I(e_I\mathrm{j})$ and

$$\begin{aligned}
\widetilde{x} \wedge (x\mathrm{j}) &= \sum_{J,I} \lambda_J\lambda_I\widetilde{e}_J \wedge (e_I\mathrm{j}) \\
&= \sum_I \lambda_I^2\widetilde{e}_I \wedge (e_I\mathrm{j}) \\
&= \sum_I \lambda_I^2 q(e_I)\mathrm{j} = q(x)\mathrm{j}.
\end{aligned}$$

We have used that $\widetilde{e}_J \wedge (e_I \mathfrak{j}) = 0$ for $J \neq I$, which is clear because in this case the blades $\widetilde{e}_J$ and $e_I \mathfrak{j}$ share a factor e_i for any $i \in J - I$.

> Río Duero, río Duero, / nadie a acompanarte baja; / nadie se detiene a oír / tu eterna estrofa de agua. [River Duero, river Duero, / no one comes down to join you; / no one stops to listen / to your eternal verse of water.]
>
> GERARDO DIEGO, first stanza of *Romance del Duero.*

2. The Wessel Algebra: $\mathcal{W} = \mathcal{G}_2$

Let us start by casting a closer look on the Euclidean plane $\mathcal{E}_2$ and its geometric algebra $\mathcal{W} = \mathcal{G}\mathcal{E}_2$, which for historical reasons we will call *Wessel algebra*. In fact Caspar Wessel was the first to introduce (1799) a geometrical representation of the complex algebra $\mathbf{C}$, a discovery for which several others, including Gauss and Argand, are also recognized [2, p. 5]. As we will see below, $\mathcal{W}$ refines this traditional representation while capturing its nature in clear geometric terms. Mastering $\mathcal{W}$ is a long stretch toward mastering $\mathcal{G}_q(E)$ in general.

$\diamond$**6** (The unit areas $\pm\,\boldsymbol{i}$)**.** Given an orthonormal basis $\boldsymbol{e} = e_1, e_2$ of $\mathcal{E}_2$, set $\boldsymbol{i} = e_1 \wedge e_2 = e_1 e_2$. It is a basis of $\mathcal{W}^2$ and $1, e_1, e_2, \boldsymbol{i}$ is a basis of $\mathcal{W}$. The bivector $\boldsymbol{i}$ stands for an oriented unit area, as $q(\boldsymbol{i}) = A(e_1, e_2)^2 = 1$. Note that $\pm\,\boldsymbol{i} \in \mathcal{W}^2$ are the only unit bivectors. Indeed, if $\boldsymbol{i}' \in \mathcal{W}^2$ is a unit bivector, then $\boldsymbol{i}' = \lambda\,\boldsymbol{i}$ for some nonzero scalar λ and we have

$$1 = q(\boldsymbol{i}') = q(\lambda\,\boldsymbol{i}) = \lambda^2 q(\boldsymbol{i}) = \lambda^2,$$

so that $\lambda = \pm 1$. The choice of one of the two unit bivectors is tantamount as choosing an orientation of $\mathcal{E}_2$. Since $\boldsymbol{i}$ anticommutes with e_1 and e_2, it anticommutes with all vectors: $v\,\boldsymbol{i} = -\,\boldsymbol{i}v$ for all $v \in \mathcal{E}_2$.

The relations $e_1\,\boldsymbol{i} = e_1 e_1 e_2 = e_2$ and $e_2\,\boldsymbol{i} = e_1 e_2 e_1 = -e_1 e_1 e_2 = -e_2$ show that the map $\mathcal{E}_2 \to \mathcal{E}_2$, $v \mapsto v\,\boldsymbol{i}$, is a $\frac{\pi}{2}$-rotation with respect to the orientation $o(\boldsymbol{e})$ defined by $\boldsymbol{e}$. Similarly, $v \mapsto \boldsymbol{i}v$ is $\frac{\pi}{2}$-rotation with respect to $o(\widetilde{\boldsymbol{e}})$, where $\widetilde{\boldsymbol{e}} = e_2, e_1$, or a $-\frac{\pi}{2}$-rotation with respect to $o(\boldsymbol{e})$. Furthermore, since $\boldsymbol{i}^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1$, we see that $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\,\boldsymbol{i}$ is a two-dimensional subalgebra of $\mathcal{W}$ and that it is isomorphic to $\mathbf{C}$ (by mapping $\boldsymbol{i} \mapsto i$). Due to the semantics of $\boldsymbol{i}$, $\mathbb{C}$ deserves to be called *geometric complex numbers*, while the historical $\mathbf{C}$ may be called *algebraic complex numbers*, to connote that the status of $i = \sqrt{-1}$ is purely algebraic. In our context, the nonzero elements of $\mathbb{C}$ are called *spinors*, and unit spinors, *rotors* (another name for unit geometric complex numbers). Thus rotors have the form $\cos\alpha + \sin\alpha\ \boldsymbol{i} = e^{\alpha\,\boldsymbol{i}}$, and the map $\mathcal{E}_2 \to \mathcal{E}_2$, $v \mapsto ve^{\alpha\,\boldsymbol{i}}$ is a rotation of $\mathcal{E}_2$ through α in the sense of the $\boldsymbol{i}$-orientation. Spinors have the form $re^{\alpha\,\boldsymbol{i}}$ (r a positive scalar) and they act on $\mathcal{E}_2$ as similarities.

$\diamond$**7** (Law of cosines)**.** To keep the traditional notations (a, b, c) for the sides of a triangle ABC, here we denote vectors with bold italic lower case letters, such as $\boldsymbol{v}$

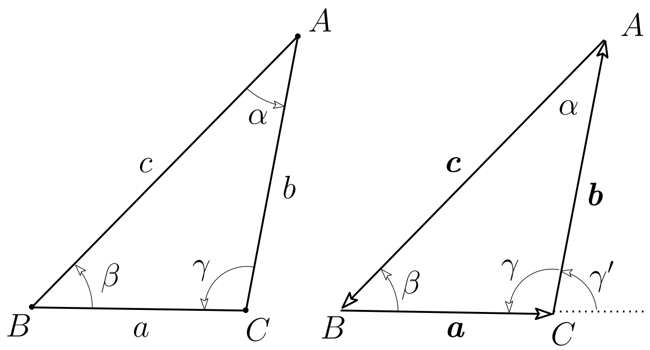

Fig. 1. A triangle and a vector representation of it. Note that $\gamma' + \gamma = \pi$.

and by v the length of this vector, so that $\boldsymbol{v}^2 = q(\boldsymbol{v}) = v^2$. Taking Fig. 1 as a clue, we have

$$c^2 = (\boldsymbol{a} + \boldsymbol{b})^2 = a^2 + b^2 + \boldsymbol{ab} + \boldsymbol{ba} = a^2 + b^2 + 2\boldsymbol{a} \cdot \boldsymbol{b}.$$

Since $\boldsymbol{a} \cdot \boldsymbol{b} = ab \cos \gamma' = -ab \cos \gamma$, we get $c^2 = a^2 + b^2 - 2ab \cos \gamma$.

$\diamond$**8** (Law of sines)**.** We have $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = 0$. Wedging with $\boldsymbol{a}$, and then with $\boldsymbol{b}$, we get the relations

$$\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{b} \wedge \boldsymbol{c} = \boldsymbol{c} \wedge \boldsymbol{a} = \boldsymbol{A}, \tag{19}$$

where the bivector $\boldsymbol{A}$ is the double of the oriented area of the triangle, that is, the area of the corresponding parallelogram. Using the angles, and the fact that supplementary angles have the same sine,

$$ab\,\boldsymbol{i} \sin \gamma = bc\,\boldsymbol{i} \sin \alpha = ca\,\boldsymbol{i} \sin \beta$$

and this clearly implies the law of sines (divide by $abc\,\boldsymbol{i}$).

$\diamond$**9** (Example: The orthocenter and the Euler line of a triangle)**.** We illustrate the use of $\mathcal{W}$ to obtain an analytic expression for the orthocenter of a triangle ABC in the (affine) Euclidean plane (see Fig. 2). We also review Euler's GOH theorem, namely that the homothety with center at G and ratio -2 maps O to H and hence we also get an analytic expression for O.

Since $\boldsymbol{a\,i}$, $\boldsymbol{b\,i}$, $\boldsymbol{c\,i}$ are orthogonal to $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, respectively, relative to G the points of the heights of A, B, C have the form, respectively,

$$\boldsymbol{x} + \lambda \boldsymbol{a\,i}, \quad \boldsymbol{y} + \mu \boldsymbol{b\,i}, \quad \boldsymbol{z} + \rho \boldsymbol{c\,i} \quad (\lambda, \mu, \rho \in \mathbb{R}).$$

The intersection of the first two heights, say H_{AB}, can be obtained by solving for λ and μ the equation $\boldsymbol{x} + \lambda \boldsymbol{a\,i} = \boldsymbol{y} + \mu \boldsymbol{b\,i}$, or

$$\lambda \boldsymbol{a\,i} = \boldsymbol{c} + \mu \boldsymbol{b\,i}. \tag{20}$$

Multiplying by $\boldsymbol{a}$ on the left, we get $\lambda \boldsymbol{a}^2\,\boldsymbol{i} = \boldsymbol{ac} + \mu \boldsymbol{ab\,i}$. The scalar part of this equation is $0 = \boldsymbol{a} \cdot \boldsymbol{c} + \mu(\boldsymbol{a} \wedge \boldsymbol{b})\,\boldsymbol{i} = \boldsymbol{a} \cdot \boldsymbol{c} + \mu \boldsymbol{A\,i}$, where the meaning of the bivector

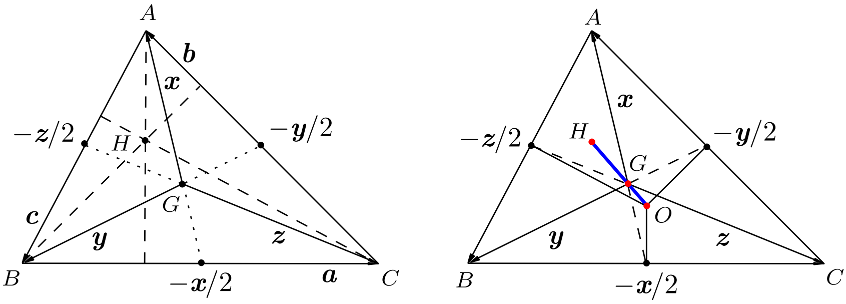

Fig. 2. (Color online) Two views of a triangle ABC. The point G is the *barycenter*, or *centroid*, which by definition is $(A+B+C)/3$ (in affine geometry terms). Let $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ denote the position vectors of A, B, C with respect to G, so that $\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}=0$. Hence, the midpoint of BC is $(\boldsymbol{y}+\boldsymbol{z})/2=-\boldsymbol{x}/2$, and similarly for the other sides, which shows that the *medians* meet at G. O is the *circumcenter*, the point where the *bisectors* of the sides meet, which is the center of the circumscribed circle, and H is the *orthocenter*, the point where the heights meet. We also have $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=0$, where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ have the same meaning as in Fig. 1: $\boldsymbol{a}=\boldsymbol{z}-\boldsymbol{y}$, $\boldsymbol{b}=\boldsymbol{x}-\boldsymbol{z}$, $\boldsymbol{c}=\boldsymbol{y}-\boldsymbol{x}$.

$\boldsymbol{A}$ is as in the law of sines, Eq. (19). Thus $\mu\,\boldsymbol{i}=-(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{A}^{-1}$, which yields $H_{AB}=\boldsymbol{y}-(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}\boldsymbol{A}^{-1}$ (seen as a point on the height of B). To get the expression of H_{AB} as a point of the height of A, multiply Eq. (20) by $\boldsymbol{b}$ on the left and get, after a similar calculation, $H_{AB}=\boldsymbol{x}-(\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a}\boldsymbol{A}^{-1}$ (we have used $\boldsymbol{A}=\boldsymbol{y}\wedge\boldsymbol{z}$, as found in the discussion of the law of sines). So we have

$$H_{AB}=\boldsymbol{x}-(\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a}\boldsymbol{A}^{-1}=\boldsymbol{y}-(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}\boldsymbol{A}^{-1}.$$

Permuting cyclically once, we also have

$$H_{BC}=\boldsymbol{y}-(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}\boldsymbol{A}^{-1}=\boldsymbol{z}-(\boldsymbol{b}\cdot\boldsymbol{a})\boldsymbol{c}\boldsymbol{A}^{-1},$$

which shows that $H_{AB}=H_{BC}$ and hence that the three heights are concurring at this point. This proves the existence of the orthocenter, H, and that it can be expressed as

$$H=\boldsymbol{x}-(\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a}\boldsymbol{A}^{-1}=\boldsymbol{y}-(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}\boldsymbol{A}^{-1}=\boldsymbol{z}-(\boldsymbol{b}\cdot\boldsymbol{a})\boldsymbol{c}\boldsymbol{A}^{-1}.$$

Finally, summing the three expressions and dividing by 3 we get

$$H=-\tfrac{1}{3}\big((\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}+(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}+(\boldsymbol{c}\cdot\boldsymbol{a})\boldsymbol{b}\big)\boldsymbol{A}^{-1}. \tag{21}$$

From Fig. 2, *Right*, we see that the homothety with center G and modulus -2 maps the midpoints of the sides to their opposite vertices, so it maps the bisectors to the heights and therefore it maps O to H: $H=-2O$ (Euler), which provides an expression for $O=-\frac{1}{2}H$ in terms of the vertices.

If the vertices of the triangle were referred to an arbitrary origin, the final formula for the orthocenter is

$$H = G - \tfrac{1}{3}((\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} + (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a} + (\boldsymbol{c}\cdot\boldsymbol{a})\boldsymbol{b})\boldsymbol{A}^{-1}.$$

$\diamond$**10** (The Levi-Civita spinor regularization for the Kepler problem in $\mathcal{E}_2$)**.** Fix a unit vector $\boldsymbol{a} \in \mathcal{E}_2$ (various authors take $\boldsymbol{a} = e_1$, where $e_1, e_2 \in \mathcal{E}_2$ is an orthonormal basis, but we will have no use for e_2). We can represent the position vector $\boldsymbol{r}$ of a particle in polar form (with $r = |\boldsymbol{r}|$):

$$\boldsymbol{r} = \boldsymbol{a} r e^{\boldsymbol{i}\theta} = r e^{-\boldsymbol{i}\theta}\boldsymbol{a}.$$

To conform to the way rotations work in dimensions 3 (or higher), the complex number

$$U = \sqrt{r} e^{-\boldsymbol{i}\theta/2} \quad (U\bar{U} = r),$$

may be expected to play a relevant role in the kinematics and dynamics or $\boldsymbol{r}$. By definition, $\boldsymbol{r} = \boldsymbol{a}\bar{U}^2$, but since $\boldsymbol{i}$ anticommutes with vectors, we have

$$\boldsymbol{r} = \boldsymbol{a}\bar{U}^2 = U\boldsymbol{a}\bar{U} = U^2\boldsymbol{a}. \tag{22}$$

As we will see, the expression $U\boldsymbol{a}\bar{U}$ is analogous to the way rotations are expressed in dimension 3 (or higher), but the other two stem from the commutativity of $\mathbb{C} = \mathcal{W}^+$, a property that does not occur in higher dimensions. Now, we can say that the main point of this subsection is to show that the dynamics of $\boldsymbol{r}$ subject to central newtonian potential (the 2-body Kepler problem) can be expressed in terms of the dynamics of U, which turns out to be simpler and readily solved explicitly. When the said forces are perturbed, we are forced to work in $\mathcal{E}_3$ and therefore we postpone the discussion of this problem to $\diamond$20.

From $\dot{\boldsymbol{r}} = 2\dot{U}U\boldsymbol{a}$ (use Eq. (22)) we get, multiplying by $\boldsymbol{a}\bar{U}$ on the right,

$$2r\dot{U} = \dot{\boldsymbol{r}}\boldsymbol{a}\bar{U} = \dot{\boldsymbol{r}}U\boldsymbol{a}. \tag{23}$$

Introduce a new variable s (it is usually called *fictitious time*) satisfying

$$\frac{d}{ds} = r\frac{d}{dt}.$$

Then Eq. (23) gives

$$2U_s = \dot{\boldsymbol{r}}U\boldsymbol{a} \quad \text{or} \quad U_s\boldsymbol{a} = \tfrac{1}{2}\dot{\boldsymbol{r}}U. \tag{24}$$

Taking the derivative with respect to s in Eq. (24), the following relation can be obtained:

$$2U_{ss} = U\left(\ddot{\boldsymbol{r}}\boldsymbol{r} + \tfrac{1}{2}\dot{\boldsymbol{r}}^2\right). \tag{25}$$

Indeed, for the derivative with respect to s of the right-hand side of Eq. (24) we get the sum $r\ddot{\boldsymbol{r}}U\boldsymbol{a} + \dot{\boldsymbol{r}}U_s\boldsymbol{a}$. The first summand can be transformed as follows (in

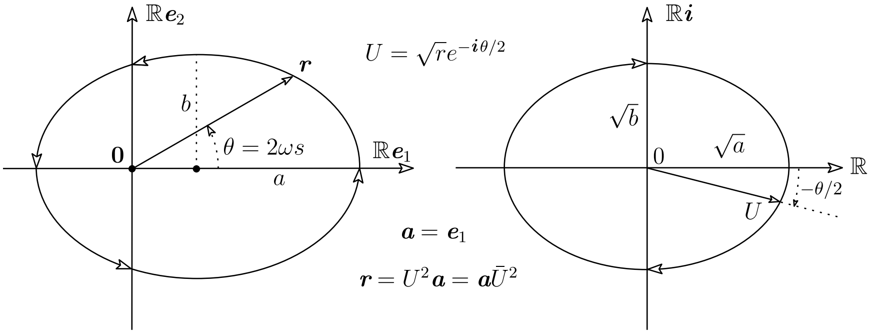

Fig. 3. (Color online) Left: Kepler ellipse in $\mathcal{E}_2$ with focus at $\mathbf{0}$ and semi-axes a and b. *Right*: Orbit of U in $\mathbb{C}$, an ellipse with center at 0 with semi-axis $\sqrt{a}$ and $\sqrt{b}$. In terms of the fictitious time s, $\theta = 2\omega s$: for every clockwise cycle of U, the vector $\boldsymbol{r}$ completes two counterclockwise cycles.

the last equality use Eq. (22)):

$$r\ddot{\boldsymbol{r}}U\boldsymbol{a} = U\bar{U}\ddot{\boldsymbol{r}}U\boldsymbol{a} = U\ddot{\boldsymbol{r}}U^2\boldsymbol{a} = U\ddot{\boldsymbol{r}}\boldsymbol{r}.$$

Similarly, the second summand is $\dot{\boldsymbol{r}}U_s\boldsymbol{a} = \frac{1}{2}\dot{\boldsymbol{r}}\dot{\boldsymbol{r}}U = \frac{1}{2}U\dot{\boldsymbol{r}}^2$ (we have used Eq. (24)). This proves the stated formula for $U_{ss} = d^2U/ds^2$.

Kepler orbits. In the case of a central inverse-square gravitational force we have

$$\ddot{\boldsymbol{r}} = -\frac{\kappa}{r^3}\boldsymbol{r},$$

where κ is positive (as the gravitational force is attractive) and the particle mass is 1. Then $\ddot{\boldsymbol{r}}\boldsymbol{r} = -\kappa/r = V$, where V is the gravitational energy per unit mass, and $\frac{1}{2}\dot{\boldsymbol{r}}^2 = K$, where K the kinetic energy of a unit mass. Since $V + K = E$, the total energy (per unit mass), by Eq. (25) we conclude that

$$\frac{d^2U}{ds^2} = \frac{E}{2}U, \tag{26}$$

which gives the Kepler dynamics in terms of U. This equivalent spinor dynamics is a linear differential equation (a linearized form of the newtonian equation $\ddot{\boldsymbol{r}} = -\kappa\boldsymbol{r}/r^3$), and does not depend on $1/r$, as the newtonian potential does, so the singularity at $r = 0$ has been eliminated, which is expressed by saying that the Kepler problem becomes *regularized* when expressed in terms of the spinor U rather than in terms of the position vector $\boldsymbol{r}$.

Assume that the Kepler orbit of the particle is an *ellipse* with focus at $\mathbf{0} \in \mathcal{E}_2$, which happens if and only if $E < 0$. Then the last equation shows that U *behaves, as a function of* s*, as a harmonic oscillator* with angular frequency $\omega = \sqrt{-E/2}$. Figure 3 depicts the situation.

The above account is essentially due to Levi-Civita in [3], and it was not until 1964 that a similar regularization was achieved for $\mathcal{E}_3$, as we will document in $\diamond$20.

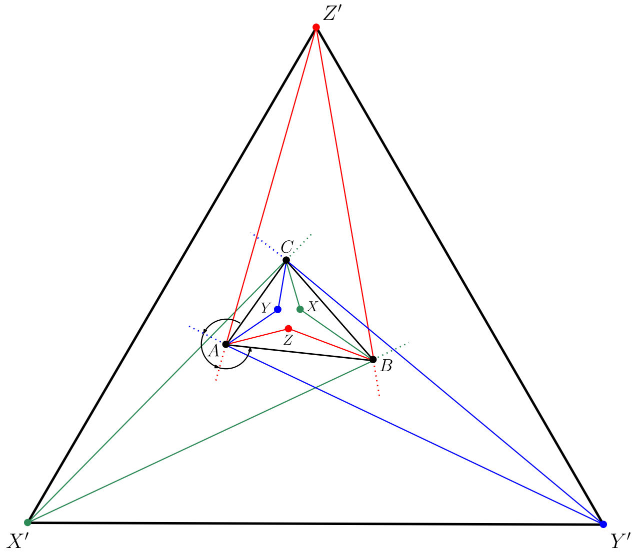

Fig. 4. Inner and outer Morley triangles (XYZ and $X'Y'Z'$) of a triangle ABC.

$\diamond$**11** (Another application of $\mathcal{W}$). *Morley's theorem.* In [4], Morley's theorem is revisited by means of Wessel's algebra, producing 27 equilateral triangles associated to any triangle. Figure 4 depicts two of them.

> Quien pudiera como tu, / a la vez quieto y en marcha, / cantar siempre el mismo verso / pero con distinta agua. [River Duero! Who, like you, / both still and yet in motion, / could sing the very same verse / with ever renewed water.]
>
> Gerardo Diego, fifth stanza of *Romance del Duero.*

3. The Pauli Algebra: $\mathcal{P} = \mathcal{G}_3$

The algebra $\mathcal{G}_3 = \mathcal{G}\mathcal{E}_3$ of the Euclidean three-dimensional space $\mathcal{E}_3$ is named after W. Pauli because he came up with it, as we will see later in this section, under the guise of 2×2 complex matrices (the algebra $\mathbf{C}(2)$) in his research about the spin of the electron. Like the Wessel algebra, it is an excellent case, for its simplicity and surprising applications, to hone geometric algebra proficiency, an endeavor that is rewarded by experiencing the shining of concepts, old and new, under a propitious light.

Notations: We keep the notation q for the metric of $\mathcal{E}_3$ and we let $\boldsymbol{e} = e_1, e_2, e_3$ denote an arbitrary orthonormal basis of $\mathcal{E}_3$.

$\diamond$**12** (Basics about $\mathcal{P} = \mathcal{G}\mathcal{E}_3$). In $\mathcal{P}^3$ there are exactly two unit pseudoscalars, $\pm\mathfrak{i}$. For the existence, set $\mathfrak{i} = e_1 \wedge e_2 \wedge e_3 = e_1e_2e_3 = e_{123}$: it satisfies $q(\mathfrak{i}_{\boldsymbol{e}}) = 1$ by Eq. (1). And if $\mathfrak{i}'$ is another unit pseudoscalar, then $\mathfrak{i}' = \pm\mathfrak{i}$ (same argument as for $\mathcal{W}$, $\diamond$6, or just a special case of $\diamond$5). Orienting $\mathcal{E}_3$ consist of choosing one of the two unit pseudoscalars $\pm\mathrm{i}$. Given any other basis $\boldsymbol{v} = v_1, v_2, v_3$ of $\mathcal{E}_3$, $v_1 \wedge v_2 \wedge v_3 = \lambda\mathrm{i}$, $\lambda \neq 0$. If $\lambda > 0$, $\boldsymbol{v}$ is said to be i-oriented, otherwise it is $-\mathrm{i}$-oriented.

Artin's formula, or a direct check, gives that $\mathrm{i}^2 = -1$. Thus $\mathbb{C} = \mathcal{P}^0 \oplus \mathcal{P}^3 = \mathbb{R} \oplus \mathbb{R}\mathrm{i}$ is another geometric version of the complex numbers. The commutation rule (Eq. (9)) tells us that i commutes with e_1, e_2, e_3 and therefore i commutes with any multivector (we say that i is a *central* element of $\mathcal{P}$). In addition we have $e_1\mathrm{i} = e_{23}$, $e_2\mathrm{i} = e_{31}$, $e_3\mathrm{i} = e_{12}$. It is thus clear that $\mathcal{P}$ multivectors have the following form:

$$x = \alpha + u + v\mathrm{i} + \beta\mathrm{i} \quad (\alpha, \beta \in \mathbb{R},\ u, v \in \mathcal{E}_3). \tag{27}$$

In terms of this expression, we have

$$\begin{aligned} \hat{x} &= \alpha - u + v\mathrm{i} - \beta\mathrm{i}, \\ \widetilde{x} &= \alpha + u - v\mathrm{i} - \beta\mathrm{i}, \\ \bar{x} &= \alpha - u - v\mathrm{i} + \beta\mathrm{i}. \end{aligned} \tag{28}$$

On the other hand, $\mathcal{E}_3 = \mathcal{G}_3^1 \to \mathcal{G}_3^2$, $v \mapsto v\mathrm{i}$, is an *isometry*, because $q(v\mathrm{i}) = \widetilde{v\mathrm{i}}v\mathrm{i} = -\mathrm{i}vv\mathrm{i} = v^2 = q(v)$. The inverse isometry $G_3^2 \to \mathcal{G}_3^1 = \mathcal{E}_3$ is given by $b \mapsto -b\mathrm{i}$ (these statements are instances of the Hodge duality, $\diamond 5(2)$).

$\diamond$**13** (Geometric quaternions)**.** Just as the geometric complex numbers were found to be the even Wessel algebra, $\mathbb{C} = \mathcal{W}^+ = \{\alpha + \beta\,\boldsymbol{i}\}_{\alpha,\beta \in \mathbb{R}}$, the geometric quaternions are the even Pauli algebra, $\mathbb{H} = \mathcal{P}^+ = \{h = \alpha + v\mathrm{i}]\}_{\alpha \in \mathbb{R}, v \in \mathcal{E}_3}$. Given $h = \alpha + v\mathrm{i}$, $\bar{h} = \alpha - v\mathrm{i}$ and $h\bar{h} = \alpha^2 + v^2 = q(h)$ (cf. Eq. (16); here $\bar{h} = \widetilde{h}$). It follows that any $h \neq 0$ is invertible, with $h^{-1} = \bar{h}/q(h)$, which means that $\mathbb{H}$ is a (skew) field.

If $h = \alpha + v\mathrm{i}$ and $h' = \alpha' + v'\mathrm{i}$, then $hh' - h'h = v'v - vv' = 2v \wedge v'$. Therefore h and h' commute if and only if $v \wedge v' = 0$. This happens if either h or h' is a scalar or, otherwise, if $v \sim v'$.

Let $\boldsymbol{i}_k = (-1)^{k-1} e_k \mathrm{i}$ $(k = 1, 2, 3)$, that is, $\boldsymbol{i}_1 = e_2 e_3$, $\boldsymbol{i}_2 = e_1 e_3$, $\boldsymbol{i}_3 = e_1 e_2$. These bivectors satisfy Hamilton's relations, namely

$$\boldsymbol{i}_k^2 = -1,\ \ \boldsymbol{i}_1\,\boldsymbol{i}_2\,\boldsymbol{i}_3 = -1\ (k = 1, 2, 3). \tag{29}$$

Therefore there is an obvious (non-canonical) linear isomorphism between $\mathbb{H} = \langle 1,\, \boldsymbol{i}_1,\, \boldsymbol{i}_2,\, \boldsymbol{i}_3 \rangle$ and $\mathbf{H} = \langle 1,\, \boldsymbol{i},\, \boldsymbol{j},\, \boldsymbol{k} \rangle$ (Hamilton's algebraic quaternions), and this linear isomorphism is an algebra isomorphism because of Eq. (29).

As for $\mathbb{C} = \mathcal{W}^+$, and for reasons that will be clear below ($\diamond 15$), the nonzero geometric quaternions h are called *spinors*, and *rotors* are spinors satisfying $h\bar{h} = 1$. Spinors form a multiplicative group, $H^\times = \{h \in \mathbb{H} : h \neq 0\}$, which is also denoted by Spin_3, and $U\mathbb{H} = \{h \in \mathbb{H}^\times : h\bar{h} = 1\}$, the set of rotors, is a subgroup of $\mathbb{H}^\times$.

Canonical form of a rotor. Let $h = \alpha + v\mathrm{i}$ be a rotor, so that

$$q(h) = \alpha^2 + v^2 = \alpha^2 + |v|^2 = 1.$$

Since $|v| \geq 0$, there is a unique $\varphi \in [0, \pi]$ such that $\alpha = \cos\varphi$ and $|v| = \sin\varphi$. Thus $v = 0$ is equivalent to $\varphi = 0, \pi$ (or to $h = \pm 1$). Assuming that $h \neq \pm 1$, we can write

$h = \cos\varphi + \sin\varphi\, v^\star \mathfrak{i}$, with $v^\star = v/|v| = v/\sin\varphi$. Even better, $\cos\varphi + \sin\varphi\, v^\star\mathfrak{i} = e^{\varphi\, v^\star \mathfrak{i}}$, which follows from the fact that $(v^\star\mathfrak{i})^2 = -1$. In sum, if $h \neq \pm 1$, we have $h = R_{v,\varphi}$, where

$$R_{v,\varphi} = \cos\varphi + \sin\varphi\, v^\star\mathfrak{i} = e^{\varphi\, v^\star\mathfrak{i}}. \tag{30}$$

In this expression φ may be allowed to belong to $(-\pi, 0)$, but since $\cos(-\varphi) = \cos\varphi$ and $\sin(-\varphi) = -\sin\varphi$, we obtain $R_{v,-\varphi} = \bar{R}_{v,\varphi}$.

$\diamond$**14** (Vector algebra: cross product). The *cross product* $v \times v'$ of two vectors $v, v' \in E_3$ can be defined via the Hodge dual of $v \wedge v'$, namely

$$v \times v' = -\mathfrak{i}(v \wedge v'), \quad \text{or} \quad \mathfrak{i}(v \times v') = v \wedge v'. \tag{31}$$

It will be of use to know that it can also be defined as follows:

$$v \times v' = -\mathfrak{i}v \cdot v' = -v \cdot \mathfrak{i}v'. \tag{32}$$

This is immediate if we recall that $\mathfrak{i}x = \mathfrak{i} \cdot x$ if x is scalar-free, i.e. $x_0 = 0$ (Eq. (14)). Indeed, $v \times v' = -\mathfrak{i}(v \wedge v') = -\mathfrak{i} \cdot (v \wedge v') = -(\mathfrak{i} \cdot v) \cdot v' = -\mathfrak{i}v \cdot v'$ (we have used the recursive rule of the dot product, $\diamond$3) and also $v \times v' = \mathfrak{i}(v' \wedge v) = \mathfrak{i}v' \cdot v = -v \cdot \mathfrak{i}v'$ (by the commutation rule of the dot product).

The definition of the cross product using geometric algebra favors natural deductions of its properties. For example,

$$|v \wedge v'| = A(v, v'), \tag{33}$$

where $A(v, v')$ denotes the *area* (2-volume) of the parallelogram defined by v and v'. Indeed,

$$|v \times v'|^2 = (v \times v')^2 = q(v \times v') = q(v \wedge v') = \Delta_q(v, v') = A(v, v')^2,$$

where the second equality is given by the contraction rule, the third by the fact that the Hodge duality is an isometry, and the fourth by the expression of $q(v \wedge v')$ by means of the Gram determinant (cf. $\diamond$2). In this case, the Gram determinant, with $\alpha = \alpha(v, v') \in [0, \pi]$ (the Euclidean angle between v and v'), is given by

$$\Delta_q(v, v') = \begin{vmatrix} v^2 & v \cdot v' \\ v' \cdot v & v'^2 \end{vmatrix} = |v|^2|v'|^2(1 - \cos^2\alpha) = |v|^2|v'|^2\sin^2\alpha.$$

Consequently, $A(v, v') = |v||v'|\sin\alpha$.

Other properties are the *mixed product* and *double cross product*: If $v, v', v'' \in \mathcal{E}_3$, then

$$(v \times v') \cdot v'' = \det{}_\mathfrak{i}(v, v', v''), \tag{34}$$

$$(v \times v') \times v'' = (v \cdot v'')v' - (v' \cdot v'')v, \tag{35}$$

where $\det_\mathfrak{i}$ is the determinant with respect to an $\mathfrak{i}$-oriented orthonormal basis. Recalling that $\mathfrak{i}x = \mathfrak{i} \cdot x$ if $x_0 = 0$ (Eq. (14)), we have

$$(v \times v') \cdot v'' = -\mathfrak{i}(v \wedge v') \cdot v'' = -(\mathfrak{i} \cdot (v \wedge v')) \cdot v''$$
$$= -\mathfrak{i} \cdot (v \wedge v' \wedge v'') = -\mathfrak{i} \det{}_{\mathfrak{i}}(v, v', v'')\mathfrak{i} = \det{}_{\mathfrak{i}}(v, v', v'').$$

For the second identity we have: $(v \times v') \times v'' = -\mathfrak{i}(v \times v') \cdot v'' = -(v \wedge v') \cdot v'' = (v'' \cdot v)v' - (v' \cdot v'')v$.

Note that Eq. (34) tells us that $v \times v'$ is orthogonal to v and v':

$$v \cdot (v \times v') = v' \cdot (v \times v') = 0. \tag{36}$$

In addition, the triple $v, v', v \times v'$ is $\mathfrak{i}$-oriented if $v \wedge v' \neq 0$, because

$$(v \times v') \wedge v \wedge v' = (v \times v') \wedge (v \times v')\mathfrak{i} = (v \times v')^2\mathfrak{i} = A(v, v')^2\mathfrak{i}. \tag{37}$$

In the second equality we have used Eq. (18), and (33) in the third.

For practical computations the following instances may be helpful:

$$e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2.$$

For example,

$$e_1 \times e_2 = -(e_1 \wedge e_2)\mathfrak{i} = -(e_3\mathfrak{i})\mathfrak{i} = e_3.$$

$\diamond$**15** (Symmetries and rotations)**.** Assume that $\mathcal{E}_3$ is oriented by the pseudoscalar $\mathfrak{i}$. If $u \in \mathcal{E}_3$, $u \neq 0$, we let s_u denote the *axial symmetry* about $\langle u \rangle$; m_u the *reflection* in the direction u (or across the plane $u^\perp$); and r_u the ($\mathfrak{i}$-oriented) *rotation* about $\langle u \rangle$ through an angle $\alpha = |u|$. We also set $R_u = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\, u^\star\mathfrak{i}$, with $u^\star = u/|u|$, and note that $\bar{R}_u = \cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\, u^\star\mathfrak{i}$ and $R_u\bar{R}_u = 1$ (with the notations of Eq. (30), R_u is the rotor $R_{u,\alpha/2}$). A short calculation also shows that $R_u^2 = \cos\alpha + \sin\alpha\, u^\star\mathfrak{i}$ (this is even easier to see on recalling that $R_u = e^{\frac{\alpha}{2}u^\star\mathfrak{i}}$).

Then for any $v \in \mathcal{E}_3$ we have the following:

(1) $s_u(v) = uvu^{-1}$. Indeed, the expression on the right is linear in v, it yields u when $v = u$, and $-v$ when $v \in u^\perp$, for in this case $uv = -vu$.
(2) $m_u(v) = -uvu^{-1} = \hat{u}vu^{-1}$, for $m_u(v) = -s_u(v)$.
(3) $r_u(v) = \bar{R}_u v R_u$. Indeed, the expression on the right is linear in v and $\bar{R}_u u R_u = u\bar{R}_u R_u = u = r_u(u)$, while for $v \in u^\perp$ we have

$$\bar{R}vR_u = vR_u^2 = v(\cos\alpha + \sin\alpha\, u^\star\mathfrak{i}) = r_u(v).$$

For the last equality, we may assume that $v \neq 0$ and use that $v^\star u^\star\mathfrak{i} = -(u^\star \wedge v^\star)\mathfrak{i} = u^\star \times v^\star$ to conclude that

$$v(\cos\alpha + \sin\alpha\, u^\star\mathfrak{i}) = |v|(\cos\alpha\ v^\star + \sin\alpha\ v^\star u^\star\mathfrak{i})$$
$$= |v|(\cos\alpha\ v^\star + \sin\alpha\ u^\star \times v^\star)$$

and we know that the basis $u^\star, v^\star, u^\star \times v^\star$ is $\mathfrak{i}$-oriented (cf. Eq. (37)).

(4) Let $u, u' \in \mathcal{E}_3$ be unit vectors. Then, according to (2),

$$(m_{u'} \circ m_u)(v) = -u'(-uvu)u' = (u'u)v(uu') = \bar{R}vR,$$

where $R = uu'$. Note that $\bar{R}R = u'uuu' = 1$, so R is a rotor. Assuming $u \wedge u' \neq 0$, and setting $\alpha = \alpha(u, u') \in (0, \pi)$,

$$uu' = u \cdot u' + u \wedge u' = \cos\alpha + (u \times u')\mathrm{i} = \cos\alpha + \sin\alpha\ (u \times u')^\star\,\mathbf{i},$$

because $|u \times u'|^2 = A(u, u')^2 = \sin^2\alpha$ and $\sin\alpha > 0$. This shows that $R = R_{(u\times u')^\star,\,\alpha}$ and therefore that $m_{u'} \circ m_u$ is a rotation about $\langle u \times u' \rangle$ through an angle 2α. Note that $\langle u \times u' \rangle = u^\perp \cap u'^\perp$, i.e. the intersection of the reflection planes of m_u and $m_{u'}$.

Example. Since $\langle e_2, e_3 \rangle^\perp = \langle e_1 \rangle$ and $\alpha(e_2, e_3) = \pi/2$, the rotation produced by the rotor $\boldsymbol{i}_1$ has axis $\langle e_1 \rangle$ and amplitude $2\alpha = \pi$. In other words, it is the axial symmetry with respect to the axis $\langle e_1 \rangle$. In a similar way we find that the rotors $\boldsymbol{i}_2$ and $\boldsymbol{i}_3$ yield the axial symmetries with respect to the axes $\langle e_2 \rangle$ and $\langle e_3 \rangle$, respectively.

$\diamond$**16** (The Olinde Rodrigues formulas). Given two unit vectors $u, u' \in \mathcal{E}_3$, the composition of the rotations $r_{\alpha u}$ and $r_{\alpha' u'}$, $r_{\alpha' u'} \circ r_{\alpha u}$ is a rotation, and the problem is to find a unit vector u'' and an angle α'' such that $r_{\alpha' u'} \circ r_{\alpha u} = r_{\alpha'' u''}$. By the description $\diamond$15(3), it suffices to solve for u'' and α'' the relation

$$\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\,u\mathrm{i}\right)\left(\cos\frac{\alpha'}{2} + \sin\frac{\alpha'}{2}\,u'\mathrm{i}\right) = \cos\frac{\alpha''}{2} + \sin\frac{\alpha''}{2}\,u''\mathrm{i}.$$

The expansion of the left-hand side (setting $u \cdot u' = \cos\gamma$ and using the relation $uu' = \cos\gamma + u \wedge u' = \cos\gamma + (u \times u')\mathrm{i}$) yields

$$\begin{aligned}
&\cos\frac{\alpha}{2}\cos\frac{\alpha'}{2} + \cos\frac{\alpha}{2}\sin\frac{\alpha'}{2}\,u'\mathrm{i} + \cos\frac{\alpha'}{2}\sin\frac{\alpha}{2}\,u\mathrm{i} + \sin\frac{\alpha}{2}\sin\frac{\alpha'}{2}\,u\mathrm{i}\,u'\mathrm{i} \\
&\quad = \cos\frac{\alpha}{2}\cos\frac{\alpha'}{2} - \sin\frac{\alpha}{2}\sin\frac{\alpha'}{2}\cos\theta \\
&\qquad + \left(\cos\frac{\alpha}{2}\sin\frac{\alpha'}{2}\,u' + \cos\frac{\alpha'}{2}\sin\frac{\alpha}{2}\,u - u \times u'\right)\mathrm{i}.
\end{aligned}$$

Equating scalar and bivector components of this expression with those of $\cos\frac{\alpha''}{2} + \sin\frac{\alpha''}{2}\,u''\mathfrak{i}$, we obtain the sought after equations:

$$\cos\frac{\alpha''}{2} = \cos\frac{\alpha}{2}\cos\frac{\alpha'}{2} - \sin\frac{\alpha}{2}\sin\frac{\alpha'}{2}\cos\theta$$
$$u''\sin\frac{\alpha''}{2} = \cos\frac{\alpha}{2}\sin\frac{\alpha'}{2}\,u' + \sin\frac{\alpha}{2}\cos\frac{\alpha'}{2}\,u - \sin\frac{\alpha}{2}\sin\frac{\alpha'}{2}\,u\times u'.$$

$\diamond$**17** (Two-body dynamics)**.** The newtonian mechanics of a system of point masses m_k located at position vectors $\boldsymbol{r}_k \in \mathcal{E}_3$, with velocities $\dot{\boldsymbol{r}}_k$ and accelerations $\ddot{\boldsymbol{r}}_k$, can be adapted easily to the formalism of GA by replacing the traditional cross product in the definitions of angular momentum L and torque N by the wedge product: $L = \sum \boldsymbol{r}_k \wedge \boldsymbol{p}_k$ ($\boldsymbol{p}_k = m_k\dot{\boldsymbol{r}}_k$) and $N = \sum \boldsymbol{r}_k \wedge \boldsymbol{f}_k$ ($\boldsymbol{f}_k$ the external force acting on m_k). Thus L and N are, in this approach, bivectors. The equation $\dot{L} = N$ can be derived with only slight modifications of the presentation in classical mechanics texts and in particular we get that L is conserved if there are no external forces, in which case the total linear momentum $\boldsymbol{p} = \sum \boldsymbol{p}_k$ is also conserved and the position vector $\boldsymbol{z}$ of the center of mass, namely $\boldsymbol{z} = \frac{1}{\sum m_k}\sum m_k \boldsymbol{r}_k$, moves with uniform velocity ($\ddot{\boldsymbol{z}} = 0$).

For our purposes, we only need to have a closer look at the simplest case: two distinct point particles M and m with no external forces (Fig. 5) in action-reaction interaction.

Set $\boldsymbol{r} = \boldsymbol{r}_m - \boldsymbol{r}_M = r\boldsymbol{u}$ ($\boldsymbol{u} = \boldsymbol{r}^\star$, $r > 0$). Let us assume that the interaction force of M on m has the form $\boldsymbol{f} = f\boldsymbol{u}$, where $f = f(r)$ (*central interaction*). It is attractive if $f < 0$, repulsive if $f > 0$. Then the interaction force of m on M is $-f\boldsymbol{u}$, so that

$$m\ddot{\boldsymbol{r}}_m = \boldsymbol{f}, \quad M\ddot{\boldsymbol{r}}_M = -\boldsymbol{f}.$$

From these relations it follows that

$$Mm\ddot{\boldsymbol{r}} = Mm\ddot{\boldsymbol{r}}_m - Mm\ddot{\boldsymbol{r}}_M = (M+m)\boldsymbol{f}$$

or, setting $\mu = Mm/(M+m)$, which is called the *reduced mass*,

$$\mu\ddot{\boldsymbol{r}} = \boldsymbol{f}, \tag{38}$$

the *equation of motion* of $\boldsymbol{r}$. Note that if $m \ll M$, then $\mu \approx m$.

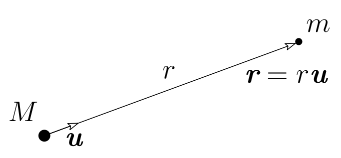

Fig. 5. Two point particles.

Intrinsic angular momentum. Let $\boldsymbol{z}$ be the position vector of the center of mass. From its definition, we have $(M+m)\boldsymbol{z} = M\boldsymbol{r}_M + m\boldsymbol{r}_m$. Thus

$$(M+m)\boldsymbol{z} + m\boldsymbol{r} = M\boldsymbol{r}_M, \quad (M+m)\boldsymbol{z} - M\boldsymbol{r} = (M+m)\boldsymbol{r}_M$$

or, after dividing by $M+m$, and multiplying by M the first equation and by m the second,

$$M\boldsymbol{r}_M = M\boldsymbol{z} - \mu\boldsymbol{r}, \quad m\boldsymbol{r}_m = m\boldsymbol{z} + \mu\boldsymbol{r}. \tag{39}$$

Using these formulas, and the relation $(m+M)\dot{\boldsymbol{z}} = m\dot{\boldsymbol{r}}_m + M\dot{\boldsymbol{r}}_M$, we get

$$\begin{aligned} L &= m\boldsymbol{r}_m \wedge \dot{\boldsymbol{r}}_m + M\boldsymbol{r}_M \wedge \dot{r}_M \\ &= (m\boldsymbol{z} + \mu\boldsymbol{r}) \wedge \dot{\boldsymbol{r}}_m + (M\boldsymbol{z} - \mu\boldsymbol{r}) \wedge \dot{\boldsymbol{r}}_M \\ &= \boldsymbol{z} \wedge (m\dot{\boldsymbol{r}}_m + M\dot{\boldsymbol{r}}_M) + \mu\boldsymbol{r} \wedge (\dot{\boldsymbol{r}}_m - \dot{\boldsymbol{r}}_M) \\ &= \boldsymbol{z} \wedge ((m+M)\dot{\boldsymbol{z}}) + \mu\boldsymbol{r} \wedge \dot{\boldsymbol{r}} \\ &= \boldsymbol{z} \wedge \boldsymbol{p} + \boldsymbol{r} \wedge \mu\dot{\boldsymbol{r}}, \end{aligned} \tag{40}$$

where $\boldsymbol{p} = (m+M)\dot{\boldsymbol{z}}$ is the linear momentum of a (fictitious) particle of mass $m+M$ moving with the center of mass, $L^{\mathrm{t}} = \boldsymbol{z} \wedge \boldsymbol{p}$ is its angular momentum (*translational angular momentum*) and $L^{\mathrm{i}} = \boldsymbol{r} \wedge \mu\dot{\boldsymbol{r}}$ is the angular momentum relative to M of a particle of mass μ (*intrinsic angular momentum*). From the definitions it is easy to find that $\boldsymbol{r}_M - \boldsymbol{z} = -\rho\boldsymbol{r}$ and $\boldsymbol{r}_m - \boldsymbol{z} = (1-\rho)\boldsymbol{r}$, where $\rho = m/(M+m)$, and then to deduce that L^{i} coincides with the angular momentum about the center of mass, that is

$$L^{\mathrm{i}} = M(-\rho\boldsymbol{r}) \wedge (-\rho\dot{\boldsymbol{r}}) + m(1-\rho)\boldsymbol{r} \wedge (1-\rho)\dot{\boldsymbol{r}},$$

which itself boils down to check that $M\rho^2 + m(1-\rho)^2 = \mu$. Furthermore, using that $\dot{\boldsymbol{r}} = \dot{r}\boldsymbol{u} + r\dot{\boldsymbol{u}}$ and that $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$ are orthogonal, we get (this is the first place in this context where the geometric product is used):

$$\boldsymbol{r} \wedge \mu\dot{\boldsymbol{r}} = \mu r\boldsymbol{u} \wedge (\dot{r}\boldsymbol{u} + r\dot{\boldsymbol{u}}) = \mu r^2\boldsymbol{u}\dot{\boldsymbol{u}} = -\mu r^2\dot{\boldsymbol{u}}\boldsymbol{u}. \tag{41}$$

Since we assume that that there are no external forces acting on m and M, the linear moment $\boldsymbol{p}$ is constant ($\dot{\boldsymbol{p}} = M\ddot{\boldsymbol{r}}_M + m\ddot{\boldsymbol{r}}_m = -\boldsymbol{f} + \boldsymbol{f} = 0$) and so is the translational angular momentum ($\dot{L}^{\mathrm{t}} = \dot{\boldsymbol{z}} \wedge \boldsymbol{p} + \boldsymbol{z} \wedge \dot{\boldsymbol{p}} = 0$, as $\dot{\boldsymbol{z}} \sim \boldsymbol{p}$ and $\dot{\boldsymbol{p}} = 0$). Therefore the intrinsic angular momentum, which heceforth will be denoted by L, is also constant and the movement of $\boldsymbol{r}$ takes place in the oriented plane $\langle L\rangle$. Now the area swept out by $\boldsymbol{r}$ per unit of time is $A = \frac{1}{2}\boldsymbol{r} \wedge \dot{\boldsymbol{r}} \sim L$, so A is constant, that is, in the plane $\langle L\rangle$ the vector $\boldsymbol{r}$ sweeps out area at a constant rate (*Kepler's second law*).

The case of inverse-square forces. This is the case $f = -\kappa/r^2$, by far the most important for *Newtonian gravity*, *celestial mechanics*, *astronautics* and *Rutherford* *scattering*. It is attractive if $\kappa > 0$ and repulsive if $\kappa < 0$. The equation of motion (Eq. 38) becomes

$$\mu\ddot{\boldsymbol{r}} = -\frac{\kappa}{r^2}\boldsymbol{u}. \tag{42}$$

In the solution there will be six constants of integration corresponding to the initial position and velocity.

The Laplace-Runge-Lenz vector and the excentricity vector. We know (see (41)) that $L = \mu r^2\boldsymbol{u}\dot{\boldsymbol{u}} = -\mu r^2\dot{\boldsymbol{u}}\boldsymbol{u}$, and that this is a constant of motion. From this we get that $L\boldsymbol{u} = -\mu r^2\dot{\boldsymbol{u}}$ and

$$L\ddot{\boldsymbol{r}} = L\left(-\frac{\kappa}{\mu r^2}\boldsymbol{u}\right) = -\frac{\kappa}{\mu r^2}(-\mu r^2\dot{\boldsymbol{u}}) = \kappa\dot{\boldsymbol{u}}.$$

This says that

$$L\dot{\boldsymbol{r}} - \kappa\boldsymbol{u} \tag{43}$$

is a conserved vector (*Laplace-Runge-Lenz vector*). This vector plays a key part in what follows.

The *excentricity vector* is defined as the vector $\boldsymbol{\epsilon}$ such that

$$L\dot{\boldsymbol{r}} - \kappa\boldsymbol{u} = \kappa\boldsymbol{\epsilon} \;\Leftrightarrow\; L\dot{\boldsymbol{r}} = \kappa(\boldsymbol{u} + \boldsymbol{\epsilon}). \tag{44}$$

It is also a conserved vector and it lies in the oriented plane $\langle L\rangle$, so that L and $\boldsymbol{\epsilon}$ together amount to 5 constants of integration.

$\diamond$**18** (Kepler orbits). Multiply both sides of $L\dot{\boldsymbol{r}} = \kappa(\boldsymbol{u} + \boldsymbol{\epsilon})$ by $\boldsymbol{r}$ on the right. The right-hand side yields $\kappa r(1 + \boldsymbol{\epsilon}\boldsymbol{u})$ and the left-hand side becomes

$$L\dot{\boldsymbol{r}}\boldsymbol{r} = L(\dot{\boldsymbol{r}} \wedge \boldsymbol{r} + \dot{\boldsymbol{r}} \cdot \boldsymbol{r}) = \frac{1}{\mu}L\widetilde{L} + (\dot{\boldsymbol{r}} \cdot \boldsymbol{r})L = \frac{\ell^2}{\mu} + (\dot{\boldsymbol{r}} \cdot \boldsymbol{r})L.$$

Equating the scalar part of both sides, we obtain that

$$\frac{\ell^2}{\mu} = \kappa r(1 + \boldsymbol{\epsilon} \cdot \boldsymbol{u}) = kr(1 + \epsilon\cos\theta),$$

where $\epsilon = |\boldsymbol{\epsilon}|$ and θ is the angle between $\boldsymbol{\epsilon}$ and $\boldsymbol{u}$ (or $\boldsymbol{r}$). Finally, the trajectory, or *orbit*, is the geometric locus given by

$$r(\theta) = \frac{\ell^2}{\kappa\mu(1 + \boldsymbol{\epsilon} \cdot \boldsymbol{u})} = \frac{\ell^2}{\kappa\mu(1 + \epsilon\cos\theta)} = \frac{\lambda}{1 + \epsilon\cos\theta} \quad (\lambda = \ell^2/\kappa\mu). \tag{45}$$

Kepler's first law. This orbit is a conic with principal axis the oriented line $\langle\boldsymbol{\epsilon}\rangle$, focus at the origin ($M$) and eccentricity ϵ. Note that $\lambda = r(\frac{\pi}{2})$, which explains why λ is (traditionally) called the (semi-)*latus rectum*. The conic is a circle for $\epsilon = 0$, an ellipse for $0 < \epsilon < 1$, a parabola for $\epsilon = 1$ and an hyperbola branch if $\epsilon > 1$. This fact can be (usefully) ascertained analytically, as in the immemorial lore on the geometry of conics: $\triangleright\boxed{\textbf{6}}$ for a brief afresh exploration of this avenue.

Here it seems convenient to collect a few observations on the orbit (45) to provide a direct grasp of some of its properties. If $\epsilon = 0$, the orbit is a *circle* of radius λ

toured at constant velocity (by Kepler's second law). Thus, we can assume that $\epsilon > 0$. Since $\cos(-\theta) = \cos(\theta)$, the locus is symmetric about the axis $\langle \boldsymbol{\epsilon} \rangle$, which is called the *principal axis* of the orbit. Thus, we may restrict our attention to the locus on or above the principal axis, that is, to $\theta \in [0, \pi]$.

The points where the orbit meets the principal axis can only occur for $\theta = 0$ or $\theta = \pi$. For $\theta = 0$, we get the point P whose abscissa is $\frac{\lambda}{1+\epsilon}$. This is the point where m gets closest to M and it is called the *periapsis*. For $\theta = \pi$, $0 < 1 + \epsilon \cos \pi = 1 - \epsilon$, which happens only when $\epsilon < 1$, and in this case we get the point A on the principal axis whose abscissa is $-\frac{\lambda}{1-\epsilon}$. This is the point where m gets farthest from M and is called the *apoapsis*: the orbit is a bounded closed curve, an *ellipse* (see Fig. 6 for an illustration of this case). For $\epsilon = 1$, θ can be any value in $[0, \pi)$ and $r(\theta)$ tends to ∞ when θ tends to π: the orbit is a *parabola*. For $\epsilon > 1$, the condition $0 < 1 + \epsilon \cos \theta$ says that $\cos \theta > -\frac{1}{\epsilon}$, which means that $\theta \in [0, \alpha)$, $\alpha = \arccos(-\frac{1}{\epsilon})$ and $r(\theta)$ tends to ∞ when θ tens to α: the orbit is an *hyperbola branch*.

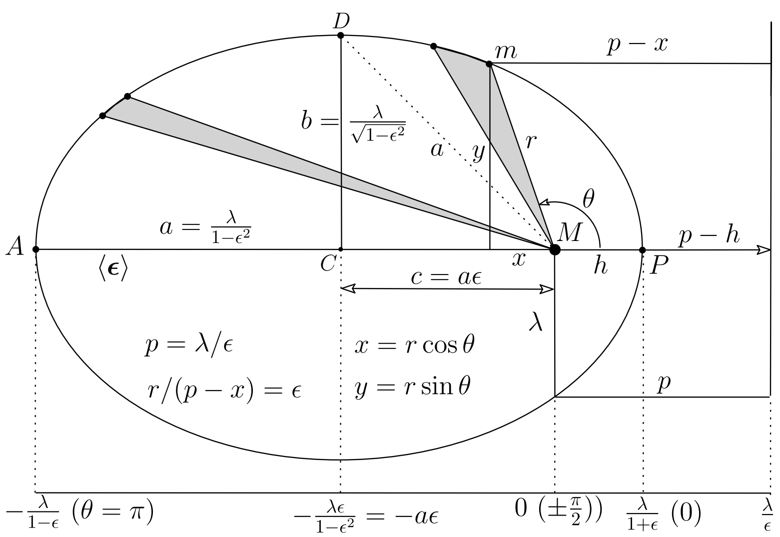

Fig. 6. Illustration of an elliptic Kepler orbit ($\epsilon \in (0,1)$) of m (a planet) around M (a sun), which is a focus of the ellipse. The gray triangles are meant to suggest Kepler's second law. The principal axis is the oriented line $\langle \boldsymbol{\epsilon} \rangle$ and θ is the angle between this axis and $\boldsymbol{r}$. The orbit crosses the principal axis for $\theta = 0$ (at the point P with abscissa $\lambda/(1+\epsilon)$, called the *periapsis*) and for $\theta = \pi$ (at the point A with abscissa $-\lambda/(1-\epsilon)$, called the *apoapsis*). The distance between these two points is $\lambda(\frac{1}{1+\epsilon} + \frac{1}{1-\epsilon}) = \frac{2\lambda}{1-\epsilon^2}$, and so $a = \frac{\lambda}{1-\epsilon^2}$ is the semi-major axis. The abscissa of the orbit's center is $\frac{1}{2}(-\frac{\lambda}{1-\epsilon} + \frac{\lambda}{1+\epsilon}) = -a\epsilon$. The line $x = p$, where $p = \lambda/\epsilon$ (the *focal parameter*), is the *directrix* of the focus M of the ellipse and its role is that the relation $r/p - x = \epsilon$ yields an equation of the ellipse in terms of the rectangular coordinates $x = r\cos\theta$, $y = r\sin\theta$ ($\triangleright$ $\boxed{\mathbf{6}}$). This equation implies that the distance of D from M is a and hence the distance b from C to D (the transversal semi-axis) is $\frac{\lambda}{\sqrt{1-\epsilon^2}}$.

Energy. Since for $V = -\kappa/r$ we have $-dV/dr = -\kappa/r^2 = f$, we may take V as the potential. In this way, we fix $V = 0$ at infinity. Since the kinetic energy is $\frac{1}{2}\mu\dot{\boldsymbol{r}}^2$, we see that

$$E = \frac{\mu}{2}\dot{\boldsymbol{r}}^2 - \frac{\kappa}{r}.$$

The following computation, in which we use $L\dot{\boldsymbol{r}} = \kappa(\boldsymbol{u}+\boldsymbol{\epsilon})$ (see (44)) and $\boldsymbol{u}\cdot\boldsymbol{\epsilon} = \frac{\lambda}{r}-1$ (see (45)) leads to a more useful expression for E:

$$L\dot{\boldsymbol{r}}\dot{\boldsymbol{r}}\widetilde{L} = \ell^2\dot{\boldsymbol{r}}^2 = \kappa^2(\boldsymbol{u}+\boldsymbol{\epsilon})^2 = \kappa^2(\epsilon^2+1+2\boldsymbol{u}\cdot\boldsymbol{\epsilon}) = \kappa^2(\epsilon^2-1+2\lambda/r)$$

and hence (remember that $\lambda = \ell^2/\kappa\mu$)

$$\frac{\mu}{2}\dot{\boldsymbol{r}}^2 = \frac{\mu\kappa^2}{2\ell^2}(\epsilon^2-1+2\lambda/r) = \frac{\mu\kappa^2}{2\ell^2}(\epsilon^2-1) + \frac{\kappa}{r}, \tag{46}$$

$$E = \frac{\mu\kappa^2}{2\ell^2}(\epsilon^2-1). \tag{47}$$

Classification. The sign of E is the same as the sign of $\epsilon^2 - 1$. Therefore

- $E < 0 \;\Leftrightarrow\; \epsilon < 1$: elliptical orbit.
- $E = 0 \;\Leftrightarrow\; \epsilon = 1$: parabolic orbit.
- $E > 0 \;\Leftrightarrow\; \epsilon > 1$: hyperbolic orbit.

If $\epsilon < 1$, for example, $a = p\epsilon/(1-\epsilon^2) = \lambda/(1-\epsilon^2)$, or $1-\epsilon^2 = \lambda/a$, so that

$$E = -\frac{\mu\kappa^2}{2\ell^2}\frac{\lambda}{a} = -\frac{\kappa}{2a}. \tag{48}$$

$\diamond$**19** (Ruherford scattering in a Coulomb field)**.** The positive charged particle q moves in the Coulomb potential $\gamma Q/r$ of a fixed positive charged particle Q (see Fig. 7), where $\gamma \approx 9{\cdot}10^9\,\mathrm{N{\cdot}m^2/C^2}$ is Coulomb's constant. The *scattering angle* is the angle α between the asymptotic initial and final velocities, $\boldsymbol{v}$ and $\boldsymbol{v}'$, respectively.

In this example, we will use GA to derive Rutherford's formula expressing α in terms of the especific energy E of q (energy per unit mass) and the *impact parameter* b:

$$\tan\frac{\alpha}{2} = \frac{\gamma}{2Eb}. \tag{49}$$

Indeed, the (specific) angular momentum L (angular momentum per unit mass) and the excentricity vector $\boldsymbol{\epsilon}$ are constant, and they are related by the formula $L\dot{\boldsymbol{r}} - \gamma\boldsymbol{u} = \gamma\boldsymbol{\epsilon}$, where $\boldsymbol{u} = \boldsymbol{r}^\star$.

For $t = -\infty$, we have $\boldsymbol{u} = -\boldsymbol{v}^\star$ and hence $(Lv+\gamma)\boldsymbol{v}^\star = \gamma\boldsymbol{\epsilon}$.

If time 0 is set when q is at a minimum distance of Q, then we also have $L\boldsymbol{v}_0 - \gamma\boldsymbol{u}_0 = \gamma\boldsymbol{\epsilon}$, and so $(Lv+\gamma)\boldsymbol{v}^\star = L\boldsymbol{v}_0 - \gamma\boldsymbol{u}_0$. But $L\boldsymbol{v}_0 - \gamma\boldsymbol{u}_0 = (Lv_0 + \gamma\,\boldsymbol{i})\boldsymbol{v}_0^\star$, for $\boldsymbol{i}\boldsymbol{v}_0^\star = -\boldsymbol{u}_0$ ($\boldsymbol{i}$ is the unit oriented area in the plane of the orbit), and therefore $(Lv+\gamma)\boldsymbol{v}^\star = (Lv_0+\gamma\,\boldsymbol{i})\boldsymbol{v}_0^\star$. But $L = \ell\,\boldsymbol{i}$, and so $(\ell v\,\boldsymbol{i}+\gamma)\boldsymbol{v}^\star = (\ell v_0+\gamma)\,\boldsymbol{i}\boldsymbol{v}_0^\star$, or

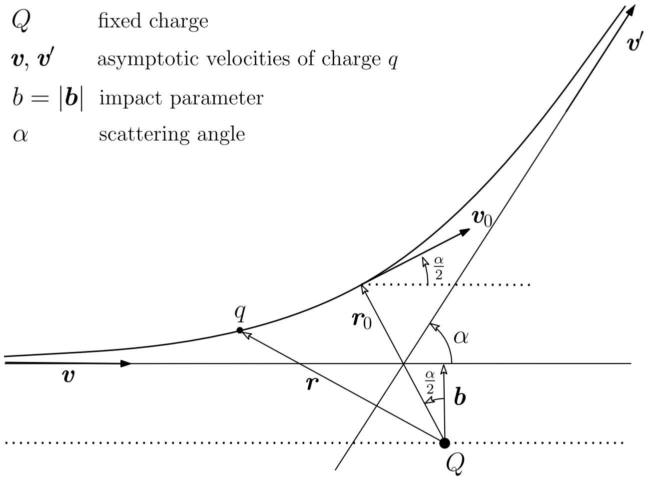

Fig. 7. Rutherford scattering.

$\boldsymbol{v}_0^\star \boldsymbol{v}^\star = \frac{\ell v - \gamma \boldsymbol{i}}{l v_0 + \gamma}$. Finally, since $\boldsymbol{v}_0^\star \boldsymbol{v}^\star = \cos(\alpha/2) - \boldsymbol{i}\sin(\alpha/2)$, we get, with E the specific energy of q,

$$\tan\frac{\alpha}{2} = \frac{\gamma}{\ell v} = \frac{\gamma}{bv^2} = \frac{\gamma}{2Eb}.$$

$\diamond$**20** (The Kustaanheimo–Stiefel spinor regularization of the Kepler problem in $\mathcal{E}_3$)**.** We aim to attune within $\mathcal{E}_3$ the spinor analysis presented in $\diamond$10 for a point particle moving under a central newtonian potential, this time also taking perturbations into account. Initially this discovery was due to Paul Kustaanheimo [5] and then it was presented in [6] in matrix form.

The first treatment with geometric algebra appeared in [7], which was included in the treatise [8], and later in its second edition (1999). It is also included in [9]. The book [10], which is focused on rather thorough applications to Celestial Mechanics, includes a foreword by D. Hestenes that recounts the story of those discoveries, with the following remark about the book (italics added):

> Vrbik tells me he chose to formulate his theory in terms of quaternions rather than geometric algebra because the former has an established tradition in celestial mechanics. On the other hand, *geometric algebra enhances the value of quaternions by clearly integrating them with vector algebra and spinors, thus resolving a history of confusion and controversy and consolidating their place in the broader context of mathematical physics* [here Hestenes cites [9]]. It is an easy and instructive exercise to relate the quaternion formulation in this book to the spinor formulation in mine [i.e. in [8]]. The serious student will be in for some surprises.

The aim of this subsection is to present, to some extend as a 'surprised student', a sketch of the geometric algebra version. However, for further inquiries it is worth taking note of the chapter titles of Vrbik's book (116 p): (1) Linearized Kepler problem; (2) Iterative solution of perturbed problem; (3) Perturbing forces; (4) Solar system; (5) Oblateness perturbations; (6) Lunar problem; (7) Resonances; (8) Other Perturbations.

Let $\boldsymbol{a} \in \mathcal{E}_3$ be a fixed unit vector (authors usually choose the first vector e_1 of an orthonormal basis), U a variable spinor, and let $r = q(U) = U\bar{U}$. Then the vector $\boldsymbol{r} = U\boldsymbol{a}\bar{U}$ satisfies $r = |\boldsymbol{r}|$, because $\boldsymbol{r}^2 = U\boldsymbol{a}\bar{U}U\boldsymbol{a}\bar{U} = q(U)^2 = r^2$. Thus, we have

$$\boldsymbol{r} = U\boldsymbol{a}\bar{U} \;\;\Leftrightarrow\;\; \boldsymbol{r}U = rU\boldsymbol{a}. \tag{50}$$

The map $\mathbb{H}^\times \to \mathcal{E}_3 - \{0\}$, $U \mapsto \boldsymbol{r}$, is surjective: for any $\boldsymbol{r} \in \mathcal{E}_3 - \{0\}$, $\boldsymbol{r} = U\boldsymbol{a}\bar{U}$, where $U = \sqrt{r}R$ with R any rotor such that $\bar{R}\boldsymbol{a}R = \boldsymbol{r}^\star$. But this time there are infinitely many U mapping to a given $\boldsymbol{r}$. To circumvent this redundancy, we may try to restrict U to a suitable subset Z of $\mathbb{H}^\times$ such that the map $Z \to E_3$ is still surjective.

To find such Z, we use the first form of Eq. (50) to get

$$\dot{\boldsymbol{r}} = \dot{U}\boldsymbol{a}\bar{U} + U\boldsymbol{a}\dot{\bar{U}}. \tag{51}$$

We would like to have

$$U\boldsymbol{a}\dot{\bar{U}} = \dot{U}\boldsymbol{a}\bar{U} \tag{$*$}$$

in which case $\dot{\boldsymbol{r}} = 2\dot{U}\boldsymbol{a}\bar{U}$ and so we could adapt to $\mathcal{E}_3$ the analysis presented in ⋄10 for $\mathcal{E}_2$ (in $\mathcal{E}_2$, the analogous relation is easily derived from Eq. (22)).

To proceed, first note that the condition ($*$) is equivalent to $-\overline{\dot{U}\boldsymbol{a}\bar{U}} = \dot{U}\boldsymbol{a}\bar{U}$, which itself is equivalent to $(\dot{U}\boldsymbol{a}\bar{U})_3 = 0$, because $\dot{U}\boldsymbol{a}\bar{U}$ is odd and its conjugation only changes the sign of the grade 1 part. So we will assume that this condition is enforced, and hence that we can rely on the relation $\dot{\boldsymbol{r}} = 2\dot{U}\boldsymbol{a}\bar{U}$. Multiplying this relation by $U\boldsymbol{a}$ on the right, and taking into account that $\bar{U}U = r$, we obtain

$$2U_s = \dot{\boldsymbol{r}}U\boldsymbol{a} \;\;\Leftrightarrow\;\; U_s\boldsymbol{a} = \frac{1}{2}\dot{\boldsymbol{r}}U. \tag{52}$$

Now the second derivative with respect to s works as in the $\mathcal{E}_2$ case:

$$2U_{ss} = \left(\ddot{\boldsymbol{r}}\boldsymbol{r} + \frac{1}{2}\dot{\boldsymbol{r}}^2\right)U. \tag{53}$$

Let's recall the argument. By (52), $2U_{ss}$ is equal to $r\ddot{\boldsymbol{r}}U\boldsymbol{a} + \dot{\boldsymbol{r}}U_s\boldsymbol{a}$. By (50), the first term in this sum is equal to $\ddot{\boldsymbol{r}}\boldsymbol{r}U$, while the second is, by (52) again, $\frac{1}{2}\dot{\boldsymbol{r}}^2U$. These two facts together confirm the claim.

For an inverse-square force law, we get the same harmonic oscillator equation as for $\mathcal{E}_2$ ($2U_{ss} = EU$, cf. Eq. (26)) and the evolution of $\boldsymbol{r}$ takes place in a plane. In the presence of a perturbing specific force $\boldsymbol{f}$ (i.e. force per unit mass), the equation is modified as folows (in the second step use Eq. (50)):

$$2U_{ss} - EU = \boldsymbol{f}\boldsymbol{r}U = r\boldsymbol{f}U\boldsymbol{a}. \tag{54}$$

We refer to [8, §8.4] for iterative solutions of this problem framed in GA. See also [10, Chap. 2].

$\diamond$**21** (The Pauli representation of $\mathcal{P} = \mathcal{G}_3$). An element of $\mathcal{G}_3$ can be uniquely written as $x = \boldsymbol{\xi}_0 + \boldsymbol{\xi}_1\boldsymbol{e}_1 + \boldsymbol{\xi}_2\boldsymbol{e}_2 + \boldsymbol{\xi}_3\boldsymbol{e}_3$, where $\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3 \in \mathbb{C}$. Indeed, in terms of Eq. (27), with $u = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ and $v = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3$, we have

$$\boldsymbol{\xi}_0 = \alpha + \beta\mathfrak{i}, \quad \boldsymbol{\xi}_1 = \alpha_1 + \beta_1\mathfrak{i}, \quad \boldsymbol{\xi}_1 = \alpha_2 + \beta_2\mathfrak{i}, \quad \boldsymbol{\xi}_3 = \alpha_3 + \beta_3\mathfrak{i}.$$

Thus, we have an $\mathbb{R}$-linear isomorphism $\mathcal{G}_3 \to \mathbf{C}(2)$,

$$x = \boldsymbol{\xi}_0 + \boldsymbol{\xi}_1\boldsymbol{e}_1 + \boldsymbol{\xi}_2\boldsymbol{e}_2 + \boldsymbol{\xi}_3\boldsymbol{e}_3 \mapsto X = \xi_0\sigma_0 + \xi_1\sigma_1 + \xi_2\sigma_2 + \xi_3\sigma_3, \tag{55}$$

where $\xi_i \in \mathbf{C}$ denotes the algebraic complex number corresponding to $\boldsymbol{\xi}_i$ ($\alpha + \beta\mathfrak{i} \mapsto \alpha + \beta i$, where i is the conventional imaginary unit) and

$$\sigma_0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the so-called *Pauli matrices*. Note that $\mathfrak{i} = e_1e_2e_3 \mapsto \sigma_1\sigma_2\sigma_3 = iI_2$. Note also that the inverse map $X \mapsto x$ can be immediately determined from the relations

$$2\xi_0 = X_{11} + X_{22}, \quad 2\xi_1 = X_{12} + X_{21}, \quad 2\xi_2 = i(X_{12} - X_{22}), \quad 2\xi_3 = X_{11} - X_{22}.$$

Since the Pauli matrices satisfy Clifford's relations, namely

$$\sigma_j^2 = \sigma_0, \quad \sigma_j\sigma_k + \sigma_k\sigma_j = 0 \text{ if } j \neq k,$$

it follows that the linear map given in Eq. (55) is actually an algebra isomorphism.

Remark. The Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are hermitian, their eigenvalues are ± 1, and their eigenvectors are $(1, \pm 1)$, $(1, \mp i)$ and $\{(1,0),(0,1)\}$, respectively. In quantum mechanics parlance, these matrices correspond to spin observables in the directions e_1, e_2, e_3, respectively: the eigenvalues ± 1 indicate that the observation of the spin in the direction e_k can produce just to values: $\pm e_k$. For a thorough study of spin 1/2 particles (the electron, for instance) by means of the geometric quaternions we refer to [11].

The Pauli represention of $\mathcal{G}_3$, or any other matrix representation for that matter, is not needed to understand $\mathcal{G}_3$ and its applications. The advantages of working directly with $\mathcal{G}_3$, which can be regarded as the 'true' *Pauli algebra*, have been noticed already and will be further highlighted in the considerations that follow, particularly when studying the Dirac algebra in following section.

Indiferente o cobarde, / la ciudad vuelve la espalda. / No quiere ver en tu espejo / su muralla desdentada. [Indifferent or coward, / the city turns its back. / It loathes your mirror image / of its odd edentated wall.]

Gerardo Diego, second stanza of *Romance del Duero*.

4. The Dirac Algebra: $\mathcal{D} = \mathcal{G}_{1,3}$

> Relativity describes Nature from quark to cosmos. Relativity empowers its user to ponder deeply, to analyze widely, to predict accurately. It is a theory of fantastic innocence, simplicity and power.
>
> E.F. Taylor & J.A. Wheeler, *Space-time Physics*.

In this section, we provide a brief overview of $\mathcal{D} = \mathcal{G}_{1,3} = \mathcal{G}E_{1,3}$ and of its capacity to deal with core relativity topics, including Lorentz transformations, Maxwell's electromagnetism and Dirac's equation for the relativistic electron. The algebra $\mathcal{D}$ is named after Dirac because he discovered it in the guise of a 16-dimensional subalgebra of $\mathbf{C}(4)$ in his endeavor to get a relativistic theory of the electron.

$\diamond$**22** (Notations, definitions, conventions)**.** The *Minkowski space*, whose points are called *events*, is an affine space $\mathcal{M} = \mathcal{M}_{1,3}$ ($\triangleright \boxed{\mathbf{7}}$) with associated vector space $E = E_{1,3}$. As usual in affine geometry, for a fixed $O \in \mathcal{M}$ the map $\mathcal{M} \to E$, $X \mapsto x = X - O$, is bijective and its inverse is the map $E \to \mathcal{M}$, $x \mapsto O + x$.

Although the quadratic form of $E_{1,3}$ is frequently denoted by η, we will instead keep using the symbol q. A vector $v \in E$ is called *time-like* or *space-like* according to whether $q(v)$ is positive or negative, respectively, and it is said to be *isotropic* or *null* if $q(v) = 0$. Recall from Eq. (17) that for any blade $x \in \mathcal{D}^j$ we have $q(x) = \tilde{x}x = (-1)^{j/\!\!/2}x^2$, that is, $q(x) = x^2$ for $j = 0, 1, 4$ and $q(x) = -x^2$ for $j = 2, 3$.

A *frame* of E is an orthonormal basis $\mathfrak{e} = e_0, e_1, e_2, e_3$ of E such that

$$q(e_0) = e_0^2 = 1 \quad \text{and} \quad q(e_j) = e_j^2 = -1 \quad \text{for } j = 1, 2, 3. \tag{56}$$

Conventionally, it is also denoted by e_t, e_x, e_y, e_z and we set

$$E' = \langle e_1, e_2, e_3 \rangle = \langle e_x, e_y, e_z \rangle. \tag{57}$$

The $\mathfrak{e}$-components of a vector x are denoted by x^μ ($\mu = 0, 1, 2, 3$); using Einstein's summation convention, $x = x^\mu e_\mu$. The frame $e_0, -e_1, -e_2, -e_3$ is called the *reciprocal* of $\mathfrak{e}$ and is denoted by e^0, e^1, e^2, e^3. It satisfies (and is determined by) the conditions $e^\mu \cdot e_\nu = \delta^\mu_\nu$. The *reciprocal components* of a vector x are denoted by x_μ: $x = x_\mu e^\mu = x^\mu e_\mu$; clearly, $x_0 = x^0$ and $x_j = -x^j$ ($j = 1, 2, 3$). Henceforth we will follow the convention that indices denoted by Greek letters, like μ and ν above, will run in $0, 1, 2, 3$, while indices denoted by Latin letters, like j in the previous sentence, will run in $1, 2, 3$.

Set $\mathfrak{i} = e_{0123}$ to denote the pseudoscalar corresponding to a frame $\mathfrak{e}$. It anticommutes with vectors and satisfies $q(\mathfrak{i}) = \mathfrak{i}^2 = -1$. For any multivector $x \in \mathcal{D}$, its (Hodge) *dual* is $x^* = x\mathfrak{i}$, and we know from $\diamond$5(4) that $\mathcal{D}^\mu \to \mathcal{D}^{4-\mu}$, $x \mapsto x^*$, is an anti-isometry. In particular, $e_0^*, e_1^*, e_2^*, e_3^* \in \mathcal{D}^3$ is a basis of $\mathcal{D}^3$; $-1 = q(e_0^*) = -(e_0^*)^2$; and $1 = q(e_j^*) = -(e_j^*)^2$.

As for $\mathcal{D}^2$, let $\boldsymbol{\sigma}_j = e_j e_0$. Then $q(\boldsymbol{\sigma}_j) = e_j e_0 e_0 e_j = -1$ and $\boldsymbol{\sigma}_j^2 = 1$, while $q(\boldsymbol{\sigma}_j^*) = 1$ and $(\boldsymbol{\sigma}_j^*)^2 = -1$. Thus $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \boldsymbol{\sigma}_1^*, \boldsymbol{\sigma}_2^*, \boldsymbol{\sigma}_3^*$ in an orthonormal basis of $\mathcal{D}^2$. See Fig. 8 for a synopsis of the considerations so far about a frame.

$$1$$

$$e_0 \quad e_1 \quad e_2 \quad e_3 \qquad\qquad x^* = x\,\mathbf{i}$$

$$\boldsymbol{\sigma}_1 \quad \boldsymbol{\sigma}_2 \quad \boldsymbol{\sigma}_3 \quad \boldsymbol{\sigma}_1^* \quad \boldsymbol{\sigma}_2^* \quad \boldsymbol{\sigma}_3^* \qquad\qquad q(x) = +1$$

$$e_0^* \quad e_1^* \quad e_2^* \quad e_3^* \qquad\qquad q(x) = -1$$

$$1^*$$

Fig. 8. (Color online) Basis of $\mathcal{D}$ corresponding to a frame $\mathfrak{e}$ of $E_{1,3}$. The value of $q(x)$ is $+1$ for the red branes and -1 for the blue branes. For the bivectors and trivectors, the value of x^2 is the opposite of $q(x)$, namely -1 for the red branes and $+1$ for the blue branes. Note that the signature of $\mathcal{D}$, relative to q, is $(8,8)$, whereas it would be $(6,10)$ if referred to the signs of the squares.

Relative space. It is the space $\boldsymbol{\mathcal{E}} = \langle \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3 \rangle$, so that $\mathcal{D}^2 = \boldsymbol{\mathcal{E}} \oplus \boldsymbol{\mathcal{E}}\mathbf{i}$. Thus the elements of $\mathcal{D}^2$ have the form $\boldsymbol{x} + \boldsymbol{y}\,\mathbf{i}$, where $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\mathcal{E}}$ are arbitrary, and $\mathcal{D}^+ = \mathbb{C} \oplus \boldsymbol{\mathcal{E}} \oplus \boldsymbol{\mathcal{E}}\mathbf{i}$, where $\mathbb{C} = \{\alpha + \beta\mathbf{i}\}_{\alpha,\beta\in\mathbb{R}}$ (another instance of geometric complex numbers). It is clear that $\mathcal{D}^+$ is a $\mathbb{C}$-vector space and that its $\mathbb{C}$-dimension is 4, as $\mathcal{D}^+ = \langle 1, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3 \rangle_{\mathbb{C}}$. Similarly, $\mathcal{D}^- = \mathcal{D}^1 \oplus \mathcal{D}^3 = \mathcal{D}^1 \oplus \mathcal{D}^1\mathbf{i} = E \oplus E\,\mathbf{i}$, which shows that $\mathcal{D}^-$ is a $\mathbb{C}$-vector space of dimension 4 as well: $\mathcal{D}^- = \langle e_0, e_1, e_2, e_3 \rangle_{\mathbb{C}}$. In sum, an arbitrary element $x \in \mathcal{D}$ has the following form:

$$x = (\alpha + \beta\mathbf{i}) + (u + v\mathbf{i}) + (\boldsymbol{x} + \boldsymbol{y}\mathbf{i}), \quad \alpha, \beta \in \mathbb{R}, \quad u, v \in E, \quad \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\mathcal{E}}, \tag{58}$$

that is, the sum of a *complex scalar* $(\alpha + \beta\mathbf{i})$, a *complex vector* $(u + v\mathbf{i})$, and a *complex bivector* $(\boldsymbol{x} + \boldsymbol{y}\mathbf{i})$. The grade, reverse and conjugation involutions of $\mathcal{D}$ work as follows:

$$\widehat{x} = (\alpha + \beta\mathbf{i}) - (u + v\mathbf{i}) + (\boldsymbol{x} + \boldsymbol{y}\,\mathbf{i}),$$

$$\widetilde{x} = (\alpha + \beta\mathbf{i}) + (u - v\mathbf{i}) - (\boldsymbol{x} + \boldsymbol{y}\,\mathbf{i}),$$

$$\bar{x} = (\alpha + \beta\mathbf{i}) - (u - v\mathbf{i}) - (\boldsymbol{x} + \boldsymbol{y}\,\mathbf{i}).$$

Note, in particular, that $\widetilde{\boldsymbol{z}} = -\boldsymbol{z}$ for any complex bivector.

$\diamond$**23** (Relative vectors and relative algebra)**.** Given $x \in E$ and a frame $\mathfrak{e}$, we have $x = x^\mu e_\mu = x^0 e_0 + x^j e_j$. The bivector $\boldsymbol{x} = x \wedge e_0 = x^j \boldsymbol{\sigma}_j \in \boldsymbol{\mathcal{E}}$ will be called the *relative vector* of $x \in E$ with respect to $\mathfrak{e}$. Note that $xe_0 = x^j e_j$, for $\boldsymbol{\sigma}_j e_0 = e_j$; that

$$xe_0 = x \cdot e_0 + x \wedge e_0 = t + \boldsymbol{x},$$

where we write $t = x^0$ (units are chosen so that $c = 1$, c the speed of light in empty space; otherwise the equation would be $ct = x^0$); and that

$$\begin{aligned} x^2 &= xe_0e_0x = (x \cdot e_0 + x \wedge e_0)(x \cdot e_0 + e_0 \wedge x) \\ &= (t + \boldsymbol{x})(t - \boldsymbol{x}) = t^2 - \boldsymbol{x}^2 \end{aligned}$$

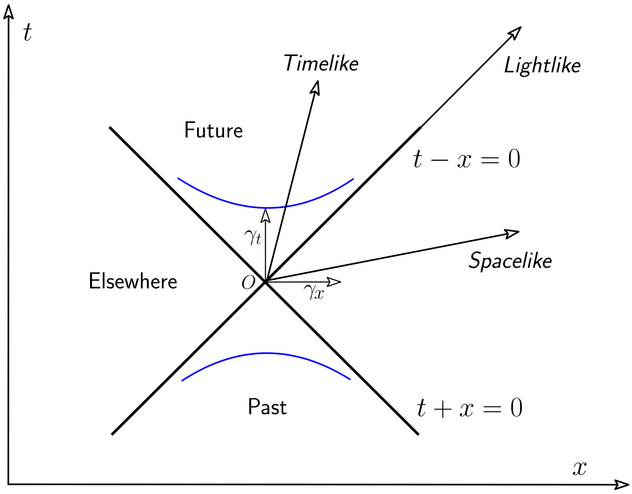

Fig. 9. (Color online) An observer's view of the temporal axis t and one spacial coordinate x. The equations of the light cone $q(x) = 0$ and the Lorentz sphere $q(x) = 1$ (in blue) are $t^2 - x^2 = 0$ and $t^2 - x^2 = 1$, respectively. The events in the observer's Past and Future are defined by $t^2 - x^2 > 0$ and $t < 0$ or $t > 0$, respectively. Elsewhere is defined by the condition $t^2 < x^2$.

is the relative expression of the quadratic form q. See Fig. 9 for a picture of $\mathcal{M}$ relative to a reference $(O, \mathfrak{e})$ —which in this context is called an *observer*— including the light-cone $q(x) = 0$, the Lorentz sphere $q(x) = 1$, and the classification of events $X = O + x$ as Past, Future and Elsewhere.

Now let us consider the subalgebra $\mathcal{R}$ of $\mathcal{D}^+$ generated by $\boldsymbol{\mathcal{E}}$. It is graded, with $\mathcal{R}^0 = \mathbb{R}$, $\mathcal{R}^1 = \boldsymbol{\mathcal{E}}$, $\mathcal{R}^2 = \langle \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3, \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 \rangle$, $\mathcal{R}^3 = \langle \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 \rangle$. If jkl is a cyclic permutation of 123, $\boldsymbol{\sigma}_k\boldsymbol{\sigma}_l = -e_k e_l = \boldsymbol{\sigma}_j \mathfrak{i} = \boldsymbol{\sigma}_j^*$; hence $\mathcal{R}^2 = \boldsymbol{\mathcal{E}}\mathfrak{i}$. We also have $\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_1^* = \mathfrak{i}$; therefore $\mathcal{R}^3 = \mathbb{R}\mathfrak{i}$. Consequently, $\mathcal{R} = \mathbb{R} \oplus \boldsymbol{\mathcal{E}} \oplus \boldsymbol{\mathcal{E}}\mathfrak{i} \oplus \mathbb{R}\mathfrak{i} = \mathcal{D}^+$. The argument shows that $\mathcal{D}^+$ is the Pauli algebra of $\boldsymbol{\mathcal{E}}$, but note that the metric $q|_{\boldsymbol{\mathcal{E}}}$ is the negative of the Euclidean metric: for any $\boldsymbol{x} \in \boldsymbol{\mathcal{E}}$, $q(\boldsymbol{x}) = -\boldsymbol{x}^2$.

$\diamond$**24** (Spinors and Lorentz transformations)**.** Invertible even multivectors are called (Dirac) *spinors*. Actually, $\psi \in \mathcal{D}^+$ is a spinor if and only if $\psi\tilde{\psi} \neq 0$. This condition is clearly necessary, and it sufficient as well, for $\psi\tilde{\psi}$ is its own reverse, hence a complex scalar and so $\psi^{-1} = \tilde{\psi}/(\psi\tilde{\psi})$. A spinor R is called a (Dirac) *rotor* if $R\widetilde{R} = 1$.

A bivector $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}\,\mathfrak{i} \in \mathcal{D}^2$ is spinor if and only if $\boldsymbol{z}^2 \neq 0$, for $\tilde{\boldsymbol{z}} = -\boldsymbol{z}$. We will say that $\boldsymbol{z} \in \mathcal{D}^2$ is a *Lorentz spinor* if $\boldsymbol{z}^2 = \pm 1$. For example, if $\boldsymbol{u}$ is a unit relative vector ($\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3$ among them), then $\boldsymbol{u}$ and $\boldsymbol{u}^* = \boldsymbol{u}\mathfrak{i}$ are Lorentz spinors, as $\boldsymbol{u}^2 = 1$ and $(\boldsymbol{u}^*)^2 = -1$. If $\boldsymbol{z}$ is a Lorentz spinor, let us write $\boldsymbol{z}^2 = \epsilon$, where $\epsilon = \epsilon(\boldsymbol{z}) = \pm 1$ (we say that $\boldsymbol{z}$ is *positive* or *negative* according to whether $\epsilon = 1$ or $\epsilon = -1$, respectively). Define

$$R = R_{\boldsymbol{z},\alpha} = e^{\frac{1}{2}\alpha \boldsymbol{z}} = \cos_\epsilon\left(\tfrac{1}{2}\alpha\right) + \boldsymbol{z}\sin_\epsilon\left(\tfrac{1}{2}\alpha\right),$$

where $\cos_\epsilon, \sin_\epsilon$ denote the functions $\cosh, \sinh$ or $\cos, \sin$ if $\epsilon = 1$ or $\epsilon = -1$, respectively. Since $\tilde{\boldsymbol{z}} = -\boldsymbol{z}$, it is clear that $R\tilde{R} = 1$, that is, R is a rotor. We will say that it is the ***z**-rotor* of amplitude α.

Theorem. *Let $\boldsymbol{z} \in \mathcal{D}^2$ be a Lorentz spinor and $R = R_{\boldsymbol{z},\alpha}$. Let $\underline{R}$ be the (inner) automorphism of $\mathcal{D}$ defined by $\underline{R}(x) = RxR^{-1}$. Then (recall that $\mathcal{D}^1 = E$) $\underline{R}(E) = E$ and $\underline{R} : E \to E$ is an isometry with* $\det(\underline{R}) = 1$.

Indeed, let $v \in E = \mathcal{D}^1$ and set $x = \underline{R}(v) = RvR^{-1}$. To show that $x \in \mathcal{D}^1$, it is enough to see that $\hat{x} = -x$ and $\tilde{x} = x$ (the first relation insures that $x = x_1 + x_3$ and the second that $x_3 = 0$). But these relations can be seen quite easily: $\hat{x} = \hat{R}\hat{v}\hat{R}^{-1} = -RvR^{-1} = -x$ (as $\hat{R} = R$ and $\hat{v} = -v$), while $\tilde{x} = (\tilde{R})^{-1}\tilde{v}\tilde{R} = RvR^{-1} = x$ (as $\tilde{R} = R^{-1}$ and $\tilde{v} = v$).

To see that $\underline{R}$ is an isometry, it suffices to check that it preserves the quadratic form q. Let $v \in \mathcal{D}^1$ and $w = \underline{R}(v) = RvR^{-1}$. Then

$$w^2 = RvR^{-1}RvR^{-1} = Rv^2R^{-1} = v^2.$$

Finally, $\underline{R}(\mathrm{i}) = R\,\mathrm{i}R^{-1} = \mathrm{i}$, as i commutes with the even elements, and on the other hand $\underline{R}(\mathrm{i}) = \det(\underline{R})\mathrm{i}$, so $\det(\underline{R}) = 1$.

Theorem (Lorentz boosts and rotations). *Let $\boldsymbol{v} \in \boldsymbol{\mathcal{E}}$ satisfy $\boldsymbol{v}^2 = 1$ and $v = \boldsymbol{v}e_0 \in E'$ (note that $\boldsymbol{v}$ is the relative vector of v, as $v \wedge e_0 = ve_0 = \boldsymbol{v}$). Then (1) $\underline{R}_{\boldsymbol{v},\alpha}$ is the Lorentz boost in the direction v of velocity $u = \tanh\alpha$ (definition recalled in the proof) and (2) $\underline{R}_{\boldsymbol{v}^*,\alpha}$ is the rotation in E' about v of amplitude α.*

For the proof of (1) we will use (a) $wv = -vw \;\Leftrightarrow\; w\boldsymbol{v} = \boldsymbol{v}w$ for all $w \in \langle e_0, v\rangle^\perp$, and ($b$) $e_0v = -ve_0 \;\Leftrightarrow\; e_0\boldsymbol{v} = -\boldsymbol{v}e_0$. From ($a$) we get that $\underline{R}_{\boldsymbol{v},\alpha}(w) = w$ for all $w \in \langle e_0, v\rangle^\perp$, as w commutes with $R_{\boldsymbol{v},\alpha}$. Thus it suffices to investigate the action of $\underline{R}_{\boldsymbol{v},\alpha}$ in the plane $\langle e_0, v\rangle$. Using (b), we see that $e^{\frac{\alpha}{2}\boldsymbol{v}}e_0e^{-\frac{\alpha}{2}\boldsymbol{v}} = e^{\alpha\boldsymbol{v}}e_0 = e_0\cosh\alpha + v\sinh\alpha$ and $e^{\frac{\alpha}{2}\boldsymbol{v}}ve^{-\frac{\alpha}{2}\boldsymbol{v}} = e^{\alpha\boldsymbol{v}}v = v\cosh\alpha + \boldsymbol{v}v\sinh\alpha = e_0\sinh\alpha + v\cosh\alpha$. In other words, the matrix of $\underline{R}_{\boldsymbol{v},\alpha}$ acting on $\langle e_0, v\rangle$ is $\begin{pmatrix}\cosh\alpha & \sinh\alpha\\ \sinh\alpha & \cosh\alpha\end{pmatrix} = \gamma(u)\begin{pmatrix}1 & u\\ u & 1\end{pmatrix}$, where we set $u = \tanh\alpha \in (-1, 1)$, hence $\cosh\alpha = (1-u^2)^{-1/2} = \gamma(u)$, which is the Lorentz γ-factor for the velocity u. Thus the vector $\tau e_0 + \xi v$ is transformed into the vector $\tau' e_0 + \xi' v$, where $\tau' = \gamma(u)(\tau + u\xi)$ and $\xi' = \gamma(u)(\xi + u\tau)$, which agrees with the usual definition of the Lorentz boost in the direction v and velocity u. Since $d\tau'/d\tau = \gamma(u) > 0$, the boost preserves the temporal orientation and this is expressed by saying that $\underline{R}_{\boldsymbol{v},\alpha}$ is *orthochronus.*

For the proof of (2), we will use $(\boldsymbol{v}^*)^2 = -1$ and $e_0\boldsymbol{v}^* = \boldsymbol{v}^*e_0$. The latter relation implies $\underline{R}_{\boldsymbol{v}^*,\alpha}(e_0) = e_0$ and $\underline{R}_{\boldsymbol{v}^*,\alpha}$ induces an isometry of $e_0^\perp = E'$. Now it is easy to check that $v\boldsymbol{v}^* = \boldsymbol{v}^*v$ and therefore $\underline{R}_{\boldsymbol{v}^*,\alpha}(v) = v$. Choose any unit vector $w \in E'$ orthogonal to v, so that $wv = -vw \;\Leftrightarrow\; w\boldsymbol{v}^* = -\boldsymbol{v}^*w$, and let $w' = wv$. Then it is easy to see that v, w, w' is an orthonormal basis of E', with the same orientation as e_1, e_2, e_3 and the following computation ends the proof of the claim:

$$\begin{aligned}\underline{R}_{\boldsymbol{v}^*,\alpha}(w) &= e^{\frac{\alpha}{2}\boldsymbol{v}^*} w e^{-\frac{\alpha}{2}\boldsymbol{v}^*} = e^{\alpha \boldsymbol{v}^*} w = w\cos\alpha + \boldsymbol{v}^* w \sin\alpha \\ &= w\cos\alpha + w'\sin\alpha.\end{aligned}$$

Relativistic composition of velocities. With the same notations as in the theorem, since $e^{\frac{\alpha}{2}\boldsymbol{v}} e^{\frac{\alpha'}{2}\boldsymbol{v}} = e^{\frac{\alpha+\alpha'}{2}\boldsymbol{v}}$, we see that the composition of two Lorentz boosts along the same direction v is a Lorentz boost along v and that its velocity is

$$u = \tanh(\alpha + \alpha') = \frac{u + u'}{1 + uu'},$$

where $u = \tanh\alpha$ and $u' = \tanh\alpha'$. This proves the relativistic composition of velocities along the same direction.

Recovery of $R_{\boldsymbol{v},\alpha}$. The rotor $R = R_{\boldsymbol{v},\alpha}$ can be recovered explicitly from $e_0' = \underline{R}(e_0)$. Since $e_0\boldsymbol{v} = -\boldsymbol{v}e_0$, $e_0' = Re_0\tilde{R} = R^2 e_0 = (C + \boldsymbol{v}S)e_0$, where $C = \cosh\alpha$ and $S = \sinh\alpha$, and therefore $e_0'e_0 = C + \boldsymbol{v}S$. But $e_0'e_0 = e_0'\cdot e_0 + e_0'\wedge e_0$, so $e_0'\cdot e_0 = C \geq 1$ and $\boldsymbol{v}S = e_0' \wedge e_0$. Now we can show that $R_{\boldsymbol{v},\alpha} = \frac{1+e_0'e_0}{\sqrt{2(1+e_0'\cdot e_0)}}$. Indeed, $R = c + s\boldsymbol{v}$, where $c = \cosh\frac{\alpha}{2}$ and $s = \sinh\frac{\alpha}{2}$, hence $2cR = 2c^2 + 2sc\boldsymbol{v} = 1 + C + S\boldsymbol{v} = 1 + e_0'e_0$ and $2c = \sqrt{2(C+1)} = \sqrt{2(1 + e_0'\cdot e_0)}$.

There is a bit more that can be said about the recovering formula. Suppose e_0' is a vector such that $(e_0')^2 = 1$. Then $e_0' \cdot e_0 \neq 0$, as otherwise e_0' would belong to $e_0^\perp = \langle e_1, e_2, e_3\rangle$ and therefore $(e_0')^2$ would be negative. If $e_0'\cdot e_0 > 0$, then $R = \frac{1+e_0'e_0}{\sqrt{2(1+e_0'\cdot e_0)}}$ is a Dirac rotor: it belongs to $\mathcal{D}^+$ and it satisfies $R\tilde{R} = 1$ because $(1 + e_0'e_0)(1 + e_0e_0') = 2 + e_0e_0' + e_0'e_0 = 2 + 2e_0\cdot e_0'$. Now the observation is that $\underline{R}(e_0) = e_0'$, which shows that the upper half of the e_0-Lorentz sphere (that is $\{e_0' \in E : (e_0')^2 = 1, e_0'\cdot e_0 > 0\}$) is the set $\{\underline{R}(e_0) : \underline{R} \in \mathrm{SO}^+_{1,3}\}$, where $\mathrm{SO}^+_{1,3}$ is the group of proper orthochronous Lorentz transformations (orthochronous means that time orientation is preserved, that is, $e_0'\cdot e_0 > 0$).

To prove that $\underline{R}(e_0) = e_0'$ we will use that there is a unique positive real number α such that $e_0'\cdot e_0 = \cosh\alpha$, for which we refer to [12, 3.1.3], where the notion of hyperbolic angle and its basic properties are established. We can assume that $e_0' \neq e_0$, which allows us to write $e_0' = Ce_0 + Sv$, where $v \in E' - \{0\}$, $S = \sinh\alpha$ and $v^2 = -1$. Then we have $e_0'e_0 = C + S\boldsymbol{v}$, with $\boldsymbol{v} = v\wedge e_0 = ve_0$. So $R = \frac{1+C+S\boldsymbol{v}}{\sqrt{2(1+C)}} = c + s\boldsymbol{v} = R_{\boldsymbol{v},\alpha}$, where $c = \cosh\frac{\alpha}{2}$ and $s = \sinh\frac{\alpha}{2}$, and with a short calculation we find that $\underline{R}(e_0) = e_0'$.

Composition of two boosts. The preceding interpretation of the recovery formula provides an expedient tool with which to prove that *the composition of two boosts is a boost composed with a rotation.* Suppose B and B' are the rotors of two boosts. Then the rotor $B'B$ produces the composition of the two boosts. Let us seek a boost rotor B'' and a rotation rotor R such that $B'B = B''R$. Setting $e_0'' = B'Be_0\widetilde{B}\widetilde{B}'$, we have $e_0'' = B''Re_0\widetilde{R}\widetilde{B}'' = B''e_0\widetilde{B}''$ and therefore $B'' = \frac{1+e_0''e_0}{\sqrt{2(1+e_0''\cdot e_0)}}$. Finally, $R = \widetilde{B}''B'B = \frac{B'B+e_0B'Be_0}{\sqrt{2(1+e_0''\cdot e_0)}}$.

$\diamond$**25** (Space-time kinematics). Let $X(s) = O + x(s)$ be a *parameterized curve*, or *path*, in $\mathcal{M} = \mathcal{M}_{1,3}$. First observe that *the sign of* $(dx/ds)^2$ *is invariant under strictly monotonous reparameterizations* $s = s(\tau)$. Indeed, from $dx/d\tau = (dx/ds)(ds/d\tau)$, and the fact that $ds/d\tau$ is a nonzero scalar, the relation $(dx/d\tau)^2 = (ds/d\tau)^2(dx/ds)^2$ shows that the signs of $(dx/d\tau)^2$ and $(dx/ds)^2$ are the same.

If we regard (as we will) two paths differing in a strictly monontonous reparameterization as the same (geometric) *curve* (or *path*), the claim above says that there is a well defined *sign* associated to any curve. A path $X(s)$ is said to be *time-like*, *light-like*, or *space-like* according to whether $(dx/ds)^2$ is positive, zero, or negative.

Time-like paths and proper time. If $X(s)$ is a timelike curve, the quantity

$$\tau(s) = \int_0^s \left(\frac{dx}{ds}(\xi)\cdot\frac{dx}{ds}(\xi)\right)^{1/2} d\xi$$

does not depend on the parametrization of the curve and will be called *proper time* on the curve.

Since $\tau(s)$ is a strictly increasing function of s, it has an inverse, $s = s(\tau)$. Then we can consider the parametrization $x(\tau) = x(s(\tau))$ by proper time. We will denote $dx/d\tau$ by $\dot{x}$ and say that it is the *proper velocity*. Note that $\dot{x}^2 = 1$: if $\alpha(\xi) = \left(\frac{dx}{ds}(\xi)\cdot\frac{dx}{ds}(\xi)\right)^{1/2}$, then $d\tau/ds = \alpha(s)$ and $ds/d\tau = 1/\alpha(s(\tau))$ and hence

$$\dot{x}^2 = \left(\frac{dx}{d\tau}\right)^2 = \left(\frac{ds}{d\tau}\right)^2\left(\frac{dx}{ds}(s(\tau))\right)^2 = \alpha(s(\tau))^{-2}\alpha(s(\tau))^2 = 1. \tag{59}$$

Example. The path $X(\tau) = O + \tau e_0$ represents the space-time path of a particle at rest at the origin of the observer's space $O + \langle e_1, e_2, e_3\rangle$. Since $\dot{x} = e_0$ and $e_0^2 = 1$, τ is the proper time of that particle.

Let us find the consequences in relative terms. From

$$\dot{x}e_0 = \frac{d}{d\tau}(xe_0) = \frac{d}{d\tau}(t + \boldsymbol{x}), \tag{60}$$

we get

$$\frac{dt}{d\tau} = \dot{x}\cdot e_0, \quad \frac{d\boldsymbol{x}}{d\tau} = \dot{x}\wedge e_0. \tag{61}$$

Let $\boldsymbol{v}$ be the *relative velocity*, so $\boldsymbol{v} = d\boldsymbol{x}/dt$. Then we have

$$\boldsymbol{v} = d\boldsymbol{x}/dt = (d\boldsymbol{x}/d\tau)(d\tau/dt) = \frac{\dot{x}\wedge e_0}{\dot{x}\cdot e_0}. \tag{62}$$

Since $q(\dot{x}\wedge e_0) = 1 - (\dot{x}\cdot e_0)^2$, it follows that

$$-\boldsymbol{v}^2 = q(\boldsymbol{v}) = (\dot{x}\cdot e_0)^{-2} - 1 \;\Leftrightarrow\; \dot{x}\cdot e_0 = 1/\sqrt{1-\boldsymbol{v}^2} \tag{63}$$

(the Lorentz γ-factor of $\boldsymbol{v}$) and

$$\dot{x} = \dot{x}e_0e_0 = (\dot{x}\cdot e_0 + \dot{x}\wedge e_0)e_0 = \frac{1}{\sqrt{1-\boldsymbol{v}^2}}(1+\boldsymbol{v})e_0. \tag{64}$$

$\diamond$**26** (The gradient operator and the electromagnetic field)**.** For applications, particularly to physics, we also need the gradient operator $\partial = \partial_{\mathcal{D}}$ (there is not a generally accepted notation for this operator; among the symbols used there are ∇ and $\square$). It is defined by $\partial = e^\mu\partial_\mu$, $\partial_\mu = \partial/\partial x^\mu$, where x^μ are the $\mathfrak{e}$-components of vector $x \in E$. Owing to the Schwartz rule, the ∂_μ behave as if they were scalars and so ∂ behaves as if it were a vector. Beyond this intuition, the precise rules to work with ∂ are as follows. If $F : E \to \mathcal{D}$ is a multivector field, $\partial F = e^\mu\partial_\mu F$, $\partial\cdot F = e^\mu\cdot\partial_\mu F$ and $\partial\wedge F = e^\mu\wedge\partial_\mu F$. Since $e^\mu\partial_\mu F = e^\mu\cdot\partial_\mu F + e^\mu\wedge\partial_\mu F$, we also have $\partial F = \partial\cdot F + \partial\wedge F$.

The operator ∂ is a square root of the *dalembertian* operator $\square$:

$$\partial^2 = \partial\cdot\partial = e^\mu\cdot e^\nu\partial_\mu\partial_\nu = e^\mu\cdot e^\mu\partial_\mu^2 = \partial_0^2 - (\partial_1^2+\partial_2^2+\partial_3^2) = \square.$$

As we will see at the beginning of $\diamond$27, this fact is critically related to Dirac's discovery of a relativistic theory of the electron.

Relative version of ∂:

$$\boldsymbol{\partial} = \partial\wedge e_0 = e^j\wedge e_0\partial_j = -\boldsymbol{\sigma}_j\partial_j = -\boldsymbol{\nabla},$$

where $\boldsymbol{\nabla} = \boldsymbol{\sigma}_j\partial_j$ is the gradient operator of the relative space $\boldsymbol{\mathcal{E}}$. Then

$$\begin{aligned}\partial e_0 &= \partial\cdot e_0 + \partial\wedge e_0 = \partial_0 + \boldsymbol{\partial} = \partial_0 - \boldsymbol{\nabla},\\ e_0\partial &= e_0\cdot\partial + e_0\wedge\partial = \partial_0 - \boldsymbol{\partial} = \partial_0 + \boldsymbol{\nabla}.\end{aligned}$$

The electromagnetic field is represented by a bivector field $F : E \to \mathcal{D}^2$. In relative terms, $F = \boldsymbol{E} + \boldsymbol{B}\,\mathrm{i}$, where $\boldsymbol{E}$ and $\boldsymbol{B}$ are denote the electric and magnetic fields, respectively.

In what follows, we use units such that $\epsilon_0 = \mu_0 = 1$, and hence also $c = 1$.

Theorem (Riesz form of Maxwell equations). *Let $\rho = \rho(\boldsymbol{x}, t)$ represent a charge density in $\boldsymbol{\mathcal{E}}$, and $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{x}, t) \in \boldsymbol{\mathcal{E}}$, a current density. Define the current vector $J = J(x,t)$ by the formula $J = (\rho + \boldsymbol{j})e_0 \in E$. Then the equation (Riesz [13])*

$$\partial F = J$$

is equivalent to the four Maxwell's equations for the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ created by the charge density ρ and the current density vector $\boldsymbol{j}$.

We reproduce the proof included in [12, § 3.3.1], with some improvements and adapting the notations to those used here. Since $Je_0 = \rho + \boldsymbol{j}$, we have $\rho = J\cdot e_0$ and $\boldsymbol{j} = J\wedge e_0$. As a direct consequence, $e_0J = \rho - \boldsymbol{j}$. We also have $(\rho + \boldsymbol{j})e_0 = e_0(\rho - \boldsymbol{j})$, as e_0 anticommutes with $\boldsymbol{j}$. Multiplying the equation $\partial F = J$ by e_0 on the left, we obtain the equivalent relation $(\partial_0 + \boldsymbol{\nabla})(\boldsymbol{E} + \boldsymbol{B}\,\mathrm{i}) = \rho - \boldsymbol{j}$. So

$$\partial_0 \boldsymbol{E} + \boldsymbol{\nabla} \cdot \boldsymbol{E} + \boldsymbol{\nabla} \wedge \boldsymbol{E} + (\partial_0 \boldsymbol{B} + \boldsymbol{\nabla} \cdot \boldsymbol{B} + \boldsymbol{\nabla} \wedge \boldsymbol{B})\mathrm{i} = \rho - \boldsymbol{j}.$$

On equating the corresponding grades of both sides, we see that this equation is equivalent to the four equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho, \quad \partial_0 \boldsymbol{E} + (\boldsymbol{\nabla} \wedge \boldsymbol{B})\mathrm{i} = -\boldsymbol{j}, \quad (\partial_0 \boldsymbol{B})\mathrm{i} + \boldsymbol{\nabla} \wedge \boldsymbol{E} = 0, \quad (\boldsymbol{\nabla} \cdot \boldsymbol{B})\mathrm{i} = 0.$$

Now it suffices to observe that $\boldsymbol{\nabla}\cdot$ is the divergence operator of the relative space and that $(\boldsymbol{\nabla} \wedge \boldsymbol{B})\mathrm{i} = -\boldsymbol{\nabla} \times \boldsymbol{B} = -\mathrm{curl}(\boldsymbol{B})$ to conclude that these equations are equivalent to

$$\mathrm{div}(\boldsymbol{E}) = \rho \qquad \text{(Gauss law for } \boldsymbol{E}\text{)}, \tag{65}$$

$$\mathrm{curl}(\boldsymbol{B}) - \partial_t \boldsymbol{E} = \boldsymbol{j} \qquad \text{(Ampère-Maxwell law)}, \tag{66}$$

$$\partial_t \boldsymbol{B} + \mathrm{curl}(\boldsymbol{E}) = 0 \qquad \text{(Faraday's induction law)}, \tag{67}$$

$$\mathrm{div}(\boldsymbol{B}) = 0 \qquad \text{(Gauss law for } \boldsymbol{B}\text{)}, \tag{68}$$

which are Maxwell's equations in differential form for the electromagnetic field caused by ρ and $\boldsymbol{j}$.

The charge conservation equation. On multiplying $\partial F = J$ by ∂ on the left, we get $\Box F = \partial \cdot J + \partial \wedge J$. Since the left side is a bivector ($\Box$ preserves grades), the scalar part of the right-hand side expression must vanish: $\partial \cdot J = 0$. In relative terms, this equation becomes $\partial_t \rho + \mathrm{div}(\boldsymbol{j}) = 0$ as a short calculation shows: $J = (\rho + \boldsymbol{j})e_0 = \rho e_0 + j^k e_k$ and $\partial \cdot J = (e^\mu \partial_\mu) \cdot (\rho e_0 + j^k e_k) = \partial_0 \rho + \partial_k j^k$, which can be written as $\partial_t \rho + \mathrm{div}(\boldsymbol{j})$.

Potentials. Using similar techniques, together with the Poincaré lemma and the existence of scalar solutions f of the equation $\Box f = g$ (g given), it can be seen (cf. [12, pp. 68–69]) that there is a vector field A (*electromagnetic potential*) such that $F = \partial \wedge A$ and $\partial \cdot A = 0$ (Lorentz gauge condition). In relative terms, $F = \boldsymbol{E} + \boldsymbol{B}\,\mathrm{i}$, $Ae_0 = \phi + \boldsymbol{A}$ (ϕ and $\boldsymbol{A}$ are the *scalar* and *vector potentials*, respectively), and $F = \partial \wedge A$ gets translated into the familiar equations that determine the electric and magnetic fields in terms of the potentials:

$$\boldsymbol{E} = -(\boldsymbol{\nabla}\phi + \partial_t \boldsymbol{A}), \quad \boldsymbol{B} = \mathrm{curl}(\boldsymbol{A}). \tag{69}$$

Monochromatic electromagnetic waves in empty space. An electromagnetic field $F : E \to \mathcal{D}^2$ of the form $F(x) = F_0 e^{\mathrm{i}(k\cdot x)}$, where $F_0 \in \mathcal{D}^2$ is a constant nonzero bivector and $k \in \mathcal{D}^1$ a constant nonzero vector, is said to be a *circularly polarized monochromatic wave* of momentum k. In absence of charges and currents, the Riesz-Maxwell equation for F is $\partial F = 0$, which can be analyzed as follows. It is immediate that $\partial(k \cdot x) = k$ and $\partial F = k\mathrm{i}F_0 e^{\mathrm{i}(k\cdot x)}$ and hence $\partial F = 0$ is equivalent to $kF_0 = 0$. Writing $ke_0 = \omega + \boldsymbol{k}$ in the relative formalism, so that $e_0 k = \omega - \boldsymbol{k}$, we have $k^2 = ke_0e_0k = \omega^2 - \boldsymbol{k}^2$. Multiplying $kF_0 = 0$ by k, we get $\omega^2 = \boldsymbol{k}^2$, or $k^2 = 0$ (this is called the *dispersion relation*). Let $F_0 = \boldsymbol{E} + \mathrm{i}\,\boldsymbol{B}$, with $\boldsymbol{E}, \boldsymbol{B} \in \mathcal{E}$. Then we have $(\omega - \boldsymbol{k})(E + \mathrm{i}\,\boldsymbol{B}) = 0$, which is equivalent to

$$\omega\boldsymbol{E} + \mathrm{i}\omega\boldsymbol{B} = \boldsymbol{k} \cdot \boldsymbol{E} + \boldsymbol{k} \wedge \boldsymbol{E} + \mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{B}) + \mathrm{i}(\boldsymbol{k} \wedge \boldsymbol{B}).$$

Looking at the scalar part, we get $\boldsymbol{k} \cdot \boldsymbol{E} = 0$. Similarly, the pseudoscalar part yields $\boldsymbol{k} \cdot \boldsymbol{B} = 0$. So we have $\omega\boldsymbol{E} + \mathrm{i}\omega\boldsymbol{B} = \boldsymbol{k} \wedge \boldsymbol{E} + \mathrm{i}(\boldsymbol{k} \wedge \boldsymbol{B})$, which allows us to conclude that $\omega\boldsymbol{E} = -\boldsymbol{k} \times \boldsymbol{B}$ and $\omega\boldsymbol{B} = \boldsymbol{k} \times \boldsymbol{E}$. This shows that $\boldsymbol{k}, \boldsymbol{E}, \boldsymbol{B}$ is an orthogonal system, and also that $\boldsymbol{E}^2 = \boldsymbol{B}^2$. At a given point x, in relative terms we have $k \cdot x = (\omega - \boldsymbol{k}) \cdot (t + \boldsymbol{x}) = \omega t - \boldsymbol{k} \cdot \boldsymbol{x}$ and hence the electric and magnetic fields are given by $\boldsymbol{E}\cos(\omega t - \boldsymbol{k} \cdot \boldsymbol{x}) - \boldsymbol{B}\sin(\omega t - \boldsymbol{k} \cdot \boldsymbol{x})$ and $\boldsymbol{E}\sin(\omega t - \boldsymbol{k} \cdot \boldsymbol{x}) + \boldsymbol{B}\cos(\omega t - \boldsymbol{k} \cdot \boldsymbol{x})$, respectively, which show that they turn in the plane $\langle \boldsymbol{E}, \boldsymbol{B} \rangle$ with angular frequency ω. Moreover, these fields move along $\boldsymbol{k}$ with speed $c = 1$. Indeed, the increment $\Delta t = 2\pi/\omega$ repeats the value of the fields at x, but this increment can be absorbed by $\boldsymbol{x}$ as $\Delta\boldsymbol{x} = -\frac{2\pi}{\omega^2}\boldsymbol{k}^\star$, for $-\boldsymbol{k} \cdot \Delta\boldsymbol{x} = -\omega\,\boldsymbol{k}^\star \cdot \Delta\boldsymbol{x} = \frac{2\pi}{\omega}$, and this implies that the speed with which the waves propagate is $\frac{2\pi}{\omega} / \frac{2\pi}{\omega} = 1$.

To find a typical linearly polarized monochromatic electromagnetic wave we can start with $F = (E_0\boldsymbol{\sigma}_1 + \mathrm{i}B_0\boldsymbol{\sigma}_2)\cos(k \cdot x)$, where $k \in E$ is a fixed vector. So we are trying $\boldsymbol{E} = E_0\boldsymbol{\sigma}_1$ and $\boldsymbol{B} = B_0\boldsymbol{\sigma}_2$, which are constant orthogonal vectors. Arguments similar to those presented above show that $\partial F = 0$ implies that $k = \omega(e_0 + e_3)$, or $\boldsymbol{k} = \omega\boldsymbol{\sigma}_3$, that $B_0 = E_0$ and that $k \cdot x = \omega(t - z)$, where $z = x^3$. The resulting field $F = E_0(\boldsymbol{\sigma}_1 + \mathrm{i}\boldsymbol{\sigma}_2)$ satisfies $\partial F = 0$ and stands for an electromagnetic wave propagating along z-axis with velocity c (1 in the units we have been using; see Fig. 10).

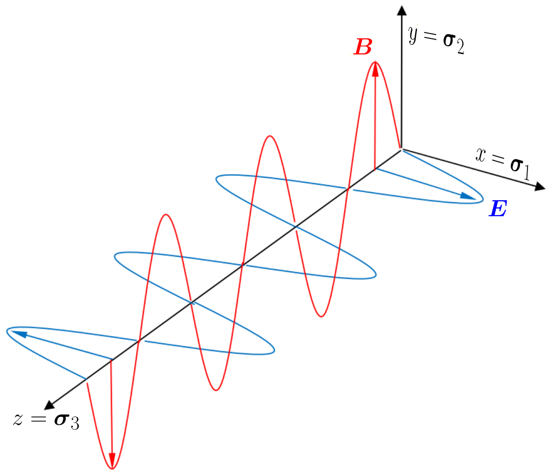

Fig. 10. (Color online) Linearly polarized electromagnetic wave moving along the z-axis at the speed of light.

$\diamond$**27** (Dirac's equation)**.** The original Dirac equation appeared in his landmark paper [14]. In it he introduced 4×4 complex matrices in his endeavor to build a relativistic theory of the electron that are usually denoted by γ_μ, after a trend initiated by Gregor Wentzel and Wolfgang Pauli in the mid 1930's:

$$\gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \tag{70}$$

where σ_μ are the Pauli matrices ($\diamond$21). The matrices $\gamma_\mu \in \mathbf{C}(4)$ satisfy, as it is easily checked, the Clifford relations for the signature $(+,-,-,-)$:

$$\frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \begin{cases} 0 & \text{if } \mu \neq \nu, \\ I_4 & \text{if } \mu = \nu = 0, \\ -I_4 & \text{if } \mu = \nu \neq 0. \end{cases} \tag{71}$$

These were the properties required to get $(\gamma_\mu \partial_\mu)^2 = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$, that is, to get a square root of the d'Alambertian operator (we refer to Dirac's book [15, §67] for the physical reasoning that led to this result). Anyway, (71) gives rise to an algebra homomorphism $\mathcal{D} \to \mathbf{C}(4)$ such that $e_\mu \mapsto \gamma_\mu$. Note, however, that $\mathcal{D}$ has dimension 16 while $\mathbf{C}(4)$ has dimension 32.

The space on which the γ_μ act, $\mathbf{C}^4$, was the space of *Dirac spinors*; the *wave function* was map $\psi : E_{1,3} \to \mathbf{C}^4$; and the Dirac equation was derived as a "relativistic Schrödinger equation for the electron wave function". It turns out, however, that GA shows that *the complex matrices are superfluous, as the only crucial fact required is that they satisfy Clifford's relations.* And after that, the analysis reveals that the role of $\mathbf{C}^4$ is better played by the space $\mathcal{D}^+$ (which has complex dimension 4) and hence that the wave function is to be thought as a *spinor field*, the name for a function $\psi : \mathcal{M} \to \mathcal{D}^+$ such that $\psi(x)$ is a Dirac spinor as defined in $\diamond$24. Thanks to fundamental work of David Hestenes, beginning with [16] and followed by many other papers, particularly [17], *Dirac's equation* was morphed into the following equation for the spinor field ψ:

$$\partial\psi\, \mathbf{i}\hbar - eA\psi = m_e \psi e_0, \tag{72}$$

where e is the electron charge and m_e its mass. In this equation $\mathbf{i}$ is not $\sqrt{-1}$, but the bivector $\mathbf{i} = e_{21}$, and A is the *electromagnetic potential.* As expressed by D. Hestenes, this equation "reveals geometric structure in the Dirac theory that is so deeply hidden [even inaccessible] in the matrix version that it remains unrecognized by QED experts to this day". Note that $\mathbf{i} = e_2e_1 = \mathrm{i}e_3e_0 = \mathrm{i}\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$: this unit area element replaces the (ungeometric) imaginary unit $\sqrt{-1}$ in the original Dirac equation. In what follows of this section we distill a few results from [17] that illustrate the goods that can be gleaned from Eq. (72).

The first important advantage of the GA formulation of the Dirac equation stems from the nature of $\psi = \psi(x)$, as it implies that it has a unique decomposition of the following form:

$$\psi = \rho^{1/2} e^{\mathrm{i}\beta/2} R,$$

where $\rho = \rho(x)$ is a positive real number, $\beta = \beta(x) \in [0, 2\pi)$ and $R = R(x)$ is a Dirac *rotor* (that is, $R\widetilde{R} = 1$). Indeed, from ⋄24 we know that $\psi\tilde{\psi}$ is a nonzero complex scalar, which we can write in polar form: $\psi\tilde{\psi} = \rho e^{\beta \mathrm{i}}$, with $\rho > 0$ and $\beta \in [0, 2\pi)$. Let $R = \rho^{-1/2}\psi e^{-\beta\mathrm{i}/2}$. Then R is a rotor, because $\tilde{R} = \rho^{-1/2}e^{-\beta\mathrm{i}/2}\tilde{\psi}$ and $R\tilde{R} = \rho^{-1}\psi e^{-\beta\mathrm{i}}\tilde{\psi} = 1$, as i commutes with even multivectors. Uniqueness is also clear, for ρ and β are uniquely determined by $\psi\tilde{\psi}$, and R is uniquely determined by the relation $R = \psi\rho^{-1/2}e^{-\beta\mathrm{i}/2}$.

Comoving frame, Dirac current and spin. Define $e'_\mu = e'_\mu(x) = Re_\mu\tilde{R}$. Since R is a rotor, this is an orthonormal frame field in $E_{1,3}$ with the same orientation and temporal orientation as the reference frame e_μ; it is the *comoving frame*. Note that $\psi e_\mu\widetilde{\psi} = \rho e'_\mu$, because i anticommutes with vectors and $\tilde{\mathrm{i}} = \mathrm{i}$:

$$\psi e_\mu\widetilde{\psi} = \rho e^{\mathrm{i}\beta/2} R e_\mu \widetilde{R} e^{\mathrm{i}\beta/2} = \rho e^{\mathrm{i}\beta/2} e^{-\mathrm{i}\beta/2} R e_\mu \widetilde{R} = \rho e'_\mu.$$

In particular, $\psi e_0\widetilde{\psi} = \rho e'_0$, which is the *Dirac current*. The vector

$$s = \frac{\hbar}{2} R e_3 \widetilde{R} = \frac{\hbar}{2} e'_3 \tag{73}$$

is the *spin vector*. The rotor R transforms the unit $\mathbf{i}$ to $\mathbf{i}' = R\,\mathbf{i}\widetilde{R} = e'_2 e'_1$ and $\boldsymbol{s} = \frac{\hbar}{2}\,\mathbf{i}'$ is the *spin bivector*. We have $\boldsymbol{s} = \mathrm{i}se'_0$:

$$\mathrm{i}se'_0 = \frac{\hbar}{2}\mathrm{i}Re_3\widetilde{R}Re_0\widetilde{R} = \frac{\hbar}{2}R\mathrm{i}e_3e_0\widetilde{R} = \frac{\hbar}{2}R\,\mathbf{i}\widetilde{R} = \frac{\hbar}{2}\,\mathbf{i}' = \boldsymbol{s}.$$

> Tú, viejo Duero, sonríes / entre tus barbas de plata, / moliendo con tus romances / las cosechas mal logradas. [You, old Duero, are smiling / within your silver beard, / grinding with your romances / the harvests gone awry.]
>
> Gerardo Diego, third stanza of *Romance del Duero*.

5. Ending Remarks and Further Topics

Today GA is a vast subject, if only because of the number and variety of the publications that have seen the light for over one hundred years [18]. Here we just mention a few additional prominent topics and some references in which they can be pursued.

To a great extend, the value and beauty of the Pauli and Dirac algebras is their capacity to deal with isometries of E_3 and $E_{1,3}$, the latter being precisely the Lorentz transformations [12, §3.1]. This capacity is general for all geometric algebras $\mathcal{G}E_{r,s}$ (see [19, Chaps. 4 and 5] and the references cited there).

We have also seen that the Pauli and Dirac algebras were discovered through matrix representations. This turns out to be valid in general in the following form:

if $\nu = s - r \bmod 8$, the algebra $\mathcal{G}E_{r,s}$ is either isomorphic to a matrix algebra $\mathbf{F}_\nu(m)$ (where $\mathbf{F}_\nu = \mathbb{R}$ for $\nu = 0, 6$; $\mathbf{F}_\nu = \mathbf{C}$ for $\nu = 1, 5$; $\mathbf{F}_\nu = \mathbf{H}$ for $\nu = 2, 4$) or to a double matrix algebra $2\mathbf{F}_\nu(m) = \mathbf{F}_\nu(m) \oplus \mathbf{F}_\nu(m)$ (where $\mathbf{F}_\nu = \mathbb{R}$ for $\nu = 7$; $\mathbf{F}_\nu = \mathbf{H}$ for $\nu = 3$). In each case the value of m is determined by equating 2^{r+s} to the dimension of the corresponding matrix or double-matrix algebra. Examples: for the Wessel algebra, $\nu = 6$, hence $\mathcal{W} \simeq \mathbb{R}(2)$; for the Pauli algebra, $\nu = 5$, so $\mathcal{P} \simeq \mathbf{C}(2)$; for the Dirac algebra we have $\nu = 2$, so $\mathcal{D} \simeq \mathbf{H}(2)$ (as $\mathcal{D}$ has dimension 16 and $\mathbf{H}(m)$ has dimension $4m^2$). These representations are closely related to the representations of the spinorial groups ([19, §6.3] and the references cited there).

One important technique in GA is that $\mathcal{G}_{4,1}$ allows to encode conformal transformations of $\mathcal{E}_3 = E_{3,0}$ with the same spinorial procedures used for dealing with orthogonal transformations. This is generically called *conformal geometric algebra* (CGA), an idea introduced by D. Hestenes in [20]. Mathematical details of this model, and key references, are provided in [12, Chap. 2]. To note that the model generalizes to a CGA of $E_{r,s}$ by means of $E_{r+1,s+1}$.

Appendix A. Background

$\boxed{1}$ **($\mathbf{P}(E)$ and $\mathrm{Gr}_k(E)$).** For any vector space E, the points of the projective space $\mathbf{P}(E)$ are, by definition, the one-dimensional vector subspaces of E, that is $\mathbf{P}(E) = \mathrm{Gr}_1(E)$. So the inclusion $\mathrm{Gr}_1(E)$ in $\mathbf{P}(\mathcal{G}^1E) = \mathbf{P}(E)$ is actually an equality.

Let us see that $\mathrm{Gr}_k(E)$ can be covered by sets each of which is bijective with a linear space of dimension $k(n-k)$. Given $V \in \mathrm{Gr}_k(E)$, fix any subspace $W \subseteq E$ such that $E = V \oplus W$ and let $U_W = \{V' \in \mathrm{Gr}_k(E) : E = V' \oplus W\}$. Let $\Pi_W(E) = \{f \in L(E, W) : f|_W = \mathrm{Id}_W\}$, i.e. the set of projections of E onto W. The map $\Pi_W(E) \to U_W$, $f \mapsto \ker(f)$, is bijective and it is easy to see that $\Pi_W(E) \to \mathcal{L}(V, W)$, $f \mapsto f|V$, is bijective. Thus we have a bijection $\mathcal{L}(V, W) \to U_W$, which shows that U_W is bijective with a linear space of dimension $k(n-k)$. These linear patches turn out to confer a manifold structure on $\mathrm{Gr}_k(E)$ of dimension $k(n-k)$.

The idea of the smooth compatibility of two patches U_W and $U_{W'}$ has a simple expression in the case of $\mathbf{P}(E)$. Indeed, let $\langle v_0 \rangle \in \mathbf{P}(E)$ play the role of V. Then for any decomposition $E = \langle v_0 \rangle \oplus W$ there is a unique linear map $\omega : E \in \mathbb{R}$ such that $\omega(v_0) = 1$, and $W = \ker(\omega)$ and the bijection of U_W to $\mathcal{L}(\langle v_0 \rangle, W) = W$ is given by $\langle v \rangle \mapsto \frac{v}{\omega(v)} - v_0$. Similarly, if we have another decomposition $E = \langle v_0 \rangle \oplus W'$, and ω' is the linear map such that $W' = \ker(\omega')$ and $\omega'(v_0) = 1$, then $\langle v \rangle \mapsto \frac{v}{\omega'(v)} - v_0$ is the bijection of $U_{W'}$ to W'. Since the inverse of the first bijection is $w \mapsto \langle v_0 + w \rangle$, the map $W \to W'$ relating the two bijections is

$$w \mapsto w' = \frac{v_0 + w}{\omega'(v_0 + w)} - v_0 = \frac{w - \omega'(w)v_0}{1 + \omega'(w)},$$

which is smooth function of w provided $1 + \omega'(w) \neq 0$.

A final remark is that $\mathrm{Gr}_k(E)$ is closed in $\mathbf{P}(\mathcal{G}^k E)$, as shown in introductory texts to algebraic geometry (it is defined by the *Plüker relations*) and hence it is a compact manifold. Note that the dimension of $\mathbf{P}(\mathcal{G}^k E)$, $\binom{n}{k} - 1$, is in general much higher than $k(n-k)$. Indeed, it is known that for large m the binomial $\binom{2m}{m}$ grows as $4^m/\sqrt{\pi m}$, whereas $\dim \mathrm{Gr}(2m, m) = m^2$. For example, for $n = 10$ and $k = 5$, the two dimensions are 251 and 25, respectively, while the asymptotic expression $4^5/\sqrt{\pi 5} \approx 258$. The values $n = 4$ and $k = 2$ are the lowest for which the second dimension (4) is less the first dimension (5). $\lhd$

$\boxed{\mathbf{2}}$ (Quadratic spaces)**.** We define a *metric* on E to be a non-degenerate symmetric bilinear map $q : E \times E \to \mathbb{R}$. A *quadratic space*, (E, q) in symbols, is a vector space E endowed with a metric q.

Instead of $q(v, v)$ we will simply write $q(v)$. The function $q(v)$ is the *quadratic form* associated to q. It determines q by the *polarization relation*

$$2q(v, v') = q(v + v') - q(v) - q(v'). \tag{A.1}$$

A vector v is *positive*, *negative*, or *null* if $q(v) > 0$, $q(v) < 0$, or $q(v) = 0$, respectively (null vectors are also said to be *isotropic*). The set of all non-null vectors will be denoted by $E^{\times}$. A vector u is said to be a *unit* vector if $q(u) = \pm 1$.

Two vectors v and v' are said to be *q-orthogonal* if and only if $q(v, v') = 0$. Two sets of vectors F and F' are said to be q-orthogonal if $q(v, v') = 0$ for all $v \in F$ and $v' \in F'$.

If F is a set of vectors, $F^{\perp}$ denotes the set of vectors that are q-orthogonal to F. The bilinearity of q implies that $F^{\perp}$ is a vector subspace of E. The fact that q is non-degenerate is equivalent to say that $E^{\perp} = \{0\}$. Note also that $\dim F^{\perp} = n - \dim F$ for all F.

A special example of quadratic space is $\mathcal{E}_n$, the Euclidean space of dimension n. In this case the metric is $v \cdot v'$ and the quadratic form, $v \cdot v = |v|^2$. The angle $\alpha = \alpha(v, v') \in [-\pi, \pi]$ between to nonzero vectors os defined by the relacion $\cos \alpha = (v \cdot v')/|v||v'|$.

A basis $\mathfrak{e} = e_1, \ldots, e_n$ of a quadratic space is *orthogonal* if $q(e_i, e_j) = 0$ for all $i \neq j$. It is easy to see that orthogonal bases exist. In this case, $q(e_j) \neq 0$ for all j, as otherwise the metric would be degenerate. It is also immediate, by suitably rescaling the e_j, that there are orthogonal bases such that $q(e_j) = \pm 1$. Such bases are said to be *orthonormal*. Let us also recall that the number r of positive $q(e_j)$ (and hence the number s of negative $q(e_j)$) of an orthogonal basis is the same for all orthogonal bases. The pair (r, s) is the *signature* of q and by $E_{r,s}$ we understand a quadratic space of signature (r, s). The existence of orthonormal basis shows that spaces having the same signature are *isometric*, that is, isomorphic as quadratic spaces.

Remark. If q is singular, it is still true that it admits orthogonal bases, but in this case the signature is the triple (r, s, t), where r and s are as above and t is the number of isotropic vectors in a orthogonal basis. By $E_{r,s,t}$ we denote a quadratic space of signature (r, s, t), which is well defined up to isometry. ◁

$\boxed{\mathbf{3}}$ (Dot product formula). Let us prove that for $k > 1$ we have

$$(v_1 \wedge \cdots \wedge v_k) \cdot (v'_1 \wedge \cdots \wedge v'_k) = (-1)^{k/\!\!/2} q(v_1 \wedge \cdots \wedge v_k, v'_1 \wedge \cdots \wedge v'_k).$$

Indeed, by the recursive rule, the left-hand side is equal to

$$\sum_{j=1}^{k} (-1)^{j-1} q(v_k, v'_j)(v_1 \wedge \cdots \wedge v_{k-1}).(v'_1 \wedge \cdots \wedge \widehat{v'_j} \wedge \cdots \wedge v'_k).$$

By induction on k, this is equal to

$$(-1)^{(k-1)/\!\!/2} \sum_{j=1}^{k} (-1)^{j-1} q(v_k, v'_j) q(v_1 \wedge \cdots \wedge v_{k-1}, v'_1 \wedge \cdots \wedge \widehat{v'_j} \wedge \cdots \wedge v'_k),$$

which in turn is equal to

$$(-1)^{(k-1)/\!\!/2}(-1)^{k+1} q(v_1 \wedge \cdots \wedge v_k, v'_1 \wedge \cdots \wedge v'_k),$$

as seen by developing the Gram determinant by the kth row. The claim follows on checking that $(k-1)/\!\!/2 + k + 1 \equiv k/\!\!/2 \bmod 2$. ◁

$\boxed{\mathbf{4}}$ (Involutions of a dot product). We may assume that $x \in \mathcal{G}_q^j$ and $x' \in \mathcal{G}_q^k$ and also, by virtue of the commutation rule ◇3(3), that $j \leq k$, so that the grade $x \cdot x'$ is $k - j$. Moreover, we only need to consider the grade and reverse involutions.

Now we have $\widehat{x \cdot x'} = (-1)^{k-j} x \cdot x'$, while $\widehat{x} \cdot \widehat{x}' = (-1)^j (-1)^k x \cdot x'$, and both expressions coincide because $k - j$ and $k + j$ have the same parity.

Next we have $\widetilde{x \cdot x'} = (-1)^{(k-j)/\!\!/2}$, on one hand and

$$\widetilde{x}' \cdot \widetilde{x} = (-1)^{k/\!\!/2}(-1)^{j/\!\!/2} x' \cdot x = (-1)^{k/\!\!/2}(-1)^{j/\!\!/2}(-1)^{(k-j)j} x \cdot x',$$

on the other. Now it is easy to check that the parity of $(k-j)/\!\!/2$ coincides with the parity of $k/\!\!/2 + j/\!\!/2 + 1$ if j is odd and k even, and with the parity of $k/\!\!/2 + j/\!\!/2$ otherwise, which agrees with the parity of $k/\!\!/2 + j/\!\!/2 + (k-j)j$ in both cases. ◁

$\boxed{\mathbf{5}}$ (Involutions of a geometric product). By bilinearity, it suffices to consider the case of a product $e_I e_J$ as in Artin's formula in the form of Eq. (10): $e_I e_J = (-1)^{t(I,J)} q_{IJ} e_{I+J}$. Set $i = |I|$, $j = |J|$, $c = |IJ|$, so that $|I + J| = i + j - 2c$. In the case of the grade involution, we have

$$\widehat{e_I e_J} = (-1)^{t(I,J)}(-1)^{i+j-2c} q_{IJ} e_{I+J} = (-1)^{i+j} e_I e_J = \widehat{e}_I \widehat{e}_J.$$

In the case of the reverse involution, we have

$$\widetilde{e_I e_J} = (-1)^{t(I,J)}(-1)^{(i+j)-c)/\!\!/2} q_{IJ} e_{I+J} = (-1)^{(i+j)/\!\!/2-c} e_I e_J$$

On the other hand, by the commuting rule,

$$\widetilde{e_J}\widetilde{e_I} = (-1)^{j/\!\!/2}(-1)^{i/\!\!/2} e_J e_I = (-1)^{j/\!\!/2}(-1)^{i/\!\!/2}(-1)^{c+ij} e_I e_J.$$

To show equality of the two expressions, it suffices to see that $(i + j)/\!\!/2 - c$ and $i/\!\!/2 + j/\!\!/2 + c + ij$ have the same parity, which is equivalent to see that $(i+j)/\!\!/2$ and

$i /\!/ 2 + j /\!/ 2 + ij$ have the same parity. This is immediate if i and j are not both odd, for under this assumption ij is even. And if $i = 2r+1$ and $j = 2s+1$, then the first expression is $r+s+1$ and the second is $r+s+(2r+1)(2s+1) \equiv r+s+1 \bmod 2$. $\lhd$

$\boxed{6}$ (Analytics of Kepler orbits)**.** Let us examine Eq. (45), which describes a Kepler orbit, by an analytic geometry approach. Start by noting that it can be written in the form $r + \epsilon x = \lambda$, where x, y are the Cartesian coordinates associtated to the polar coordinates (r, θ) centered at M. For $\epsilon \neq 0$, this equation is equivalent to $\frac{r}{p-x} = \epsilon$, with $p = \lambda/\epsilon$ (p is called *focal parameter*), and agrees with one of the classical definitions of conics.

Now, squaring the relation $r = \lambda - \epsilon x$, and relying on some simiple algebra, we get the following equivalent equations (in the fourth equation we set $a = \frac{\lambda}{1-\epsilon^2}$):

$$\begin{aligned} x^2 + y^2 = \lambda^2 - 2\lambda\epsilon x + \epsilon^2 x^2 \quad &\Leftrightarrow \quad (1-\epsilon^2)x^2 + 2\lambda\epsilon x + y^2 = \lambda^2 \\ &\Leftrightarrow (1-\epsilon^2)(x^2 + \frac{2\lambda\epsilon}{1-\epsilon^2}x) + y^2 = \lambda^2 \\ &\Leftrightarrow (1-\epsilon^2)(x + a\epsilon)^2 + y^2 = \lambda^2 + a^2\epsilon^2(1-\epsilon^2) \\ &\Leftrightarrow (1-\epsilon^2)(x + a\epsilon)^2 + y^2 = \frac{\lambda^2}{1-\epsilon^2} \\ &\Leftrightarrow \frac{(1-\epsilon^2)^2}{\lambda^2}(x + a\epsilon)^2 + \frac{1-\epsilon^2}{\lambda^2}y^2 = 1. \end{aligned} \tag{A.2}$$

If $\epsilon < 1$, this represents the ellipse with semi-axes $\frac{\lambda}{1-\epsilon^2} = a$, $\frac{\lambda}{\sqrt{1-\epsilon^2}} = b$ and center at the point $(-a\epsilon, 0)$.

If $\epsilon = 1$, the equation simplifies to $y^2 + 2\lambda x = \lambda^2$, which is equivalent to $y^2 + 2p(x - \frac{p}{2})$ (as $p = \lambda$ in this case), so the orbit is a parabola with periapsis at $x = p/2$.

The case $\epsilon > 1$ can be leisurely dealt with on noting that the last equation in (A.2) can be written in the following form:

$$\frac{(\epsilon^2-1)^2}{\lambda^2}(x - a\epsilon)^2 - \frac{\epsilon^2-1}{\lambda^2}y^2 = 1,$$

where now $a = \frac{\lambda}{\epsilon^2-1}$, and this leads to the equation $\frac{(x-a\epsilon)^2}{a^2} - \frac{y^2}{b^2} = 1$, with $b = \frac{\lambda}{\sqrt{\epsilon^2-1}}$, which defines a hyperbola with semiaxes a and b and center at $x = a\epsilon$. As an orbit, only the left branch is toured. $\lhd$

$\boxed{7}$ (Affine space)**.** The ingredients of an *affine space* are a set $\mathbb{A}$, whose elements are called *points*, a vector space V, whose elements are called *vectors*, and a map $\mathbb{A} \times V \to \mathbb{A}$, $(P, v) \mapsto P + v$, satisfying the following axioms: (1) For any fixed vector v, the map $T_v : \mathbb{A} \to \mathbb{A}$, $P \mapsto P + v$ is bijective (T_v is called the *translation* of $\mathbb{A}$ associated to v); (2) for any fixed P, the map $V \to \mathbb{A}$, $v \mapsto T_v(P) = P + v$, is bijective; and (3) for any point P and any vectors v, v', $(P+v)+v' = P+(v+v')$, or $T_{v'} \circ T_v = T_{v+v'}$.

From (1) it follows that $P + v = P' + v \iff P = P'$ and that for any point Q there is a unique point $Q - v$ such that $Q = (Q - v) + v$, which means that $T_v \circ T_{-v} = \mathrm{Id}_{\mathbb{A}}$. The condition (2) says that $P + v = P + v' \iff v = v'$ and that for any points P, Q, there is a unique vector $Q - P$ such that $P + (Q - P) = Q$. Finally (3) implies, using (1) and (2), that $P + 0 = P$ for all P, or $T_0 = \mathrm{Id}_{\mathbb{A}}$; that $T_{v'} \circ T_v = T_v \circ T_{v'} = T_{v+v'}$; and that $R - P = (R - Q) + (Q - P)$ for all $P, Q, R \in \mathbb{A}$.

The structure of an affine space is the price to pay in order that in $\mathbb{A}$ there is no distinguished point and that relative to any point P the space $\mathbb{A}$ still looks like V, that is, that the map $\mathbb{A} \to V$, $Q \mapsto Q - P$, is bijective for any point P.

An (affine) *reference* for $\mathbb{A}$ is a pair $(O, \mathfrak{u})$, where $O \in \mathbb{A}$ and $\mathfrak{u} = u_1, \ldots, u_n$ is a basis of V. We call O and $\mathfrak{u}$ the *origin* and the *frame* of the reference. The (affine) coordinates of $X \in \mathbb{A}$ in the reference $(O, \mathfrak{u})$ are the components x^k of the vector $x = X - O$ in the frame $\mathfrak{u}$, that is, $X = O + x^k u_k$. $\lhd$

Appendix B. Notations and Conventions

Notations and conventions. The identity map of any set X will be denoted by I, or I_X if X is not clear from the context, and I_n stands for the identity square matrix of order n.

For a given positive integer n, $1..n$ or $[n]$ denote the range $1, \ldots, n$.

List of frequently used symbols

$(\mathcal{A}, \circ)$	algebra with bilinear product $\circ$
$\mathcal{A}(m)$	algebra of $m \times m$ matrices with entries in $\mathcal{A}$
$A^\dagger$	adjoint of the matrix A
$\mathbf{C}, \mathbb{C}$	algebraic and geometric complex numbers
$\mathbf{C}(m)$	see $\mathcal{A}(m)$
Δ_q	Gram determinants
$\mathcal{D} = \mathcal{G}_{1,3}$	Dirac algebra
(E, q)	quadratic vector space with metric q
$E_{r,s}$	quadratic vector space of signature (r, s)
$\mathcal{E}_n$	Euclidean space of dimension n
$\boldsymbol{\mathcal{E}}$	relative space of the Dirac algebra $\mathcal{D}$
$\mathcal{G}E, \mathcal{G}^k E$	Grassmann algebra and kth exterior power of E
$\mathcal{G}^+ E, \mathcal{G}^- E$	even subalgebra and odd subspace of $\mathcal{G}E$
$\mathcal{G}_q E$	metric Grassmann algebra of (E, q)
$\mathcal{G}_{r,s} = \mathcal{G}E_{r,s}$	geometric algebra of $E_{r,s}$
$\mathrm{Gr}_k E$	Grassmann manifold of k-dim linear subspaces of E
$\mathbf{H}, \mathbb{H}$	algebraic and geometric quaternion fields
$\mathbf{H}(m)$	see $\mathcal{A}(m)$
$\mathcal{L}(V, W)$	vector space of linear maps $V \to W$
$\mathcal{M} = \mathcal{M}_{1,3}$	Minkowski space
$\mathrm{O}_q E$	Orthogonal group of $(E, q) = \{q\text{-isometries of } E\}$

List of frequently used symbols

$\mathrm{SO}_q E$	Proper orthogonal group (isometries with det $= 1$)
$\mathrm{SO}^+_{1,3} E$	Group of proper orthochronous Lorentz transformations
$\mathbf{P}E$	projective space of E
$\mathcal{P} = \mathcal{G}_3$	Pauli algebra
$\mathbb{R}(m)$	see $\mathcal{A}(m)$
$\mathcal{W} = \mathcal{G}_2$	Wessel algebra
$\alpha(v, v')$	angle between $v, v' \in \mathcal{E}_n$
$\mathfrak{i}$	Pauli and Dirac pseudoscalars
$\mathfrak{j}_{\boldsymbol{e}} = e_1 \wedge \cdots \wedge e_n$	pseudoscalar of basis $\boldsymbol{e} = e_1, \ldots, e_n$
$\sigma_1, \sigma_2, \sigma_3$	Pauli matrices
$\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3$	Pauli bivectors in $\mathcal{D}$
$v \sim v'$	$v, v' \in E - \{0\}$ and there is $\lambda \in \mathbb{R}$ such that $v' = \lambda v$
$v \perp v'$	$v, v' \in E$ are orthogonal $\Leftrightarrow$ $v \cdot v' = 0$
$v \times v'$	cross product of $v, v' \in \mathcal{E}_3$
$\langle v_1, \ldots, v_k \rangle$	linear span of $v_1, \ldots, v_k$ with real coefficients
$\langle v_1, \ldots, v_k \rangle_{\mathbb{C}}$	likewise, but with coefficients in $\mathbb{C}$
$x \wedge x' \in \mathcal{G}E$	wedge product of multivectors x and x'
$x \cdot x' \in \mathcal{G}_q E$	inner product of multivectors x and x'
$xx' \in \mathcal{G}_q E$	geometric product of multivectors x and x'
$\hat{x}, \tilde{x}, \bar{x}$	grade, reversal and (Clifford) conjugation of $x \in \mathcal{G}E$

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